SHARP A_1 WEIGHTED ESTIMATES FOR VECTOR VALUED OPERATORS

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ABSTRACT. Given $1 \leq q < p < \infty$, quantitative weighted L^p estimates, in terms of A_q weights, for vector valued maximal functions, Calderón-Zygmund operators, commutators, and maximal rough singular integrals are obtained. The results for singular operators will rely upon suitable convex body domination results, which in the case of commutators will be provided in this work, obtaining as a byproduct a new proof for the scalar case as well.

1. Introduction

We recall that a weight, namely, a non negative locally integrable function w belongs to A_p for $1 < p < \infty$ if

$$[w]_{A_p} = \sup_Q \fint_Q w(x)dx \left(\fint_Q w(y)^{-\frac{p'}{p}} dy \right)^{\frac{p}{p'}} < \infty$$

where as customary, $\fint_Q$ denotes the unweighted average over Q. The A_p class of weights characterizes the $L^p(w)$ boundedness of the maximal function as B. Muckenhoupt established in the 70s. Subsequent works of B. Muckenhoupt himself, R. Wheeden, R. Hunt, R. R. Coifman, and C. Fefferman were devoted to exploring the connection between the A_p class and weighted estimates for singular integrals. However, it was not until the 2000s that the quantitative dependence on the so called A_p constant, namely $[w]_{A_p}$, became a trending topic. Probably the paradigmatic question in that line of research was the A_2 theorem finally established by T. Hytönen [10].

Now we recall that the A_p classes are increasing, so it is natural to define $A_\infty = \bigcup_{p \geq 1} A_p$. T. Hytönen and C. Pérez [12] proved that

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)(x)dx < \infty$$

is the smallest constant characterizing A_∞, at least up until now, and provided a number of quantitative estimates in terms of $[w]_{A_\infty}$. Nevertheless, it is worth noting that essentially the same constant had already appeared in works by Fujii [7] and Wilson [38]. After [12] several papers have been devoted to the study of quantitative weigthed estimates in terms of the A_p and the A_∞ constants.

Among the possible extensions of the classical scalar theory of Calderón-Zygmund operators (see Section 3 for the precise definition), vector valued extensions have received an increasing degree of attention in the last years. Let $W : \mathbb{R}^d \to \mathbb{C}^{n \times n}$ be a matrix weight, namely, a matrix function such that $W(x)$ is positive definite a.e. Given $\vec{f} : \mathbb{R}^d \to \mathbb{C}^n$ and $1 < p < \infty$, we define

$$\|\vec{f}\|_{L^p(W)} = \left(\int_{\mathbb{R}^d} \left| W^{\frac{1}{p}}(x)\vec{f}(x) \right|^p dx \right)^{\frac{1}{p}}.$$

Let $1 < p < \infty$. We say that a matrix weight W is an A_p weight if

$$[W]_{A_p} = \sup_Q \fint_Q \left(\fint_Q \left\| W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y) \right\|^{p'} dy \right)^{\frac{p}{p'}} dx < \infty.$$

Note that in the identities above powers of matrices are involved. Those powers shall be defined via funtional calculus. We refer the interested reader to [11] for more details on this.

Matrix A_p weights were introduced by S. Treil and A. Volberg in [36]. In the late 90s it was shown in a series of works by M. Goldberg [8], F. Nazarov and S. Treil [27], and A. Volberg [37] that if W is a matrix A_p weight then certain classes of Calderón-Zygumund operators (see Section 3 for the precise definition) acting componentwise on $\mathbb{C}^n$ valued functions are bounded on on $L^p(W)$. The definition of A_p that we have presented here is due to S. Roudenko [35] and is equivalent to the definitions in the aforementioned works.

Contrary to what happpens in the scalar setting, the A_2 conjecture remains an open problem in the vector valued setting. In [1], K. Bickel, S. Petermichl, and B. Wick proved that the dependence of the norm of the martingale and Hilbert transform on the A_2 constant of the weight W is at most $[W]_{A_2}^{\frac{3}{2}}\log([W]_{A_2})$. The second author and A. Stoica [32] established that the dependence of the norm of all Calderón-Zygmund operators with cancellation on $[W]_{A_2}$ coincides with the one for the matrix martingale transform, hence reducing the A_2 conjecture for those operators to the proof of the linear bound for the latter.

Given $1\leq q<\infty$, we say that $W\in A^{sc}_{q,\infty}$ if

$$[W]_{A^{sc}_{q,\infty}}=\sup_{e\in\mathbb{C}^n}\left[\left|W^{\frac{1}{q}}e\right|^q\right]_{A_\infty}<\infty.$$

Quite recently F. Nazarov, S. Petermichl, S. Treil, and A. Volberg [25] established the following quantitative estimate for $W\in A_2$,

$$\|T\vec{f}\|_{L^2(W)}\leq c_{n,d,T}[W]_{A_2}^{\frac{1}{2}}[W]_{A^{sc}_{2,\infty}}^{\frac{1}{2}}[W^{-1}]_{A^{sc}_{2,\infty}}^{\frac{1}{2}}\|\vec{f}\|_{L^2(W)}\leq c_{n,T}[W]_{A_2}^{\frac{3}{2}}\|f\|_{L^2(W)}. \tag{1.1}$$

The preceding estimate is obtained using the so called convex body domination. In that work the linear dependence on the A_2 constant is conjectured. In the case of maximal rough singular integrals with $\Omega\in L^\infty(\mathbb{S}^{n-1})$, the following estimate, in the case $p=2$, was quite recently provided by F. Di Plinio, K. Li and T. Hytönen [4],

$$\left\|\sup_{\delta>0}|W^{\frac{1}{2}}T_{\Omega,\delta}\vec{f}|\right\|_{L^2(\mathbb{R}^d)}\leq c_{n,d,T}[W]_{A_2}^{\frac{5}{2}}\|W^{\frac{1}{2}}\vec{f}\|_{L^2(\mathbb{R}^d)}$$

where the scalar operator $T_{\Omega,\delta}$ is defined as follows

$$T_{\Omega,\delta}f(x)=\int_{|x-y|>\delta}\frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^d}f(y)dy.$$

Very recently D. Cruz-Uribe, J. Isralowitz, and K. Moen [3] extended (1.1) to every $1<p<\infty$, providing the following estimate

$$\|T\vec{f}\|_{L^p(W)}\leq c_{n,d,T}\,[W]_{A_p}\left[W^{-\frac{p'}{p}}\right]_{A^{sc}_{p',\infty}}^{\frac{1}{p}}[W]_{A^{sc}_{p,\infty}}^{\frac{1}{p'}}\|f\|_{L^p(W)}\leq c_{n,d,T}\,[W]_{A_p}^{1+\frac{1}{p-1}-\frac{1}{p}}\|\vec{f}\|_{L^p(W)}.$$

Some sharp estimates have been obtained as well in the vector valued setting. T. Hytönen, S. Petermichl and A. Volberg [14], and Isralowitz, Kwon, and the first author [15] established the linear upper bound on $[W]_{A_2}$ for the matrix-weighted square function and the matrix-weighted maximal function, respectively (namely, $M_{W,p}$ defined as in Section 2).

We recall that given a linear operator G and a locally integrable function b, the commutator $[b,G]$ is defined by

$$[b,G]f(x)=b(x)Gf(x)-G(bf)(x).$$

At this point we turn our attention back to the scalar setting. A. Lerner, S. Ombrosi and C. Pérez [19, 21] established the following result for Calderón-Zygmund operators. Given a Calderón-Zygmund operator T and $w\in A_1$ we have that

$$\|Tf\|_{L^p(w)}\leq c_{n,T}pp'[w]_{A_1}^{\frac{1}{p}}[w]_{A_\infty}^{\frac{1}{p'}}\|f\|_{L^p(w)}. \tag{1.2}$$

In the case of commutators, for $b\in BMO$ and T a Calderón-Zygmund operator, C. Ortiz-Caraballo [29] proved that

$$\|[b,T]f\|_{L^p(w)}\leq c_{n,T}\|b\|_{BMO}\left(pp'\right)^2[w]_{A_1}^{\frac{1}{p}}[w]_{A_\infty}^{1+\frac{1}{p'}}\|f\|_{L^p(w)}. \tag{1.3}$$

One of the motivations to obtain such a precise estimate for Calderón-Zygmund operators was to provide a proof of the A_2 constant. Assume that for every $w \in A_1$

$$\|Tf\|_{L^{1,\infty}(w)} \leq c\varphi([w]_{A_1})\|f\|_{L^1(w)}$$

where φ is a submultiplicative, increasing function such that $\varphi(1) \geq 1$. Then we also have that for every $1 < p < \infty$ and every $w \in A_p$ ([19])

$$\|Tf\|_{L^{p,\infty}(w)} \leq c\varphi([w]_{A_p})\|f\|_{L^p(w)}.$$

We observe that in [19] it was proved that $\varphi(t) \leq t\log(e+t)$ using (1.2) as a main ingredient and it was also conjectured that $\varphi(t) \simeq t$. If true, the latter would have led to a proof of the A_2 conjecture, since in [31] it was established that

$$\|T\|_{L^2(w)} \leq c_{n,T}[w]_{A_2} + c_{n,T}\left(\|T\|_{L^2(w)\to L^{2,\infty}(w)} + \|T\|_{L^2(w^{-1})\to L^{2,\infty}(w^{-1})}\right).$$

However, the fact that $\varphi(t) \simeq t$ was disproved in [26], furthermore, in [18] it was established that $\varphi(t) \simeq t\log(e+t)$, and consequently the estimate in [19] is sharp.

2. Main Results

One of the main purposes of this paper is to provide vector valued counterparts of (1.2) and (1.3). To provide that kind of estimates we rely upon the definition of the matrix A_1 class that M. Frazier and S. Roudenko introduced in [6]. Namely, we say that a weight $W \in \mathrm{A}_1$ if

$$[W]_{A_1} = \sup_Q \operatorname{ess\,sup}_{y\in Q} \fint_Q \|W(x)W^{-1}(y)\|dx < \infty.$$

Before presenting our first result we would like to discuss briefly the definition of the maximal function. Due to the non-linearity of the maximal function, when it comes to study weighted estimates for it, the approach that has been mainly considered in the literature is to study weighted variants of it (see [8, 15]). In what follows we will deal with the following weighted maximal functions.

$$M_{W,p}(\vec{f})(x) = \sup_{x\in Q}\fint_Q \left|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\vec{f}(y)\right| dy$$

$$M'_{W,p}(\vec{f})(x) = \sup_{x\in Q}\fint_Q \left|\mathcal{W}_{Q,p}W^{-\frac{1}{p}}(y)\vec{f}(y)\right| dy$$

We direct the reader to Section 4 for the definition of $\mathcal{W}_Q$.

Theorem 1. *Let $W \in A_1$ and $1 < p < \infty$. Then*

$$\|M_{W,p}\|_{L^p(\mathbb{R}^d;\mathbb{C}^n)\to L^p(\mathbb{R}^d)} \leq c_{n,p}[W]_{A_1}^{\frac{1}{p}} \tag{2.1}$$

$$\|M'_{W,p}\|_{L^p(\mathbb{R}^d;\mathbb{C}^n)\to L^p(\mathbb{R}^d)} \leq c_{n,p}[W]_{A_1}^{\frac{1}{p}} \tag{2.2}$$

Let T be a Calderón-Zygmund operator, $b \in BMO$, and $\Omega \in L^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}}\Omega = 0$. If $W \in A_1$ and $1 < p < \infty$, then

$$\|T\|_{L^p(W)\to L^p(W)} \leq c_{n,p,T}[W]_{A_1}^{\frac{1}{p}}[W]_{A^{sc}_{1,\infty}}^{\frac{1}{p'}} \leq c_{n,p,T}[W]_{A_1} \tag{2.3}$$

$$\left\|T^*_{\Omega,W,p}\right\|_{L^p(\mathbb{R}^d;\mathbb{C}^n)\to L^p(\mathbb{R}^d)} \leq c_{n,d,\Omega,p}[W]_{A_1}^{\frac{1}{p}}[W]_{A^{sc}_{1,\infty}}^{1+\frac{1}{p'}} \leq c_{n,d,\Omega,p}[W]_{A_1}^2 \tag{2.4}$$

$$\|[b,T]\|_{L^p(W)\to L^p(W)} \leq c_{n,p,T}\|b\|_{BMO}[W]_{A_1}^{\frac{1}{p}}[W]_{A^{sc}_{1,\infty}}^{1+\frac{1}{p'}} \leq c_{n,p,T}[W]_{A_1}^2 \tag{2.5}$$

where $T^*_{\Omega,W,p}f = \sup_{\delta>0}\left|W^{\frac{1}{p}}T_{\Omega,\delta}\left(W^{-\frac{1}{p}}\vec{f}\right)\right|$.

Coming back once again to the scalar setting, it is a known fact that an extrapolation argument [5, Corollary 4.3] allows one to prove that if we have that

$$\|Gf\|_{L^p(w)} \leq c_{T,p,q}\varphi([w]_{A_1})\|f\|_{L^p(w)},$$

for every A_1 weight, then the same dependence holds as well for every $w \in A_q$ with $1 \leq q < p$, namely,

$$\|Gf\|_{L^p(w)} \leq c_{T,p,q}\varphi([w]_{A_q})\|f\|_{L^p(w)}.$$

Extrapolation arguments, in case of being feasible, have not been developed yet in this setting so we provide a direct proof of the preceding result in the cases considered in Theorem 1. We observe that we recover again the linear dependence already available in the scalar case. We wonder whether it is possible to provide some estimate analogous to the one supremum estimates obtained in [23] and [34].

Theorem 2. *Let $1 < q < p < \infty$ and $W \in A_q$. Then*

$$\|M_{W,p}\|_{L^p(\mathbb{R}^d;\mathbb{C}^n)\to L^p(\mathbb{R}^d)} \leq c_{n,p,T}[W]_{A_q}^{\frac{1}{p}} \tag{2.6}$$

$$\|M'_{W,p}\|_{L^p(\mathbb{R}^d;\mathbb{C}^n)\to L^p(\mathbb{R}^d)} \leq c_{n,p,T}[W]_{A_q}^{\frac{1}{p}} \tag{2.7}$$

Let T be a Calderón-Zygmund operator, $b \in BMO$, and $\Omega \in L^\infty(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}}\Omega = 0$. If $W \in A_1$ and $1 < p < \infty$, then

$$\|T\|_{L^p(W)\to L^p(W)} \leq c_{n,p,q,T}[W]_{A_q}^{\frac{1}{p}}[W]_{A_{q,\infty}^{sc}}^{\frac{1}{p'}} \leq c_{n,p,T}[W]_{A_q} \tag{2.8}$$

$$\left\|T^*_{\Omega,W,p}\right\|_{L^p(\mathbb{R}^d;\mathbb{C}^n)\to L^p(\mathbb{R}^d)} \leq c_{n,d,\Omega,q,p}[W]_{A_q}^{\frac{1}{p}}[W]_{A_{q,\infty}^{sc}}^{1+\frac{1}{p'}} \leq c_{n,d,\Omega,q,p}[W]_{A_q}^2 \tag{2.9}$$

$$\|[b,T]\|_{L^p(W)\to L^p(W)} \leq c_{n,p,q,T}\|b\|_{BMO}[W]_{A_q}^{\frac{1}{p}}[W]_{A_{q,\infty}^{sc}}^{1+\frac{1}{p'}} \leq c_{n,p,q,T}\|b\|_{BMO}[W]_{A_q}^2 \tag{2.10}$$

where $T^*_{\Omega,W,p}f = \sup_{\delta>0}\left|W^{\frac{1}{p}}T_{\Omega,\delta}\left(W^{-\frac{1}{p}}f\right)\right|$.

We would like to note that both in Theorems 1 and 2 the dependencies obtained are the same as the best known ones in the scalar case, and therefore, besides the case of the maximal rough singular integral, are sharp.

The rest of the paper is organized as follows. In Section 3 we present a convex body domination result for commutators. We provide some extra facts about matrix A_p weights in Section 4. Finally, in Section 5 we settle Theorems 1 and 2.

3. Convex body domination for Commutators

We begin the section borrowing some definitions from [17]. We say that a family of cubes $\mathcal{D}$ is a dyadic lattice if it satisfies the following properties

(1) If $Q \in \mathcal{D}$ then every dyadic child of Q belongs to $\mathcal{D}$. In other words, if $\mathcal{D}(Q)$ is the standard grid of dyadic cubes of Q and $Q \in \mathcal{D}$ then $\mathcal{D}(Q) \subseteq \mathcal{D}$.
(2) If $Q_1, Q_2 \in \mathcal{D}$ then there exists a common ancestor in $\mathcal{D}$. That is there exists $Q \in \mathcal{D}$ such that $Q_1, Q_2 \in \mathcal{D}(Q)$.
(3) For every compact set $K \subseteq \mathbb{R}^d$ there exists $Q \in \mathcal{D}$ such that $K \subseteq Q$.

Given $\eta \in (0,1)$ we say that $\mathcal{S} \subset \mathcal{D}$ is a η-sparse family if for every $Q \in \mathcal{S}$ there exists a measurable subset $E_Q \subset Q$ such that

(1) $\eta|Q| \leq |E_Q|$.
(2) The sets E_Q are pairwise disjoint.

Further, given $\Lambda > 1$ we say that $\mathcal{S} \subset \mathcal{D}$ is a Λ Carleson family if for every $Q \in \mathcal{S}$,

$$\sum_{P\in\mathcal{S}, P\subseteq Q}|P| \leq \Lambda|Q|.$$

Clearly every η-sparse family is η^{-1} Carleson, since

$$\sum_{P\in\mathcal{S},P\subseteq Q}|P|\leq\eta^{-1}\sum_{P\in\mathcal{S},P\subseteq Q}|E_P|\leq\Lambda^{-1}|Q|.$$

Though less obvious, the converse is true. Every Λ Carleson family is Λ^{-1} sparse [17, Lemma 6.3]. We will also use without further comment the fact that every Λ Carleson family can be written as a union of m Carleson families, each of which is $1+\frac{\Lambda-1}{m}$ Carleson [17, Lemma 6.6]. Hereafter we will sometimes refer to a family as sparse or Carleson without reference to η or Λ if the specific values of these constants are unimportant.

Convex body domination was introduced by F. Nazarov, S. Petermichl, and A. Volberg in [25]. That notion provides a suitable counterpart to sparse domination in the vector-valued setting. Let $\vec{f}:\mathbb{R}^d\longrightarrow\mathbb{C}^n$. Given a cube Q if additionally $\vec{f}\in L^r(Q)$ where $1\leq r<\infty$ and $r'=\infty$ if $r=1$, we define

$$\langle\langle\vec{f}\rangle\rangle_{r,Q}=\left\{\fint_Q\vec{f}\varphi dx\,:\,\varphi:Q\to\mathbb{R},\,\varphi\in B_{L^{r'}}(Q)\right\}$$

where $B_{L^{r'}(Q)}=\left\{\phi\in L^{r'}(Q)\,:\,\|\phi\|_{L^{r'}}\leq1\right\}$. We will drop the subscript r in the case $r=1$. In [25] it was established that $\langle\langle\vec{f}\rangle\rangle_Q$ is a symetric, convex and compact set in $\mathbb{C}^n$ and in [4] that property was extended to the case $r>1$.

We recall that given T a linear operator, the grand-maximal operator M_T was defined for first as follows in [16]

$$M_Tf(x)=\sup_{Q\ni x}\text{ess sup}_{y\in Q}|T(f\chi_{\mathbb{R}^d\setminus3Q})(y)|.$$

In [25] the authors proved the following result (see also [11]).

Theorem 3. *Let $T:L^1(\mathbb{R}^d)\to L^{1,\infty}(\mathbb{R}^d)$ be a linear operator such that also $M_T:L^1(\mathbb{R}^d)\to L^{1,\infty}(\mathbb{R}^d)$. For $\vec{f}\in L^\infty_c(\mathbb{R}^d;\mathbb{C}^n)$ and $\varepsilon\in(0,1)$ there exists 3^d $(1-\varepsilon)$-sparse collections $\mathcal{S}_j$ of dyadic cubes such that*

$$T\vec{f}(x)\in\frac{c_{d,n}c_T}{\varepsilon}\sum_{j=1}^{3^d}\sum_{Q\in\mathcal{S}_j}\langle\langle\vec{f}\rangle\rangle_Q\chi_Q(x)$$

where $c_T=\|T\|_{L^1\to L^{1,\infty}}+\|M_T\|_{L^1\to L^{1,\infty}}$. More precisely, there exist functions $k_Q\in B_{L^\infty(Q\times Q)}$ such that

$$T\vec{f}(x)=\frac{c_{d,n}c_T}{\varepsilon}\sum_{j=1}^{3^d}\sum_{Q\in\mathcal{S}_j}\left(\fint_Q k_Q(x,y)\vec{f}(y)dy\right)\chi_Q(x).\tag{3.1}$$

Our purpose in this section is to establish the following vector-valued counterpart for commutators extending [22, Theorem 1.1].

Theorem 4. *Let $T:L^1(\mathbb{R}^d)\to L^{1,\infty}(\mathbb{R}^d)$ be a linear operator such that also $M_T:L^1(\mathbb{R}^d)\to L^{1,\infty}(\mathbb{R}^d)$. For $\vec{f}\in L^\infty_c(\mathbb{R}^d;\mathbb{C}^n)$, every $b\in L^1_{loc}$ and $\varepsilon\in(0,1)$ there exist 3^d $(1-\varepsilon)$-sparse collections $\mathcal{S}_j$ of dyadic cubes such that*

$$[b,T]\vec{f}(x)\in\frac{c_{d,n}c_T}{\varepsilon}\sum_{j=1}^{3^d}\sum_{Q\in\mathcal{S}_j}\left[(b(x)-\langle b\rangle_Q)\langle\langle\vec{f}\rangle\rangle_Q\chi_Q(x)+\langle\langle(b-b_Q)\vec{f}\rangle\rangle_Q\chi_Q(x)\right]$$

where $\langle b\rangle_Q$ is the unweighted average of b over Q, each $c_{d,n}$ is a constant depending on n and d, and $c_T=\|T\|_{L^1\to L^{1,\infty}}+\|M_T\|_{L^1\to L^{1,\infty}}$. More precisely, there exist functions k_Q, $k_Q^*\in B_{L^\infty(Q\times Q)}$

such that

$$[b,T]\vec{f}(x)=\frac{c_{d,n}c_T}{\varepsilon}\sum_{j=1}^{3^d}\sum_{Q\in\mathcal{S}_j}(b(x)-\langle b\rangle_Q)\left(\fint_Q k_Q(x,y)\vec{f}(y)dy\right)\chi_Q(x)$$
$$-\frac{c_{d,n}c_T}{\varepsilon}\sum_{j=1}^{3^d}\sum_{Q\in\mathcal{S}_j}\left(\fint_Q k_Q^*(x,y)(b(y)-b_Q)\vec{f}(y)dy\right)\chi_Q(x).$$

At this point we would like recall the definition of Calderón-Zygmund operator. We say that T is a Calderón-Zygmund operator if T is a linear operator bounded on L^2 that admits a representation

$$Tf(x)=\int_{\mathbb{R}^d}K(x,y)f(y)dy \qquad x\notin \operatorname{supp} f$$

for $f\in L_c^\infty(\mathbb{R}^d)$ where $K:\mathbb{R}^d\times\mathbb{R}^d\setminus\Delta\to\mathbb{R}$ with $\Delta=\{(x,x):x\in\mathbb{R}^d\}$ is a locally integrable kernel satisfying the following conditions.

(1) Size condition.
$$|K(x,y)|\leq\frac{c_K}{|x-y|^d}.$$
(2) Smoothness condition. If $|x-y|\geq 2|x-z|$ then
$$|K(x,y)-K(z,y)|+|K(y,x)-K(y,z)|\leq\omega\left(\frac{|x-z|}{|x-y|}\right)\frac{1}{|x-y|^d}$$
where $\omega:[0,1]\to[0,\infty)$ is an increasing, subadditive function with $\omega(0)=0$ satisfying some additional size condition. In the sequel we will assume that the latter is a Dini condition, namely, that
$$\|\omega\|_{\text{Dini}}=\int_0^1\omega(t)\frac{dt}{t}<\infty.$$
Note that this assumption is weaker than the classical Hölder-Lipschitz condition given by the choice $\omega(t)=ct^\delta$ with $c,\delta>0$ that is, probably, the most frequent in the literature.

We observe that A. Lerner [16] proved for Calderón-Zygmund operators that

$$\|M_T\|_{L^1\to L^{1,\infty}}\leq c_d\left(\|T\|_{L^2\to L^2}+c_K+\|\omega\|_{\text{Dini}}\right)$$

and it is also a known fact that

$$\|T\|_{L^1\to L^{1,\infty}}\leq c_d\left(\|T\|_{L^2\to L^2}+\|\omega\|_{\text{Dini}}\right).$$

Consequently Theorems 3 and 4 hold in the case that T is a Calderón-Zygmund operator with

$$c_T=\|T\|_{L^2\to L^2}+c_K+\|\omega\|_{\text{Dini}}.$$

3.1. **Proof of Theorem 4.** Let $\vec{f}\in L_c^\infty(\mathbb{R}^d,\mathbb{C}^n)$ and $b\in L^1_{\text{loc}}$. We further assume that $b\in L^\infty$. We define the $\mathbb{C}^{2n}$ valued function $\tilde{f}$ by

$$\tilde{f}(x)=\begin{pmatrix}\vec{f}(x)\\ \vec{f}(x)\end{pmatrix}$$

and define the $2n\times 2n$ block matrix $\Phi(x)$ by

$$\Phi(x)=\begin{pmatrix}1_{n\times n} & b\otimes 1_{n\times n}\\ 0 & 1_{n\times n}\end{pmatrix}$$

so that

$$\Phi^{-1}(x)=\begin{pmatrix}1_{n\times n} & -b\otimes 1_{n\times n}\\ 0 & 1_{n\times n}\end{pmatrix}.$$

Then we have that

$$\Phi^{-1}(y)\tilde{f}(y)=\begin{pmatrix}1_{n\times n} & -b(y)\otimes 1_{n\times n}\\ 0 & 1_{n\times n}\end{pmatrix}\begin{pmatrix}\vec{f}(y)\\ \vec{f}(y)\end{pmatrix}=\begin{pmatrix}\vec{f}(y)-\vec{f}(y)b(y)\\ \vec{f}(y)\end{pmatrix}.$$

By assumption, $\Phi^{-1}\tilde{f}$ is bounded with compact support. A direct computation shows that

$$\Phi(x)(T\Phi^{-1}\tilde{f})(x) = \begin{pmatrix} T\vec{f}(x) + [b,T]\vec{f}(x) \\ T\vec{f}(x) \end{pmatrix}.$$

Let us plug in $\Phi^{-1}(y)\tilde{f}(y)$ into (3.1) and equate components. Namely

$$\Phi^{-1}(y)\tilde{f}(y) = \begin{pmatrix} 1_{n\times n} & -b(y)\otimes 1_{n\times n} \\ 0 & 1_{n\times n} \end{pmatrix} \begin{pmatrix} \vec{f}(y) \\ \vec{f}(y) \end{pmatrix} = \begin{pmatrix} \vec{f}(y) - \vec{f}(y)b(y) \\ \vec{f}(y) \end{pmatrix}$$

so that

$$\begin{aligned}\Phi(x)(T\Phi^{-1}\tilde{f})(x) &= c_{d,n}c_T \sum_{j=1}^{3^d} \sum_{Q\in\mathcal{S}_j} \Phi(x) \begin{pmatrix} \langle k_Q(x,\cdot)(\vec{f}-\vec{f}b)\rangle_Q \\ \langle k_Q(x,\cdot)\vec{f}\rangle_Q \end{pmatrix} \chi_Q(x) \\ &= c_{d,n}c_T \sum_{j=1}^{3^d} \sum_{Q\in\mathcal{S}_j} \begin{pmatrix} \langle k_Q(x,\cdot)(\vec{f}-\vec{f}b)\rangle_Q + b(x)\langle k_Q(x,\cdot)\vec{f}\rangle_Q \\ \langle k_Q(x,\cdot)\vec{f}\rangle_Q \end{pmatrix} \chi_Q(x).\end{aligned}$$

However, adding and subtracting $\langle k_Q(x,\cdot)\vec{f}\rangle_Q\langle b\rangle_Q$ to the first component, we get

$$\begin{aligned}&\Phi(x)(T\Phi^{-1}\tilde{f})(x) \\ &= c_{d,n}c_T \sum_{j=1}^{3^d} \sum_{Q\in\mathcal{S}_j} \begin{pmatrix} \langle k_Q(x,\cdot)(\vec{f}-\vec{f}b)\rangle_Q + b(x)\langle k_Q(x,\cdot)\vec{f}\rangle_Q \\ \langle k_Q(x,\cdot)\vec{f}\rangle_Q \end{pmatrix} \chi_Q(x) \\ &= c_{d,n}c_T \sum_{j=1}^{3^d} \sum_{Q\in\mathcal{S}_j} \begin{pmatrix} \langle k_Q(x,\cdot)\vec{f}\rangle_Q + \langle k_Q(x,\cdot)\vec{f}(\langle b\rangle_Q - b)\rangle_Q + (b(x)-\langle b\rangle_Q)\langle k_Q(x,\cdot)\vec{f}\rangle_Q \\ \langle k_Q(x,\cdot)\vec{f}\rangle_Q \end{pmatrix} \chi_Q(x).\end{aligned}$$

Hence,

$$\begin{aligned}[b,T]\vec{f}(x) = c_{d,n}c_T \sum_{j=1}^{3^d} \sum_{Q\in\mathcal{S}_j} &\fint_Q k_Q(x,y)(b(y)-\langle b\rangle_Q)\vec{f}(y)\,dy \\ &+ (b(x)-\langle b\rangle_Q)\left(\fint_Q f(y)k_Q(x,y)\vec{f}(y)\,dy\right)\end{aligned}$$

and we are done.

Remark 1. The proof presented above works as well in the case $n = 1$, hence providing a new proof for the scalar case that was settled in [22].

4. The reverse Hölder inquality. A_1, A_q and $A_{q,\infty}^{sc}$ weights

We recall that if $\rho(x)$ is a norm on $\mathbb{C}^n$ then there exists a positive matrix A that we call a reducing operator of ρ where

$$\rho(x) \simeq |Ax| \qquad x \in \mathbb{C}^n.$$

If $1 \le p < \infty$ we will call $\mathcal{W}_{Q,p}$ a reducing operator for

$$\rho_{W,p,Q}(x) = \left(\fint_Q \left|W^{\frac{1}{p}}(t)x\right|^p dt\right)^{\frac{1}{p}}.$$

In the case $1 < p < \infty$ we shall call $\mathcal{W}'_{Q,p}$ a reducing operator for

$$\rho^*_{W,p',Q}(x) = \left(\fint_Q \left|W^{-\frac{1}{p}}(t)x\right|^{p'} dt\right)^{\frac{1}{p'}}.$$

It follows from the proof of Roudenko's characterization [35, Lemma 1.3] that

$$[W]_{A_p} \simeq \|\mathcal{W}_{Q,p}\mathcal{W}'_{Q,p}\|^p \qquad 1 < p < \infty.$$

Now we observe that if we call $V=W^{-\frac{1}{p-1}}$, we have that

$$\rho_{V,p',Q}(x)=\left(\fint_Q\left|V^{\frac{1}{p'}}(t)x\right|^{p'}dt\right)^{\frac{1}{p'}}=\left(\fint_Q\left|W^{-\frac{1}{p}}(t)x\right|^{p'}dt\right)^{\frac{1}{p'}}=\rho^*_{W,p',Q}(x).$$

This yields that we can take $\mathcal{V}_{Q,p'}=\mathcal{W}'_{Q,p}$. Analogously

$$\rho^*_{V,p,Q}(x)=\left(\fint_Q\left|V^{-\frac{1}{p'}}(t)x\right|^{p}dt\right)^{\frac{1}{p}}=\left(\fint_Q\left|W^{\frac{1}{p}}(t)x\right|^{p}dt\right)^{\frac{1}{p}}=\rho_{W,p,Q}(x)$$

and we can choose $\mathcal{V}'_{Q,p'}=\mathcal{W}_{Q,p}$. Consequently we have that

$$[V]_{A_{p'}}\simeq\|\mathcal{V}_{Q,p'}\mathcal{V}'_{Q,p'}\|^{p'}=\|\mathcal{W}_{Q,p}\mathcal{W}'_{Q,p}\|^{p'}.$$

The preceding discussion can be summarized in the following proposition.

Proposition 1. *Let $1<p<\infty$. Then*

$$[W]_{A_p}\simeq\left[W^{-\frac{1}{p-1}}\right]_{A_{p'}}^{\frac{1}{p-1}}.$$

In our next result we show that the A_1 type conditions constants control the corresponding A_∞ constants. We include in the statement the case of the A_q constant that was already established in [3] for the sake of completeness.

Proposition 2. *If $1\leq q<\infty$ and $W\in A_q$, then $[W]_{A^{sc}_{q,\infty}}\leq c_n[W]_{A_q}$.*

Proof. We just settle the case $q=1$. We observe that for every cube Q, a.e $y\in Q$, and every $\vec{e}\in\mathbb{C}^n$

$$\fint_Q\left\|W^{-1}(y)W(x)\right\|dx=\fint_Q\sup_{\vec{f}\neq0}\frac{|W(x)W^{-1}(y)\vec{f}|}{|\vec{f}|}dx\geq\fint_Q\frac{|W(x)\vec{e}|}{|W(y)\vec{e}|}dx$$

or equivalently

$$[W]_{A_1}|W(y)\vec{e}|\geq\fint_Q|W(x)\vec{e}|dx.$$

Hence

$$[W]_{A_1}|W(y)\vec{e}|\geq\sup_{z\in Q}M(\chi_Q|W\vec{e}|)(z)$$

and integrating in y over Q,

$$\int_Q M(\chi_Q|W\vec{e}|)(y)dy\leq|Q|\sup_{z\in Q}M(\chi_Q|W\vec{e}|)(z)\leq[W]_{A_1}\int_Q|W(y)\vec{e}|dy.$$

Consequently

$$\frac{1}{\int_Q|W(y)\vec{e}|dy}\int_Q M(\chi_Q|W\vec{e}|)(y)dy\leq[W]_{A_1}$$

and since the preceding estimate holds for every cube Q and every $\vec{e}$ we have that $[W]_{A^{sc}_{1,\infty}}\leq[W]_{A_1}$. □

Now we recall the quantitative version of the reverse Hölder inequality. This estimate was obtained originally in [12] (see [13] for another proof).

Lemma 1. *Let $w\in A_\infty$ then if $0<\delta\leq\frac{1}{2^{d+11}[w]_{A_\infty}}$ for every cube $Q\subseteq\mathbb{R}^d$ we have that*

$$\left(\fint_Q w^{1+\delta}\right)^{\frac{1}{1+\delta}}\leq2\fint_Q w.$$

We would like to finish the section presenting a technical result that will be crucial for the proof of the main results.

Lemma 2. *Let $1 \leq q < p < \infty$. Assume that $W \in A_q$ and let $r = 1 + \frac{1}{2^{d+11}[W]_{A^{sc}_{q,\infty}}}$. Then we have that for a.e $y \in Q$,*

$$\left(\fint_Q \|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}} \leq c_{n,p,q}\left(\fint_Q \|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^{q}dx\right)^{\frac{1}{p}}.$$

Proof. Choosing $\vec{e}_j(y)$ to be an orthonormal basis of eigenvectors corresponding to the eigenvalues $\lambda_j(y)$ of $W(y)$, we have by the classical Hölder-McCarthy inequality (see [2, Lemma 2.1]) that

$$\begin{aligned}\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\| &\lesssim \sum_{j=1}^{n}\left|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\vec{e}_j(y)\right| = \sum_{j=1}^{n}\lambda_j(y)^{-\frac{1}{p}}\left|W^{\frac{1}{p}}(x)\vec{e}_j(y)\right|\\ &\leq \sum_{j=1}^{n}\lambda_j(y)^{-\frac{1}{p}}\left|W^{\frac{1}{q}}(x)\vec{e}_j(y)\right|^{\frac{q}{p}} = \sum_{j=1}^{n}\left|W^{\frac{1}{q}}(x)\lambda_j(y)^{-\frac{1}{q}}\vec{e}_j(y)\right|^{\frac{q}{p}}\\ &= \sum_{j=1}^{n}\left|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\vec{e}_j(y)\right|^{\frac{q}{p}} \lesssim \left\|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\right\|^{\frac{q}{p}}.\end{aligned}$$

Hence

$$\left(\fint_Q \|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}} \lesssim \left(\fint_Q \|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^{qr}dx\right)^{\frac{1}{rp}}.$$

Now, since $r = 1 + \frac{1}{2^{d+11}[W]_{A^{sc}_{q,\infty}}}$, taking account that $W \in A_q \subset A^{sc}_{q,\infty}$, by reverse Hölder inequality we have that choosing any basis $\{\vec{e}_i\}_{i=1}^n$ of $\mathbb{C}^n$,

$$\begin{aligned}\left(\fint_Q \|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^{qr}dx\right)^{\frac{1}{rp}} &\lesssim \sum_{j=1}^{n}\left(\fint_Q \left|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\vec{e}_i\right|^{qr}dx\right)^{\frac{1}{rp}}\\ &\leq \sum_{j=1}^{n}\left(\fint_Q \left|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\vec{e}_i\right|^{q}dx\right)^{\frac{1}{p}}\\ &\lesssim \left(\fint_Q \|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^{q}dx\right)^{\frac{1}{p}}\end{aligned}$$

and we are done. □

5. Proofs of A_1 and A_q estimates

5.1. Proof of Theorems 1 and 2 for $M'_{W,p}$ and $M_{W,p}$.

5.1.1. *Estimates for $M'_{W,p}$.* First we deal with (2.2) and (2.7). We proceed as follows. Notice that

$$\begin{aligned}\fint_Q\left|\mathcal{W}_{Q,p}W^{-\frac{1}{p}}(y)\vec{f}(y)\right|dy &\leq \fint_Q\left\|\mathcal{W}_{Q,p}W^{-\frac{1}{p}}(y)\right\|\left|\vec{f}(y)\right|dy\\ &\lesssim \fint_Q\left(\fint_Q\left\|W^{\frac{1}{p}}(z)W^{-\frac{1}{p}}(y)\right\|^p dz\right)^{\frac{1}{p}}\left|\vec{f}(y)\right|dy.\end{aligned}$$

If $q = 1$ then it suffices to use Lemma 2 and the definition of A_1 weights to see that

$$\left(\fint_Q\left\|W^{\frac{1}{p}}(z)W^{-\frac{1}{p}}(y)\right\|^p dz\right)^{\frac{1}{p}} \leq c_{n,d}[W]^{\frac{1}{p}}_{A_1}.$$

In that case

$$\fint_Q\left|\mathcal{W}_{Q,p}W^{-\frac{1}{p}}(y)\vec{f}(y)\right|dy \leq c_{n,d}[W]^{\frac{1}{p}}_{A_1}\fint_Q|\vec{f}(y)|dy$$

and using the strong type (p,p) boundedness of the scalar maximal function we are done.

If $q>1$,

$$\begin{aligned}&\fint_Q\left(\fint_Q\left\|W^{\frac{1}{p}}(z)W^{-\frac{1}{p}}(y)\right\|^p dz\right)^{\frac{1}{p}}\left|\vec{f}(y)\right|dy\\&\leq\left(\fint_Q\left(\fint_Q\left\|W^{\frac{1}{p}}(z)W^{-\frac{1}{p}}(y)\right\|^p dz\right)^{\frac{q'}{p}}dy\right)^{\frac{1}{q'}}\left(\fint_Q\left|\vec{f}(y)\right|^q dy\right)^{\frac{1}{q}}.\end{aligned}$$

Now we notice that taking into account Lemma 2,

$$\left(\fint_Q\left(\fint_Q\left\|W^{\frac{1}{p}}(z)W^{-\frac{1}{p}}(y)\right\|^p dz\right)^{\frac{q'}{p}}dy\right)^{\frac{1}{q'}}\leq\left(\fint_Q\left(\fint_Q\left\|W^{\frac{1}{q}}(z)W^{-\frac{1}{q}}(y)\right\|^q dz\right)^{\frac{q'}{p}}dy\right)^{\frac{1}{q'}}.$$

To end the estimate, observe that if we call $V=W^{-\frac{1}{q-1}}$

$$\begin{aligned}&\left[\fint_Q\left(\fint_Q\|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^q dx\right)^{\frac{1}{p}q'}dy\right]^{\frac{1}{q'}}\\&=\left[\fint_Q\left(\fint_Q\|V^{-\frac{1}{q'}}(x)V^{\frac{1}{q'}}(y)\|^q dx\right)^{\frac{q}{p}\frac{q'}{q}}dy\right]^{\frac{1}{q'}}\\&\leq\left[\fint_Q\left(\fint_Q\|V^{-\frac{1}{q'}}(x)V^{\frac{1}{q'}}(y)\|^q dx\right)^{\frac{q'}{q}}dy\right]^{\frac{q}{q'p}}\\&=\left([V]_{A_{q'}}^{q-1}\right)^{\frac{1}{p}}\simeq[W]_{A_q}^{\frac{1}{p}}.\end{aligned}\tag{5.1}$$

where the last step is a direct application of Proposition 1. Then

$$\fint_Q\left|\mathcal{W}_{Q,p}W^{-\frac{1}{p}}(y)\vec{f}(y)\right|\leq c_{n,d}[W]_{A_q}^{\frac{1}{p}}\left(\fint_Q\left|\vec{f}(y)\right|^q dy\right)^{\frac{1}{q}}$$

and using the strong type (p,p) for the scalar operator $M_q(f)=M(|f|^q)^{\frac{1}{q}}$, we are done.

5.1.2. *Estimates for $M_{W,p}$.* We are going to settle (2.1) and (2.6) at the same time. First we note that by the proof of Lemma 2 we have that

$$\|\mathcal{W}_{Q,q}^{\frac{q}{p}}W^{-\frac{1}{p}}(y)\|\lesssim\|\mathcal{W}_{Q,q}W^{-\frac{1}{q}}(y)\|^{\frac{q}{p}}\tag{5.2}$$

for any $Q\in\mathscr{D}$. Fix $J\in\mathscr{D}$. Let $\mathcal{J}(J)$ denote the maximal cubes $L\in\mathcal{D}(J)$ (if any exist) where

$$\left\langle|\mathcal{W}_{J,q}^{\frac{q}{p}}W^{-\frac{1}{p}}\vec{f}|\right\rangle_L>4\left\langle|\mathcal{W}_{J,q}^{\frac{q}{p}}W^{-\frac{1}{p}}\vec{f}|\right\rangle_J.\tag{5.3}$$

By maximality, as usual, we have

$$\begin{aligned}\sum_{L\in\mathcal{J}(J)}|L|&\leq\frac{1}{4\left\langle|\mathcal{W}_{J,q}^{\frac{q}{p}}W^{-\frac{1}{p}}\vec{f}|\right\rangle_J}\sum_{L\in\mathcal{J}(J)}\int_L|\mathcal{W}_{J,q}^{\frac{q}{p}}W^{-\frac{1}{p}}(y)\vec{f}(y)|\,dy\\&\leq\frac{1}{4\left\langle|\mathcal{W}_{J,q}^{\frac{q}{p}}W^{-\frac{1}{p}}\vec{f}|\right\rangle_J}\int_J|\mathcal{W}_{J,q}^{\frac{q}{p}}W^{-\frac{1}{p}}(y)\vec{f}(y)|\,dy\\&=\frac{|J|}{4}.\end{aligned}$$

Now let $\mathcal{F}(J)$ be the collection of cubes in $\mathcal{D}(J)$ that are not a subset of any cube $I \in \mathcal{J}(J)$. Furthermore, for ease of notation let $\cup\mathcal{J}(J) = \cup_{L\in\mathcal{J}(J)}L$. Let

$$M_{J,W}\vec{f}(x) = \sup_{\substack{Q\ni x\\ Q\in\mathcal{D}(J)}} \fint_Q |W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\vec{f}(y)|\,dy.$$

We pointwise dominate $M_{J,W}\vec{f}(x)$ by looking at three cases. First, assume $Q \in \mathcal{F}(J)$ and assume $x \in \cup\mathcal{J}(J)$. Thus, let $x \in Q \in \mathcal{F}(J)$ and $x \in I \in \mathcal{J}(J)$. Then by definition of $\mathcal{F}(J)$ we must have $I \subsetneq Q \subseteq J$ so that in this case, (5.2) and (5.3) gives us

$$\begin{aligned}
\fint_Q |W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\vec{f}(y)|\,dy &\le \sup_{I\in\mathcal{J}(J)}\ \sup_{J\supseteq Q\supsetneq I\ni x} \fint_Q |W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\vec{f}(y)|\,dy\\
&\le \|W^{\frac{1}{p}}(x)\mathcal{W}_{J,q}^{-\frac{q}{p}}\| \sup_{I\in\mathcal{J}(J)}\ \sup_{J\supseteq Q\supsetneq I} \fint_Q |\mathcal{W}_{J,q}{}^{\frac{q}{p}}W^{-\frac{1}{p}}(y)\vec{f}(y)|\,dy\\
&\le 4\|W^{\frac{1}{p}}(x)\mathcal{W}_{J,q}{}^{-\frac{q}{p}}\| \fint_J |\mathcal{W}_{J,q}^{\frac{q}{p}}W^{-\frac{1}{p}}(y)\vec{f}(y)|\,dy\\
&\le 4\|W^{\frac{1}{p}}(x)\mathcal{W}_{J,q}{}^{-\frac{q}{p}}\| \fint_J \|\mathcal{W}_{J,q}^{\frac{q}{p}}W^{-\frac{1}{p}}(y)\||\vec{f}(y)|\,dy\\
&\le 4\|W^{\frac{1}{q}}(x)\mathcal{W}_{J,q}{}^{-1}\|^{\frac{q}{p}} \fint_J \|\mathcal{W}_{J,q}W^{-\frac{1}{q}}(y)\|^{\frac{q}{p}}|\vec{f}(y)|\,dy\\
&= A.
\end{aligned}$$

At this point, if $q = 1$, we have that

$$\begin{aligned}
A &\le 4\|W(x)\mathcal{W}_{J,1}^{-1}\|^{\frac{1}{p}} \fint_J \|\mathcal{W}_{J,1}W^{-1}(y)\|^{\frac{1}{p}}|\vec{f}(y)|\,dy\\
&\le 4c_n[W]_{\mathrm{A}_1}^{\frac{1}{p}}\|W(x)\mathcal{W}_{J,1}^{-1}\|^{\frac{1}{p}}\left\langle|\vec{f}|\right\rangle_J
\end{aligned}$$

and if $1 < q < \infty$,

$$\begin{aligned}
A &\le 4c_n\|W^{\frac{1}{q}}(x)\mathcal{W}_{J,q}^{-1}\|^{\frac{q}{p}} \fint_J \left(\fint_J \|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^q dx\right)^{\frac{1}{q}\frac{q}{p}} |\vec{f}(y)|\,dy\\
&\le 4c_n\|W^{\frac{1}{q}}(x)\mathcal{W}_{J,q}^{-1}\|^{\frac{q}{p}} \left(\fint_J \left(\fint_J \|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^q dx\right)^{\frac{q'}{p}} dy\right)^{\frac{1}{q'}} \left(\fint_J |\vec{f}(y)|^q\,dy\right)^{\frac{1}{q}}\\
&\le 4c_n\|W^{\frac{1}{q}}(x)\mathcal{W}_{J,q}^{-1}\|^{\frac{q}{p}}[W]_{A_q}^{\frac{1}{p}} \left(\fint_J |\vec{f}(y)|^q\,dy\right)^{\frac{1}{q}}.
\end{aligned}$$

Next, assume $Q \in \mathcal{F}(J)$ and $x \notin \cup\mathcal{J}(J)$. Pick a sequence L_k^x of nested dyadic cubes where

$$\{L_k^x\} = \{L \in \mathcal{F}(J) : x \in L\} = \{L \in \mathcal{D}(J) : x \in L\}.$$

But if

$$\sup_k \left\langle |\mathcal{W}_{J,q}^{\frac{q}{p}}W^{-\frac{1}{p}}\vec{f}|\right\rangle_{L_k^x} > 4\left\langle |\mathcal{W}_{J,q}^{\frac{q}{p}}W^{-\frac{1}{p}}\vec{f}|\right\rangle_J$$

then for some k we have

$$\left\langle |\mathcal{W}_{J,q}^{\frac{q}{p}}W^{-\frac{1}{p}}\vec{f}|\right\rangle_{L_k^x} > 4\left\langle |\mathcal{W}_{J,q}^{\frac{q}{p}}W^{-\frac{1}{p}}\vec{f}|\right\rangle_J$$

which means that $x \in L_k^x \subseteq L$ for some $L \in \mathcal{J}(J)$. Thus, bearing the computation above in mind,

$$\begin{aligned}\fint_Q |W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\vec{f}(y)|\,dy &\le \sup_k \fint_{L_k^x} |W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\vec{f}(y)|\,dy\\ &\le \|W^{\frac{1}{p}}(x)\mathcal{W}_{J,q}^{-\frac{q}{p}}\| \sup_k \left\langle |\mathcal{W}_{J,q}^{\frac{q}{p}} W^{-\frac{1}{p}}\vec{f}|\right\rangle_{L_k^x}\\ &\le 4\|W^{\frac{1}{p}}(x)\mathcal{W}_{J,q}^{-\frac{q}{p}}\| \fint_J |\mathcal{W}_{J,q}^{\frac{q}{p}} W^{-\frac{1}{p}}(y)\vec{f}(y)|\,dy\\ &\le 4c_n \|W^{\frac{1}{q}}(x)\mathcal{W}_{J,q}^{-1}\|^{\frac{q}{p}} [W]_{A_q}^{\frac{1}{p}} \left(\fint_J |\vec{f}(y)|^q\,dy\right)^{\frac{1}{q}}\end{aligned}$$

in the case $1 < q < \infty$ and

$$\fint_Q |W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\vec{f}(y)|\,dy \lesssim [W]_{\mathrm{A}_1}^{\frac{1}{p}} \|W(x)\mathcal{W}_{J,1}^{-1}\|^{\frac{1}{p}} \left\langle |\vec{f}|\right\rangle_J$$

in the case $q = 1$.

Lastly, if $Q \notin \mathcal{F}(J)$ then $Q \subseteq L$ for some $L \in \mathcal{J}(J)$ so obviously if $x \in Q$ then $x \in \cup\mathcal{J}(J)$. Combining all this gives

$$M_{J,W}\vec{f}(x) \le \max\left\{4c_n \|W^{\frac{1}{q}}(x)\mathcal{W}_{J,q}^{-1}\|^{\frac{q}{p}} [W]_{A_q}^{\frac{1}{p}} \left(\fint_J |\vec{f}(y)|^q\,dy\right)^{\frac{1}{q}},\ \chi_{\cup\mathcal{J}(J)}(x) \sup_{L\in\mathcal{J}(J)} M_{L,W}\vec{f}(x)\right\}.$$

Thus, for $1 \le q < \infty$,

$$\begin{aligned}&\int_J (M_{J,W}\vec{f}(x))^p\,dx = (4c_n)^p [W]_{\mathrm{A}_q} \left\langle |\vec{f}|\right\rangle_{J,q}^p \int_J \|W^{\frac{1}{q}}(x)\mathcal{W}_{J,q}^{-1}\|^q + \sum_{L\in\mathcal{J}(J)} \int_L (M_{L,W}\vec{f}(x))^p\,dx\\ &= (4c_n)^p [W]_{\mathrm{A}_q} \left\langle |\vec{f}|\right\rangle_{J,q}^p |J| \left(\fint_J \|W^{\frac{1}{q}}(x)\mathcal{W}_{J,q}^{-1}\|^q\right) + \sum_{L\in\mathcal{J}(J)} \int_L (M_{L,W}\vec{f}(x))^p\,dx\\ &\lesssim (4c_n)^p [W]_{\mathrm{A}_q} \left\langle |\vec{f}|\right\rangle_{J,q}^p |J| + \sum_{L\in\mathcal{J}(J)} \int_L (M_{L,W}\vec{f}(x))^p\,dx.\end{aligned}$$

If as usual $\mathcal{J}_k(J) = \{L \in \mathcal{J}(Q) : Q \in \mathcal{J}_{k-1}(J)\}$ with $\mathcal{J}_0(J) = \{J\}$ and $\mathcal{S} = \cup_k \mathcal{J}_k(J)$ then $\mathcal{S}$ is sparse and iteration gives us

$$\begin{aligned}\int_J (M_{J,W}\vec{f}(x))^p\,dx &\lesssim [W]_{\mathrm{A}_q} \sum_{L\in\mathcal{S}} |L| \inf_{x\in L} (M_q(|\vec{f}|)(x))^p\\ &\lesssim [W]_{\mathrm{A}_q} \sum_{L\in\mathcal{S}} |E_L| \inf_{x\in L} (M_q(|\vec{f}|)(x))^p\\ &\lesssim [W]_{\mathrm{A}_q} \sum_{L\in\mathcal{S}} \int_{E_L} (M_q(|\vec{f}|)(x))^p\,dx\\ &\lesssim [W]_{\mathrm{A}_q} \|M_q(|\vec{f}|)\|_{L^p}^p\\ &\lesssim [W]_{\mathrm{A}_q} \|\vec{f}\|_{L^p}^p.\end{aligned}$$

5.2. **Proof of Theorems 1 and 2 for singular operators.**

5.2.1. *A reduction to bump conditions.* We recall that $A : [0,\infty) \to [0,\infty)$ is Young function if $A(0) = 0$ and it is a convex and increasing function. Given a function f and a measurable set E with finite measure, we can define the average on E of f associated to A by

$$\|f\|_{A,E} = \inf\left\{\lambda > 0 : \fint_E A\left(\frac{|f|}{\lambda}\right) \le 1\right\}.$$

From that definition it readily follows that if

$$\lambda_1 \leq \|f\|_{A,Q} \leq \lambda_2$$

then

$$\fint_Q A\left(\frac{|f|}{\lambda_2}\right) \leq 1 \quad \text{and} \quad \fint_Q A\left(\frac{|f|}{\lambda_1}\right) \geq 1. \tag{5.4}$$

Given a Young function A it is natural to define a maximal operator M_A hinging upon the preceding definition of average as follows

$$M_A f(x) = \sup_{Q \ni x} \|f\|_{A,Q}.$$

The boundedness of those operators on L^p spaces was thoroughly studied by C. Pérez [30] under the additional condition that A is doubling, which was proved to be superfluous by Liu and Luque [24]. The condition is the following

$$\|M_A\|_{L^p \to L^p} \leq c_d \left(\int_1^\infty \frac{A(t)}{t^p}\frac{dt}{t}\right)^{\frac{1}{p}}. \tag{5.5}$$

Associated to each Young function we can define the so called associated Young function $\overline{A}$ by

$$\overline{A}(t) = \sup_{s>0}\{st - A(s)\}.$$

This function has some interesting properties. The first of them is that

$$t \leq A^{-1}(t)\overline{A}^{-1}(t) \leq 2t.$$

The second one, that will be very interesting for us, is the following generalized Hölder inequality

$$\fint_Q |fg| \leq 2\|f\|_{A,Q}\|g\|_{\overline{A},Q}.$$

For more details about Young functions we refer the reader to [28, 33].

Let T be a Calderón-Zygmund operator and W, V be matrix weights. If we let

$$T_{\mathcal{S}}^{W,V}\phi(x) = \sum_{Q \in \mathcal{S}} \fint_Q \left\| W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\right\| \phi(y)dy\chi_Q(x)$$

and

$$\begin{aligned}[b,T]_{\mathcal{S}}^{W,V}\phi(x) = &\sum_{Q \in \mathcal{S}} |b(x) - \langle b\rangle_Q| \fint_Q \left\| W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\right\| \phi(y)dy\chi_Q(x)\\ &+ \sum_{Q \in \mathcal{S}} \fint_Q |b(y) - \langle b\rangle_Q| \left\| W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\right\| \phi(y)dy\chi_Q(x)\end{aligned}$$

then Theorems 3 and 4 immediately give us that

$$\|T\|_{L^p(V)\to L^p(W)} \lesssim \sup_{\mathcal{S}} \|T_{\mathcal{S}}^{W,V}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)}$$

and

$$\|[b,T]\|_{L^p(V)\to L^p(W)} \lesssim \sup_{\mathcal{S}} \|[b,T]_{\mathcal{S}}^{W,V}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)}.$$

Armed with the preceding definitions and results and arguing in the spirit of [3] we can prove a lemma that will be fundamental for our purposes.

Lemma 3. *Let A, B be Young functions. Then*

$$\|T_{\mathcal{S}}^{W,V}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \lesssim \|M_{\overline{A}}\|_{L^{p'}}\|M_{\overline{B}}\|_{L^p}\min\{\kappa_1,\kappa_2\}$$

where $\kappa_1 = \sup_Q \|\|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}$ *and* $\kappa_2 = \sup_Q \|\|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{B_y,Q}\|_{A_x,Q}$.

Proof. Without loss of generality we may assume that $f,g\geq 0$. Then taking into account generalized Hölder inequality

$$\begin{aligned}
&\sum_{Q\in\mathcal{S}}\fint_Q\int_Q\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|f(y)g(x)dydx\\
&=\sum_{Q\in\mathcal{S}}\fint_Q\int_Q\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|g(x)f(y)dxdy\\
&\leq 2\sum_{Q\in\mathcal{S}}\|g\|_{\overline{A},Q}\int_Q\|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{A_x,Q}f(y)dy\leq\\
&\leq c\sum_{Q\in\mathcal{S}}\|f\|_{\overline{B},Q}\|g\|_{\overline{A},Q}|E_Q|\|\|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\\
&\leq c\sup_S\|\|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,S}\sum_{Q\in\mathcal{S}}\|f\|_{\overline{B},Q}\|g\|_{\overline{A},Q}|E_Q|\\
&\leq c\|M_{\overline{A}}\|_{L^{p'}}\|M_{\overline{B}}\|_{L^p}\sup_Q\|\|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{p'}(\mathbb{R}^d)}.
\end{aligned}$$

The other estimate is obtained arguing analogously. □

In the case of commutators we can provide the following counterpart

Lemma 4. *Let A,B,C,D be Young functions. Then*

$$\|[b,T]_{\mathcal{S}}^{W,V}\|_{L^p(\mathbb{R}^d;\mathbb{C}^n)\to L^p(\mathbb{R}^d;\mathbb{C}^n)}\lesssim(\Lambda_1+\Lambda_2)$$

where $\Lambda_1=\|M_{\overline{A}}\|_{L^{p'}}\|M_{\overline{B}}\|_{L^p}\min\{\kappa_1,\kappa_2\}$ *with*

$$\begin{aligned}
\kappa_1&=\sup_Q\|\|\|b(x)-\langle b\rangle_Q\|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\\
\kappa_2&=\sup_S\||b(x)-\langle b\rangle_Q|\|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{B_y,Q}\|_{A_x,Q}
\end{aligned}$$

and $\Lambda_2=\|M_{\overline{C}}\|_{L^{p'}}\|M_{\overline{D}}\|_{L^p}\min\{\kappa_3,\kappa_4\}$ *with*

$$\begin{aligned}
\kappa_3&=\sup_Q\||b(y)-\langle b\rangle_Q|\|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{C_x,Q}\|_{D_y,Q}\\
\kappa_4&=\sup_Q\|\|\|b(y)-\langle b\rangle_Q\|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{D_y,Q}\|_{C_x,Q}.
\end{aligned}$$

Proof. We recall that

$$\begin{aligned}
[b,T]_{\mathcal{S}}^{W,V}\phi(x)=&\sum_{Q\in\mathcal{S}}|b(x)-\langle b\rangle_Q|\fint_Q\left\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\right\|\phi(y)dy\chi_Q(x)\\
&+\sum_{Q\in\mathcal{S}}\fint_Q|b(y)-\langle b\rangle_Q|\left\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\right\|\phi(y)dy\chi_Q(x).
\end{aligned}$$

Without loss of generality we may assume that $f,g\geq 0$. For the first term we can argue as follows

$$\begin{aligned}
&\sum_{Q\in\mathcal{S}}\fint_Q |b(x)-\langle b\rangle_Q|\int_Q \|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|f(y)g(x)dydx\\
&=\sum_{Q\in\mathcal{S}}\fint_Q\int_Q |b(x)-\langle b\rangle_Q|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|g(x)f(y)dxdy\\
&\leq 2\sum_{Q\in\mathcal{S}}\|g\|_{\overline{A},Q}\int_Q \||b(x)-\langle b\rangle_Q|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{A_x,Q}f(y)dy\leq\\
&\leq c\sum_{Q\in\mathcal{S}}\|f\|_{\overline{B},Q}\|g\|_{\overline{A},Q}|E_Q|\|\||b(x)-\langle b\rangle_Q|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\\
&\leq c\sup_S \|\||b(x)-b_Q|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,S}\sum_{Q\in\mathcal{S}}\|f\|_{\overline{B},Q}\|g\|_{\overline{A},Q}|E_Q|\\
&\leq c\|M_{\overline{A}}\|_{L^{p'}}\|M_{\overline{B}}\|_{L^p}\sup_Q \|\||b(x)-\langle b\rangle_Q|\|W^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{p'}(\mathbb{R}^d)}.
\end{aligned}$$

Arguing analogously we obtain the rest of the estimates. □

5.2.2. *Proof of estimates* (2.3) *and* (2.5). Again we deal first with (2.3). We will use Lemma 3. Let us choose $\overline{B}(t)=t^{\frac{p+1}{2}}$ and $A(t)=t^{rp}$ with $r=1+\frac{1}{2^{d+11}[W]_{A^{sc}_{1,\infty}}}$. We observe that $B(t)\simeq t^{\frac{p+1}{p-1}}$. It is not hard to check that $\|M_{\overline{B}}\|_{L^p\to L^p}\leq c_d(p')^{\frac{1}{p}}$ and that $\|M_{\overline{A}}\|_{L^{p'}\to L^{p'}}\leq c_d p^{\frac{1}{p'}}[W]^{\frac{1}{p'}}_{A^{sc}_{1,\infty}}$. We observe that using Lemma 2 and the definition of A_1 weight,

$$\begin{aligned}
&\|\|\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\\
&=\left[\fint_Q\left(\fint_Q \|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}\frac{p+1}{p-1}}dy\right]^{\frac{p-1}{p+1}}\\
&\leq c_{n,p}\left[\fint_Q\left(\fint_Q \|W(x)W^{-1}(y)\|dx\right)^{\frac{1}{p}\frac{p+1}{p-1}}dy\right]^{\frac{p-1}{p+1}} \qquad (5.6)\\
&\leq c_{n,p}\left[\fint_Q([W]_{A_1})^{\frac{1}{p}\frac{p+1}{p-1}}dy\right]^{\frac{p-1}{p+1}}=c_{n,p}[W]^{\frac{1}{p}}_{A_1}
\end{aligned}$$

and we are done.

Now we turn our attention to (2.5). We use Lemma 4. First we choose $\overline{B}(t)=t^{\frac{p+1}{2}}$ and $A(t)=t^{sp}$ with $s=\frac{r+1}{2}$ and $r=1+\frac{1}{2^{d+11}[W]_{A^{sc}_{1,\infty}}}$. For that choice of s we have that $\left(\frac{r}{s}\right)'=2r'$. Notice that again $B(t)\simeq t^{\frac{p+1}{p-1}}$, $\|M_{\overline{B}}\|_{L^p\to L^p}\leq c(p')^{\frac{1}{p}}$, and that $\|M_{\overline{A}}\|_{L^{p'}\to L^{p'}}\leq c_d p^{\frac{1}{p'}}[W]^{\frac{1}{p'}}_{A^{sc}_{1,\infty}}$. On the other hand,

$$\begin{aligned}
&\|\||b(x)-\langle b\rangle_Q|\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\\
&=\left[\fint_Q\left(\fint_Q |b(x)-\langle b\rangle_Q|^{sp}\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{sp}dx\right)^{\frac{1}{sp}\frac{p+1}{p-1}}dy\right]^{\frac{p-1}{p+1}}\\
&\leq\left(\fint_Q |b(x)-\langle b\rangle_Q|^{sp\left(\frac{r}{s}\right)'}dx\right)^{\frac{1}{sp\left(\frac{r}{s}\right)'}}\left[\fint_Q\left(\fint_Q \|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}\frac{p+1}{p-1}}dy\right]^{\frac{p-1}{p+1}}\\
&\leq c_d sp\left(\frac{r}{s}\right)'\|b\|_{BMO}\left[\fint_Q\left(\fint_Q \|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}\frac{p+1}{p-1}}dy\right]^{\frac{p-1}{p+1}}.
\end{aligned}$$

From this point arguing as in (5.6) we have that

$$\begin{aligned}&\|\||b(x)-\langle b\rangle_Q|\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\\&\leq c_{n,d}sp\left(\frac{r}{s}\right)'\|b\|_{BMO}[W]_{A_1}^{\frac{1}{p}}\\&\leq c_{n,d,p}\|b\|_{BMO}[W]_{A_{1,\infty}^{sc}}[W]_{A_1}^{\frac{1}{p}}.\end{aligned}$$

For the other term, we choose $\overline{D}(t)=t^{\frac{p+1}{2}}$ and $C(t)=t^{rp}$ with $r=1+\frac{1}{2^{d+11}[W]_{A_{1,\infty}^{sc}}}$. Then

$$\begin{aligned}&\||b(y)-\langle b\rangle_Q|\|\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|\|_{C_x,Q}\|_{D_y,Q}\\&=\left[\fint_Q|b(y)-\langle b\rangle_Q|^{\frac{p+1}{p-1}}\left(\fint_Q\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}\frac{p+1}{p-1}}dy\right]^{\frac{p-1}{p+1}}.\end{aligned}$$

Arguing as above,

$$\left(\fint_Q\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}\frac{p+1}{p-1}}\leq c_n[W]_{A_1}^{\frac{1}{p}\frac{p-1}{p+1}}.$$

Hence

$$\begin{aligned}&\left[\fint_Q|b(y)-\langle b\rangle_Q|^{\frac{p+1}{p-1}}\left(\fint_Q\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}\frac{p+1}{p-1}}dy\right]^{\frac{p-1}{p+1}}\\&\leq c_n[W]_{A_1}^{\frac{1}{p}}\left[\fint_Q|b(y)-\langle b\rangle_Q|^{\frac{p+1}{p-1}}dy\right]^{\frac{p-1}{p+1}}\leq c_{n,d,p}\|b\|_{BMO}[W]_{A_1}^{\frac{1}{p}}.\end{aligned}$$

Consequently gathering all the preceding estimates we obtain (2.5).

5.2.3. *Proof of estimates* (2.8) *and* (2.10). We deal first with (2.8). We rely again upon Lemma 3. We note that choosing $A(t)=t^{rp}$ with $r=1+\frac{1}{2^{d+11}[W]_{A_{q,\infty}^{sc}}}$ and $B(t)=t^{q'}$ we have that $\|M_{\overline{A}}\|_{L^{p'}\to L^{p'}}\leq c_d(r')^{\frac{1}{p'}}\leq c_d[W]_{A_{q,\infty}^{sc}}^{\frac{1}{p'}}$ and $\|M_{\overline{B}}\|_{L^p}\leq c_{d,p,q}$. On the other hand, notice that

$$\begin{aligned}&\|\|\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\\&=\left[\fint_Q\left(\fint_Q\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}q'}dy\right]^{\frac{1}{q'}}.\end{aligned}$$

By Lemma 2

$$\left(\fint_Q\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}}\leq c_{n,d,p,q}\left(\fint_Q\|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^{q}dx\right)^{\frac{1}{p}}. \tag{5.7}$$

Then

$$\|\|\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\leq c_{n,d,p,q}\left[\fint_Q\left(\fint_Q\|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^{q}dx\right)^{\frac{q'}{p}}dy\right]^{\frac{1}{q'}}$$

and by (5.1) we have that

$$\|\|\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\leq c_{n,d,p,q}[W]_{A_q}^{\frac{1}{p}}.$$

Gathering the preceding estimates and taking into account that $r=1+\frac{1}{2^{d+11}[W]_{A_{q,\infty}^{sc}}}$ we obtain that (2.8) holds.

Let us deal now with (2.10). Arguing analogously as above, we will use Lemma 4. First we choose $B(t)=t^{q'}$ and $A(t)=t^{sp}$ with $s=\frac{r+1}{2}$ and $r=1+\frac{1}{2^{d+11}[W]_{A^{sc}_{q,\infty}}}$. For that choice of s we have that $\left(\frac{r}{s}\right)'=2r'$. It is also straightforward that $\|M_{\overline{B}}\|_{L^p}\leq c_{n,p,q}$ and that $\|M_{\overline{A}}\|_{L^{p'}}\leq c_{n,d}p\,(s')^{\frac{1}{p'}}\leq c_{n,d}p[W]^{\frac{1}{p'}}_{A^{sc}_{q,\infty}}$. Then we have that

$$
\begin{aligned}
&\||\|b(x)-\langle b\rangle_Q|\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\\
&=\left[\fint_Q\left(\fint_Q|b(x)-\langle b\rangle_Q|^{sp}\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{sp}dx\right)^{\frac{1}{sp}q'}dy\right]^{\frac{1}{q'}}\\
&\leq\left(\fint_Q|b(x)-\langle b\rangle_Q|^{sp\left(\frac{r}{s}\right)'}dx\right)^{\frac{1}{sp\left(\frac{r}{s}\right)'}}\left[\fint_Q\left(\fint_Q\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}q'}dy\right]^{\frac{1}{q'}}\\
&\leq c_dr'\|b\|_{BMO}\left[\fint_Q\left(\fint_Q\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}q'}dy\right]^{\frac{1}{q'}}.
\end{aligned}
$$

Arguing as above,

$$
\||\|b(x)-\langle b\rangle_Q|\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|\|_{A_x,Q}\|_{B_y,Q}\leq c_{d,p,q,n}\|b\|_{BMO}[W]_{A^{sc}_{q,\infty}}[W]^{\frac{1}{p}}_{A_q}.
$$

For the other term we note that choosing $C(t)=t^{rp}$ with $r=1+\frac{1}{2^{d+11}[W]_{A^{sc}_{q,\infty}}}$ and $D(t)=t^{q'}$ we have that $\|M_{\overline{C}}\|_{L^{p'}}\leq c_d(r')^{\frac{1}{p'}}\leq c_{n,d}[W]^{\frac{1}{p'}}_{A^{sc}_{q,\infty}}$ and $\|M_{\overline{D}}\|_{L^p}\leq c_{d,p,q}$. We observe that

$$
\begin{aligned}
&\||b(y)-\langle b\rangle_Q|\|\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|\|_{C_x,Q}\|_{D_y,Q}\\
&=\left[\fint_Q|b(y)-\langle b\rangle_Q|^{q'}\left(\fint_Q\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}q'}dy\right]^{\frac{1}{q'}}.
\end{aligned}
$$

Taking into account (5.7), and arguing as above

$$
\begin{aligned}
&\left[\fint_Q|b(y)-\langle b\rangle_Q|^{q'}\left(\fint_Q\|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{rp}dx\right)^{\frac{1}{rp}q'}dy\right]^{\frac{1}{q'}}\\
&\leq c_n\left[\fint_Q|b(y)-\langle b\rangle_Q|^{q'}\left(\fint_Q\|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^{q}dx\right)^{\frac{q'}{p}}dy\right]^{\frac{1}{q'}}\\
&\leq c_n\left[\fint_Q|b(y)-\langle b\rangle_Q|^{q'\left(\frac{p}{q}\right)'}dy\right]^{\frac{1}{q'\left(\frac{p}{q}\right)'}}\left[\fint_Q\left(\fint_Q\|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^{q}dx\right)^{\frac{q'}{q}}dy\right]^{\frac{q}{q'p}}\\
&\leq c_{n,d,p,q}\|b\|_{BMO}\left[\fint_Q\left(\fint_Q\|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^{q}dx\right)^{\frac{q'}{q}}dy\right]^{\frac{q}{q'p}}\\
&\leq c_{n,d,p,q}\|b\|_{BMO}[W]^{\frac{1}{p}}_{A_q}.
\end{aligned}
$$

Gathering all the choices and estimates above, a direct application of Lemma 4 yields the desired estimate.

5.3. **Proof of the estimates for Maximal Rough Singular Integrals.** Arguing as in [4], we have that

$$\left\|T^*_{\Omega,W,p}\vec{f}\right\|_{L^p(\mathbb{R}^d)} \lesssim \left\|M_{W,p}\vec{f}\right\|_{L^p(\mathbb{R}^d)} + \sup_{\||g\||_{L^{p'}(\mathbb{R}^d)}=1} \inf_{\varepsilon>0} \sup_{\mathcal{S}} \frac{1}{\varepsilon} \sum_{Q\in\mathcal{S}} |Q| \left\langle\left\langle W^{-\frac{1}{p}}\vec{f}\right\rangle\right\rangle_{1+\varepsilon,Q} \left\langle\left\langle W^{\frac{1}{p}}\vec{g}\right\rangle\right\rangle_{1+\varepsilon,Q}$$

where we interpret the product in second term as the right endpoint of the Minkowski product

$$AB = \{(a,b) : a\in A, b\in B\}$$

which, in the case of $A, B\subset\mathbb{R}^d$ being convex symmetric sets, is a closed symmetric interval. The estimate for the first term is (2.1) in the case $q=1$ and (2.6) in the case $q>1$, so we are left with settling the estimate for the second term. We proceed as follows.

First we notice that if $a\in\left\langle\left\langle W^{-\frac{1}{p}}\vec{f}\right\rangle\right\rangle_{1+\varepsilon,Q}$ and $b\in\left\langle\left\langle W^{\frac{1}{p}}\vec{g}\right\rangle\right\rangle_{1+\varepsilon,Q}$ then

$$|(a,b)| \leq \fint_Q\fint_Q |(W^{-\frac{1}{p}}(x)\vec{f}(x)\varphi_{a,Q}(x), W^{\frac{1}{p}}(y)\vec{g}(y)\psi_{b,Q}(y))|dxdy$$

where $\varphi_{a,Q},\psi_{b,Q}\in L^{(1+\varepsilon)'}(Q)$. Since $W^{\frac{1}{p}}$ is positive definite and symmetric a.e. we have that

$$\begin{aligned}
&\fint_Q\fint_Q \left|\left(W^{-\frac{1}{p}}(x)\vec{f}(x)\varphi_{a,Q}(x), W^{\frac{1}{p}}(y)\vec{g}(y)\psi_{b,Q}(y)\right)\right| dxdy\\
&=\fint_Q\fint_Q \left|\left(W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\vec{f}(x)\varphi_{a,Q}(x), \vec{g}(y)\psi_{b,Q}(y)\right)\right| dxdy\\
&\leq\fint_Q\fint_Q |W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\vec{f}(x)||\varphi_{a,Q}(x)||\vec{g}(y)||\psi_{b,Q}(y)|dxdy\\
&\leq\fint_Q |\vec{g}(y)||\psi_{b,Q}(y)|\left(\fint_Q |W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\vec{f}(x)|^{1+\varepsilon}dx\right)^{\frac{1}{1+\varepsilon}}\left(\fint_Q|\varphi_{a,Q}(x)|^{(1+\varepsilon)'}dx\right)^{\frac{1}{(1+\varepsilon)'}}dy\\
&\leq\fint_Q |\vec{g}(y)|\left(\fint_Q |W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\vec{f}(x)|^{1+\varepsilon}dx\right)^{\frac{1}{1+\varepsilon}}|\psi_{b,Q}(y)|dy\\
&\leq\left(\fint_Q |\vec{g}(y)|^{1+\varepsilon}\left(\fint_Q |W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\vec{f}(x)|^{1+\varepsilon}dx\right)dy\right)^{\frac{1}{1+\varepsilon}}\left(\fint_Q|\psi_{b,Q}(y)|^{(1+\varepsilon)'}dy\right)^{\frac{1}{(1+\varepsilon)'}}\\
&\leq\left(\fint_Q\fint_Q |W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\vec{f}(x)|^{1+\varepsilon}|\vec{g}(y)|^{1+\varepsilon}dxdy\right)^{\frac{1}{1+\varepsilon}}.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{\varepsilon}\sum_{Q\in\mathcal{S}}|Q|\left\langle\left\langle W^{-\frac{1}{p}}\vec{f}\right\rangle\right\rangle_{1+\varepsilon,Q}\left\langle\left\langle W^{\frac{1}{p}}\vec{g}\right\rangle\right\rangle_{1+\varepsilon,Q}\\
&\lesssim\frac{1}{\varepsilon}\sum_{Q\in\mathcal{S}}|Q|\left(\fint_Q\fint_Q |W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\vec{f}(x)|^{1+\varepsilon}|\vec{g}(y)|^{1+\varepsilon}dxdy\right)^{\frac{1}{1+\varepsilon}}.
\end{aligned}$$

We are going to obtain a suitable control for this term providing an argument analogous to the one we gave for Calderón-Zygmund operators. Let $\tau = 8\cdot 2^{d+11}$,

$$\varepsilon=\frac{1}{p\frac{p}{p-q}\tau[W]_{A^{sc}_{q,\infty}}}\qquad s=1+\frac{1}{p'\frac{p}{p-q}(\tau-2)[W]_{A^{sc}_{q,\infty}}}$$

Then we have that for $u \in Q$

$$\begin{aligned}
&\left(\fint_Q \fint_Q |W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\vec{f}(x)|^{1+\varepsilon}|\vec{g}(y)|^{1+\varepsilon}dxdy\right)^{\frac{1}{1+\varepsilon}}\\
&\leq \left(\fint_Q |\vec{f}(x)|^{1+\varepsilon}\fint_Q \|W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\|^{1+\varepsilon}|\vec{g}(y)|^{1+\varepsilon}dydx\right)^{\frac{1}{1+\varepsilon}}\\
&\leq \left(\fint_Q |\vec{f}(x)|^{1+\varepsilon}\left(\fint_Q \|W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\|^{ps(1+\varepsilon)}dy\right)^{\frac{1}{ps(1+\varepsilon)}(1+\varepsilon)}dx\right)^{\frac{1}{1+\varepsilon}} M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(u).
\end{aligned}$$

It is not hard to check that

$$\|M_{(ps)'(1+\varepsilon)}\|_{L^{p'}(\mathbb{R}^d)} \lesssim \left([W]_{A^{sc}_{q,\infty}}\right)^{\frac{1}{p'}} \qquad 1 \leq q < p. \tag{5.8}$$

Indeed, taking into account (5.5) it suffices to prove that

$$\frac{1}{p' - (ps)'(1+\varepsilon)} \lesssim [W]_{A^{sc}_{q,\infty}}.$$

First we note that

$$\frac{1}{p' - (ps)'(1+\varepsilon)} = \frac{1}{p}\left[\frac{(ps-1)(p-1)}{(ps-1) - s(p-1)(1+\varepsilon)}\right].$$

Working on the denominator we have that

$$\begin{aligned}
&(ps-1) - s(p-1)(1+\varepsilon)\\
&= (ps-1) + (-sp+s)(1+\varepsilon) = ps - 1 - ps - ps\varepsilon + s + s\varepsilon\\
&= -1 - ps\varepsilon + s + s\varepsilon = -1 + ((1-p)\varepsilon + 1)\left(1 + (p-1)\frac{\tau}{\tau-2}\varepsilon\right)\\
&= -1 - (p-1)^2\frac{\tau}{\tau-2}\varepsilon^2 + (p-1)\frac{\tau}{\tau-2}\varepsilon - (p-1)\varepsilon + 1\\
&= -(p-1)^2\frac{\tau}{\tau-2}\varepsilon^2 + (p-1)\frac{\tau}{\tau-2}\varepsilon - (p-1)\varepsilon\\
&= (p-1)\varepsilon\left[-(p-1)\frac{\tau}{\tau-2}\varepsilon + \frac{\tau}{\tau-2} - \frac{\tau-2}{\tau-2}\right]\\
&= \frac{(p-1)\varepsilon}{\tau-2}\left[2 - (p-1)\tau\varepsilon\right].
\end{aligned}$$

It is clear that $(p-1)\tau\varepsilon \leq 1$. Combining this estimate with the identities above,

$$\begin{aligned}
\frac{1}{p' - (ps)'(1+\varepsilon)} &= \frac{1}{p}\frac{(ps-1)(p-1)}{(ps-1) - s(p-1)(1+\varepsilon)} = \frac{\tau-2}{p}\frac{(ps-1)(p-1)}{\varepsilon(p-1)\left[2-(p-1)\tau\varepsilon\right]}\\
&= \frac{(\tau-2)}{\varepsilon p}\frac{(ps-1)}{\left[2-(p-1)\tau\varepsilon\right]} \leq (ps-1)\frac{(\tau-2)}{\varepsilon p} \lesssim [W]_{A^{sc}_{q,\infty}}.
\end{aligned}$$

Now we focus on the proof of (2.4). In that case, by Lemma 2, since $(1+\varepsilon)s \leq 1 + \frac{1}{2^{d+11}[W]_{A^{sc}_{q,\infty}}}$, for $u \in Q$ we have

$$\begin{aligned}
&\left(\fint_Q |\vec{f}(x)|^{1+\varepsilon}\left(\fint_Q \|W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\|^{ps(1+\varepsilon)}dy\right)^{\frac{1}{ps(1+\varepsilon)}(1+\varepsilon)}dx\right)^{\frac{1}{1+\varepsilon}} M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(u)\\
&\leq \left(\fint_Q |\vec{f}(x)|^{1+\varepsilon}\left(\fint_Q \|W(y)W^{-1}(x)\|dy\right)^{\frac{1}{p}(1+\varepsilon)}dx\right)^{\frac{1}{1+\varepsilon}} M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(u)\\
&\leq [W]_{A_1}^{\frac{1}{p}} M_{1+\varepsilon}(|\vec{f}|)(u) M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(u).
\end{aligned}$$

This yields that

$$\begin{aligned}
&\frac{1}{\varepsilon}\sum_{Q\in\mathcal{S}}|Q|\left(\fint_Q\fint_Q|W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\vec{f}(x)|^{1+\varepsilon}|\vec{g}(y)|^{1+\varepsilon}dxdy\right)^{\frac{1}{1+\varepsilon}}.\\
&\lesssim[W]_{A_{1,\infty}^{sc}}[W]_{A_1}^{\frac{1}{p}}\sum_{Q\in\mathcal{S}}|Q|\inf_{x\in Q}M_{1+\varepsilon}(|\vec{f}|)(x)M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(x)\,dx\\
&\lesssim[W]_{A_{1,\infty}^{sc}}[W]_{A_1}^{\frac{1}{p}}\int_{\mathbb{R}^d}M_{1+\varepsilon}(|\vec{f}|)M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(x)\\
&\lesssim[W]_{A_{1,\infty}^{sc}}[W]_{A_1}^{\frac{1}{p}}\|M_{1+\varepsilon}\|_{L^p(\mathbb{R}^d)}\|M_{(ps)'(1+\varepsilon)}\|_{L^{p'}(\mathbb{R}^d)}\|\vec{f}\|_{L^p(\mathbb{R}^d;\mathbb{C}^n)}\|\vec{g}\|_{L^{p'}(\mathbb{R}^d;\mathbb{C}^n)}\\
&\lesssim[W]_{A_{1,\infty}^{sc}}^{1+\frac{1}{p'}}[W]_{A_1}^{\frac{1}{p}}\|\vec{f}\|_{L^p(\mathbb{R}^d;\mathbb{C}^n)}\|\vec{g}\|_{L^{p'}(\mathbb{R}^d;\mathbb{C}^n)}
\end{aligned}$$

by (5.8) and taking into account that from (5.5) it follows that

$$\|M_{1+\varepsilon}\|_{L^p(\mathbb{R}^d)}\lesssim(p')^{\frac{1}{p}}.$$

In the case $q>1$, namely, to settle (2.9), we argue as follows. By Lemma 2, since $(1+\varepsilon)s\leq 1+\frac{1}{2^{d+11}[W]_{A_{q,\infty}^{sc}}}$

$$\begin{aligned}
&\left(\fint_Q|\vec{f}(x)|^{1+\varepsilon}\left(\fint_Q\|W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\|^{ps(1+\varepsilon)}dy\right)^{\frac{1}{ps(1+\varepsilon)}(1+\varepsilon)}dx\right)^{\frac{1}{1+\varepsilon}}M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(u)\\
&\leq\left(\fint_Q|\vec{f}(x)|^{1+\varepsilon}\left(\fint_Q\|W^{\frac{1}{q}}(y)W^{-\frac{1}{q}}(x)\|^{q}dy\right)^{\frac{1}{p}(1+\varepsilon)}dx\right)^{\frac{1}{1+\varepsilon}}M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(u).
\end{aligned}$$

If we call $W=V^{1-q}$, then for $u\in Q$ we that

$$\begin{aligned}
&\left(\fint_Q|\vec{f}(x)|^{1+\varepsilon}\left(\fint_Q\|V^{-\frac{1}{q'}}(y)V^{\frac{1}{q'}}(x)\|^{q}dy\right)^{\frac{1}{p}(1+\varepsilon)}dx\right)^{\frac{1}{1+\varepsilon}}M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(u)\\
&\leq\left(\left(\fint_Q\|V^{-\frac{1}{q'}}(y)V^{\frac{1}{q'}}(x)\|^{q}dy\right)^{\frac{q'}{q}}dx\right)^{\frac{q}{q'p}}M_{\left(\frac{pq'}{q(1+\varepsilon)}\right)'(1+\varepsilon)}(|\vec{f}|)(u)M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(u)\\
&\leq[V]_{A_{q'}}^{\frac{q-1}{p}}M_{\left(\frac{pq'}{q(1+\varepsilon)}\right)'(1+\varepsilon)}(|\vec{f}|)(u)M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(u)\\
&\lesssim[W]_{A_q}^{\frac{1}{p}}M_{\left(\frac{pq'}{q(1+\varepsilon)}\right)'(1+\varepsilon)}(|\vec{f}|)(u)M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(u)
\end{aligned}$$

where the last estimate follows from Proposition 1. Taking that estimate into account,

$$\begin{aligned}
&\frac{1}{\varepsilon}\sum_{Q\in\mathcal{S}}|Q|\left(\fint_Q\fint_Q|W^{\frac{1}{p}}(y)W^{-\frac{1}{p}}(x)\vec{f}(x)|^{1+\varepsilon}|\vec{g}(y)|^{1+\varepsilon}dxdy\right)^{\frac{1}{1+\varepsilon}}\\
&\lesssim [W]_{A^{sc}_{q,\infty}}[W]^{\frac{1}{p}}_{A_q}\sum_{Q\in\mathcal{S}}|Q|\inf_{x\in Q}M_{\left(\frac{pq'}{q(1+\varepsilon)}\right)'(1+\varepsilon)}(|\vec{f}|)(x)M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(x)\\
&\lesssim [W]_{A^{sc}_{q,\infty}}[W]^{\frac{1}{p}}_{A_q}\int_{\mathbb{R}^d}M_{\left(\frac{pq'}{q(1+\varepsilon)}\right)'(1+\varepsilon)}(|\vec{f}|)(x)M_{(ps)'(1+\varepsilon)}(|\vec{g}|)(x)\,dx\\
&\lesssim [W]_{A^{sc}_{q,\infty}}[W]^{\frac{1}{p}}_{A_q}\left\|M_{\left(\frac{pq'}{q(1+\varepsilon)}\right)'(1+\varepsilon)}\right\|_{L^p(\mathbb{R}^d)}\|M_{(ps)'(1+\varepsilon)}\|_{L^{p'}(\mathbb{R}^d)}\|\vec{f}\|_{L^p(\mathbb{R}^d;\mathbb{C}^n)}\|\vec{g}\|_{L^{p'}(\mathbb{R}^d;\mathbb{C}^n)}\\
&\lesssim [W]^{1+\frac{1}{p'}}_{A^{sc}_{q,\infty}}[W]^{\frac{1}{p}}_{A_q}\|\vec{f}\|_{L^p(\mathbb{R}^d;\mathbb{C}^n)}\|\vec{g}\|_{L^{p'}(\mathbb{R}^d;\mathbb{C}^n)}
\end{aligned}$$

by (5.8) and taking into account that from (5.5), it is not hard to derive that

$$\left\|M_{\left(\frac{pq'}{q(1+\varepsilon)}\right)'(1+\varepsilon)}\right\|_{L^p(\mathbb{R}^d)}\lesssim\left(\frac{p}{p-q}\right)^{\frac{1}{p}}.$$

This ends the proof of (2.9).

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