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THREE-LAYER FACTORIZED DIFFERENCE SCHEMES AND PARALLEL ALGORITHMS FOR SOLVING THE SYSTEM OF LINEAR PARABOLIC EQUATIONS WITH MIXED DERIVATIVES AND VARIABLE COEFFICIENTS

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Abstract. The nonstationary problem for system of parabolic equations with first order boundary conditions is considered

$$B\frac{\partial u}{\partial t} = Lu + f,$$

where B is a positively defined, symmetric and bounded matrix, L is a strong elliptic operator with variable coefficients containing the mixed derivatives, $u = (u^{(1)}, u^{(2)}, \dots, u^{(n)})$, and $f = (f^{(1)}, f^{(2)}, \dots, f^{(n)})$ are n-dimensional vectors. The three-layer factorised scheme, whose solution does not require the inversion of matrix B, is constructed. The obtained algorithms can be effectively used on multiprocessing computing systems. For the difference scheme, the a priori estimation on layer in norm of grid space $\overset{\circ}{W}{}_2^{(1)}$ is obtained, on the basis of which the convergence of its solution to the solution of the initial problem is proved.

Keywords: Parabolic-type equation systems, Difference Schemes, Convergence, Paralell Computing, Strong-elliptic operator.

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1. Introduction

The mathematical formulation of many scientific and technical problems gives rise to differential equations with partial derivatives of a parabolic type. Such problems concern, for instance, the distribution of heat in the continuous environment, transport processes, diffusion, movement of ground waters, dissipative biological structures, dynamics of distribution of power in structures, etc. [13, 16].

The solution of such problems is one of the challenges in numerical mathematics and, as a rule, it demands a lot of computing resources. One of the possible ways to reduce the time for solving such problems is to use parallel computations on multiprocessing computing systems.

Many examples related to the construction of difference splitting schemes and applications and parallel algorithms for the solution of parabolic equations are given in [1, 3, 4, 5, 6, 7, 15].

In the present paper the system of second order linear parabolic equations with mixed derivatives is considered. For this system, the factorized difference scheme is constructed. Based on

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this scheme, the algorithm suitable for calculation on a modern parallel supercomputing system is proposed. It means and added value on this field since there is only a few literature on parallel computations for the considered problems in comparison with the results in the field of sequential computations.

The paper is focused on the following problems:

(1) To obtain the economic difference scheme at minimal requirements on the spatial operator. We only require the spatial operator to be strong elliptic.
(2) To achieve absolute stability at any $\tau < \infty$ (τ is a step on time axis) and $h < \infty$ (h_α is a step along the axis x_α, $\alpha = 1, 2, \dots, p$).
(3) To prove convergence of the difference schemes with a smaller smoothness of a solution of the initial system of differential equations, which can be achieved by refusing an estimation of local approximation.
(4) To apply a one-dimensional double-sweep algorithm to solve the obtained difference equations.
(5) To construct the algorithm of parallel computation.

2. Basic Concepts and Some Auxiliary Information

1°. Let

$$D_p = (0, l_1) \times (0, l_2) \times \cdots \times (0, l_p) = \prod_{\alpha=1}^{p} (0, l_\alpha) =$$
$$= \{x = (x_1, x_2, \dots, x_p), \quad 0 < x_\alpha < l_\alpha, \quad \alpha = 1, 2, \dots, p\}$$

be an open p-dimensional parallelepiped in p-dimensional Euclid space E_p, where Γ is its boundary and $\bar{D}_p = D_p \cup \Gamma$ is its closure.

Let $G_T = D_p \times (0, T]$ be a cylinder in $(p+1)$–dimensional Euclid space with the base in E_p; (x, t) is any point in G_T; $\bar{G}_T = \bar{D}_p \times [0, T]$ is its closure; $C^{l,k}(\bar{G}_T)$ is the set of continuous functions in $\bar{G}_T$, which have in $\bar{G}_T$ continuous derivatives to the l-th order (inclusive) with respect to x and to the k-th order (inclusive) with respect to t.

Let us introduce the difference spatial-time grid in the cylinder $\bar{G}_T$ using one-dimensional grids on intervals $[0, l_\alpha]$, $\alpha = 1, 2, \dots, p$, and $[0, T]$:

$$\bar{\omega}_\alpha = \left\{ x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha, \quad i_\alpha = 0, 1, \dots, N_\alpha, \quad N_\alpha h_\alpha = l_\alpha, \quad \alpha = 1, 2, \dots, p \right\}.$$
$$\omega_\alpha = \left\{ x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha, \quad i_\alpha = 1, \dots, N_\alpha - 1, \quad N_\alpha h_\alpha = l_\alpha, \quad \alpha = 1, 2, \dots, p \right\}.$$

In parallelepiped $\bar{D}_p$ the difference grid is constructed in the following way:

$$\bar{\omega}_h = \prod_{\alpha=1}^{p} \bar{\omega}_\alpha = \left\{ x = (i_1 h_1, \dots, i_p h_p) \in \bar{D}_p, \quad i_\alpha = 0, 1, \dots, N_\alpha, \quad N_\alpha h_\alpha = l_\alpha \right\}.$$

where $\omega_h = \prod_{\alpha=1}^{p} \omega_\alpha$ is the set of internal points of the difference grid in D_p.

Denote by $\gamma_h = \bar{\omega}_h \backslash \omega_h$ the set of knots belonging to the boundary Γ, which are called the boundary knots.

On the interval $[0, T]$ let us introduce the difference grid

$$\bar{\omega}_\tau = \{t_j = j\tau, \quad j = 0, 1, \dots, k, \quad k\tau = T\}.$$

Let $\Omega_{h\tau} = \omega_h \times \bar{\omega}_\tau$ be the partial-time grid of internal knots of the cylinder G_T, $\Gamma_{h\tau} = \gamma_h \times \bar{\omega}_\tau$ be the set of knots of difference grid on lateral surface of cylinder $\bar{G}_T$. Denote by (x, t_j) the knot of $\Omega_{h\tau}$ and by (x', t_j) the knot belonging to $\Gamma_{h\tau}$. Then

$$\bar{\Omega}_{h\tau} = \Omega_{h\tau} \cup \Gamma_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$$

is the partial-time grid in $\bar{G}_T$.

2°. We will deal with functions of discrete argument defined in knots of difference grid and named as the grid functions.

We will consider the set of grid vector-functions $y = (y^{(1)}, y^{(2)}, \dots, y^{(n)})$ defined on $\bar{\Omega}_{h\tau}$. For them, the symbol $y^{(j)}$ is the value of grid vector-function in knots (x, t_j).

We will use the following notations:

$$x^{(\pm 1_\alpha)} = (x_1, \dots, x_{\alpha-1}, x_\alpha(\pm h_\alpha), x_{\alpha+1}, \dots, x_p),$$
$$y^{(\pm 1\alpha)} = y(x^{(\pm 1_\alpha)}) = (y_1(x^{(\pm 1_\alpha)}), \dots, y_n(x^{(\pm 1_\alpha)})),$$
$$|h|^2 = \sum_{\alpha=1}^{p} h_\alpha^2.$$

Let us consider the difference relations:

$$y_{\bar{x}_\alpha} = \frac{y - y^{(-1_\alpha)}}{h_\alpha}, \qquad y_{x_\alpha} = \frac{y^{(+1_\alpha)} - y}{h_\alpha},$$
$$-\Lambda_\alpha^0 y = A_\alpha^0 y, \qquad A^0 = \sum_{\alpha=1}^{p} A_\alpha^0.$$

For the grid vector-functions defined on $\bar{\Omega}_{h\tau}$ besides the above-mentioned notations, we will use the following ones:

$$\hat{y} = y(x, t_{j+1}), \qquad y = y(x, t_j), \qquad \check{y} = y(x, t_{j-1}),$$
$$y_{\bar{t}} = \frac{y - \check{y}}{\tau}, \qquad y_t = \frac{\hat{y} - y}{\tau}, \qquad y_{\bar{t}t} = \frac{\hat{y} - 2y + \check{y}}{\tau^2}, \qquad y_{\overset{\circ}{t}} = \frac{\hat{y} - \check{y}}{2\tau}.$$

3°. The set of grid functions, defined on the grid $\bar{\omega}_h$ will be denoted by H. Its subset, consisting of grid functions, which vanish on γ_h by $\overset{\circ}{H}$.

On set $\overset{\circ}{H}$ the scalar product is defined in the following way:

$$(y, v) = \sum_{i=1}^{n} (y^i, v^i),$$

where

$$(y^i, v^i) = \sum_{x \in \omega_h} (y^i(x), v^i(x))\, h, \qquad h = \prod_{\alpha=1}^{p} h_\alpha.$$

It is obvious that this scalar product induces the norm $\|y\|_0^2 = (y, y)$.

Denote by $\omega_h^{+\alpha} = \omega_h \cup \gamma_\alpha^+$, where γ_α^+ is the set of knots of γ_h border, with $x_\alpha = l_\alpha$. Let $\gamma_{\alpha\beta}^+$ $(\alpha \neq \beta)$ be the set of knots of γ_h border with $x_\alpha = l_\alpha$, $x_\beta = l_\beta$, etc. Similarly, we will denote by γ_α^- the set of knots of γ_h border with $x_\alpha = 0$, etc.

Let us use the notation $\omega_h^+ = \omega_h \cup \gamma_{1,2,\dots,p}^+$. Here the scalar product is also required:

$$(y, v]_\alpha = \sum_{i=1}^{n} (y^i, v^i), \qquad (y^i, v^i] = \sum_{x \in \omega_h^+} y^i(x) v^i(x)\, h.$$

Let us consider the grid Sobolev's space $\overset{\circ}{W}{}_2^{(1)}(\omega_h)$ consisting of the functions from the set $\overset{\circ}{H}$. The norm in this space is defined by the following expression:

$$\|u\|_1^2 = \|u\|_{\overset{\circ}{W}{}_2^{(1)}}^2 = \sum_{\alpha=1}^{p} \|u_{\bar{x}_\alpha}\|_0^2, \qquad \|u_{\bar{x}_\alpha}\|_0^2 = (1, (u_{\bar{x}_\alpha})^2). \tag{1}$$

Note that $\overset{\circ}{W}{}_2^{(1)}$ is a complete Hilbert space with respect to norm (1).

4°. Let $A > 0$ be the linear self-conjugate operator, in Hilbert space H. We will define a new scalar product

$$(y, v)_A = (Ay, v).$$

Owing to this definition, H becomes a new Hilbert space, that will be denoted by H_A. The norm in H_A will be denoted by symbol $\|\cdot\|_A$:

$$\|u\|_A^2 = (Au, u).$$

3. Statement of the Problem

We consider the problem of finding continuous functions in $\bar{G}_T$ for the system of parabolic-type equations with mixed derivatives:

$$\sum_{j=1}^{n} b_{ij}(x,t)\frac{\partial u_j}{\partial t} = \sum_{j=1}^{n}\sum_{\alpha,\beta=1}^{p} \frac{\partial}{\partial x_\alpha}\left(K_{\alpha,\beta}^{i,j}(x,t)\frac{du_j}{dx_\beta}\right) + f_i(x,t), \qquad i = 1,2,\dots,n. \tag{2}$$

which satisfies the boundary conditions

$$u_i(x',t) = g_i(x',t), \qquad \text{when } (x',t) \in \Gamma \times [0,T], \tag{3}$$

and the initial conditions

$$u_i(x,0) = u_i^0(x), \qquad \text{when } x \in \bar{D}_p, \quad i = 1,2,\dots,n. \tag{4}$$

Rewrite the the problem (2)-(4) in a vector form:

$$B\frac{\partial u}{\partial t} = \sum_{\alpha,\beta}^{1\div p} \frac{\partial}{\partial x_\alpha}\left(K_{\alpha\beta}(x,t)\frac{du}{dx_\beta}\right) + f(x,t),$$
$$u(x',t) = g(x',t), \qquad (x',t) \in \Gamma \times [0,T],$$
$$u(x,0) = u_0(x), \qquad x \in \bar{D}_p,$$

where

$$u(x,t) = (u_1(x,t), u_2(x,t), \dots, u_n(x,t)), \quad x = (x_1, x_2, \dots, x_p),$$
$$f(x,t) = (f_1(x,t), f_2(x,t), \dots, f_n(x,t)),$$
$$B(x,t) = \begin{pmatrix} b_{11}(x,t) & \cdots & b_{1n}(x,t) \\ \vdots & & \vdots \\ b_{n1}(x,t) & \cdots & b_{nn}(x,t) \end{pmatrix}, \quad K_{\alpha\beta}(x,t) = \begin{pmatrix} K_{\alpha\beta}^{11}(x,t) & \cdots & K_{\alpha\beta}^{1n}(x,t) \\ \vdots & & \vdots \\ K_{\alpha\beta}^{n1}(x,t) & \cdots & K_{\alpha\beta}^{nn}(x,t) \end{pmatrix}.$$

Assume that the following conditions are fulfilled:

(I) Coefficients $K_{\alpha\beta}^{i,j}(x,t)$ are Lipschitz-continuous with respect to t.

(II) Matrices of coefficients of a spatial operator $K_{\alpha\beta} = (K_{\alpha\beta}^{ij})_{i,j=1}^{n}$ $(\alpha, \beta = 1,2,\dots,p)$ are symmetric.

(III) Problem (2)-(4) has a unique solution $u \equiv u(x,t)$, continuous in $\bar{G}_T$ and sufficiently differentiable.

(IV) Spatial operator of the system of equations (2) is strong-elliptic, so for any $t \in [0,T]$ the following inequality is true:

$$\nu_1 \sum_{\alpha=1}^{p}\sum_{i=1}^{n}(\xi_\alpha^i)^2 \le \sum_{\alpha,\beta}^{1\div p}\sum_{i,j}^{1\div n} K_{\alpha\beta}^{ij}\xi_\beta^j\xi_\alpha^i \le \nu_2 \sum_{\alpha=1}^{p}\sum_{i=1}^{n}(\xi_\alpha^i)^2, \tag{5}$$

where $\{\xi_\alpha^i\}$ are real numbers and ν_1 and ν_2 are positive constants.

(V) Matrix $B = (b_{ij})$ is defined as positive, symmetric and bounded

$$\|B\| \le \nu_3, \qquad B \ge \delta E,$$

where ν_3 and δ are positive constants and E is a unit matrix.

Note that the last condition is necessary for existence of the solution of problem (2)-(4). However, the algorithm of solution of the difference scheme does not require the inversion of the matrix B.

4. Three-layer Difference Scheme for Problem (2)-(4)

1°. For the problem (2)-(4), let us consider three-layer difference scheme approximation of which in a class $C^{4,2}(\bar{G}_T)$ of solutions of (2) has an order $O(\tau + |h|^2)$:

$$By_{\bar{t}} + \tau^2 R y_{\bar{t}t} + Ay = f, \qquad (x,t) \in \Omega_{h\tau}, \tag{6}$$

where $y = (y_1, y_2, \dots, y_n)$ is a n-dimensional vector, which approximates a vector function u,

$$Ay = -\frac{1}{2} \sum_{\alpha,\beta=1,\dots,p} \left[\left(K_{\alpha\beta}(x,t) y_{\bar{x}_\beta}\right)_{x_\alpha} + \left(K_{\alpha\beta}(x,t) y_{x_\beta}\right)_{\bar{x}_\alpha} \right], \tag{7}$$

and R is an operator-regulator, whose selection provides absolute stability of difference scheme (6).

Vector function $y(x,t)$ satisfies the following boundary and initial conditions:

$$\begin{cases} y(x',t) = g(x',t), & \text{when } x' \in \Gamma_{h\tau} \\ y(x,0) = u^0(x), \\ y(x,\tau) = u^1(x) \end{cases} \tag{8}$$

where the third initial condition on layer $t = \tau$ can be approximated, for example, supposing that $u^1(x) = u^0(x)$. The approximation error in this case has an order $O(\tau)$.

Let us write difference scheme (6) in a canonical form [12]:

$$By_{\overset{\circ}{t}} + \tau^2 \left(R - \frac{1}{2\tau} B\right) y_{\bar{t}t} + Ay = f. \tag{9}$$

The validity of the following relationships is shown in [10, 11]:

$$\nu_1 (A^0 y, y) \le (Ay, y) \le \nu_2 (A^0 y, y)$$

or

$$\nu_1 A^0 \le A \le \nu_2 A^0$$

for any vector $y \in \overset{\circ}{H}$, where operator A^0 is defined in Section 2.

Hence, operators A and A^0 in $\overset{\circ}{H}$ are energetically equivalent with constants ν_1 and ν_2.

In [10] it is proved that the following three-layer difference scheme:

$$By_{\overset{\circ}{t}} + \tau^2 \bar{R} y_{\bar{t}t} + Ay = \varphi(t), \qquad (x,t) \in \Omega_{h\tau}, \quad y(0) = y_0, \quad y(\tau) = y_1,$$
$$0 < t = n\tau < t_0, \qquad n = 1, 2, \dots, n_0 - 1, \qquad t_0 = n_0 \tau,$$

is absolutely stable with respect to the initial data and the right side, where y_0 and y_1 are the given arbitrary vectors of finite-dimensional space H (H - Hilbert space), $\varphi(t)$ is a given arbitrary abstract function, $t \in \overline{\omega_\tau}$, and A, B, $\bar{R}$ are linear operators on H. In addition

- $A = A(t) = A^*(t) > 0$, $\bar{R} = \bar{R}(t) = \bar{R}^*(t) > 0$ are Lipschitz-continuous in t, where $A^*(t)$ and $\bar{R}^*(t)$ are the conjugate operators of $A(t)$ and $\bar{R}(t)$, respectively,
- $B = B(t) > \delta E$ (where δ is a positive constant and E is the identity operator)
- $\bar{R}(t) = \dfrac{1+\epsilon}{4} A(t)$ for all $0 < t = n\tau < t_0$ (where $\epsilon = \text{const}$, $0 < \epsilon \le 1$, does not depend on τ and h).

Based on this result, we can conclude that difference scheme (9), for which $\bar{R} = R - \frac{1}{2\tau} B$, with regulator

$$R = \sigma A^0 + \frac{\nu_3}{2\tau} E, \tag{10}$$

where $\sigma \geq \frac{1+\epsilon}{4}\nu_2$ $(0 < \epsilon \leq 1)$, is absolutely stable in space $\overset{\circ}{H}$ (stable with respect to the initial data and the right side). So, we can assume that operator R is defined by means of equality (10).

2°. Proceeding with the construction of the economic factorised difference scheme, considering scheme (6) as the initial one, we can rewrite the difference scheme (6) as follows:

$$By_{\bar{t}} + \frac{\nu_3\tau}{2}(E + \tau R^0)y_{\bar{t}t} + Ay = f, \tag{11}$$

where

$$R^0 = \bar{\sigma}A^0 = \sum_{\alpha=1}^{p} R^0_\alpha, \qquad R^0_\alpha = \bar{\sigma}A^0_\alpha, \qquad \bar{\sigma} = \frac{2\sigma}{\nu_3}.$$

Equation (11) can be written in the following way:

$$\frac{\nu_3\tau}{2}(E + \tau R^0)y_{\bar{t}t} = f - Ay - By_{\bar{t}}. \tag{12}$$

Replacing the operator $E + \tau R^0 = E + \tau \sum_{\alpha=1}^{p} R^0_\alpha$ with factorised operator $\prod_{\alpha=1}^{p}(E + \tau R^0_\alpha)$, we obtain the three-layer factorised scheme

$$\frac{\nu_3\tau}{2}\prod_{\alpha=1}^{p}(E + \tau R^0_\alpha)y_{\bar{t}t} = f - Ay - By_{\bar{t}}. \tag{13}$$

Let us use the equality

$$\prod_{\alpha=1}^{p}(E + \tau R^0_\alpha) = E + \tau \sum_{\alpha=1}^{p} R^0_\alpha + \tau^2 Q_p,$$

where

$$Q_p = \sum_{\substack{\alpha,\beta,\gamma=1,\dots,p \\ \alpha<\beta}} R^0_\alpha R^0_\beta + \tau \sum_{\substack{\alpha,\beta,\gamma=1,\dots,p \\ \alpha<\beta<\gamma}} R^0_\alpha R^0_\beta R^0_\gamma + \cdots + \tau^{p-2}\prod_{\alpha=1}^{p} R^0_\alpha,$$

then difference scheme (13) can be rewritten in a following way:

$$By_{\bar{t}} + \frac{\nu_3\tau}{2}(E + \tau R^0 + \tau^2 Q_p)y_{\bar{t}t} + Ay = f, \tag{14}$$

or in a canonical form:

$$By_{\overset{\circ}{t}} + \tau^2 \tilde{R} y_{\bar{t}t} + Ay = f, \tag{15}$$

where

$$\tilde{R} = \frac{\nu_3}{2}R^0 + \frac{\nu_3}{2\tau}E - \frac{1}{2\tau}B + \frac{\nu_3\tau}{2}Q_p.$$

Take into account, that

$$B \leq \nu_3 E, \qquad R^0 = \bar{\sigma}A^0, \qquad \bar{\sigma} = \frac{2\sigma}{\nu_3}$$

It is evident that, if $\sigma \geq \frac{1+\epsilon}{4}\nu_2$ $(0 < \epsilon \leq 1)$, then $\tilde{R} \geq \frac{1+\epsilon}{4}A$ and factorised scheme (13) will be absolutely stable in $\overset{\circ}{H}$, which directly follows from sufficient condition of stability of three layer schemes.

3°. It is important to note that for the difference schemes (13) it is possible to construct the effective computing algorithm by means of decomposition of the operator $\prod_{\alpha=1}^{p}(E+\tau R_\alpha^0)$.

We can rewrite difference scheme (13) as follows:

$$\prod_{\alpha=1}^{p} S_\alpha y_{\bar{t}t} = F,$$

where $S_\alpha = E + \tau\bar{\sigma}A_\alpha^0$, $F = \frac{2}{\nu_3\tau}(f - Ay - By_{\bar{t}})$.

The previously defined difference scheme provides the opportunity to construct the following computing algorithm:

$$\begin{aligned} S_1\nu_{(1)} &= F, \\ S_\alpha\nu_{(\alpha)} &= \nu_{(\alpha-1)}, \qquad (\alpha = 2,3,\dots,p) \\ \nu_{(p)} &= y_{tt}, \\ y^{j+1} &= \tau^2\nu_{(p)} + 2y^j - y^{j-1}, \qquad j = 1,2,3,\dots \end{aligned} \tag{16}$$

In order to define the functions $\nu_{(\alpha)}$ at $x \in \Gamma_{h\tau}$, it is necessary to use the following formula:

$$\nu_{(\alpha)} = S_{\alpha+1}\cdots S_p g_{tt}(x,t),$$

where $x \in \gamma_\alpha^+ \cup \gamma_\alpha^- \cup [0,T]$.

Thus, the difference equation (12) is reduced to the system of equations (16).

The obtained problems (16) can be solved by applying one of the one-dimensional double-sweep algorithms (for example, Thomas algorithm [8]). This algorithm is especially convenient for solving the considered problem when function $g(x,t)$ does not depend on t. In this case we obtain homogeneous boundary conditions for functions $\nu_{(\alpha)}$ $(\alpha = 1,2,\dots,p)$.

So, the initial problem (12) can be reduced to a number of problems, each of which can be solved using parallel algorithms [2, 14, 17].

5. Convergence of Difference Scheme

1°. During the construction of initial difference scheme (6), we have required the solution of system of differential equations (2) to belong to the class $C^{4,2}(\bar{G}_T)$. We can represent an approximation error of factorised scheme (14) in a following way $\psi = \psi_0 + \psi_1$, where ψ_0 is an error of approximation of initial difference scheme (6), with an order $O(\tau + |h|^2)$ in the considered class and $\psi_1 = \tau^3\frac{\nu_3}{2}Q_p u_{\bar{t}t}$. From here, it follows that if $p > 2$ requirement $\psi_1 = O(\tau + |h|^2)$ imposes additional restrictions on the smoothness of the solutions of the system of equations (2). However, the a priori estimation and the theorem of convergence in norm of space $\overset{\circ}{W}{}_2^{(1)}$ can be obtained under weaker restrictions than the condition of local approximation of order $O(\tau + |h|^2)$.

2°. To prove the convergence of factorised scheme (14) we will consider vector function $z = y - u$, $(x,t) \in \bar{\Omega}_{h\tau}$, where u is a solution of the initial problem (2)-(4), and y is a solution of the difference scheme (14), (8).

For the grid function z, the following problem is obtained:

$$\begin{cases} Bz_{\overset{\circ}{t}} + \tau^2\tilde{R}z_{\bar{t}t} + Az = \psi, \\ z(0) = 0 \\ z(\tau) = \tau\mu, \end{cases} \tag{17}$$

where $\psi = \psi_0 + \psi_1$ is an error of approximation of the difference scheme (14), and $|\mu(x)| \le \nu_4 =$ const.

We will consider $z = z(t)$ as an abstract function of time argument $t \in \bar{\omega}_\tau$ with values in $\overset{\circ}{H}$, where $\overset{\circ}{H}$ denotes a set of vector functions defined on $\bar{\omega}_h$ and equal to zero on γ_h border of $\bar{\omega}_h$ grid.

To obtain an a priori estimation, it is convenient to reduce the problem (17) to two problems:

$$\begin{cases} Bz_{\overset{\circ}{t}} + \tau^2 \tilde{R} z_{\bar{t}t} + Az = 0, \\ z(0) = 0, \\ z(\tau) = \tau\mu, \end{cases} \tag{18}$$

and

$$\begin{cases} Bz_{\overset{\circ}{t}} + \tau^2 \tilde{R} z_{\bar{t}t} + Az = \psi, \\ z(0) = 0, \\ z(\tau) = 0, \end{cases} \tag{19}$$

The solution of the problem (17) can be represented as $z = z_1 + z_2$, where z_1 is a solution of problem (18), and z_2 is a solution of the problem (19). It is evident that $z_1, z_2 \in \overset{\circ}{H}$.

$3°$. To obtain an a priori estimation, we will use the energetic identity corresponding to factorised difference scheme (17). We will take the scalar product of both sides of the equation (17) on $2\tau z_{\overset{\circ}{t}} = \tau(z_t + z_{\bar{t}}) = \hat{z} - \breve{z}$. Taking into account the expression of operator $\tilde{R}$ and the relationship $\tau z_{\bar{t}t} = z_t - z_{\bar{t}}$, we will get

$$2\tau\left(Bz_{\overset{\circ}{t}}, z_{\overset{\circ}{t}}\right) + \tau^2\left(\left(\frac{\nu_3}{2\tau}E - \frac{1}{2\tau}B\right)(z_t - z_{\bar{t}}), (z_t + z_{\bar{t}})\right) + \\ + \tau^2\left(\left(\frac{\nu_3}{2}R^0 + \frac{\nu_3}{2}\tau Q_p - \frac{1}{2}A\right)(z_t - z_{\bar{t}}), (z_t + z_{\bar{t}})\right) + \\ + \frac{1}{2}\left(A(\hat{z} + \breve{z}), \hat{z} - \breve{z}\right) = 2\tau\left(\psi, z_{\overset{\circ}{t}}\right).$$

After some simple transformations the following expression is obtained:

$$2\tau\left(Bz_{\overset{\circ}{t}}, z_{\overset{\circ}{t}}\right) + \tau^2\left(\left(\frac{\nu_3}{2\tau}E - \frac{1}{2\tau}B\right)(z_t - z_{\bar{t}}), z_t + z_{\bar{t}}\right) + \frac{1}{4}\left(A(\hat{z} + z), \hat{z} + z\right) + \\ + \tau^2\left(\left(\frac{\nu_3}{2}R^0 - \frac{1}{4}A\right) z_t, z_t\right) + \tau^3\frac{\nu_3}{2}(Q_p z_t, z_t) = \\ = \frac{1}{4}(A(z + \breve{z}), z + \breve{z}) + \tau^2\left(\left(\frac{\nu_3}{2}R^0 - \frac{1}{4}A\right) z_{\bar{t}}, z_{\bar{t}}\right) + \tau^3\frac{\nu_3}{2}(Q_p z_{\bar{t}}, z_{\bar{t}}) + 2\tau(\psi, z_{\overset{\circ}{t}}).$$

The previous equality allows us to obtain the energetic identity, which in the case of the problem (17) has the following form:

$$2\tau\left(Bz_{\overset{\circ}{t}}, z_{\overset{\circ}{t}}\right) + \|\hat{z}\|^2_{(2)} = \|z\|^2_{(2)} + \tau F + 2\tau\left(\psi, z_{\overset{\circ}{t}}\right), \tag{20}$$

where

$$\|\hat{z}\|^2_{(2)} = \|z(t+\tau)\|^2_{(2)} = \frac{1}{4}\|\hat{z} + z\|^2_A + \tau^2\left(\left(\frac{\nu_3}{2}R^0 - \frac{1}{4}A\right) z_t, z_t\right) + \\ + \tau^3\frac{\nu_3}{2}(Q_p z_t, z_t) + \tau^2\left(\left(\frac{\nu_3}{2\tau}E - \frac{1}{2\tau}B\right) z_t, z_t\right),$$

$$\|z\|^2_{(2)} = \|z(t)\|^2_{(2)} = \frac{1}{4}\|z + \breve{z}\|^2_A + \tau^2\left(\left(\frac{\nu_3}{2}R^0 - \frac{1}{4}\breve{A}\right) z_{\bar{t}}, z_{\bar{t}}\right) + \\ + \frac{\tau^3\nu_3}{2}(Q_p z_{\bar{t}}, z_{\bar{t}}) + \tau^2\left(\left(\frac{\nu_3}{2\tau}E - \frac{1}{2\tau}B\right) z_{\bar{t}}, z_{\bar{t}}\right),$$

$$F = \frac{1}{4}\left(A_{\bar{t}}(z + \hat{z}), z + \breve{z}\right) + \tau^2\left(\left(\frac{\nu_3}{2}R^0 - \frac{1}{4}A\right)_{\bar{t}} z_{\bar{t}}, z_t\right)$$

and $(Q_p)_{\bar{t}} = 0$, as operator Q_p does not depend on t.

Taking into account the condition of absolute stability of the difference scheme in space $\overset{\circ}{H}$, the following inequality is obtained

$$R = \sigma A^0 + \frac{\nu_3}{2\tau} E \geq \frac{1+\epsilon}{4} A.$$

This condition is valid if

$$R = \sigma A^0 + \frac{\nu_3}{2\tau} E \geq \frac{1+\epsilon}{4} A, \qquad \text{or} \qquad \frac{\nu_3}{2} R^0 \geq \frac{1+\epsilon}{4} A \quad \left(R^0 = \bar{\sigma} A^0, \bar{\sigma} = \frac{2\tau}{\nu_3} \right). \tag{21}$$

The operator $A = A(t)$ is self-conjugate, positive and Lipschitz-continuous with respect to t:

$$\begin{gathered} A^*(t) = A(t) > 0, \\ |(A_t y, y)| \leq \nu_5 |(\breve{A} y, y)|, \\ |(Ay, y)| \leq (1 + \nu_5 \tau)(\breve{A} y, y), \\ \breve{A} = A(t - \tau), \end{gathered} \tag{22}$$

where ν_5 is the Lipschitz constant for operator A.

Moreover, following [9] and taking into account relations (21), (22), we obtain the following estimation:

$$|F| \leq \nu_5 \|z\|^2_{(2)}$$

From here, the energetic inequality follows:

$$2\tau \left(B z_{\overset{\circ}{t}}, z_{\overset{\circ}{t}} \right) + \|\hat{z}\|^2_{(2)} \leq (1 + \nu_5 \tau) \|z\|^2_{(2)} + 2\tau \left(\psi, z_{\overset{\circ}{t}} \right), \tag{23}$$

which allows us to obtain the required a priori estimation.

4°. For the further statement, it is useful to introduce the following notations:

$$\begin{gathered} A^0_\alpha = -T_\alpha T^*_\alpha \qquad \text{or} \qquad A^0_\alpha u = -T_\alpha T^*_\alpha u, \qquad T_\alpha u = u_{\bar{x}_\alpha}, \quad T^*_\alpha u = u_{x_\alpha}, \\ \|y\|^2_{q_{2,p}} = \sum_{\substack{\alpha,\beta=1,\dots,p \\ \alpha<\beta}} \|T_\alpha T_\beta y\|^2_0, \quad \|y\|^2_{q_{3,p}} = \sum_{\substack{\alpha,\beta,\gamma=1,\dots,p \\ \alpha<\beta<\gamma}} \|T_\alpha T_\beta T_\gamma y\|^2_0, \\ \dots, \|y\|^2_{q_{p,p}} = \sum_{\substack{\alpha,\beta=1,\dots,p \\ \alpha<\beta}} \|T_1 T_2 \cdots T_p y\|^2_0. \end{gathered} \tag{24}$$

Theorem 5.1. *Let conditions (I)-(V) from Section 3 be satisfied, regularisator R be defined by equality (10) and $\sigma \geq \frac{1+\epsilon}{4}\nu_2$ ($\epsilon = const > 0$). Then for solution z_1 of the problem (18) on any sequence of grids $\bar{\Omega}_{h\tau}$, the a priori estimation is true*

$$\|z_1^{j+1}\|_{(2)} \leq M_1 \tau \left\{ \|\mu(x)\|^2_0 + \tau \|\mu(x)\|^2_{A^0} + \sum_{s=2}^{p} \tau^s \|\mu(x)\|^2_{q_{s,p}} \right\}^{1/2} \tag{25}$$

where $M_1 > 0$ is a constant and does not depend on the grid, and the norms $\|\mu(x)\|^2_{q_{s,p}}$ are defined by equalities (24).

Proof. Following [9], the inequality expressing the stability of factorised scheme (18) with respect to the initial data is obtained

$$\|z_1^{j+1}\|_{(2)} \leq M_1' \|z(\tau)\|_{(2)}, \qquad j = 1, 2, \dots,$$

where $M_1' > 0$ is constant, and does not depend on the grid.

Let us transform the right side of the inequality:

$$\|z(\tau)\|^2_{(2)} = \frac{1}{4}\|\tau\mu(x)\|^2_{\breve{A}} + \left(\left(\frac{\nu_3}{2}R^0 - \frac{1}{4}\breve{A}\right)\tau\mu(x), \tau\mu(x)\right) +$$

$$+ \frac{\tau^3\nu_3}{2}\left(Q_p\mu(x), \mu(x)\right) + \tau^2\left(\left(\frac{\nu_3}{2\tau}E - \frac{1}{2\tau}B\right)\mu(x), \mu(x)\right) \le$$

$$\le \frac{\nu_3}{2}\tau^2\left(R^0\mu(x), \mu(x)\right) + \frac{\tau^3\nu_3}{2}\left(Q_p\mu(x), \mu(x)\right) + \tau\frac{\nu_3}{2}\left(\mu(x), \mu(x)\right).$$

Furthermore, for transforming expression $(Q_p\mu(x), \mu(x))$ we will take into account Green's first difference formula [11], with $\mu(x) \in H_0$. From this, the validity of inequality (25) follows.

The theorem is proved. □

The following Lemma is required.

Lemma 5.2. *If $a > 0$, $0 < b < \min(1/T, 1)$, $a = const$, $b = const$ and $0 \le \tau \le T$, then*

$$\frac{1 + a\tau}{1 - b\tau} \le 1 + k\tau, \tag{26}$$

where

$$k \ge \frac{a + b}{1 - b\tau} \tag{27}$$

.

Proof. From the inequality (26) can be obtained

$$k \ge \frac{a + b}{1 - b\tau}.$$

The denominator of this fraction has its smallest value at $\tau = T$. Thus, $\max k$ is reached at $\tau = T$, and if k is defined by (27), the inequality (26) will be valid for any $0 < \tau \le T$. □

Theorem 5.3. *Let conditions 3-3 be satisfied. We choose the regulator as follows:*

$$R = \sigma A_0 + \frac{\nu_3}{2\tau}E,$$

where $\sigma \ge \frac{1+\epsilon}{4}\nu_2$, $0 < \epsilon < \min(1, \frac{1}{T})$; ϵ does not depend on the grid. Then a priori estimation for solution z_2 of the problem (20) is valid on any sequence of grids.

$$\|z_2^{j+1}\|_2 \le M_2 \max_{0<t'<t_j}\|\psi_0(t')\|_0 + M_3\tau \max_{0<t'<t_j}\left\{\sum_{s=2}^{p}\tau^{s-2}\|u_{\bar{t}}(t')\|^2_{q_{s,p}}\right\}^{1/2}, \tag{28}$$

where $M_2 > 0$, $M_3 > 0$ are constants, which depend only on ϵ, ν_1, ν_2, ν_5, T and area $\bar{D}_p$.

Proof. Let us rewrite energetic inequality (23) as follows:

$$2\tau\left(Bz_{\overset{\circ}{t}}, z_{\overset{\circ}{t}}\right) + \|\hat{z}\|^2_{(2)} \le (1 + \nu_5\tau)\|z\|^2_{(2)} + 2\tau(\psi_0, z_{\overset{\circ}{t}}) + 2\tau(\psi_1, z_{\overset{\circ}{t}}). \tag{29}$$

In order to obtain the a priori estimation, we have to transform the expression $2\tau(\psi_1, z_{\overset{\circ}{t}})$. Using the Green's first difference formula and ε–inequality [11], we obtain:

$$2\tau(\psi_1, z_{\overset{\circ}{t}}) = \frac{\nu_3}{2}\tau^4 \left\{(Q_p u_{\bar{t}t}, z_t) + (Q_p u_{\bar{t}t}, z_{\bar{t}})\right\} \leq$$
$$\leq M_2'(\bar{\sigma})\frac{\nu_3}{2}\tau^4 \frac{1}{2\epsilon}\left(\sum_{s=2}^{p} \tau^{s-2}\|u_{\bar{t}t}\|^2_{q_{s,p}}\right) + \frac{\nu_3}{2}\tau^4\epsilon\left\{(Q_p z_t, z_t) + (Q_p z_{\bar{t}}, z_{\bar{t}})\right\} \leq$$
$$\leq M_2'(\bar{\sigma})\frac{\nu_3}{4\epsilon}\tau^4 \left(\sum_{s=2}^{p} \tau^{s-2}\|u_{\bar{t}t}\|^2_{q_{s,p}}\right) + \epsilon\tau\|\hat{z}\|^2_{(2)} + \epsilon\tau\|z\|^2_{(2)} \leq$$
$$\leq M''\tau^3 \left(\sum_{s=2}^{p} \tau^{s-2}\|u_{\bar{t}}\|^2_{q_{s,p}}\right) + \epsilon\tau\|\hat{z}\|^2_{(2)} + \epsilon\tau\|z\|^2_{(2)}. \quad (30)$$

where M_2'' is a constant and does not depend on the grid.

On the other hand, the following inequality is valid:

$$2\tau\left(\psi_0, z_{\overset{\circ}{t}}\right) \leq 2\tau\delta\|z_{\overset{\circ}{t}}\|^2_0 + \frac{\tau}{2\delta}\|\psi_0\|^2_0. \quad (31)$$

Substituting estimations (30), (31) in inequality (29), we obtain:

$$(1-\epsilon\tau)\|\hat{z}_2\|^2_{(2)} \leq (1 + (\nu_5 + \epsilon)\tau)\|z\|^2_{(2)} + \frac{\tau}{2\delta}\|\psi_0\|^2_0 + M_2''\tau^3\sum_{s=2}^{p}\tau^{s-2}\|u_{\bar{t}}(t)\|^2_{q_{s,p}}. \quad (32)$$

On the basis of the Lemma 5.2, the inequality (32) can be rewritten as follows:

$$\|\hat{z}_2\|^2_{(2)} \leq (1+\nu_6\tau)\|z\|^2_{(2)} + \tau\nu_7\|\psi_0\|^2_0 + \nu_8\tau^3\sum_{s=2}^{p}\tau^{s-2}\|u_{\bar{t}}(t)\|^2_{q_{s,p}},$$

where ν_6, ν_7, ν_8 are positive constants which do not depend on the grid.

If we summarise the last inequality by $t' = 2\tau, 3\tau, \ldots, T$ and apply the difference analogue of Grnwall's lemma [9], on its basis the a priori estimation (28) can be obtained.

The theorem is proved. □

Merging the results of the Theorems 5.1 and 5.3, we can show validity of the following theorem.

Theorem 5.4. *Let the conditions of the Theorem 5.3 be fulfilled. Then, on any sequence of grids $\bar{\Omega}_{h\tau}$ for the solution of problem (17) the a priori estimation is true:*

$$\|z^{j+1}\|_{(2)} \leq M_1\tau\left\{\|\mu(x)\|^2_0 + \tau\|\mu(x)\|^2_{A^0} + \sum_{s=2}^{p}\tau^s\|\mu(x)\|^2_{q_{s,p}}\right\}^{1/2} +$$
$$+ M_2\max_{0<t'\leq t_j}\|\psi_0(t')\|_0 + M_3\tau\max_{0<t'\leq t_j}\left\{\sum_{s=2}^{p}\tau^{s-2}\|u_{\bar{t}}(t')\|^2_{q_{s,p}}\right\}^{1/2},$$

where M_1, M_2, M_3 are positive constants, which do not depend on the grid .

On the basis of Theorem 5.4, using the estimation $\|z^{j+1}\|_{(2)} \geq \nu_9\|z^{j+1}\|_1$, where $\nu_9 > 0$ is a constant, which does not depend on the grid, it is easy to obtain an a priori estimation for the solution of problem (17) in space $\overset{\circ}{W}{}^{(1)}_2$.

Theorem 5.5. *Let the conditions of Theorem 5.3 be satisfied. Then the a priori estimation for the solution of the problem (17) is true on any sequence of grids* $\bar{\Omega}_{h\tau}$*:*

$$\|z^{j+1}\|_{(1)} \le M_1'\tau \left\{ \|\mu(x)\|_0^2 + \tau\|\mu(x)\|_{A^0}^2 + \sum_{s=2}^{p} \tau^s \|\mu(x)\|_{q_{s,p}}^2 \right\}^{1/2} + \\ + M_2' \max_{0<t'\le t_j} \|\psi_0(t')\|_0 + M_3'\tau \max_{0<t'\le t_j} \left\{ \sum_{s=2}^{p} \tau^{s-2} \|u_{\bar{t}}(t')\|_{q_{s,p}}^2 \right\}^{1/2},$$

where M_1'*,* M_2'*,* M_3' *are positive constants that do not depend on the grid.*

Theorem 5.6. *Let the conditions of the Theorem 5.3 be satisfied. Besides, let the solutions of system of equations (2)* $u \in C^{4,2}$ *when* $p \le 4$ *and* $u \in C^{p,2}$ *when* $p \ge 4$*. Then the solutions of the difference problem (14), (8) converge to the solution of problem (2)-(4) with order* $O(\tau + |h|^2)$ *in norm* $\overset{\circ}{W}{}_2^{(1)}$*.*

6. Algorithm and results of numerical calculations

1°. To describe the algorithm we consider an example of two equations. In this case formulae (16) become:

$$S_1 S_2 y_{\bar{t}t} = F,$$

that is equivalent to

$$\begin{cases} S_1\nu^{(1)} = F \\ S_2\nu^{(2)} = \nu^{(1)} \\ \nu^{(2)} = y_{\bar{t}t}, \text{ or } y^{n+1} = \tau^2\nu^{(2)} + 2y^n - y^{n-1}, \qquad n = 1, 2, \ldots \end{cases} \tag{33}$$

where

$$\nu^{(1)} = (\vec{\nu}_1^{(1)}, \vec{\nu}_2^{(1)}), \qquad \nu^{(2)} = (\vec{\nu}_1^{(2)}, \vec{\nu}_2^{(2)}), \qquad y_{\bar{t}t} = ((\vec{y}_1)_{\bar{t}t}, (\vec{y}_2)_{\bar{t}t}), \qquad F = (\vec{F}_1, \vec{F}_2),$$
$$S_i = E - \tau\bar{\sigma}\Lambda_i^0, \quad \text{where } \Lambda_i^0 y = -y_{\bar{x}_i x_i}, \quad i = 1, 2.$$

In expanded form for $m = 1, 2$; $j = 1, 2, \ldots, N_2 - 1$, the first equation (33) can be written as follows:

$$\tau\bar{\sigma}\nu_m^{(1)}\left((i-1)h, jh, t^{n+1}\right) - (h^2 + 2\tau\bar{\sigma})\nu_m^{(1)}(ih, jh, t^{n+1}) + \tau\bar{\sigma}\nu_m^{(1)}((i+1)h, jh, t^{n+1}) = \\ = -h^2 F_m(ih, jh, t^n), \qquad i = 1, 2, \ldots, N_1 - 1, \tag{34}$$

with given initial and boundary conditions:

$$\nu_m^{(1)}(ih, jh, 0), \nu_m^{(1)}(ih, jh, \tau) \text{ and } \nu_m^{(1)}(0, jh, t^{n+1}), \nu_m^{(1)}(N_1 h, jh, t^{n+1}). \tag{35}$$

In equation (34)-(35) index j is fixed. With respect to the index i we have the linear system of equations with a three-diagonal matrix. Calculations are carried out in parallel with respect to j.

Similarly, the second equation (33) in expanded form for $m = 1, 2$, $i = 1, 2, \ldots, N_1 - 1$, can be written as follows:

$$\tau\bar{\sigma}\nu_m^{(2)}(ih, (j-1)h, t^{n+1}) - (h^2 + 2\tau\bar{\sigma})\nu_m^{(2)}(ih, jh, t^{n+1}) + \tau\bar{\sigma}\nu_m^{(2)}(ih, (j+1)h, t^{n+1}) = \\ = -h^2\nu_m^{(2)}(ih, jh, t^{n+1}), \qquad j = 1, 2, \ldots, N_2 - 1, \tag{36}$$

with given initial and boundary conditions:

$$\nu_m^{(2)}(ih, jh, 0), \quad \nu_m^{(2)}(ih, jh, \tau) \qquad \text{and} \qquad \nu_m^{(2)}(ih, 0, t^{n+1}), \quad \nu_m^{(2)}(ih, N_2 h, t^{n+1}). \tag{37}$$

In equation (36)-(37) index i is fixed. With respect to the index j we have the linear system of equations with a three-diagonal matrix. Calculations are carried out in parallel with respect to i.

Systems with three-diagonal matrices (34) and (36) are solved, using one of the algorithms of one-dimensional double-sweep method. However, the solution of these systems can be carried out in parallel, using parallel computing algorithms (see for example [14, 17]).
2°. The pseudocode of parallel algorithm is given in Algorithm 1:

$$N_1 = (d_2^{(1)} - d_1^{(1)})/h, \quad N_2 = (d_2^{(2)} - d_1^{(2)})/h, \quad b = \tau\bar{\sigma}, \quad a = (h^2 + 2\tau\bar{\sigma}), \quad c = \tau\bar{\sigma}$$

where $x_1 \in [d_1^{(1)}, d_2^{(1)}]$, $x_2 \in [d_1^{(2)}, d_2^{(2)}]$.

Results of calculations are y_1 and y_2 for all (i,j).

"**for**... **do par**" means that calculations can be carried out in parallel, "**for** $i = 1 : N_1 - 1$" means the cycle, when i changes from 1 to $N_1 - 1$ with step 1.

For illustration of the method we consider the following problem. Let $D_2 = (0,1) \times (0,1)$. Γ is its boundary and $\bar{D}_2 = D_2 \cup \Gamma$ its closure, $x = (x_1, x_2)$. Let $G_T = D_2 \times (0,T]$ be a cylinder in 3-dimensional Euclid space and $\bar{G}_T = D_2 \times [0,T]$ is its closure.

We considered the following system:

$$\begin{cases} 2\dfrac{\partial u_1}{\partial t} + \dfrac{\partial u_2}{\partial t} = 5\dfrac{\partial^2 u_1}{\partial x_1^2} + 2\dfrac{\partial^2 u_1}{\partial x_2^2} + 3\dfrac{\partial^2 u_2}{\partial x_1 \partial x_2} + f_1 \\ \dfrac{\partial u_1}{\partial t} + 2\dfrac{\partial u_2}{\partial t} = 2\dfrac{\partial^2 u_2}{\partial x_1^2} + 5\dfrac{\partial^2 u_2}{\partial x_2^2} + 3\dfrac{\partial^2 u_1}{\partial x_1 \partial x_2} + f_2 \end{cases}$$

where

$$f_1(t,x_1,x_2) = 2x_1x_2(x_1-1)(x_2-1)(t+1) - 10tx_2(x_2-1) - 4tx_1(x_1-1) - 3t^2(2x_1-1)(2x_2-1),$$
$$f_2(t,x_1,x_2) = x_1x_2(x_1-1)(x_2-1)(4t+1) - 4t^2x_2(x_2-1) - 10t^2x_1(x_1-1) - 3t(2x_1-1)(2x_2-1).$$

with initial and boundary conditions

$$u_i(x',t) = 0, \qquad (x',t) \in \Gamma \times [0,T],$$
$$u_i(x_1,x_2,0) = 0, \qquad (x_1,x_2) \in (0,1) \times (0,1), \times i = 1,2.$$

In the appropiate scheme constants have the values $\bar{\sigma} = 5/3$, $\nu_3 = 3$, and the exact solution of the system is

$$\begin{cases} u_1(x_1,x_2,t) = tx_1x_2(x_1-1)(x_2-1), \\ u_2(x_1,x_2,t) = t^2x_1x_2(x_1-1)(x_2-1), \end{cases}$$

Results of calculations are shown in Figure 1. Experiments were carried out in the quad-core computer Intel 2 Quad 2.4GHz. Time calculation (sec) is nearly 3 times less in case of quad cores with shared memory, that in case of one core.

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Algorithm 1: Pseudocode of algorithm

```
/* input boundary condition */
for m = 1,2 do
    y_m^(1)(0, 0 : N_2),  y_m^(1)(N_1, 0 : N_2),  y_m^(1)(0 : N_1, 0),  y_m^(1)(0 : N_1, N_2)
    y_m^(2)(0, 0 : N_2),  y_m^(2)(N_1, 0 : N_2),  y_m^(2)(0 : N_1, 0),  y_m^(2)(0 : N_1, N_2)
/* input initial condition */
y_1^(1)(1 : N_1 − 1, 1 : N_2 − 1)
y_2^(1)(1 : N_1 − 1, 1 : N_2 − 1)
/* y^(1)(i, j) = (y_1^(1)(i, j), y_2^(1)(i, j)) - value of solution on t = nτ layer at x_1 = ih, x_2 = jh */
y_1^(2)(1 : N_1 − 1, 1 : N_2 − 1)
y_2^(2)(1 : N_1 − 1, 1 : N_2 − 1)
/* y^(2)(i, j) = (y_1^(2)(i, j), y_2^(2)(i, j)) - value of solution on t = (n − 1)τ layer at x_1 = ih,
   x_2 = jh */
α(2) = c/a
for i = 2 : max(N_1 − 2, N_2 − 2) do
    α(i + 1) = −c/(b ∗ α(i) − a)
for t = 2τ : T, step τ do
    for m = 1,2 do
        /* Solution of the first system */
        for j = 1 : N_2 − 1 do par
            β(2) = h^2 · F_m(h_1, jh_2, t)/a
            for i = 2 : N_1 − 2 do
                β(i + 1) = (−h^2 · F_m(ih_1, jh_2, t) − b · β(i))/(b · α(i) − a)
            ν^(1)(N_1 − 1, j) = (−h^2 · F_m((N_1 − 1)h_1, jh_2, t) − b · β(N_1 − 1))/(b · α(N_1 − 1) − a)
            for i = N_1 − 2 : 1, step −1 do
                ν^(1)(i, j) = α(i + 1) · ν^(1)(i + 1, j) + β(i + 1)
        /* Solution of the second system */
        for i = 1 : N_1 − 1 do par
            β(2) = h^2 · ν^(1)(i, 1)/a
            for j = 2 : N_2 − 1 do
                β(j + 1) = (−h^2 · ν^(1)(i, j) − b · β(j))/(b · α(j) − a)
            ν^(2)(i, N_2 − 1) = (−h^2 · ν^(1)(i, N_2 − 1) − b · β(N_2 − 1))/(b · α(N_2 − 1) − a)
            for j = N_2 − 2 : 1, step −1 do
                ν^(2)(i, j) = α(j + 1) · ν^(2)(i, j + 1) + β(j + 1)
        for i = 1 : N_1 − 1 do par
            for j = 1 : N_2 − 1 do par
                y_m(i, j) = τ^2 · ν^(2)(i, j) + 2y_m^(1)(i, j) − y_m^(2)(i, j)
        y_m^(2) = y_m^(1)
        y_m^(1) = y_m
```


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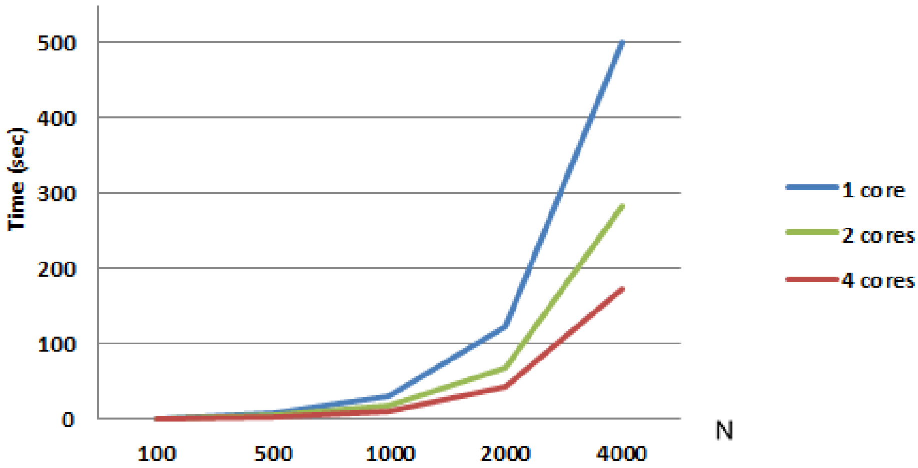

FIGURE 1. Results of calculations, $\tau = 0.01$, $err = 1.2 \times 10^{-2}$

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