Aggregation of indistinguishability operators

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Abstract

The paper studies the aggregation of pairs of T-indistinguishability operators. More concretely we address the question whether $A(E, E')$ is a T-indistinguishability operator if E, E' are T-indistinguishability operators. The answer depends on the aggregation function, the t-norm T, and the chosen T-indistinguishability operators.

It is well-known that an aggregation function preserves T-transitive relations if and only if it dominates the t-norm T. We show the important role of the minimum t-norm T_M in this preservation problem. In particular we develop weaker forms of domination that are used to provide characterizations of T_M-indistinguishability preservation under aggregation. We also prove that the existence of a single strictly monotone aggregation that satisfies the indistinguishability operator preservation property guarantees all aggregations to have the same preservation property.

Keywords: Triangular norm, Minimum t-norm, Aggregation function, Strictly monotone aggregation function, Indistinguishability operator, Domination.

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1. Introduction

Aggregation is a fundamental process in many different disciplines where data fusion is necessary (see for instance [13], [34]). Aggregation functions are

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very useful in this approach when the input data present vagueness or uncertainty and a method to combine it is required.

Recently Mayor et al (see [19]) and Martin et al (see [17]) have studied the aggregation of asymmetric distances in the context of Computer Science, providing a general description of how to combine a collection of asymmetric distances in order to obtain a single one as output. To this end, they introduced the notion of asymmetric distance aggregation function that generalizes the well-known one for distance spaces given by Borsik and Doboš (see [6], [25]). Extensive information about aggregation operators can be found in [7, 8]

Aggregation of fuzzy relations also provides an efficient tool in many fields of Applied sciences (see [23]). Special attention is paid to the classes of transitive fuzzy binary relations. One of the first papers in this direction is [22], where the author gives a geometric characterization of t-norms for which the arithmetic mean is a well-defined aggregation procedure for preserving transitive fuzzy binary relations. Since then, many other authors have obtained important results about the preservation of transitivity of fuzzy relations by aggregation processes (see [14], [15], [18], [28])

Indistinguishability operators are transitive relations essentially interpreted as measures of similarity (in contrast to dissimilarity modeled by pseudo-metrics). The problem of how to combine a collection of indistinguishability operators into a single one has been addressed in [26], and more recently it has been studied from different perspectives (see [1], [9], [30])

The aggregation functions that preserve T-transitivity of fuzzy relations were determined by Saminger et al. in [28, Theorem 3.1]. They stated that an aggregation operator A preserves T-transitivity of fuzzy relations if and only if the aggregation A dominates the t-norm T. This result generalizes another well-known one which was obtained in [10] for triangular norms. The idea of Saminger et al. was to develop other types of aggregation processes preserving T-transitivity when the introduction of different weights (degrees of importance) were required (see [5]). Observe that since reflexivity and symmetry are trivially preserved by aggregation functions, their result can be reformulated as a characterization in terms of domination of the aggregation functions which preserve indistinguishability operators.

The notion of domination was introduced in the context of probabilistic metric spaces (see [32], [33]) when constructing the Cartesian products of such spaces. In the framework of t-norms, domination relation is needed for building fuzzy equivalence relations and fuzzy orderings (see [3],[4],[10], [11]), but it was in [28], where the concept of domination was extended to the framework of aggregation operators. From that point onward, many authors have studied domination and particularly the domination of important t-norms (see for instance [12, 21, 24, 29, 31]).

The main motivation of this paper is to answer the following question. Can the condition of domination be removed or at least weakened in the aggregation of some special classes of T-indistinguishability operators? We find that considering T_M-indistinguishability operators is a sensible option, taking into account one of the first new results in this paper, in which we prove that a T-indistinguishability operator is always preserved under self-aggregation if and only if $T = T_M$ whenever the cardinal of the set is greater or equal to 3.

Moreover it is enough to consider the case $T = T_M$ to show how strong the condition of domination is and the convenience of relaxing it whenever this is possible. This is illustrated in Proposition 3.11, which states that an aggregation A dominates T_M if and only if $A(x, y) = \min(A(x, 1), A(1, y))$ for all x, y in $[0, 1]$. One of our main results shows that the domination condition can be weakened, and even removed under a mild relation between the T_M-indistinguishability operators that we write $E \approx E'$. More concretely we show in Theorem 3.17 that the following statements are equivalent:

(1) $E \approx E'$.

(2) For all aggregation function A, $A(E, E')$ is a T_M-indistinguishability operator.

(3) For all aggregation function A, A dominates T_M with respect to (E, E').

We have focused our work on binary aggregations but we note that Saminger also shows that proving T-transitivity preservation for binary associative aggregations implies the result for n-ary aggregations in some particular cases (see [28, Proposition 2.8]).

We present a necessary and sufficient condition on T_M-indistinguishability operators E and E' to ensure that $A(E, E')$ preserve the T_M-indistinguishability structure for any aggregation function A. We prove that given a pair of indistinguishability operators (E, E'), finding a strictly monotone aggregation function B such that $B(E, E')$ is a T_M-indistinguishability operator is enough to ensure that for any other aggregation function A, $A(E, E')$ is also a T_M-indistinguishability operator.

The manuscript is organized as follows: In section 2 it begins introducing some necessary preliminaries. Section 3 is devoted to the results of aggregation functions that preserve T_M-indistinguishability operators. In Section 4, we explain the role of strictly monotone aggregations in the preservation of the T_M-indistinguishability structure.

2. Preliminaries

In this section some necessary concepts will be presented. Extensive information about t-norms can be found in [16]. They were first introduced by Menger in [20].

Definition 2.1. *A triangular norm, t-norm for short, is a binary operation* $T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ *such that for all* $a, b, c \in [0, 1]$ *the following axioms are satisfied:*

- *Commutativity:* $T(a, b) = T(b, a)$.
- *Associativity:* $T(a, T(b, c)) = T(T(a, b), c)$.
- *Monotonicity:* $T(a, b) \leq T(a, c)$ *whenever* $b \leq c$.
- *Neutral element:* $T(a, 1) = a$.

Example 2.2. *The following binary operations are the four basic examples of t-norms (see* [16]*).*

- *Minimum t-norm* T_M*, i.e.,* $T_M(a,b) = \min(a,b)$ *for all* $a,b \in [0,1]$.
- *Product t-norm* T_P*, i.e.,* $T_P(a,b) = a \cdot b$ *for all* $a,b \in [0,1]$.
- *Łukasiewicz t-norm* T_L*, i.e.,* $T_L(a,b) = \max(a+b-1,0)$ *for all* $a,b \in [0,1]$.
- *Drastic t-norm* T_D*, i.e.,* $T_D(a,b) = \begin{cases} 0 & \text{if} \quad (a,b) \in [0,1)^2 \\ \min(a,b) & \text{otherwise.} \end{cases}$

Definition 2.3 ([27])**.** *Let* X *be a set,* T *a t-norm and* $E : X \times X \longrightarrow [0,1]$ *a fuzzy relation on* X*. The fuzzy relation* E *is called a* T*-indistinguishability operator if it satisfies:*

($E1$) *Reflexivity:* $E(x,x) = 1$*, for all* $x \in X$.

($E2$) *Symmetry:* $E(x,y) = E(y,x)$*, for all* $x,y \in X$.

($E3$) T*-transitivity:* $E(x,z) \geq T(E(x,y), E(y,z))$*, for all* $x,y,z \in X$.

Given a set X and a fuzzy relation $E : X \times X \longrightarrow [0,1]$, for each $\alpha \in [0,1]$, the level set E_α is defined by

$$E_\alpha = \{(x,y) \in X \times X \mid E(x,y) \geq \alpha\}.$$

Zadeh studied and characterized similarities in terms of their level sets in [35, Proposition 3], that is, T-indistinguishability operators considering $T = T_M$.

Proposition 2.4. *Let* E *be a fuzzy relation on a set* X*. The fuzzy relation* E *is a* T_M*-indistinguishability operator on* X *if and only if* E_α *is an equivalence relation, for each* $0 \leq \alpha \leq 1$.

Definition 2.5 ([7])**.** *A function* $A : [0,1] \times [0,1] \longrightarrow [0,1]$ *is called an aggregation function if for all* $r_1, r_2, s_1, s_2 \in [0,1]$ *the following properties hold:*

($A1$) *Boundary conditions:* $A(0,0) = 0$ *and* $A(1,1) = 1$.

($A2$) *Monotonicity:* $A(r_1,s_1) \leq A(r_2,s_2)$ *whenever* $r_1 \leq r_2$ *and* $s_1 \leq s_2$.

Definition 2.6 ([2]). *An aggregation function A is strictly monotone if for all $r_1 \leq r_2$ and $s_1 \leq s_2$ with $(r_1, s_1) \neq (r_2, s_2)$, we have that $A(r_1, s_1) < A(r_2, s_2)$.*

Let us take an aggregation function A and two fuzzy relations E and E' on a set X. The fuzzy relation $A(E, E') : X \times X \longrightarrow [0, 1]$ is defined pointwise as

$$A(E, E')(x, y) = A(E(x, y), E'(x, y))$$

for all $x, y \in X$. We say that $A(E, E')$ is the aggregation of E and E' by A.

3. Aggregation of T_M-indistinguishability operators

We start recalling the definition of domination given by Saminger et al.

Definition 3.1 ([28]). *Let A and B be two aggregation functions. We say that A dominates B if for all $x, y, z, u \in [0, 1]$*

$$A(B(x, z), B(y, u)) \geq B(A(x, y), A(z, u)).$$

We introduce two new weaker generalizations of the concept of domination that we will use in the sequel.

Definition 3.2. *Let A and B be aggregation functions.*

- *A weakly dominates B if for all $x, y, z \in [0, 1]$*

  $$A(B(x, y), B(y, z)) \geq B(A(x, y), A(y, z)).$$

- *Let E, and E' be indistinguishabilities in X, A dominates B with respect to (E, E') if for all $x, y, z \in X$*

  $$A\big(B(E(x, y), E(y, z)),\ B(E'(x, y), E'(y, z))\big) \geq B(A(E(x, y), E'(x, y)),\ A(E(y, z), E'(y, z))).$$

Note that from the definitions, the following implications hold.

A dominates B $\implies$ A weakly dominates B.

A dominates B $\implies$ for all pair of indistinguishability operators (E, E') defined on a non empty set X, A dominates B with respect to (E, E').

For small sets with two elements or less $A(E, E')$ turns out to always be an indistinguishability operator for any aggregation, t-norm and pair of indistinguishability operators. More precisely:

Proposition 3.3. *Let X be a set with cardinal less or equal than two, A an aggregation function and T a t-norm. If E and E' are two T-indistinguishability operators, then $A(E, E')$ is a T-indistinguishability operator.*

Proof. Straightforward. □

The following example shows that the previous proposition is not true if $|X| \geq 3$, even for self-aggregations.

Example 3.4. *We provide an example of a self-aggregation of a T_D-indistinguishability operator that it is not a T_D-indistinguishability operator.*

Let X be a set with cardinal greater or equal to 3, and $a, b \in [0,1]$ satisfying $b < a < 1$. Fixed three elements $x, y, z \in X$, we define the fuzzy relation $E : X \times X \longrightarrow [0,1]$ as follows

$$E(r,s) = \begin{cases} 1 & \text{if} \quad r = s, \\ b & \text{if} \quad (r,s) \in \{(x,z),(z,x)\}, \\ a & \quad\;\;\, \text{otherwise.} \end{cases}$$

It is easy to check that E is an indistinguishability operator with respect to the drastic t-norm T_D. However, if we consider any aggregation function A satisfying $A(a,a) = 1$ and $A(b,b) < 1$, then $A(E,E)$ is not T_D-transitive because

$$\begin{aligned} A(E,E)(x,z) = b \quad & < 1 \\ & = T_D(1,1) \\ & = T_D(A(a,a), A(a,a)) \\ & = T_D(A(E,E)(x,y), A(E,E)(y,z)). \end{aligned}$$

In the following theorem Saminger et al. showcased the important role of domination in T-transitivity preservation under aggregation. This result was later adapted by Drewniak et al. for the definition of aggregation of fuzzy relations used in the present manuscript.

Theorem 3.5. ([12, Corollary 1], [28, Theorem 3.1]) *Let X be a set of cardinal greater than or equal to 3 and let T be an arbitrary t-norm. An aggregation operator A preserves the T-transitivity of fuzzy relations on X if and only if A dominates T.*

Since reflexivity and symmetry are trivially preserved by aggregation functions, theorem 3.5 can be restated as which conditions on A ensure that $A(E, E')$ is T-transitive for all E, E'.

Corollary 3.6. *Let X be a set of cardinal greater than or equal to 3 and T be an arbitrary t-norm, $A(E, E')$ is a T-indistinguishability for every pair of T-indistinguishabilities E and E' if and only if A dominates T.*

The minimum t-norm T_M plays an important role in the Aggregation of T-indistinguishability operators. We proceed to show that the minimum is the only t-norm T that ensures generic preservation of T_M-indistinguishability operators under self-aggregation. This result underlines the aforementioned special role that T_M will play in subsequent results.

The following results show that the minimum t-norm is the only t-norm T satisfying that all self-aggregations of T-indistinguishability operators are T-indistinguishability operators.

Proposition 3.7. *Let X be a set with cardinal greater or equal to 3. For each T_M-indistinguishability operator E and each aggregation function A, the fuzzy relation $A(E, E)$ is a T_M-indistinguishability operator.*

Proof. It is enough to prove the T_M-transitivity. Take $r = E(x, y)$, $s = E(y, z)$ and $t = E(x, z)$. Without loss of generality consider $r \leq s$. We have that

$$T_M(A(E, E)(x, y), A(E, E)(y, z)) = T_M(A(r, r), A(s, s)) = A(r, r)$$

and $$A(E, E)(x, z) = A(t, t) \geq A(T_M(r, s), T_M(r, s)) = A(r, r).$$

Hence $T_M(A(E, E)(x, y), A(E, E)(y, z)) \leq A(E, E)(x, z)$. □

The following technical lemma allows to shorten the proof of theorem 3.9.

Lemma 3.8. *Let T be a t-norm such that $T \notin \{T_D, T_M\}$. If for all $a, b \in (0,1)$, $T(a,b) \in \{0, a \wedge b\}$, then there is $k \in (0,1)$ such that for all $l > k$, $T(l,l) = l$ and for all $l < k$, $T(l,l) = 0$.*

Proof. Consider the following sets:

$$N_D = \{c \in (0,1) \mid T(c,c) = 0\} \quad \text{and} \quad N_M = \{c \in (0,1) \mid T(c,c) = c\}.$$

Let us check that they are non-empty sets. If $N_M = \emptyset$, then $T(c,c) = 0$ for all $c \in (0,1)$. By monotonicity, $T(a,b) \leq T(a \vee b, a \vee b) = 0$ for all $a, b < 1$. This implies that $T = T_D$, i.e., a contradiction, hence $N_M \neq \emptyset$. If $N_D = \emptyset$, then $T(c,c) = c$ for all $c \in (0,1)$. By monotonicity, $a \wedge b = T(a \wedge b, a \wedge b) \leq T(a,b) \leq a \wedge b$ for all $a, b < 1$. This implies that $T = T_M$, i.e., a contradiction, hence $N_D \neq \emptyset$.

Obviously, $N_D \cap N_M = \emptyset$ and $N_D \cup N_M = (0,1)$. Moreover, by monotonicity of T, the elements of N_D are lower than the ones of N_M. Take $k = \sup N_D$, then for all $a \in N_D$ and $b \in N_M$, we have that

$$a \leq k \leq b.$$

Therefore, for all $l < k$, $T(l,l) = 0$ and for all $l > k$, $T(l,l) = l$. $\square$

Theorem 3.9. *Let X be a set with cardinal greater or equal to 3 and T a t-norm. The following assertions are equivalent:*

(1) $T = T_M$.

(2) *For each T-indistinguishability operator E and each aggregation function A, the fuzzy relation $A(E,E)$ is a T-indistinguishability operator.*

Proof. (1) $\implies$ (2) is Proposition 3.7.

Conversely, we will prove that if (1) is false, then (2) is also false. If $T = T_D$, there is a T_D-indistinguishability operator E and an aggregation function A such that $A(E,E)$ is not a T_D-indistinguishability operator (see Example 3.4). Otherwise, let T be a t-norm satisfying $T \notin \{T_D, T_M\}$. We consider the following two cases depending on the t-norm. For each case we construct a T-indistinguishability operator E and an aggregation function A, such that $A(E,E)$ is not a T-indistinguishability operator.

i) There are $a, b \in (0,1)$ satisfying $T(a,b) \notin \{0, a \wedge b\}$.

ii) For all $a, b \in (0,1)$, $T(a,b) \in \{0, a \wedge b\}$.

Case *i)*. Suppose that $a \leq b$.

We will use that

$$0 < T(a,b) < a \leq b.$$

In order to construct E, take three different elements $x_0, y_0, z_0 \in X$ and define the fuzzy relation $E : X \times X \longrightarrow [0,1]$ as follows:

- $E(x,x) = 1$ for all $x \in X$,
- $E(x_0, y_0) = E(y_0, x_0) = a$,
- $E(z_0, y_0) = E(y_0, z_0) = b$ and
- $E(x,y) = T(a,b)$ otherwise.

It is clear that E is reflexive and symmetric. Let us check that E is T-transitive. Take $x, y, z \in X$. If two of them are equal, then $(E3)$ is satisfied.

If they are different elements, then $E(x,y), E(y,z)$ and $E(x,z)$ belong to $\{a, b, T(a,b)\}$. We consider two disjoint cases: $E(x,z) = b$ and $E(x,z) \in \{a, T(a,b)\}$.

1) Suppose $E(x,z) = b$.

By monotonicity of the t-norm,

$$T(E(x,y), E(y,z)) \leq T(b,b) \leq b = E(x,z).$$

2) Suppose $E(x,z) \in \{a, T(a,b)\}$.

Suppose for a contradiction that

$$T(E(x,y), E(y,z)) > E(x,z) \geq T(a,b).$$

This together with the definition of E implies that $E(x,y) = b = E(y,z)$. Therefore, the pairs (x,y) and (y,z) belong to the set $\{(y_0, z_0), (z_0, y_0)\}$. This is a contradiction because x, y and z are three different elements.

We conclude that E is a T-indistinguishability operator.

Consider now the aggregation function $A : [0,1]^2 \longrightarrow [0,1]$ given by

$$A(r,s) = \begin{cases} 0 & \text{if} \quad r \leq T(a,b) \text{ and } s \leq T(a,b) \\ 1 & \qquad \text{otherwise.} \end{cases}$$

However, since

$$A(E,E)(x_0,z_0) = A(E(x_0,z_0),E(x_0,z_0)) = A(T(a,b),T(a,b)) = 0$$

and

$$T(A(E,E)(x_0,y_0),A(E,E)(y_0,z_0)) = T(A(a,a),A(b,b)) = T(1,1) = 1,$$

we conclude that $A(E,E)$ is not a T-indistinguishability operator.

Case ii). Since $T(a,b) \in \{0, a \wedge b\}$ for all $a,b \in (0,1)$, by Lemma 3.8 there is $k \in (0,1)$ satisfying that $T(l,l) = l$ for all $l > k$ and $T(l,l) = 0$ for all $l < k$.

Let us take three different elements $x_0, y_0, z_0 \in X$ and consider the fuzzy relation $E : X^2 \longrightarrow [0,1]$ defined by

$$E(x,y) = \begin{cases} 1 & \text{if} \quad x = y \\ \frac{k}{2} & \text{if} \quad (x,y) = (x_0,y_0) \text{ or } (x,y) = (y_0,x_0) \\ \frac{k}{3} & \text{if} \quad (x,y) = (y_0,z_0) \text{ or } (x,y) = (z_0,y_0) \\ \frac{k}{4} & \text{if} \quad (x,y) = (x_0,z_0) \text{ or } (x,y) = (z_0,x_0) \\ 0 & \qquad \text{otherwise.} \end{cases}$$

The fuzzy relation E is reflexive and symmetric. T-transitivity is easy to check taking into account that $T(l,l) = 0$ for all $l < k$. Hence, E is a T-indistinguishability operator.

Let us consider the aggregation function $A : [0,1]^2 \longrightarrow [0,1]$ defined by

$$A(r,s) = \begin{cases} 1 & \text{if} \quad r \geq \frac{k}{3} \text{ and } s \geq \frac{k}{3} \\ 0 & \qquad \text{otherwise.} \end{cases}$$

Since

$$A(E,E)(x_0,z_0) = A(E(x_0,z_0),E(x_0,z_0)) = A(\frac{k}{4},\frac{k}{4}) = 0$$

and

$$T(A(E,E)(x_0,y_0),A(E,E)(y_0,z_0)) = T(A(\frac{k}{2},\frac{k}{2}),A(\frac{k}{3},\frac{k}{3})) = T(1,1) = 1,$$

we conclude that $A(E,E)$ is not a T-indistinguishability operator. $\square$

We include a simple example illustrating the fact that the aggregation of two T_M-indistinguishability operators may not be a T_M-indistinguishability operator.

Example 3.10. *Consider the set* $X = \{x, y, z\}$ *and the following fuzzy relations* E *and* E' *by:*

E	x	y	z
x	1	0.5	0.7
y	0.5	1	0.5
z	0.7	0.5	1

E'	x	y	z
x	1	0.3	0.3
y	0.3	1	0.5
z	0.3	0.5	1

Since all the levels of E *and* E' *are equivalence relations,* E *and* E' *are* T_M*-indistinguishability operators.*

Consider now the arithmetic mean $A(r,s) = \frac{r+s}{2}$ *on the interval* $[0,1]$*, which is an aggregation function. We have that* $A(E,E')$ *is the following fuzzy relation.*

$A(E,E')$	x	y	z
x	1	0.4	0.5
y	0.4	1	0.5
z	0.5	0.5	1

The fuzzy relation $A(E,E')$ *is not a* T_M*-indistinguishability operator because the level* $A(E,E')_{0.5}$ *is not transitive.*

The minimum t-norm dominates any aggregation function and in particular any other t-norm (see [28]). We see in the following proposition that the domination of the minimum by any binary aggregation function admits several characterizations. The equivalence of the first three conditions in the following proposition appear either explicitly in the statement or implicitly in the proof of [28, Proposition 5.1]. We show that in this particular case an alternative characterization can be done in terms of weak domination.

Proposition 3.11. *Let* A *be a binary aggregation, the following are equivalent:*

(1) *A dominates* T_M.

(2) $A(x, y) = \min(A(x, 1), A(1, y))$.

(3) *There exist* f_1, f_2 *with* $f_i : [0, 1] \to [0, 1]$, *non decreasing, with* $f_1(1) = f_2(1) = 1$ *and* $f_1(0) \cdot f_2(0) = 0$, *such that* $A(x, y) = \min(f_1(x), f_2(y))$.

(4) *A weakly dominates* T_M.

Proof. Proof is included for the sake of completeness.

(2) $\implies$ (3).

Define $f_1(x) = A(x, 1)$ and $f_2(y) = A(1, y)$ functions from $[0, 1]$ to $[0, 1]$. Note that by hypothesis

$$A(x, y) = \min(A(x, 1), A(1, y)) = \min(f_1(x), f_2(y))$$

hence we just need to check that f_1 and f_2 satisfy the conditions required in 3. Firstly note that by definition we have $f_1(1) = f_2(1) = 1$.

Secondly $A(0, 0) = 0$ we have by hypothesis that $0 = A(0, 0) = \min(A(0, 1), A(1, 0))$. This implies that $A(0, 1) = 0$ or $A(1, 0) = 0$ and hence $f_1(0) \cdot f_2(0) = 0$.

Finally since $A(x, y)$ is non decreasing $A(x, 1)$ and $A(1, y)$ are also non decreasing.

(3) $\implies$ (1).

We now have that $A(x, y) = \min(f_1(x), f_2(y))$ with f_1, f_2 non decreasing functions from [0,1] to [0,1] satisfying $f_1(1) = f_2(1) = 1$ and $f_1(0) \cdot f_2(0) = 0$. Then

$$\begin{aligned}
\min(A(x_1, x_2), A(y_1, y_2)) &= \min(\, \min(f_1(x_1), f_2(x_2)), \min(f_1(y_1), f_2(y_2))) \\
&= \min(f_1(x_1), f_1(y_1), f_2(x_2), f_2(y_2)) \\
&\leq \min(f_1(\min(x_1, y_1)), f_2(\min(x_2, y_2))) \\
&= A(\min(x_1, y_1), \min(x_2, y_2))
\end{aligned}$$

where the inequality comes from the fact that the minimum of a subset is always greater or equal than the minimum of the set containing it. Notice that this inequality is really an equality since the functions are non decreasing.

(1) $\implies$ (4) follows immediately from the definitions.

(4) $\implies$ (2).

Suppose now that A weakly dominates T_M. Considering $z = 1$ in the definition, we can write the following:

$$A(x, y) = A(\min(x, 1), \min(1, y)) \geq \min(A(x, 1), A(1, y))$$

but $A(x, y) \leq \min(A(x, 1), A(1, y))$ since A is non decreasing and hence $A(x, y)$ is less or equal than both $A(x, 1)$ and $A(1, y)$. Therefore we have that $A(x, y) = \min(A(x, 1), A(1, y))$, which completes the proof. □

We have just showed that A dominates $T_M \iff A$ weakly dominates T_M. An alternative characterization is possible in terms of domination with respect to a pair of indistinguishability operators.

Proposition 3.12. *Let A be an aggregation function and X a non-empty set.*

A dominates T_M if and only if for all pair (E, E') of T_M-indistinguishability operators in X, A dominates T_M with respect to (E, E').

Proof. Suppose that A does not dominate T_M, then there exists $a, b, c, d \in [0, 1]$ such that

$$A(T_M(a, b), T_M(c, d)) < T_M(A(a, c), A(b, d)).$$

Let us show that there exists $X \neq \emptyset$ and a pair of indistinguishabilities (E, E') such that A does not dominate T_M with respect to (E, E').

Let us suppose without lost of generality that $a \leq b$, $c \leq d$ and define in $X = \{x, y, z\}$ the following fuzzy relations.

E	x	y	z
x	1	a	a
y	a	1	b
z	a	b	1

E'	x	y	z
x	1	c	c
y	c	1	d
z	c	d	1

E and E' are T_M-indistinguishability operators and

$$\begin{aligned}&A(T_M(E(x,y),E(y,z)),T_M(E'(x,y),E'(x,z)))=\\&=A(T_M(a,b),T_M(c,d))<T_M(A(a,c),A(b,d))=\\&=T_M(A(E(x,y),E'(x,y)),A(E(y,z),E'(y,z)))\end{aligned}$$

where the inequality is our initial supposition. Therefore, A does not dominate T_M with respect to (E,E') which completes the proof.

The converse is straightforward. □

The following Lemma will be used in the proofs of Theorem 3.17, and in Corollaries 4.2 and 4.4.

Lemma 3.13. *Let X be a set, $E : X \times X \longrightarrow [0,1]$ a T_M-indistinguishability operator and $x, y, z \in X$. If $E(x,y) \neq E(y,z)$, then*

$$E(x,z) = T_M(E(x,y),E(y,z)).$$

Proof. Suppose for a contradiction that $E(x,z) > T_M(E(x,y),E(y,z))$. Without loss of generality, we can take $E(x,y) < E(y,z)$. Under this assumption, we have $E(x,z) > T_M(E(x,y),E(y,z)) = E(x,y)$. Since E is a T_M-indistinguishability operator, we have

$$E(x,y) \geq T_M(E(x,z),E(z,y)) = T_M(E(x,z),E(y,z)).$$

This is a contradiction because $E(x,y) < E(y,z)$ and $E(x,y) < E(x,z)$. □

The following theorem weakens the domination requirement on the Aggregation in Corollary 3.6 to one depending on the indistinguishability operators to be aggregated.

Theorem 3.14. *Let X be a set, A an aggregation function and E and E' two T_M-indistinguishability operators. The fuzzy relation $A(E,E')$ is a T_M-indistinguishability operator if and only if A dominates T_M with respect to (E,E').*

Proof. Suppose $A(E,E')$ is a T_M-indistinguishability operator. Given $x,y,z\in X$, it is enough to prove that

$$A(T_M(E(x,y),E(y,z)),T_M(E'(x,y),E'(y,z)))\geq T_M(A(E,E')(x,y),A(E,E')(y,z))$$

Without loss of generality, let us consider $E(x,y)\leq E(y,z)$.

Suppose first that $E'(x,y)\leq E'(y,z)$. We obtain that

$$A(T_M(E(x,y),E(y,z)),T_M(E'(x,y),E'(y,z)))=A(E(x,y),E'(x,y))\geq$$

$$T_M(A(E,E')(x,y),A(E,E')(y,z)).$$

Suppose now that $E'(x,y)>E'(y,z)$.

If $E(x,y)=E(y,z)$, we have that

$$A(T_M(E(x,y),E(y,z)),T_M(E'(x,y),E'(y,z)))=A(E(y,z),E'(y,z))\geq$$

$$T_M(A(E,E')(x,y),A(E,E')(y,z)).$$

If $E(x,y)<E(y,z)$, using Lemma 3.13, we have that $E(x,y)=E(x,z)$ and $E'(y,z)=E'(x,z)$. Then,

$$A(T_M(E(x,y),E(y,z)),T_M(E'(x,y),E'(y,z)))=A(E(x,z),E'(x,z))$$

and since $A(E,E')$ is a T_M-indistinguishability,

$$A(E(x,z),E'(x,z))\geq T_M(A(E,E')(x,y),A(E,E')(y,z)).$$

Conversely, take $x,y,z\in X$. Since E and E' are T_M-indistinguishability operators, we have that

$$A(E,E')(x,z)\geq A(T_M(E(x,y),E(y,z)),T_M(E'(x,y),E'(y,z))).$$

Since A dominates T_M with respect to (E,E'), we conclude that

$$A(E,E')(x,z)\geq T_M(A(E,E')(x,y),A(E,E')(y,z)).$$

□

We conclude this section with a characterization of pairs of T_M indistinguishability operators which when aggregated always produce a T_M indistinguishability operator. The following new concept will be crucial for this purpose.

Definition 3.15. *Let X be a set, E and E' two fuzzy relations on X. We denote $E \approx E'$ whenever*

$$E(x,y) < E(y,z) \implies E'(x,y) \leq E'(y,z)$$

for all $x, y, z \in X$.

Observe that $\approx$ is reflexive and symmetric, but it is not an equivalence relation as the following example shows.

Example 3.16. *Let $\bar{k}$ be the T_M-indistinguishability operator defined by $\bar{k}(x,x) = 1$ for all $x \in X$ and $\bar{k}(x,y) = k$ if $x \neq y$, with $k \in [0,1]$. Take the two T_M-indistinguishability operators E and E' of Example 3.10. We have $E \approx \bar{1}$ and $\bar{1} \approx E'$, but $E \not\approx E'$ due to*

$$E(y,z) = 0.3 < 0.5 = E(z,x)$$

and

$$E'(y,z) = 0.5 > 0.3 = E'(z,x).$$

We conclude with the main result of this section which provides sufficient and necessary conditions on pairs of T_M-indistinguishability operators to ensure that their aggregation is also a T_M-indistinguishability operator.

Theorem 3.17. *Let X be a non-empty set and consider two T_M-indistinguishability operators E and E' on X. The following assertions are equivalent:*

(1) $E \approx E'$.

(2) *For any aggregation function A, $A(E, E')$ is a T_M-indistinguishability operator.*

(3) *For any aggregation function A, A dominates T_M with respect to (E, E').*

Proof. Taking into account Lemma 3.14, we have that (2) is equivalent to (3).

(1) $\implies$ (3). Suppose $E \approx E'$, and take an arbitrary aggregation function A. It is enough to prove that for any x, y, z, if we call $E(x,y) = r$, $E(y,z) = s$, $E'(x,y) = r'$, $E'(y,z) = s'$, then

$$A\big(T_M(r,s),\, T_M(r',s')\big) \geq T_M(A(r,r'),\, A(s,s'))$$

Without loss of generality, let us suppose that $r \leq s$.

1. If $r < s$. Since $E \approx E'$, $r' \leq s'$,

$$A(T_M(r,s), T_M(r',s')) = A(r,r') \geq T_M(A(r,r'), A(s,s')).$$

2. If $r = s$. We have $r' \leq s'$ or $r' \geq s'$. Let us suppose $r' \leq s'$, otherwise the argument is analogous. Again, since E and E' are T_M-indistinguishability operators and A is a non-decreasing function,

$$A(T_M(r,s), T_M(r',s')) = A(r,r') \geq T_M(A(r,r'), A(s,s')).$$

   and again we obtain the desired inequality.

(2) $\implies$ (1). Let us prove that if $E \not\approx E'$, then there is an aggregation function A such that $A(E,E')$ is not a T_M-indistinguishability operator.

Since $E \not\approx E'$, there are $x, y, z \in X$ satisfying $E(x,y) < E(y,z)$ and $E'(x,y) > E'(y,z)$. Consider the aggregation function $A : [0,1]^2 \longrightarrow [0,1]$ defined as follows:

$$A(a,b) = \begin{cases} 1 & \text{if} \quad a \geq E(x,y) \text{ and } b \geq E'(x,y) \\ 1 & \text{if} \quad a \geq E(y,z) \text{ and } b \geq E'(y,z) \\ 0 & \quad\;\; \text{otherwise.} \end{cases}$$

Note that:

$$A(E'(x,y), E'(x,y)) = 1 = A(E(y,z), E'(y,z)) \text{ and } A(E(x,y), E'(y,z)) = 0.$$

Since $E(x,y) < E(y,z)$ and $E'(x,y) > E'(y,z)$, using Lemma 3.13, we have that $E(x,y) = E(x,z)$ and $E'(y,z) = E'(x,z)$. Therefore,

$$A(E,E')(x,z) = A(E(x,z), E'(x,z)) = A(E(x,y), E'(y,z)) = 0$$

and by definition of A,

$$T_M(A(E,E')(x,y), A(E,E')(y,z)) = T_M(1,1) = 1.$$

Therefore, $A(E,E')$ is not a indistinguishability operator. $\square$

4. The role of strictly monotone aggregation functions

This section is devoted to show the role of strictly monotone aggregation functions on the aggregation of T_M-indistinguishability operators.

Our next result shows that the existence of a single strictly monotone aggregation that satisfies the indistinguishability operator preservation property guarantees all aggregations to have the same preservation property. Accordingly we give an extended version of Theorem 3.17, that includes this fact.

Theorem 4.1. *Let X be a set and consider two T_M-indistinguishability operators E and E' on X. The following assertions are equivalent:*

(1) $E \approx E'$.

(2) *For all aggregation function A, $A(E,E')$ is a T_M-indistinguishability operator.*

(3) *For all aggregation function A, A dominates T_M with respect to (E,E').*

(4) *There exists a strictly monotone aggregation function B, such that $B(E,E')$ is a T_M-indistinguishability operator.*

Proof. The first three conditions are equivalent by Proposition 3.17.

(2) $\implies$ (4) is straightforward, so we just need to prove (4) $\implies$ (3).

Suppose for a contradiction that there is an aggregation function A satisfying that $A(E,E')$ is not a T_M-indistinguishability operator. Then, by Theorem 3.17, there are $x,y,z \in X$ such that

$$E(x,y) < E(y,z) \text{ and } E'(x,y) > E'(y,z).$$

Moreover, by Lemma 3.13, we know that

$$E(x,z) = E(x,y) < E(y,z) \text{ and } E'(x,y) > E'(y,z) = E'(x,z).$$

Take $k = \min\{B(E,E')(x,y), B(E,E')(y,z)\}$ and consider the corresponding level set $B(E,E')_k$. By hypothesis, $B(E,E')$ is a T_M-indistinguishability operator, hence $B(E,E')_k$ is an equivalence relation on X. Since (x,y) and (y,z) belong to $B(E,E')_k$, we have that $(x,z) \in B(E,E')_k$. Now by the inequalities right above this paragraph and the fact that B is a strictly monotone function we have the following:

$$B(E,E')(x,z) = B(E(x,y),E'(y,z)) < B(E(y,z),E'(y,z)) = B(E,E')(y,z)$$

and

$$B(E,E')(x,z) = B(E(x,y),E'(y,z)) < B(E(x,y),E'(x,y)) = B(E,E')(x,y).$$

Therefore $B(E,E')(xz) < k$, which is a contradiction. □

Now, let us present a result which involves the aggregation of T_M-indistinguishability operators considering aggregation functions which are strictly monotone on $(0,1] \times (0,1]$, as for instance, strict t-norms (see [16]). The result provides a new characterization of the aggregation of T_M-indistinguishability operators whose supports are the whole set.

Corollary 4.2. *Let X be a set and consider two T_M-indistinguishability operator E and E' on X satisfying $E(x,y) > 0$ and $E'(x,y) > 0$ for all $x,y \in X$. The following assertions are equivalent:*

(1) $E \approx E'$.

(2) *For all aggregation function A, $A(E,E')$ is a T_M-indistinguishability operator.*

(3) *For all aggregation function A, A dominates T_M with respect to (E,E').*

(4) *There exists an aggregation function* $B : [0,1] \times [0,1] \longrightarrow [0,1]$ *strictly monotonous on* $(0,1] \times (0,1]$ *such that* $B(E, E')$ *is a* T_M*-indistinguishability operator.*

Proof. By Theorem 4.1, it is enough to prove (4) $\implies$ (1). Suppose that $B(E, E')$ is a T_M-indistinguishability operator. Let us suppose for a contradiction that there are $x, y, z \in X$ such that $0 < E(x, y) < E(y, z)$ and $0 < E'(y, z) < E'(x, y)$. By Lemma 3.13, we have that

$$0 < E(x, z) = E(x, y) < E(y, z) \text{ and } 0 < E'(x, z) = E'(y, z) < E'(x, y). \quad (1)$$

Take $k = \min\{B(E, E')(x, y), B(E, E')(y, z)\}$ and consider the corresponding level set $B(E, E')_k$. Since $B(E, E')$ is a T_M-indistinguishability operator, $B(E, E')_k$ is an equivalence relation on X. Moreover, (x, y) and (y, z) belong to $B(E, E')_k$, therefore $(x, z) \in B(E, E')_k$.

However, since B is strictly monotone on $(0,1] \times (0,1]$ and $E(x, y) \neq 0 \neq E'(y, z)$, taking into account Equation (1), we have

$$B(E, E')(x, z) = B(E(x, y), E'(y, z)) < B(E(y, z), E'(y, z)) = B(E, E')(y, z)$$

and

$$B(E, E')(x, z) = B(E(x, y), E'(y, z)) < B(E(x, y), E'(x, y)) = B(E, E')(x, y).$$

Therefore, $B(E, E')(x, z) < k$, which is a contradiction. □

The required condition for T_M-indistinguishability operators in the previous Corollary is necessary, as the following example shows.

Example 4.3. *Let* X *be a set with three elements* x, y *and* z*. Consider the fuzzy relation* E *and* E' *on* X *defined as follows.*

E	x	y	z
x	1	1	0
y	1	1	0
z	0	0	1

E'	x	y	z
x	1	0.5	0.75
y	0.5	1	0.5
z	0.75	0.5	1

They are T_M-indistinguishability operators because each level set is an equivalence relation.

Take the product t-norm $T_P(a,b) = ab$, which is a strictly monotone aggregation function on the domain $(0,1] \times (0,1]$. We have that $T_P(E,E')$ is the following fuzzy relation:

$T_P(E,E')$	x	y	z
x	1	0.5	0
y	0.5	1	0
z	0	0	1

We have that $T_P(E,E')$ is a T_M-indistinguishability operator because its level sets are equivalence relations on X. However, $E \not\approx E'$ because $E(z,x) < E(x,y)$ and $E'(z,x) > E'(x,y)$.

Corollary 4.4. *Let X be a set and consider two T_M-indistinguishability operators E and E' on X satisfying that $E(x,y) = 1$ if and only $x = y$ and $E'(x,y) = 1$ if and only $x = y$. The following assertions are equivalent:*

(1) *$E \approx E'$.*

(2) *For all aggregation function A, $A(E,E')$ is a T_M-indistinguishability operator.*

(3) *For all aggregation function A, A dominates T_M with respect to (E,E').*

(4) *There exists an aggregation function B which is strictly monotone on the domain $[0,1) \times [0,1)$ such that $B(E,E')$ is a T_M-indistinguishability operator.*

Proof. By Theorem 4.1, it is suffices to prove (4) $\implies$ (1). Let B be an aggregation, strictly monotone on the domain $[0,1) \times [0,1)$ such that $B(E,E')$ is a T_M-indistinguishability operator.

Suppose for a contradiction that there are $x, y, z \in X$ satisfying $E(x,y) < E(y,z)$ and $E'(y,z) < E'(x,y)$. It is important to note that the elements x, y and z are different. Therefore, $E(y,z) < 1$ and $E'(x,y) < 1$.

By Lemma 3.13,we have

$$E(x,z) = T_M(E(x,y), E(y,z)) \text{ and } E'(x,z) = T_M(E'(x,y), E'(y,z)).$$

Therefore,

$$E(x,z) = E(x,y) < E(y,z) < 1 \text{ and } E'(x,z) = E'(y,z) < E'(x,y) < 1. \qquad (2)$$

Take $k = \min\{B(E,E')(x,y), B(E,E')(y,<)\}$ and consider the corresponding level set $B(E,E')_k$. Since $B(E,E')$ is a T_M-indistinguishability operator, $B(E,E')_k$ is an equivalence relation on X. Moreover, (x,y) and (y,z) belong to $B(E,E')_k$, then $xy \in B(E,E')_k$.

However, since B is strictly monotone on $[0,1) \times [0,1)$ and $E(x,y) \neq 1 \neq E'(y,z)$, taking into account Equation (2) we have

$$B(E,E')(x,z) = B(E(x,y), E'(y,z)) < B(E(y,z), E'(y,z)) = B(E,E')(y,z)$$

and

$$B(E,E')(x,z) = B(E(x,y), E'(y,z)) < B(E(x,y), E'(x,y)) = B(E,E')(x,y).$$

Therefore, $B(E,E')(x,z) < k$, which is a contradiction. $\square$

Strict t-norms are examples of strictly monotone aggregation functions on $(0,1] \times (0,1]$. The previous corollary provides an analogous result for strictly monotone aggregation functions on $[0,1) \times [0,1)$, which include the dual t-conorms of strict t-norms.

Concluding Remarks

Given two T_M-indistinguishability operators E and E' on a set X and an aggregation function A, we have obtained the following main results:

1. We have proved that the only t-norm T such that $A(E,E)$ is a T-indistinguishability operator for each aggregation function A and each T-indistinguishability operator E is the minimum t-norm for $|X| \geq 3$.

2. We have provided a new characterization of domination for the minimum t-norm.
3. We have shown an alternative characterization to the one of Saminger et al. of the aggregations A such that $A(E, E')$ is a T_M-indistinguishability operator.
4. We have found necessary and sufficient conditions on E and E' to guarantee that $A(E, E')$ is a T_M-indistinguishability operator for all A.
5. We have revealed that the class of strictly monotone aggregation functions play a significant role. If two T_M-indistinguishability operators E and E' are aggregated by a strictly monotone aggregation function A, then they are aggregated by any aggregation function.

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