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Random sets on manifolds under an infinite–dimensional Log–Gaussian Cox process approach

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Abstract. *A new framework is introduced in this paper for modeling and statistical analysis of point sets in a manifold, that randomly arise through time. Specifically, in the characterization of these sets, the random counting measure is assumed to belong to the family of Cox processes driven by a $L^2(\mathcal{M})$–valued Log–Gaussian intensity, where $\mathcal{M}$ denotes here a compact two–point homogeneous space. The associated family of temporal covariance operators on $L^2(\mathcal{M})$ characterizes the n–order product density under stationarity in time. In particular, the pair correlation functional, the reduced second order moment measure or K function can also be constructed from this covariance operator family. Some functional summary statistics of interest are introduced, analyzing their asymptotic properties in the simulation study undertaken.*

Keywords. *Compact two–point homogeneous space; functional summary statistics; Log–Gaussian Cox processes; $\mathcal{M}$–valued Gaussian random fields.*

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1. Introduction

In point pattern analysis, different parametric, semiparametric and nonparametric models have been adopted in the estimation of deterministic and random intensities characterizing their counting functions. The nearest neighbor functions, empty space functions, and Ripley's and inhomogeneous K functions arise as classical summary functional statistics in point pattern analysis (see, e.g., [2];[4]). Particularly, a growing interest for point processes in the sphere is observed in recent contributions (see [7]; [8], among others). The goal of the present paper is located in a related more general framework involving point processes driven by log–intensities evaluated in the space $L^2(\mathcal{M})$ of square integrable functions on a compact two–point homogeneous space $\mathcal{M}$. Well–known examples of compact two–point homogeneous spaces are the sphere $\mathbb{S}_d \subset \mathbb{R}^{d+1}$, projective spaces over different algebras (see Section 2 in [5], for more details). Any one of these spaces defines a manifold $\mathbb{M}_d$, where d denotes its topological dimension. We restrict our attention here to random counting functions driven by a temporal Log–Gaussian infinite–dimensional process with values in $L^2(\mathcal{M})$ (see [3] in the Euclidean setting). The n–order joint product density is characterized, in the weak–sense, in terms of test functions lying in the n–fold tensor product $[L^2(\mathcal{M})]^{\otimes n}$ of the Hilbert space $L^2(\mathcal{M})$. Particularly, we adopt the setting of d–dimensional manifolds $\mathbb{M}_d$ embedded in $\mathbb{R}^{d+1}$. The isometric identification of $(\mathbb{S}_d, d_{\mathbb{S}_d})$ with $(\mathbb{M}_d, d_{\mathbb{M}_d})$ can then be considered via the identity $d_{\mathbb{S}_d}(\mathbf{x}_1,\mathbf{x}_2) = \arccos\left(\mathbf{x}_1^T\mathbf{x}_2\right)$, for $\mathbf{x}_1,\mathbf{x}_2 \in \mathbb{S}_d$. In the case of Riemannian manifolds, one can replace the inner product in $\mathbb{R}^{d+1}$ by the family of inner products $\mathcal{G}_{\mathbf{x}}(\cdot,\cdot)$ defined on the tangent space, smoothly varying over $\mathbf{x} \in \mathbb{M}_d$.

2. Log–Gaussian intensity with values in $L^2(\mathbb{M}_d)$

Let $\{X_t(\cdot),\ t \in \mathcal{T} \subseteq \mathbb{R}\}$ be an infinite–dimensional random process such that, for each $t \in \mathcal{T} \subseteq \mathbb{R}$, almost surely $\log(X_t) \in L^2(\mathbb{M}_d)$, $E[\log(X_t)] \underset{L^2(\mathbb{M}_d)}{=} 0$, with $\log(X_t)$ having characteristic functional

$$f_{\log(X_t)}(h) = \int_{L^2(\mathbb{M}_d)} \exp\left(i\langle h, \log(x_t)\rangle_{L^2(\mathbb{M}_d)}\right)\mu_{\log(X_t)}(d\log(x_t)) = \exp\left(-\frac{\langle \mathcal{R}_0(h), h\rangle_{L^2(\mathbb{M}_d)}}{2}\right), \quad h \in L^2(\mathbb{M}_d), \tag{1}$$

where $\mathcal{R}_0 = E[\log(X_t) \otimes \log(X_t)] \in \mathcal{L}^1(L^2(\mathbb{M}_d))$ denotes the covariance operator of $\log(X_t)$, and $\mathcal{L}^1(L^2(\mathbb{M}_d))$ denotes the space of trace or nuclear operators on $L^2(\mathbb{M}_d)$. Here, $\mu_{\log(X_t)}$ is the induced Gaussian measure by $\log(X_t)$ on $\left(L^2\left(\mathbb{M}_d\right), \mathcal{B}(L^2(\mathbb{M}_d))\right)$, with $\mathcal{B}(L^2(\mathbb{M}_d))$ being the σ–algebra generated by all cylindrical subsets of $L^2(\mathbb{M}_d)$. In the subsequent development, we will also assume that, for any $t, s \in \mathcal{T}$,

$$E[\log(X_t)(\mathbf{z})\log(X_s)(\mathbf{y})] = r_{t-s}(d_{\mathbb{M}_d}(\mathbf{z},\mathbf{y})) = \widetilde{r}(d_{\mathbb{M}_d}(\mathbf{z},\mathbf{y}), t-s), \quad \mathbf{z},\mathbf{y} \in \mathbb{M}_d, \tag{2}$$

i.e., stationarity in time and isotropy over $\mathbb{M}_d$ in the weak sense are assumed. Note that the covariance operator $\mathcal{R}_{t-s}$ with kernel $r_{t-s}(\cdot,\cdot)$ is a nuclear operator, and kernel $\widetilde{r}(d_{\mathbb{M}_d}(\mathbf{z},\mathbf{y}), t-s) = r_{t-s}(d_{\mathbb{M}_d}(\mathbf{z},\mathbf{y}))$ is assumed to be continuous.

For the special case $r_{t-s}(\cdot,\cdot) = r_{s-t}(\cdot,\cdot)$, the following series expansion is obtained from Theorems 4 and 5 in [5]:

$$\log(X_t)(\mathbf{z}) = \sum_{n=0}^{\infty} V_n(t) P_n^{(\alpha,\beta)}\left(\cos\left(d_{\mathbb{M}_d}(\mathbf{z},\mathbf{U})\right)\right),\ \mathbf{z} \in \mathbb{M}_d,\ t \in \mathbb{R}, \tag{3}$$

where $\{V_n(t),\ n \in \mathbb{N}_0\}$ is a sequence of independent stationary random processes on $\mathcal{T} \subseteq \mathbb{R}$, satisfying $E[V_n(t)] = 0$ and $E[V_n(t_1)V_n(t_2)] = a_n^2 b_n(t_1 - t_2)$, $n \in \mathbb{N}_0$. The random variable $\mathbf{U}$ is uniformly distributed on $\mathbb{M}_d$, and is independent of $\{V_n(t),\ n \in \mathbb{N}_0\}$, and $\sum_{n=0}^{\infty} b_n(0) P_n^{(\alpha,\beta)}(1)$ converges. Also, $\operatorname{cov}\left(V_n(t)P_n^{(\alpha,\beta)}\left(\cos\left(d_{\mathbb{M}_d}(\mathbf{z},\mathbf{U})\right)\right), V_m(t)P_m^{(\alpha,\beta)}\left(\cos\left(d_{\mathbb{M}_d}(\mathbf{z},\mathbf{U})\right)\right)\right) = 0$, for $m \neq n$, and $\mathbf{z} \in \mathbb{M}_d$, and $t \in \mathcal{T}$.

3. Cox processes family

Let now consider the measure $d\nu(\mathbf{z})$ induced on the homogeneous space $\mathbb{M}_d = G/K$, by the probabilistic invariant measure on G, with G being the connected component of the group of isometries of $\mathbb{M}_d$, and K be the stationary subgroup of a fixed point $\mathbf{o} \in \mathbb{M}_d$. As before, $H = L^2\left(\mathbb{M}_d, d\nu(\mathbf{x})\right)$. Consider $\mathbf{Y} = \{\mathbf{Y}_t,\ t \in \mathcal{T} \subseteq \mathbb{R}\}$ to be a family of finite random subsets of $\mathbb{M}_d$, arising at the random times in the interval family $\{[0,t],\ t \in \mathcal{T}\}$.

Let $\{N_t(\cdot),\ t\in\mathcal{T}\}$ be the family of counting measures associated with $\mathbf{Y}=\{\mathbf{Y}_t,\ t\in\mathcal{T}\subseteq\mathbb{R}\}$. For every $t\in\mathcal{T}$, and any Borel set $A\subseteq\mathbb{M}_d$, $N_t(A)$ denotes the number of points in $\mathbf{Y}_t$ falling in a region $A\subseteq\mathbb{M}_d$, at the random times specified by $\mathbf{Y}_t$ in the interval $[0,t]$. The state space for $\mathbf{Y}_t$ is then the set of all possible combinations of finite subsets of $\mathbb{M}_d$ with finite time sets of $[0,t]$, equipped with the σ–algebra $\mathcal{F}$ generated by the events $\{N_t(A)=n\}$, indicating that n points in $\mathbf{Y}_t$ falling in a region $A\subseteq\mathbb{M}_d$, at some specific times in $[0,t]$, for any Borel set $A\subseteq\mathbb{M}_d$, interval $[0,t]$, and integer $n\in\mathbb{N}$. Assume that, for each $t\in\mathcal{T}$, given a realization $\{x_t(\mathbf{z}),\ \mathbf{z}\in\mathbb{M}_d\}$, of X_t, satisfying (1)–(3), the conditional distribution of $N_t(A)/\{x_t(\mathbf{z}),\ \mathbf{z}\in\mathbb{M}_d\}$ is a Poisson distribution with parameter $\lambda_t=\int_0^t\int_A x_s(\mathbf{z})d\nu(\mathbf{z})ds$. The n–order product density $\rho^{(n)}_{t_1,\dots,t_n}(\mathbf{z}_1,\dots,\mathbf{z}_n)$ is such that $\rho^{(n)}_{t_1,\dots,t_n}(\mathbf{z}_1,\dots,\mathbf{z}_n)d\nu^{(n)}(\mathbf{z}_1,\dots,\mathbf{z}_n)dt_1,\dots,dt_n$ indicates the probability that $\mathbf{Y}_t$ has a point in each of n infinitesimally small regions on $\mathbb{M}_d$ around $\mathbf{z}_1,\dots,\mathbf{z}_n$, of surface measure $d\nu(\mathbf{z}_1)\cdots d\nu(\mathbf{z}_n)$, over the infinitesimal time intervals around $t_1,\dots,t_n$, of length $dt_1,\dots,dt_n$. From equation (3), for any $t_1,\dots,t_n\in\mathbb{R}$, one can compute $\rho^{(n)}_{t_1,\dots,t_n}$ as follows:

$$\rho^{(n)}_{t_1,\dots,t_n}(\mathbf{z}_1,\dots,\mathbf{z}_n)=E\left[\prod_{i=1}^{n}\exp\left(X_{t_i}(\mathbf{z}_i)\right)\right]=E\left[\exp\left(\sum_{i=1}^{n}X_{t_i}(\mathbf{z}_i)\right)\right]$$
$$=[\rho]^n\exp\left(\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=0}^{\infty}b_k(t_i-t_j)P_k^{(\alpha,\beta)}\left(\cos\left(d_{\mathbb{M}_d}(\mathbf{z}_i,\mathbf{z}_j)\right)\right)\right),\ \forall z_i\in\mathbb{M}_d, i=1,\dots,n.$$

In particular, for any $t\in\mathcal{T}$, and, for any $t_1,t_2\in\mathcal{T}$, the intensity function $\rho_t=\rho_0=\rho^{(1)}(t)$, and the pair correlation function $g_{t_1-t_2}\left(\cos\left(d_{\mathbb{M}_d}(\mathbf{z}_1,\mathbf{z}_2)\right)\right)$, $\mathbf{z}_1,\mathbf{z}_2\in\mathbb{M}_d$, respectively admit the following expressions:

$$\rho=\rho_0(\mathbf{z}) = \exp\left(\frac{1}{2}\sum_{n=0}^{\infty}b_n(0)P_n^{(\alpha,\beta)}(1)\right),\quad \forall\mathbf{z}\in\mathbb{M}_d,$$
$$g_{t_1-t_2}\left(\cos\left(d_{\mathbb{M}_d}(\mathbf{z}_1,\mathbf{z}_2)\right)\right) = \frac{\rho^{(2)}_{t_1-t_2}\left(\cos\left(d_{\mathbb{M}_d}(\mathbf{z}_1,\mathbf{z}_2)\right)\right)}{\rho^2}=\exp\left(\sum_{n=0}^{\infty}b_n(t_1-t_2)P_n^{(\alpha,\beta)}\left(\cos\left(d_{\mathbb{M}_d}(\mathbf{z}_1,\mathbf{z}_2)\right)\right)\right).$$

4. Functional summary statistics and simulation

A simulation study is undertaken for an asymptotic analysis of the usual functional summary statistics. We will consider the empirical counterparts of the nearest neighborhood function, given by $\widehat{G}_t(s)=\frac{1}{N_t(\mathbb{M}_d)}\sum_{(u,\mathbf{y})\in\mathbf{Y}_t}1_{\{\inf_{(v,\mathbf{z})\in\mathbf{Y}_t\setminus\{(u,\mathbf{y})\}}d_{\mathbb{M}_d}(\mathbf{z},\mathbf{y})|u-v|\leq s\}}$, and of the empty space function defined as $\widehat{F}_t(s)=\frac{1}{m(t)}\sum_{(u,\mathbf{x})\in Q_t}1_{\{\inf_{(v,\mathbf{z})\in Y_t}d_{\mathbb{M}_d}(\mathbf{z},\mathbf{x})|u-v|\leq s\}}$, for $s\in[0,\pi]$, and $t\in\mathcal{T}$, with Q_t being a finite grid on $\mathbb{M}_d\times[0,t]$, of $m(t)>0$ points. For the K_t function, its empirical version $\widehat{K}_t(s)=\frac{1}{\nu(\mathbb{M}_d)\widehat{\rho}^2}\sum_{(u,\mathbf{x})\neq(v,\mathbf{y})\in\mathbf{Y}_t}1_{\{d_{\mathbb{M}_d}(\mathbf{x},\mathbf{y})|u-v|\leq s\}}$ will also be asymptotically analyzed, with $\widehat{\rho}$ being an unbiased estimator of ρ, and assuming $\mathbf{Y}_t$ is fully observed on $\mathbb{M}_d$, provided $N_t(\mathbb{M}_d)>0$, for any $t\in\mathcal{T}$. This asymptotic analysis is achieved from simulations, under the particular model $r_{t-s}(\langle\mathbf{x},\mathbf{y}\rangle)=\sum_{l=0}^{\infty}C_l(t-s)\frac{2l+1}{4\pi}P_l(\langle\mathbf{x},\mathbf{y}\rangle)$, $t,s\in\mathcal{T}$, introduced in [1];[6] on the sphere $\mathbb{S}_2$, in terms of Legendre polynomials. Figure 1 displays the short–memory case at the first four rows, for $\tau=t-s$,

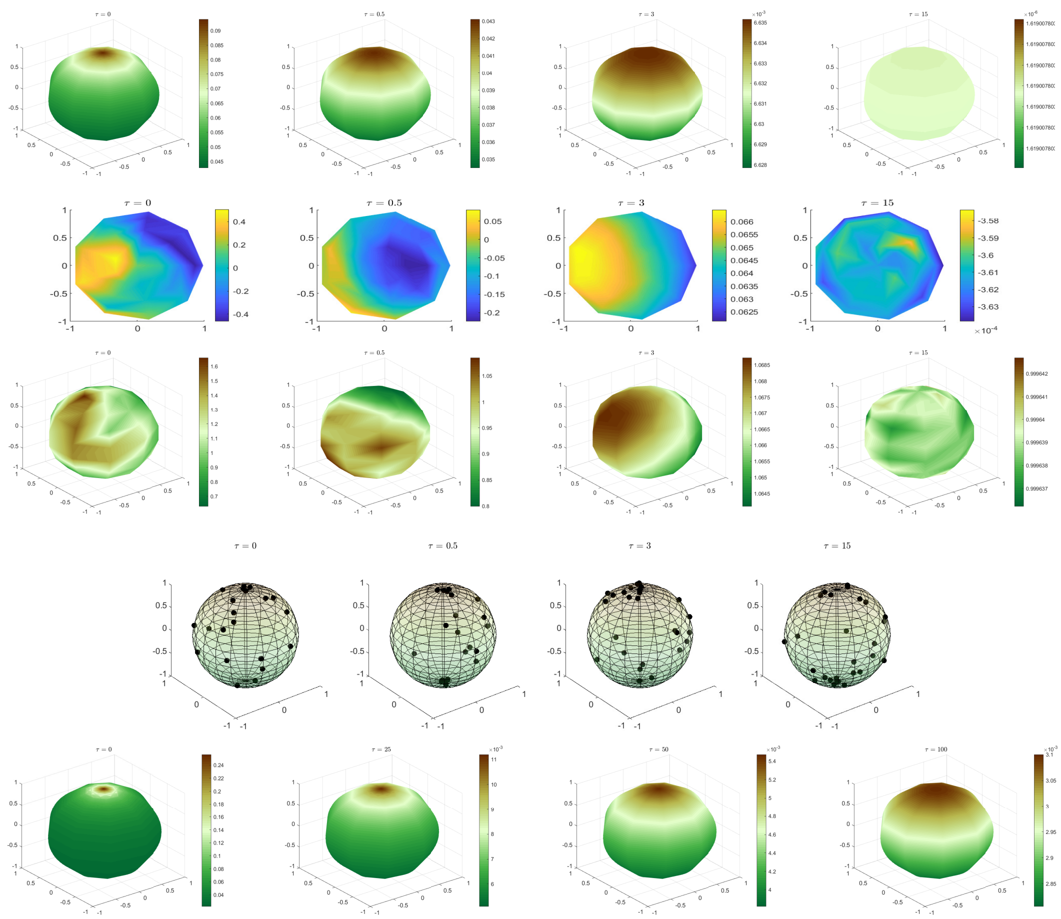

Figure 1: At the top, *short–memory* covariance operator model is displayed for different temporal lags, simulated realizations of the log-intensity and the intensity random processes are given at the second and third rows, respectively. At the fourth row, point patterns generated from the corresponding log–Gaussian Cox process are drawn for $\tau = t - s = 0, 0.5, 3, 15$. Finally, *long–memory* covariance operator model for temporal lags $\tau = t - s = 0, 25, 50, 100$ is displayed at the bottom row.

$C_l(\tau) = \frac{\phi_l^{|\tau|} C_{l;Z}}{1-\phi_l^2}$ with $\phi_l = G(l+1)^{-\alpha_\phi}$, $\alpha_\phi = 3$, $C_{l;Z} = G_Z(1+l)^{-\alpha_Z}$, $\alpha_Z = 3$, $G = G_Z = 0.5$. At the bottom of Figure 1, the long–memory case is showed, for $C_l(\tau) = G_l(\tau) g_l(\tau)$, $G_l(\tau) = G(l+1)^{-2-k_\tau \tau}$, $g_l(\tau) = (1+|l|)^{-\beta_l}$, $\beta_l = \frac{k_\beta (l+1)}{\sqrt{(l+1)^2+1}}$, $k_\beta = 0.8$, $G = 0.5$ and $k_\tau = 0.03$. In both cases Legendre series is truncated at $L_{max} = 30$ considering a spherical regular grid.

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Author Index