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Measures of inclusion and entropy based on the φ-index of inclusion*

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Abstract

Surprisingly, despite that fuzzy sets were introduced more than fifty years ago, there is not consensus yet about how to extend the notion of inclusion in such a framework. Recently, alternatively to previous methods in the literature, we introduced an approach in which we make use of the so-called φ-index of inclusion. This approach has a main difference with respect to previous ones: the degree of inclusion is identified with a function instead of with a value in $[0, 1]$, although such a feature makes it difficult to compare the φ-index of inclusion with existing axiomatic approaches concerning measures of inclusion. This is the reason why in this paper we define two different and natural measures of inclusion by means of the φ-index of inclusion and, then, show that both measures satisfy some standard axiomatic approaches about measures of inclusion in the literature. In addition, taking into account the relationship of fuzzy entropy with Young axioms for measures of inclusion, we present also a measure of entropy based on the φ-index of inclusion that is in accordance with the axioms of De Luca and Termini.

Keywords: Fuzzy sets, Measure of inclusion, Measure of fuzzy entropy, f-inclusion. φ-index of inclusion.

1. Introduction

Although fuzzy sets were introduced more than fifty years ago [31], still there is not consensus in the community about how to generalize the notion of inclusion is such a paradigm. The first attempt was done in the seminal paper of Zadeh by identifying inclusion with the point-wise ordering between membership functions. Such a proposal has been considered as a referent but, at

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the same time, was criticized *"for being rigid and for the lack of softness according to the spirit of fuzzy logic"* (quoted from [8]). For such a reason, during the last years, many other proposals have been proposed for a suitable generalization of the notion of inclusion between fuzzy sets. Those approaches can be classified mainly in three different groups: based on cardinality [9, 14, 18]; based on logic implications [1, 4, 13]; and based on axiomatic definitions [3, 11, 12, 16, 30]. The latter group plays the most important role for our standpoint, since it proposes a number of properties that "any good generalization of the notion of inclusion" should satisfy. In all the axiomatic approaches, the inclusion between two fuzzy sets is identified by a value in the unit interval $[0, 1]$; the assignment of such a value is given by means of the so-called "measure of inclusion". In consequence, every approach that proposes a constructive measure of inclusion must be compared with those axiomatic approaches.

In [22], the authors proposed a new approach to represent the inclusion of a fuzzy set into another; the so-called φ-index of inclusion. It follows the following motto, proposed in [18] and supported by many other authors, *"A 'good' measure of inclusion should measure violations of Zadeh's inclusion"*. The main differences are not in the underlying ideas, but in the mathematical structure of degrees of inclusion: whereas most of the approaches in the literature assign a value in $[0, 1]$ to represent a greater or weaker inclusion, in [22] inclusion is represented in terms of monotonic mappings from $[0, 1]$ to $[0, 1]$. In spite of the difficulties that such a feature involves for a comparison between the φ-index of inclusion and the rest of axiomatic approaches, we have shown that the φ-index of inclusion satisfies almost all the axioms required by the approaches of Sinha & Dougherty and Kitainik [25, 16] obviously, after a natural rewriting of such axioms (see [20, 21]). The only axiom that it is not satisfied is the one related to complements; namely, the one that extends the crisp relationship $A \subseteq B$ if and only if $B^c \subseteq A^c$. In the case of the φ-index of inclusion, such a relationship with the complement is given in terms of adjoint pairs [20] whereas in the Sinha & Dougherty and Kitainik [25, 16] axioms, the relationship is given by an equality. Anyway, the comparison with the respective axiom proposed by Sinha & Dougherty and Kitainik is somewhat unfair, since adjoint pairs cannot be rewritten in a natural manner in terms of specific values in $[0, 1]$.

In this paper, we present two measures of inclusion constructed from the φ-index of inclusion that assign a value in the unit interval $[0, 1]$ to each pair of fuzzy sets. These measures neither need any prior assumptions, nor external operators (e.g., such as fuzzy implications), nor consider a bizarre formula; they just integrate the φ-index of inclusion. With such measures of inclusion, the comparison with standard axiomatic approaches in the literature is much fairer. The first measure of inclusion satisfies the Fan, Xie & Pei [12] axioms and the axiom related to complement proposed by Sinha & Dougherty [25] and Kitainik [16] commented in the previous paragraph. However, and surprisingly, the proposed measure of inclusion does not satisfy another axiom proposed by both Sinha & Dougherty and Kitainik related to union and intersection between fuzzy sets. The surprise comes from the fact that such an axiom is satisfied directly by the φ-index of inclusion by simply substituting minimum and maximum operators, respectively, by infimum and supremum operators [20]. Therefore, we may conclude that all the properties that Sinha & Dougherty and Kitainik attribute to every measure of inclusion are underlying in our approach based on the φ-index of inclusion.

The second proposed measure of inclusion satisfies the four axioms proposed by Young [30] and also those proposed by Fan, Xie & Pei [12]. Since the four axioms of Young are very related to the measures of fuzzy entropy proposed by De Luca & Termini [10], we also propose the use of the φ-index of inclusion for the definition of measures of entropy via negation operators. Specifically, we show that the proposed measure of entropy satisfies the three axioms proposed by De Luca & Termini [10] and moreover, it covers the class of Knopfmacher [17] and Capocelli & De Luca [6] measures of fuzzy entropy.

The paper is structured as follows. In Section 2 we recall the axiomatic definition of measure of fuzzy entropy given by De Luca & Termini, the four most relevant axiomatic approaches about inclusion measures between fuzzy sets (namely, Kitainik [16], Sinha & Dougherty [25], Young [30], and Fan, Xie & Pie [12]) and the φ-index of inclusion together with its main properties. In Section 3, we define the first measure of inclusion and show some of their properties and relationships with the axiomatic approaches of Kitainik, Sinha & Dougherty and Fan, Xie & Pie approaches. Subsequently, in Section 4 we firstly present a measure of inclusion related to the Young axiomatic approach and secondly, a measure of fuzzy entropy related to the axioms given by De Luca & Termini. Finally, in Section 5 we provide some conclusions and prospects for future works.

2. Preliminaries

2.1. Basics on fuzzy sets

A fuzzy set A is a pair $(\mathcal{U}, \mu_A)$ where $\mathcal{U}$ is a non-empty set (called the universe of A) and μ_A is a mapping from $\mathcal{U}$ to $[0,1]$ (called membership function of A). In general, the universe is fixed by the context and, therefore, fuzzy sets are determined by their membership functions. Hence, for the sake of clarity, we identify fuzzy sets with membership functions (i.e., $A(u) = \mu_A(u)$).

We denote by $\mathcal{F}(\mathcal{U})$ the set of fuzzy sets defined on the universe $\mathcal{U}$. On $\mathcal{F}(\mathcal{U})$ we can extend the usual crisp operations of union, intersection and complement as follows: Given two fuzzy sets A and B, we define

- (union) $(A \cup B)(u) = \max\{A(u), B(u)\}$
- (intersection) $(A \cap B)(u) = \min\{A(u), B(u)\}$
- (complement) $A^c(u) = 1 - A(u)$.

The previous extensions of union, intersection and complement are the classical ones and the reader must be aware about the existence of many other options. For example, as generalization of the previous extensions, many authors use t-norms to generalize intersection, t-conorms to generalize union, negation operators to generalize the complement or other more complex structures as residuated lattices [24] or multi-adjoint lattices [23]. We consider these classical extensions in order to present a fairer comparison with approaches about standard well-known axiomatic measures of inclusion and entropy between fuzzy sets that were originally formulated in such contexts.

It is worth to recall also that any transformation $T\colon \mathcal{U} \to \mathcal{U}$ in the universe can be extended to the set of fuzzy sets $\mathcal{F}(\mathcal{U})$ by defining for each $A \in \mathcal{F}(\mathcal{U})$ the fuzzy set $T(A)(u) = A(T(u))$ for $u \in \mathcal{U}$.

2.2. Axiomatic measure of Fuzzy Entropy

De Luca & Termini provided in [10] an axiomatic definition for measures of fuzzy entropies that grouped other well-known approaches about fuzzy entropy (as Kosko or Yager approaches [18]) and that has become the referent approach concerning this topic. Let us recall that the measures of fuzzy entropy, in opposite to Shanon entropy, measures ‘the fuzziness of a fuzzy set.’

Definition 1. *A mapping* $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \to \mathbb{R}$ *is called a* measure of fuzzy entropy *if it satisfies the following axioms for all fuzzy sets* A *and* B*:*

(E1) $E(A) = 0$ *if and only if* $A(u) \in \{0, 1\}$ *for all* $u \in \mathcal{U}$*.*

(E2) $E(A)$ *reaches the maximum value if and only if* $A(u) = 0.5$ *for all* $u \in \mathcal{U}$*.*

(E3) $E(A) \leq E(B)$ *if* A *is a refinement of* B*; that is* $A(u) \leq B(u)$ *if* $B(u) \leq 0.5$ *and* $B(u) \leq A(u)$ *if* $B(u) \geq 0.5$*.*

One can find in the literature a number of different constructive techniques to define measures of fuzzy entropy, i.e. mapping satisfying the previous axioms [6, 17, 2].

2.3. Axiomatic measures of inclusion

In this section we recall, under our point of view, the four most renowned approaches concerning the axiomatic definition of measures of inclusion. It is worth mentioning the existence of other important axiomatic approaches, like [28, 19, 5, 4, 15], which in general, can be considered particular cases of some the four approaches recalled below.

2.3.1. Kitainik axioms

Leonid Kitainik [16] proposed an axiomatic definition for measures of inclusion aimed at capturing those based on fuzzy implications, and approach extensively applied in the eighties [1, 27].

Definition 2. *A mapping* $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0, 1]$ *is called a* K-measure of inclusion *if it satisfies the following axioms for all fuzzy sets* A, B *and* C*:*

(K1) $\mathcal{I}(A, B) = \mathcal{I}(B^c, A^c)$.

(K2) $\mathcal{I}(A, B \cap C) = \min\{\mathcal{I}(A, B), \mathcal{I}(A, C)\}$.

(K3) *If* $T\colon \mathcal{U} \to \mathcal{U}$ *is a bijective transformation on the universe, then* $\mathcal{I}(A, B) = \mathcal{I}(T(A), T(B))$*.*

(K4) *If* A *and* B *are crisp then* $\mathcal{I}(A, B) = 1$ *if and only if* $A \subseteq B$*.*

(K5) *If* A *and* B *are crisp then* $\mathcal{I}(A, B) = 0$ *if and only if* $A \not\subseteq B$*.*

In [13], Fodor and Yager showed that, for every K-measure of inclusion $\mathcal{I}$, there exists a fuzzy implication $\to$ such that for all fuzzy sets A and B, the following equality holds

$$\mathcal{I}(A, B) = \bigwedge_{u \in \mathcal{U}} A(u) \to B(u)\,.$$

In other words, the axiomatic definition of Kitainik aims at characterizing the measures of inclusion based on fuzzy implications.

2.3.2. Sinha & Dougherty axioms

One of the more relevant measures of inclusion was proposed by Divyendu Sinha and Edward R. Dougherty in 1993 [25]. In the original definition, they proposed nine axioms but, it was proved later (see [8, 4]) that one of them could be inferred from the others. We present below the definition with the eight independent axioms.

Definition 3. *A mapping* $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0, 1]$ *is called an* SD-measure of inclusion *if it satisfies the following axioms for all fuzzy sets* A, B *and* C*:*

(SD1) $\mathcal{I}(A, B) = 1$ *if and only if* $A(u) \le B(u)$ *for all* $u \in \mathcal{U}$.

(SD2) $\mathcal{I}(A, B) = 0$ *if and only if there exists* $u \in \mathcal{U}$ *such that* $A(u) = 1$ *and* $B(u) = 0$.

(SD3) *If* $B(u) \le C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(A, B) \le \mathcal{I}(A, C)$.

(SD4) *If* $B(u) \le C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(C, A) \le \mathcal{I}(B, A)$.

(SD5) *If* $T\colon \mathcal{U} \to \mathcal{U}$ *is a bijective transformation on the universe, then* $\mathcal{I}(A, B) = \mathcal{I}(T(A), T(B))$.

(SD6) $\mathcal{I}(A, B) = \mathcal{I}(B^c, A^c)$.

(SD7) $\mathcal{I}(A \cup B, C) = \min\{\mathcal{I}(A, C), \mathcal{I}(B, C)\}$.

(SD8) $\mathcal{I}(A, B \cap C) = \min\{\mathcal{I}(A, B), \mathcal{I}(A, C)\}$.

Sinha & Dougherty's paper did not include any comparison with Kitainik's approach, despite of the similarity between them. Actually, note that every SD-measure of inclusion is also a K-measure of inclusion.

2.3.3. Young axioms

Virginia R. Young [30] proposed in 1996 an axiomatic approach for measures of inclusion with obvious differences with respect to Kitainik's and Sinha & Dougherty's: the main idea being measuring the inclusion degree of $A \cup A^c$ into $A \cap A^c$.

Definition 4. *A mapping* $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ *is called a* Y-measure of inclusion *if it satisfies the following axioms for all fuzzy sets* A, B *and* C*:*

(Y1) $\mathcal{I}(A,B) = 1$ *if and only if* $A(u) \leq B(u)$ *for all* $u \in \mathcal{U}$*.*

(Y2) *If* $A(u) \geq 0.5$ *for all* $u \in \mathcal{U}$*, then* $\mathcal{I}(A, A^c) = 0$ *if and only if* $A = \mathcal{U}$*; i.e.,* $A(u) = 1$ *for all* $u \in \mathcal{U}$*.*

(Y3) *If* $A(u) \leq B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(C,A) \leq \mathcal{I}(B,A)$*.*

(Y4) *If* $B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(A,B) \leq \mathcal{I}(A,C)$ *for all fuzzy set* $A \in \mathcal{F}(\mathcal{U})$*.*

In the original definition [30], axioms (Y3) and (Y4) were stated as one axiom. We have decided to split the axiom in two for the sake of presentation.

2.3.4. Fan, Xie & Pei axioms

Fan, Xie & Pie [12] revised in 1999 the axiomatic definition provided by Young. As a result of such a review, they proposed three different axiomatic definitions of measure of inclusion with stronger and weaker restrictions.

Definition 5. *A mapping* $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ *is said to be a* strong FXP-inclusion measure *if it satisfies the following axioms for all fuzzy sets* A, B *and* C*:*

(sFXP1) *If* $A(u) \leq B(u)$ *for all* $u \in \mathcal{U}$*, then* $\mathcal{I}(A,B) = 1$*.*

(sFXP2) *If* $A \neq \varnothing$ *and* $A \cap B = \varnothing$ *then,* $\mathcal{I}(A,B) = 0$*.*

(sFXP3) *If* $A(u) \leq B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(C,A) \leq \mathcal{I}(B,A)$ *and* $\mathcal{I}(A,B) \leq \mathcal{I}(A,C)$*.*

Due to the restrictive condition imposed by axiom (sFXP2), Fan, Xie & Pie also proposed the following alternatives.

Definition 6. *A mapping* $\mathcal{I}\colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0,1]$ *is said to be a* FXP-inclusion measure *if it satisfies the following axioms for all fuzzy sets* A, B *and* C*:*

(FXP1) *If* $A(u) \le B(u)$ *for all* $u \in \mathcal{U}$*, then* $\mathcal{I}(A, B) = 1$.

(FXP2) $\mathcal{I}(\mathcal{U}, \varnothing) = 0$.

(FXP3) *If* $A(u) \le B(u) \le C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(C, A) \le \mathcal{I}(B, A)$ *and* $\mathcal{I}(A, B) \le \mathcal{I}(A, C)$.

Definition 7. *A mapping* $\mathcal{I} \colon \mathcal{F}(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \to [0, 1]$ *is said to be* weak FXP-inclusion measure *if it satisfies the following axioms for all fuzzy sets* A, B *and* C*:*

(wFXP1) $\mathcal{I}(\varnothing, \varnothing) = \mathcal{I}(\varnothing, \mathcal{U}) = \mathcal{I}(\mathcal{U}, \mathcal{U}) = 1$*; where* $\mathcal{U}(u) = 1$ *for all* $u \in \mathcal{U}$.

(wFXP2) $\mathcal{I}(\mathcal{U}, \varnothing) = 0$

(wFXP3) *If* $A(u) \le B(u) \le C(u)$ *for all* $u \in \mathcal{U}$ *then* $\mathcal{I}(C, A) \le \mathcal{I}(B, A)$ *and* $\mathcal{I}(A, B) \le \mathcal{I}(A, C)$.

In the original paper of Fan, Xie & Pie [12] the reader can find relationships between these measures and the theory of fuzzy implications.

2.4. The φ*-index of inclusion*

In contrast with the previous approaches concerning axiomatic measures of inclusion with range in the unit interval $[0, 1]$, Madrid and Ojeda-Aciego proposed an approach that assigns a mapping $f \colon [0, 1] \to [0, 1]$ to represent such an inclusion. Such a divergent procedure has links with the axiomatic definition of inclusion recalled above [21], but the comparison is complicated due to the essentially different nature of both approaches. In order to present here such an approach, it is necessary to define the set of φ-indexes of inclusion.

Definition 8. *The set of* φ*-indexes of inclusion, denoted by* Ω*, is the set of monotonically increasing mappings* $f \colon [0, 1] \to [0, 1]$ *such that* $f(x) \le x$ *for all* $x \in [0, 1]$.

The notion of f-inclusion [22] is defined as follows.

Definition 9. *Let* A *and* B *be two fuzzy sets and consider* $f \in \Omega$*. We say that* A is f-included in B *(denoted by* $A \subseteq_f B$*) if and only if the inequality* $f(A(u)) \le B(u)$ *holds for all* $u \in \mathcal{U}$*.*

Note that, fixed $f \in \Omega$, the relation of f-inclusion is a crisp relation and, in general, is not even an ordering relation (since transitivity fails); hence, at first view, the f-inclusion (with a fixed $f \in \Omega$) seems to be unsuitable to represent the inclusion between two fuzzy sets. However, we do not define the φ-index of inclusion by fixing an f-inclusion, but we consider all of them as different degrees of inclusion. Specifically, since each f-inclusion is determined by a mapping f in Ω, we consider mappings in Ω as indexes of inclusions. Such a consideration is described and motivated in detail in [22] and can be summarized in the following items:

- Ω has the structure of complete lattice with the natural ordering between functions; i.e., given $f, g \in \Omega$, we say that $f \leq g$ if $f(x) \leq g(x)$ for all $x \in [0,1]$. In particular, the mappings id (defined by $id(x) = x$ for all $x \in [0,1]$) and $\bot$ (defined by $\bot(x) = 0$ for all $x \in [0,1]$) are the top and bottom elements in Ω, respectively.

- Each $f \in \Omega$ determines a restriction, via the corresponding f-inclusion, that can be understood as "*how much do we have to reduce the truth values of a fuzzy set in order to be included into another in Zadeh's sense*". Let us recall that a common motto followed by many approaches concerning measures of inclusion is to measure how much Zadeh's inclusion fails to hold [18]. In this respect, each $f \in \Omega$ can be seen as a different degree of violation of the Zadeh's inclusion.

- Finally, the greater the mapping $f \in \Omega$ the stronger the restriction imposed by the f-inclusion. In particular, the id-inclusion is the most restrictive case (and is equivalent to Zadeh's inclusion) and the $\bot$-inclusion does not establish any restriction at all (below we show that $\bot$-inclusion is identified with no inclusion).

The φ-index of inclusion is based on the idea "the more f-inclusions holding between two sets, the greater is the inclusion". Fortunately, we do not need to check all the f-inclusions between two sets thanks to the following lemmas.

Lemma 1. *Let A and B be two fuzzy sets and let $f, g \in \Omega$ such that $f \leq g$. Then, $A \subseteq_g B$ implies $A \subseteq_f B$.*

Lemma 2. *Let A and B be two fuzzy sets and consider a family $\{f_i\}_{i \in I} \subseteq \Omega$. If A is f_i-included in B for all $i \in I$, then A is $\bigvee_{i \in I} f_i$-included in B.*

As a direct consequence of the previous two lemmas, given two fuzzy sets, the subset $\Lambda(A, B) = \{f \in \Omega \mid A \subseteq_f B\}$ has a maximum element in Ω. This fact allows us to introduce the following definition .

Definition 10 (φ-index of inclusion)**.** *Let A and B be two fuzzy sets, the* φ-index of inclusion *of A in B, denoted by $Inc(A, B)$, is defined as the maximum of $\Lambda(A, B)$.*

Note firstly that the φ-index of inclusion of A in B does not depend on any prior assumption or any kind of parameter [13]. Secondly, note that thanks to Lemma 1, the set $\Lambda(A, B)$ of mappings $f \in \Omega$ such A is f-included in B is characterized by $Inc(A, B)$, since:

$$\Lambda(A, B) = \{f \in \Omega \mid A \subseteq_f B\} = \{f \in \Omega \mid f \leq Inc(A, B)\}$$

In [21], the following analytical expression for $Inc(A, B)$ was given:

Theorem 1 ([21])**.** *Let A and B be two fuzzy sets, then $Inc(A, B) = f_{A,B} \wedge id$, where*

$$f_{A,B}(x) = \bigwedge_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\}.$$

We summarize below some properties that motivate the use of $Inc(A, B)$ as a suitable index of inclusion between two fuzzy sets.

Theorem 2 ([21])**.** *Let A, B and C be fuzzy sets,*

1. *(Full inclusion) $Inc(A, B) = id$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.*
2. *(Null inclusion) $Inc(A, B) = \bot$ if and only if there is a set of elements in the universe $\{u_i\}_{i \in I} \subseteq \mathcal{U}$ such that $A(u_i) = 1$ for all $i \in I$ and $\bigwedge_{i \in I} B(u_i) = 0$.*
3. *(Pseudo transitivity) $Inc(B, C) \circ Inc(A, B) \leq Inc(A, C)$.*
4. *(Monotonicity) If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $Inc(C, A) \leq Inc(B, A)$.*
5. *(Monotonicity) If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $Inc(A, B) \leq Inc(A, C)$;*
6. *(Transformation Invariance) Let A and B be two L-fuzzy sets and let $T : \mathcal{U} \to \mathcal{U}$ be a transformation on $\mathcal{U}$, then $Inc(A, B) = Inc(T(A), T(B))$.*
7. *(Relationship with intersection) $Inc(A, B \cap C) = Inc(A, B) \wedge Inc(A, C)$.*
8. *(Relationship with union) $Inc(A \cup B, C) = Inc(A, C) \wedge Inc(B, C)$.*

In order to maintain this paper self-contained, before introducing the final property of φ-indexes of inclusion, let us recall the notion of *adjoint pair*:

Given a complete lattice L, we say that a pair (f, g) of mappings $f, g : L \to L$ is an adjoint pair (or satisfies the adjoint property) in L if

$$f(x) \leq y \iff x \leq g(y) \qquad \text{for all } x \in L$$

Theorem 3 ([21]). *(Relationship with complement) Let (f, g) be an adjoint pair in the unit interval, and n and involutive negation, then*

$$A \subseteq_f B \text{ if and only if } B^c \subseteq_{nogon} A^c.$$

3. A measure of inclusion based on the φ-index of inclusion

The relationships between the φ-index of inclusion and the axiomatic approaches given by Sinha & Dougherty, Kitainik, Young, and Fan, Xie & Pie are appreciable in Theorem 2, once the respective axioms are conveniently rewritten in functional terms. The reader can note that the only axioms that do not hold are (K1), (SD6), (Y2) and (sFXP2): Axioms (K1) and (SD6) are related to the complement of fuzzy sets and the comparison with the corresponding result related to the φ-index of inclusion (Theorem 3) may be somewhat unfair. Actually, in this section we present a natural measure of inclusion from the φ-index of inclusion that satisfies (K1) and (SD6). On the other hand, axioms (Y2) and (sFXP2) are related to (fuzzy) entropy measures of fuzzy sets and in the next section we define another measure of inclusion from the φ-index of inclusion that satisfies them.

On the one hand, the most common way to measure functions is by integrals. On the other hand, since by Theorem 2 we have that the greater the φ-index of inclusion, the greater the inclusion (monotonicity), we propose the following measure of inclusion.

Definition 11. *Let A and B be two fuzzy sets, the* φ-measure of inclusion *of A in B, denoted by $M_{Inc}(A, B)$, is defined as*

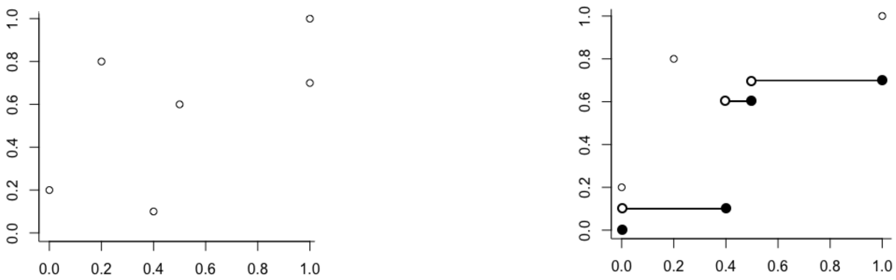
$$M_{Inc}(A, B) = 2 \int_0^1 Inc(A, B)(x)dx$$

In the previous definition we have included the scalar 2 for a normalization purpose; i.e., to make the maximum value of the measure to be equal to 1. Note also that the stronger the inclusion, the greater $Inc(A, B)$ and, then, the greater $M_{Inc}(A, B)$ as well. The following example uses a graphical representation of the φ-index of inclusion $Inc(A, B)$ to illustrate a meaning of M_{Inc} and to show how to compute the φ-measure of inclusion.

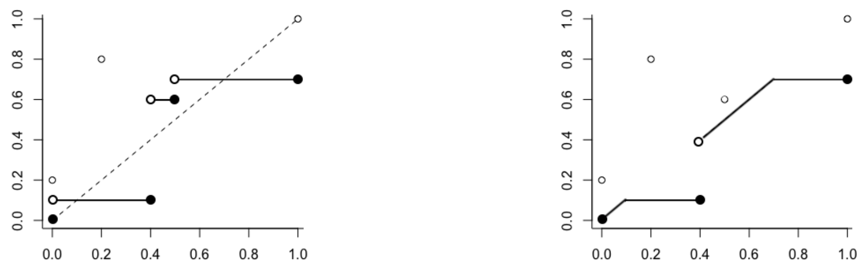
Example 1. *Let us consider the universe $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ and the two fuzzy sets given by* *the table:*

$\mathcal{U}$	u_1	u_2	u_3	u_4	u_5	u_6
A	1	0.2	0	1	0.4	0.5
B	0.7	0.8	0.2	1	0.1	0.6

In order to compute the analytical expression of $Inc(A, B)$ *we apply Theorem 1; hence, firstly, we have to compute the expression of the function* $f_{A,B}$. *To perform such a task, we represent in the unit square* $[0, 1]^2$ *the set of points* $\{(A(u), B(u)) \mid u \in \mathcal{U}\}$ *(i.e., the membership degrees of elements in* A *and* B*) and then, the graph of the function* $f_{A,B}$ *is easily determinable*



and as result, the graph of $Inc(A, B) = f_{A,B} \wedge id$ *and its expression are directly achievable:*

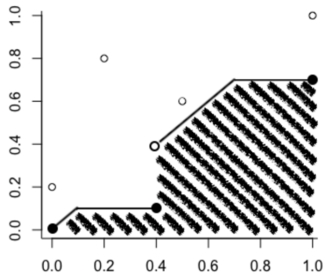


$$Inc(A, B)(x) = \begin{cases} x & \textit{if } x \leq 0.1 \\ 0.1 & \textit{if } 0.1 < x \leq 0.4 \\ x & \textit{if } 0.4 < x \leq 0.7 \\ 0.7 & \textit{if } 0.7 < x \leq 1 \end{cases}$$

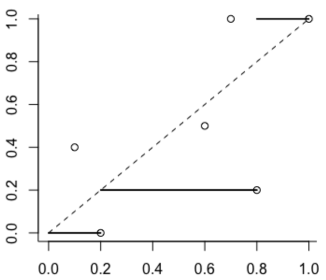
Finally, the measure of inclusion of A *into* B *is given by*

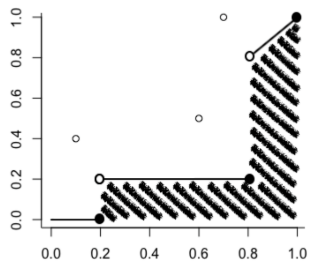
$$M_{Inc}(A, B) = 2 \int_0^1 Inc(A, B)(x) dx = 0.82$$

which represents the double of the area below the mapping $Inc(A, B)(x)$:



Note that, as expected, the measure of inclusion is not symmetric. As an example, following a similar procedure than above to measure the inclusion of B into A, we determine firstly the φ-index of inclusion $Inc(B, A)$:





and then, the measure of inclusion is

$$M_{Inc}(B, A) = 2 \int_0^1 Inc(A, B)(x)dx = 0.68.$$

As a conclusion, we can say that A is more included in B than B in A under the measure M_{Inc}.

The first theoretical result shows that the measure of inclusion M_{Inc} is an FXP-measure of inclusion.

Theorem 4. *M_{Inc} is a FXP-measure of inclusion, that is, for all fuzzy sets A, B and C:*

- *$M_{Inc}(A, B) = 1$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.*
- *$M_{Inc}(\mathcal{U}, \varnothing) = 0$.*
- *If $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then $M_{Inc}(C, A) \leq M_{Inc}(B, A)$ and $M_{Inc}(A, B) \leq M_{Inc}(A, C)$.*

Proof. Let us show that the three axioms (FXP1), (FXP2) and (FXP3) in Definition 6 are satisfied. To prove axiom (FXP1), let us assume firstly that A and B are two fuzzy sets such that $A(u) \leq B(u)$ for all $u \in \mathcal{U}$. Then, by item 1 in Theorem 2, we have that $Inc(A,B) = id$. Thus:

$$M_{Inc}(A,B) = 2\int_0^1 x\ dx = 1.$$

To prove the converse, let us assume, by reductio ad absurdum, that A and B are two fuzzy sets such that $A(u) \leq B(u)$ for all $u \in \mathcal{U}$ but $M_{Inc}(A,B) \neq 1$. Since $M_{Inc}(A,B) = 2\int_0^1 Inc(A,B)\ dx \neq 1$, we have that $Inc(A,B) \neq id$, and by item 1 in Theorem 2 there exists $u_0 \in \mathcal{U}$ such that $A(u_0) > B(u_0)$, which contradicts our hypothesis. Hence, axiom (FXP1) holds.

From item 2 in Theorem 2, we have that $Inc(\mathcal{U}, \varnothing) = \perp$. Then:

$$M_{Inc}(\mathcal{U}, \varnothing) = 2\int_0^1 0\ dx = 0.$$

Hence, axiom (FXP2) also holds.

Finally, consider A, B and C three fuzzy sets such that $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathcal{U}$. Then by items 4 and 5 we have $Inc(C,A) \leq Inc(B,A)$ and $Inc(A,B) \leq Inc(A,C)$, respectively. Therefore, by the monotonicity of integrals we have:

$$\begin{aligned} M_{Inc}(C,A) &= 2\int_0^1 Inc(C,A)\ dx \leq 2\int_0^1 Inc(B,A)\ dx \\ &= M_{Inc}(B,A) \end{aligned}$$

and

$$\begin{aligned} M_{Inc}(A,B) &= 2\int_0^1 Inc(A,B)\ dx \leq 2\int_0^1 Inc(A,C)\ dx \\ &= M_{Inc}(A,C) \end{aligned}$$

As a result, the axiom (FXP3) is also satisfied. □

As a consequence of the previous theorem, M_{Inc} is also a weak FXP-measure of inclusion. However, M_{Inc} is not a strong FXP-measure of inclusion, since axiom (sFXP2) of Definition 5 does not hold; as the following example shows.

Example 2. *Over the universe $\mathcal{U} = \{u_1, u_2\}$ we define the fuzzy sets A and B given by $A(u_1) = 0$, $A(u_2) = 0.5$, $B(u_1) = 0.5$ and $B(u_2) = 0$. The analytical expression of $Inc(A,B)$ can be obtained* *by following the procedure described in Example 1; firstly we obtain the expression of* $f_{A,B}$*:*

$$f_{A,B}(x) = \bigwedge_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\} = \begin{cases} 0 & \textit{if } x \leq 0.5 \\ 1 & \textit{otherwise} \end{cases}$$

and then, we obtain by Theorem 1 the expression of $Inc(A, B)$*:*

$$Inc(A, B)(x) = f_{A,B}(x) \wedge id = \begin{cases} 0 & \textit{if } x \leq 0.5 \\ x & \textit{otherwise} \end{cases}$$

Note that both A *and* B *are different from* $\varnothing$ *but* $A \cap B = \varnothing$*, however,*

$$M_{Inc}(A, B) = 2 \int_0^1 Inc(A, B)(x)dx = 2 \int_{0.5}^1 xdx = 0.75$$

Therefore $M_{Inc}(A, B) \neq 0$ *and* M_{Inc} *does not satisfy axiom (sFXP2).*

At this point we want to stress the fact that, under our point of view, axiom (sFXP2) (Definition 5) models the relationship between the inclusion and the empty set in a too drastic way: in fact, it goes against the inherent gradualness of fuzzy sets since, we can consider $B = \varnothing$ and a fuzzy set A as close to the empty set as desired but, by axiom (sFXP2) such an inclusion must be always equal 0. That conclusion is strange since our intuition says something different: the closer is A to B, the greater should be the inclusion of A in B.

It is worth mentioning also that, although in the crisp setting *"being different from the empty set"* is equivalent to *"the existence of an element in the set"*, both ideas are generalized differently in the fuzzy setting. Whereas the former statement is generalized as Fan, Xie & Pie do in axiom (sFXP2) by $A \neq \varnothing$; i.e., $A(u) \neq 0$ for some $u \in \mathcal{U}$, the latter one is generalized by the notion of normality; i.e., a fuzzy set A is normal if there is an element $u \in \mathcal{U}$ such that $A(u) = 1$. Finally, the measure of inclusion M_{Inc} holds the underlying idea in axiom (sFXP2) if we substitute the condition "$A \neq \varnothing$" by "A is normal", as the following result shows.

Proposition 1. *Let* A *and* B *be two fuzzy sets. If* A *is normal and* $A \cap B = \varnothing$*, then* $M_{Inc}(A, B) = 0$.

Proof. Since A is normal, then there exists $u_0 \in \mathcal{U}$ such that $A(u_0) = 1$. Since $A \cap B = \varnothing$, then necessarily $B(u_0) = 0$. By applying now item 2 of Theorem 2, we have $Inc(A, B) = \bot$ and as a consequence $M_{Inc}(A, B) = 0$. □

As we have proclaimed above, it is convenient for the measures of inclusion to be endowed with the gradualness inherent in fuzzy set theory. In this line, it is convenient to generalize the previous result in a more convenient way by establishing a smooth transition of the hypothesis from $\varnothing$ to normality.

Proposition 2. *Let A and B be two fuzzy sets and let $\alpha, \beta \in [0,1]$ such that $\beta \le \alpha$. If there exists $u_0 \in \mathcal{U}$ such that $A(u_0) \ge \alpha$ and $A \cap B(u) \le \beta$ for all $u \in \mathcal{U}$, then*

$$M_{Inc}(A,B) \le 1 - (\alpha - \beta)^2.$$

Proof. Let us consider the mapping f in Ω given by

$$f(x) = \begin{cases} \beta & \text{if } \beta \le x \le \alpha \\ x & \text{otherwise} \end{cases}$$

and let us show firstly that for all $g \in \Omega$ such that $g \not\le f$, A is not g-included in B. Since $g(x) \le x$ and $f(x) = x$ for all $x \in [0, \alpha] \cup [\beta, 1]$, there exists $x_0 \in (\beta, \alpha)$ such that $g(x_0) > \beta$. Moreover, by monotonicity, we have that $g(\alpha) \ge g(x_0) > \beta$. On the other hand, consider $u_0 \in \mathcal{U}$ such that $A(u_0) \ge \alpha$. By hypothesis, we have also that $A \cap B(u_0) \le \beta$ and then, $B(u_0) \le \beta$. Let us see that the inequality $g(A(u_0)) \le B(u)$ does not hold for $u = u_0$:

$$g(A(u_0)) > g(\alpha) = \beta \ge B(u_0).$$

Therefore, A is not g-included in B for all $g \in \Omega$ such that $g \not\le f$ and, as a result we can assert that $Inc(A,B) \le f$. Finally we have by monotonicity of integrals that

$$\begin{aligned} M_{Inc}(A,B) &= 2\int_0^1 Inc(A,B)(x)\ dx \le 2\int_0^1 f(x)\ dx \\ &= \beta^2 + 2\beta\alpha - 2\beta^2 + 1 - \alpha^2 = 1 - (\alpha-\beta)^2 \end{aligned}$$

□

The previous result can be interpreted as a fuzzy version of the axiom (sFXP2) that is satisfied by the measure of inclusion M_{Inc}. Note on the one hand, that the existence of $u_0 \in \mathcal{U}$ such that $A(u_0) \ge \alpha$ can be considered as a degree of the normality of A and, on the other hand, $A \cap B(u) < \beta$ for all $u \in \mathcal{U}$ is like a degree of emptiness of $A \cap B$. Hence, the previous result can be roughly paraphrased in natural language as "*the more normal the fuzzy set A and the more empty the intersection $A \cap B$, the lesser the inclusion of A in B*".

In the rest of this section, we focus on the properties of M_{Inc} oriented to the axiomatic definitions of Sinha & Dougherty and Kitainik. To begin with, we have that M_{Inc} satisfies axiom (SD1), and as a direct consequence axiom (K4) as well.

Proposition 3. *Let A and B be two fuzzy sets. $M_{Inc}(A,B) = 1$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.*

Proof. As a consequence of item 1 of Theorem 2, it is sufficient to prove that $M_{Inc}(A,B) = 1$ if and only if $Inc(A,B) = id$. Obviously, $Inc(A,B) = id$ implies $M_{Inc}(A,B) = 1$. To prove the converse we proceed by reductio ad absurdum and we assume that $M_{Inc}(A,B) = 1$ but $Inc(A,B) \neq id$. From $M_{Inc}(A,B) = 1$ we have that

$$2\int_0^1 Inc(A,B)\ dx = 1 = 2\int_0^1 x\ dx$$

Therefore, the difference between $Inc(A,B)$ and id is a set of measure 0; i.e., a set of isolate points. Let $\alpha \in [0,1]$ be the least value where $Inc(A,B)$ and id differ; i.e. such that $Inc(A,B)(x) = x$ for all $x \in (0,\alpha)$ but $Inc(A,B)(\alpha) \neq \alpha$. Let us assume that $Inc(A,B)(\alpha) = \beta$ for certain $\beta \in [0,1]$. Moreover, since $Inc(A,B) \in \Omega$ we have that $Inc(A,B) \leq id$ and then $\beta < \alpha$. Finally, since $\frac{\alpha+\beta}{2} \in (0,\alpha)$, we have:

$$\frac{\alpha+\beta}{2} = Inc(A,B)\left(\frac{\alpha+\beta}{2}\right) \leq Inc(A,B)\,(\alpha) = \beta$$

from which we can infer that $\alpha \leq \beta$; which is a contradiction with the choice of α. □

Concerning the 0-degree of inclusion used by axioms (SD2) and (K5) we have the following result that resembles item 2 of Theorem 2, with some differences though.

Theorem 5. *Let A and B be two fuzzy sets. If $M_{Inc}(A,B) = 0$, then there is a set of elements in the universe $\{u_i\}_{i\in I} \subseteq \mathcal{U}$ such that $\bigvee_{i\in I} A(u_i) = 1$ and $\bigwedge_{i\in I} B(u_i) = 0$.*

Proof. We proceed by contradiction, by assuming $M_{Inc}(A,B) = 0$ and that for all the subsets $\{u_i\}_{i\in I} \subseteq \mathcal{U}$, if $\bigvee_{i\in I} A(u_i) = 1$, then we have $\bigwedge_{i\in I} B(u_i) \neq 0$. In this case, we can assert the existence of $\varepsilon > 0$ such that

$$\max\{1 - A(u), B(u)\} > \varepsilon \quad \text{for all } u \in \mathcal{U}.$$

This follows by reasoning on the two possibilities existing for the whole universe $\mathcal{U}$: on the one hand, if $\bigvee_{u\in\mathcal{U}} A(u) \neq 1$, then there exists $\varepsilon_1 > 0$ such that $1 - A(u) > \varepsilon_1$ for all $u \in \mathcal{U}$; on the other hand, if $\bigvee_{u\in\mathcal{U}} A(u) = 1$, by hypothesis we have that $\bigwedge_{u\in\mathcal{U}} B(u) \neq 0$ and, again, there exists $\varepsilon_2 > 0$ such that $B(u) > \varepsilon_2$ for all $u \in \mathcal{U}$.

Let us define the mapping

$$f(x) = \begin{cases} 0 & \text{if } x \leq 1 - \varepsilon \\ \varepsilon & \text{otherwise} \end{cases}$$

which obviously belongs to Ω and let us show that $A \subseteq_f B$. Let $u \in \mathcal{U}$, if $A(u) \leq 1 - \varepsilon$ then $f(A(u)) = 0 \leq B(u)$ straightforwardly. Otherwise, if $A(u) > 1 - \varepsilon$ then, by construction of ε, we have $B(u) > \varepsilon$. Therefore, $f(A(u)) = \varepsilon < B(u)$. Hence $A \subseteq_f B$ and, as a result, $Inc(A, B) \geq f$. Finally we have:

$$M_{Inc}(A, B) = 2 \int_0^1 Inc(A, B)(x)\; dx \geq 2 \int_0^1 f(x)\; dx = 2\left(\varepsilon - \varepsilon^2\right) > 0$$

which is a contradiction. □

The converse of Theorem 6 does not hold in general, as shown by the following example:

Example 3. *On the universe $\{u_1, u_2, u_3\}$ let us consider two fuzzy sets A and B defined by $A(u_1) = 1$, $A(u_2) = 0.5$, $A(u_3) = 0.5$, $B(u_1) = 0.5$, $B(u_2) = 0.5$, and $B(u_3) = 0$. The analytical expression of $Inc(A, B)$ can be obtained by following the procedure described in Example 1; firstly we obtain the expression of $f_{A,B}$:*

$$f_{A,B}(x) = \bigwedge_{u\in\mathcal{U}} \{B(u) \mid x \leq A(u)\} = \begin{cases} 0 & \textit{if } x = 0 \\ 0.5 & \textit{otherwise} \end{cases}$$

and then, the expression of $Inc(A, B)$ is:

$$Inc(A, B)(x) = f_{A,B}(x) \wedge id = \begin{cases} x & \textit{if } x \leq 0.5 \\ 0.5 & \textit{otherwise} \end{cases}$$

which, certainly, leads to $M_{Inc}(A, B) > 0$. □

Unfortunately, the statement in Theorem 5 is not strong enough to prove axioms (K5) and (SD2), since it does not allow to find one element u such that $A(u) = 1$ and $B(u) = 0$. This is why the following alternative version is provided in terms of sequences.

Theorem 6. *Let A and B be two fuzzy sets. $M_{Inc}(A, B) = 0$ if and only if there is a sequence of elements in the universe $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\lim_{n\to\infty} A(u_n) = 1$ and $\lim_{n\to\infty} B(u_n) = 0$.*

Proof. Assume that $M_{Inc}(A,B) = 0$ and let us prove the existence of a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\lim_{n\to\infty} A(u_n) = 1$ and $\lim_{n\to\infty} B(u_n) = 0$.

From $M_{Inc}(A,B) = 0$ we have that $Inc(A,B)$ is zero almost everywhere and, moreover, by the monotonicity of elements in Ω, there exists $\alpha \in [0,1]$ such that:

$$Inc(A,B)(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ \alpha & \text{if } x = 1 \end{cases}$$

Recall that, by Theorem 1, we have $Inc(A,B) = f_{A,B} \wedge id$ and, as a result, for all $x \in [0,1)$ we have

$$f_{A,B}(x) = \bigwedge_{u\in\mathcal{U}} \{B(u) \mid x \leq A(u)\} = Inc(A,B)(x) = 0. \tag{1}$$

We can now construct $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\lim_{n\to\infty} A(u_n) = 1$ and $\lim_{n\to\infty} B(u_n) = 0$. For each $n \in \mathbb{N}$ we have, by Equation (1), that

$$\bigwedge_{u\in\mathcal{U}} \left\{B(u) \mid \frac{n-1}{n} \leq A(u)\right\} = 0.$$

As a result, for all n, there exists $u_n \in \mathcal{U}$ such that $B(u_n) \leq \frac{1}{n}$ and $\frac{n-1}{n} \leq A(u_n)$. It is clear that $\lim_{n\to\infty} A(u_n) = 1$ and $\lim_{n\to\infty} B(u_n) = 0$.

For the converse, let us assume a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{U}$ such that $\lim_{n\to\infty} A(u_n) = 1$ and $\lim_{n\to\infty} B(u_n) = 0$. Then, for all $x \in [0,1)$

$$f_{A,B}(x) = \bigwedge_{u\in\mathcal{U}} \{B(u) \mid x \leq A(u)\} = 0.$$

and, by Theorem 1,

$$Inc(A,B)(x) \leq F(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Finally, we have:

$$M_{Inc}(A,B) = 2\int_0^1 Inc(A,B)(x)\ dx = 2\int_0^1 F(x)\ dx = 0$$

□

As a consequence of the previous result, the measure of inclusion M_{Inc} satisfies axiom (K5).

Corollary 1. *Let A and B be two crisp sets. $M_{Inc}(A,B) = 0$ if and only if there exists $u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0$.*

Another consequence of Theorem 6 is that the axiom (SD2) is satisfied by the measure of inclusion M_{Inc} when the underlying universe considered is finite.

Corollary 2. *Let A and B be two fuzzy sets defined on a finite universe $\mathcal{U}$. $M_{Inc}(A, B) = 0$ if and only if there exists $u \in \mathcal{U}$ such that $A(u) = 1$ and $B(u) = 0$.*

The following result concerns the monotonicity of the measure of inclusion M_{Inc}.

Proposition 4. *Let A, B and C be three fuzzy sets such that $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then:*

- $M_{Inc}(A, B) \leq M_{Inc}(A, C)$;
- $M_{Inc}(C, A) \leq M_{Inc}(B, A)$.

Proof. This result is a consequence of items 4 and 5 of Theorem 2 and monotonicity of the integral. □

The transformation invariance of the measure M_{Inc} is also a direct consequence of Theorem 2.

Proposition 5. *Let A and B be two fuzzy sets and $T: \mathcal{U} \to \mathcal{U}$ a bijective transformation on the universe $\mathcal{U}$. Then, $M_{Inc}(A, B) = M_{Inc}(T(A), T(B))$.*

Proof. This result is a consequence of item 6 in Theorem 2. □

The only axiom of Sinha & Dougherty and Kitainik that is not satisfied by the φ-index of inclusion, under the natural interpretation of axioms in the set $[0, 1]^{[0,1]}$, is the one related to the complement of fuzzy sets. The next theorem will show that, although the relation between the φ-index of inclusion and complement of fuzzy sets is given by means of adjoint pairs (Theorem 3), the measure of inclusion M_{Inc} satisfies axioms (SD6) and (K1). But previously, let us state the following lemma in order to simplify the presentation of the proof.

Lemma 3. *Let A and B be two fuzzy sets, then the mapping τ defined by*

$$\tau(x) = \bigvee_{y<x} Inc(A, B)(y).$$

satisfies the following properties:

i) $\tau \in \Omega$;

ii) $\tau \leq Inc(A, B)$;

iii) τ *is left-continuous;*

iv) *The set of* $x \in [0, 1]$ *such that* $\tau(x) \neq Inc(A, B)(x)$ *is a null set.*

Proof. i) Note that since $Inc(A, B) \in \Omega$ we have that τ is increasing and

$$\tau(x) = \bigvee_{y<x} Inc(A, B)(y) \leq \bigvee_{y<x} y = x.$$

In other words, $\tau \in \Omega$.

ii) Let $x \in [0, 1]$ and note that, since $Inc(A, B)$ is increasing, we have $Inc(A, B)(y) \leq Inc(A, B)(x)$ for all $y \in [0, 1]$ such that $y < x$. As a result:

$$\tau(x) = \bigvee_{y<x} Inc(A, B)(y) \leq Inc(A, B)(x)$$

iii) Note firstly that since τ is increasing, left-continuity is equivalent to say that $\tau(\bigvee X) = \bigvee_{x\in X} \tau(x)$ for all $x \in [0, 1]$. Given a subset $X \subseteq [0, 1]$, taking into account that the supremum is applied in a totally order set, we have

$$\tau\left(\bigvee X\right) = \bigvee_{y<\sup X} Inc(A, B)(y) = \bigvee_{x\in X} \left(\bigvee_{y<x} Inc(A, B)(y)\right) = \bigvee_{x\in X} \tau(x).$$

iv) Finally, we proceed by reductio ad absurdum and assume that the set of $x \in [0, 1]$ such that $\tau(x) \neq Inc(A, B)(x)$ has a measure different from 0. That means that there is an interval $[a, b] \subseteq [0, 1]$ such that $\tau(x) \neq Inc(A, B)(x)$ for all $x \in [a, b]$. By adding the information from item ii) we have that $\tau(x) < Inc(A, B)(x)$ for all $x \in [a, b]$. Finally, by using item iii) (i.e., τ is left continuous), we have that:

$$\tau(b) = \bigvee_{y<b} Inc(A, B)(y) \geq \bigvee_{y<b} \tau(y) = \tau\left(\bigvee_{y<b} y\right) = \tau(b)$$

which is a contradiction with $\tau(b) < Inc(A, B)(b)$. □

Theorem 7. *Let* A *and* B *be two fuzzy sets, then* $M_{Inc}(A, B) = M_{Inc}(B^c, A^c)$.

Proof. Given the fuzzy sets A and B let us consider the mapping given in Lemma 3, that is:

$$\tau(x) = \bigvee_{y<x} Inc(A, B)(y).$$

Note that,

- Lemma 3 items i) and ii), we have that A is τ-included in B;
- Lemma 3 item iii), we have that there exists $g\colon [0,1] \to [0,1]$ such that (τ, g) is an adjoint pair. As a consequence, we can apply Theorem 3 and obtain that $B^c \subseteq_{1-g(1-x)} A^c$.
- thanks to item iv) in Lemma 3, we have that

$$2\int_0^1 \tau(x)\ dx = 2\int_0^1 Inc(A,B)(x)\ dx = M_{Inc}(A,B).$$

The rest of the proof is split into two parts: firstly, we will show that:

$$M_{Inc}(B^c, A^c) = 2\int_0^1 (1 - g(1-x))\ dx$$

and, then, we will prove

$$\int_0^1 \tau(x)\ dx = \int_0^1 (1 - g(1-x))\ dx,$$

as a consequence, we will have $M_{Inc}(A,B) = 2\int_0^1 \tau(x)\ dx = 2\int_0^1 (1 - g(1-x))\ dx = M_{Inc}(B^c, A^c)$ and the proof ends.

Let us consider the mapping

$$\gamma(x) = \bigvee_{y<x} Inc(B^c, A^c)(y),$$

which, as said above for τ (i.e., Lemma 3), satisfies that

- $2\int_0^1 \gamma(x)\ dx = M_{Inc}(B^c, A^c)$ and
- there exists f such that (γ, f) is an adjoint pair and $A \subseteq_{1-f(1-x)} B$.

Note that:

1. $(\tau(x), g(x))$, $(\gamma(x), f(x))$, $(1 - g(1-x), 1 - \tau(1-x))$ and $(1 - f(1-x), 1 - \gamma(1-x))$ are adjoint pairs;
2. by item 1 above $\tau(x)$, $\gamma(x)$, $1 - g(1-x)$ and $1 - f(1-x)$ are left-continuous.
3. $\tau(x)$ (resp. $\gamma(x)$) is, by construction, the greatest left-continuous mapping among those lesser than or equal to $Inc(A,B)$ (resp. $Inc(B^c, A^c)$).
4. by item 3 above and $B^c \subseteq_{1-g(1-x)} A^c$ we have $1 - g(1-x) \le \gamma(x)$ for all $x \in [0,1]$;
5. by item 3 above and $A \subseteq_{1-f(1-x)} B$ we have $1 - f(1-x) \le \tau(x)$ for all $x \in [0,1]$;

6. From items 1 and 5 we have that $1-\gamma(1-x)\geq g(x)$ for all $x\in[0,1]$, by using the following well-known result of adjoint pairs: if (f_1,g_1) and (f_2,g_2) are adjoint pairs such that $f_1\leq f_2$, then $g_2\leq g_1$.
7. From item 6 we can infer that $1-g(1-x)\geq\gamma(x)$ for all $x\in[0,1]$.

Joining items 4 and 7 we have that $1-g(1-x)=\gamma(x)$ for all $x\in[0,1]$ and then, $2\int_0^1(1-g(1-x))\,dx=2\int_0^1\gamma(x)\,dx=M_{Inc}(B^c,A^c)$, as we wanted to prove in this first part.

In order to prove the equality $\int_0^1\tau(x)\,dx=\int_0^1(1-g(1-x))\,dx$, note that $\int_0^1\tau(x)\,dx$ and $\int_0^1(1-g(1-x))\,dx$ coincide, respectively, with the areas of the sets $\Delta_1=\{(x,y)\in[0,1]^2\colon y<\tau(x)\}$ and $\Delta_2=\{(x,y)\in[0,1]^2\colon y<1-g(1-x)\}$ respectively. Let us show that Δ_2 is the image of the rigid transformation[1] $\Pi(u,v)=(1-v,1-u)$ applied to the set Δ_1; and therefore they both have the same area.

$$\begin{aligned}(x,y)\in\Pi(\Delta_1) &\iff (1-y,1-x)\in\Delta_1\\ &\iff 1-x<\tau(1-y)\\ &\iff g(1-x)<1-y\\ &\iff y<1-g(1-x)\\ &\iff (x,y)\in\Delta_2\end{aligned}$$

where we have used that, since $[0,1]$ is totally ordered and (τ,g) is an adjoint pair, $x<\tau(y)\iff g(x)<y$ for all $x,y\in[0,1]$. Note that, since the unit interval is linearly ordered, the statement $\tau(x)\leq y\iff x\leq g(y)$ is equivalent to $x<\tau(y)\iff g(x)<y$. □

Surprisingly, although M_{Inc} satisfies axiom (SD6) (respectively (K1)) and the φ-index of inclusion satisfies all the rest of axioms provided by Sinha & Dougherty (respectively Kitainik), M_{Inc} is not an SD-measure of inclusion (respectively nor a K-measure of inclusion). The reason is that M_{Inc} does not satisfy the axioms (SD7)=(K2) and (SD8), as the following example shows.

Example 4. *On the one hand, let us consider the fuzzy sets A, B and C defined on the universe $\mathcal{U}=\{u_1,u_2\}$, and given by $A(u_1)=0.5$, $A(u_2)=1$, $B(u_1)=1$, $B(u_2)=0.5$, $C(u_1)=0$ and $C(u_2)=1$. It is easy to check that the φ-indexes of inclusion $Inc(A,B)$ and $Inc(A,C)$ are*

[1] Also called Euclidean isometry.

respectively:

$$Inc(A,B) = \begin{cases} x & \textit{if } x \leq 0.5 \\ 0.5 & \textit{otherwise} \end{cases} \qquad Inc(A,C) = \begin{cases} 0 & \textit{if } x \leq 0.5 \\ x & \textit{otherwise} \end{cases}$$

Moreover, the φ-index of inclusion $Inc(A, B \cap C)$ can be obtained directly as a consequence of item 7 in Theorem 2:

$$Inc(A, B \cap C) = Inc(A,B) \wedge Inc(A,C) = \begin{cases} 0 & \textit{if } x \leq 0.5 \\ 0.5 & \textit{otherwise} \end{cases}$$

As a result, we have the corresponding measures of inclusion:

- $M_{Inc}(A,B) = M_{Inc}(A,C) = 0.75$ *and*
- $M_{Inc}(A, B \cap C) = 0.5$.

In other words, the equality $M_{Inc}(A, B \cap C) = \min\{M_{Inc}(A,B), M_{Inc}(A,C)\}$ (i.e., axiom (SD8)) does not hold in this case.

On the other hand, let us consider in addition the fuzzy sets D and E given by: $D(u_1) = 0$, $D(u_2) = 0.5$, $E(u_1) = 0.5$ and $E(u_2) = 0$. Then, we have the following φ-indexes of inclusion:

$$Inc(C,D) = \begin{cases} x & \textit{if } x \leq 0.5 \\ 0.5 & \textit{otherwise} \end{cases} \qquad Inc(E,D) = \begin{cases} 0 & \textit{if } x \leq 0.5 \\ x & \textit{otherwise} \end{cases}$$

and by item 8 of Theorem 2 we have:

$$Inc(C \cup E, D) = Inc(C,D) \wedge Inc(E,D) = \begin{cases} 0 & \textit{if } x \leq 0.5 \\ 0.5 & \textit{otherwise} \end{cases}$$

Finally, we get the corresponding measures of inclusion:

- $M_{Inc}(C,D) = M_{Inc}(E,D) = 0.75$
- $M_{Inc}(C \cup E, D) = 0.5$.

Hence, the equality $M_{Inc}(C \cup E, D) = \min\{M_{Inc}(C,D), M_{Inc}(E,D)\}$ (i.e., axioms (SD7) and (K2)) does not hold in this case.

Despite of the previous counter-example, thanks to properties in Theorem 2 for the φ-index of inclusion which are the basic bricks to build M_{Inc}, we may say that the measure of inclusion M_{Inc} intrinsically gathers the idea behind axioms (SD8), (SD7) and (K2). Indeed, M_{Inc} satisfies the following weak version of axioms (SD8), (SD7) and (K2).

Proposition 6. *Let A, B and C be three fuzzy sets, then*

i) $M_{Inc}(A, B \cap C) \leq \min\{M_{Inc}(A, B), M_{Inc}(A, C)\}$

ii) $M_{Inc}(A \cup B, C) \leq \min\{M_{Inc}(A, B), M_{Inc}(C, D)\}$

Proof.

i) Let A, B and C be fuzzy sets, then by item 7 in Theorem 2 we have $Inc(A, B \cap C) = Inc(A, B) \wedge Inc(A, C)$. As a result, thanks to the monotonicity of integrals:

$$\begin{aligned} M_{Inc}(A, B \cap C) &= \int_0^1 Inc(A, B \cap C)(x)\ dx \\ &= \int_0^1 Inc(A, B)(x) \wedge Inc(A, C)(x)\ dx \leq \int_0^1 Inc(A, B)(x)\ dx = M_{Inc}(A, B) \end{aligned}$$

$$\begin{aligned} M_{Inc}(A, B \cap C) &= \int_0^1 Inc(A, B \cap C)(x)\ dx \\ &= \int_0^1 Inc(A, B)(x) \wedge Inc(A, C)(x)\ dx \leq \int_0^1 Inc(A, C)(x)\ dx = M_{Inc}(A, C) \end{aligned}$$

Joining both inequalities we get $M_{Inc}(A, B \cap C) \leq \min\{M_{Inc}(A, B), M_{Inc}(A, C)\}$.

Item ii) is proved similarly. □

4. From M_{Inc} to a Young measure of inclusion and to fuzzy entropy measures.

4.1. A Young measure of inclusion from the φ-index of inclusion.

In this section we define a measure of inclusion, based on the φ-index of inclusion, that satisfies the axiom proposed by Young. The main difference between the Young approach and the rest is that the idea behind the former deals with measures of inclusion defined as mean values of an implication operator; as done by Willmott [28]. This difference is very visible in axioms (Y2), (SD2) and (K5) where the null inclusion is depicted; (SD2) and (K5) are described by the existential quantifier whereas (Y2) is described by the universal quantifier.

In accordance with Young, we propose to measure the inclusion of a fuzzy set into another fuzzy set by considering each element in the universe individually and then, computing the mean. Formally, given a fuzzy set A, for each $u \in \mathcal{U}$ we define the fuzzy set A_u on the singleton universe $\mathcal{U}_u = \{u\}$ defined by $A_u(u) = A(u)$. Thus, given two fuzzy sets A and B, the *single measure of inclusion for* $u \in \mathcal{U}$ is defined as $M_{Inc}(A_u, B_u)$. Once the notion of single measure of inclusion has been introduced, we can define the following measure of inclusion between fuzzy sets.

Definition 12. *Let A and B be two fuzzy sets defined on a finite universe. We define the mean-measure of inclusion as the value:*

$$\widehat{M}_{Inc}(A, B) = \frac{\sum_{u \in \mathcal{U}} M_{Inc}(A_u, B_u)}{Card(\mathcal{U})}$$

Theorem 8. *The mean-measure of inclusion $\widehat{M}_{Inc}$ is a Young measure of inclusion. That is:*

(Y1) $\widehat{M}_{Inc}(A, B) = 1$ *if and only if* $A(u) \leq B(u)$ *for all* $u \in \mathcal{U}$

(Y2) *If* $A(u) \geq 0.5$ *for all* $u \in \mathcal{U}$, *then* $\widehat{M}_{Inc}(A, A^c) = 0$ *if and only if* $A(u) = 1$ *for all* $u \in \mathcal{U}$.

(Y3) *If* $A(u) \leq B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\widehat{M}_{Inc}(C, A) \leq \widehat{M}_{Inc}(B, A)$.

(Y4) *If* $B(u) \leq C(u)$ *for all* $u \in \mathcal{U}$ *then* $\widehat{M}_{Inc}(A, B) \leq \widehat{M}_{Inc}(A, C)$ *for all fuzzy set* $A \in \mathcal{F}(\mathcal{U})$.

Proof. (Y1) $\widehat{M}_{Inc}(A, B) = 1$ is equivalent to

$$\sum_{u \in \mathcal{U}} M_{Inc}(A_u, B_u) = Card(\mathcal{U})$$

or equivalently $M_{Inc}(A_u, B_u) = 1$ for all $u \in \mathcal{U}$. By applying now Theorem 4, we can conclude that $\widehat{M}_{Inc}(A, B) = 1$ is equivalent to $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.

(Y2) Let A be a fuzzy set such that $A(u) \geq 0.5$ for all $u \in \mathcal{U}$. Then, $\widehat{M_{Inc}}(A, A^c) = 0$ if and only if $M_{Inc}(A_u, A_u^c) = 0$ for all $u \in \mathcal{U}$ which, by Corollary 2, is equivalent to stating that either $A(u) = 1$ and $A^c(u) = 0$ or, $A(u) = 0$ and $A^c(u) = 1$ for all $u \in \mathcal{U}$. Since $A(u) \geq 0.5$ for all $u \in \mathcal{U}$, necessarily $A(u) = 1$ for all $u \in \mathcal{U}$.

(Y3) Let $A(u) \leq B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, by Proposition 4, $M_{Inc}(C_u, A_u) \leq M_{Inc}(B_u, A_u)$ for all $u \in \mathcal{U}$. As a consequence, $\widehat{M}_{Inc}(C, A) \leq \widehat{M}_{Inc}(B, A)$.

(Y4) the proof is similar to (Y3). $\square$

Since every Young measure of inclusion satisfies the axioms proposed by Fan, Xie & Pie, we have the following result.

Corollary 3. *$\widehat{M}_{Inc}$ is a FXP-measure of inclusion.*

$\widehat{M}_{Inc}$ not only satisfies the axioms proposed by Young, the contraposition law as well; i.e., axioms (K1) and (SD6).

Proposition 7. *Let A and B be two fuzzy sets, then $\widehat{M}_{Inc}(A,B) = \widehat{M}_{Inc}(B^c, A^c)$.*

Proof. From Theorem 7 we have that $M_{Inc}(A_u, B_u) = M_{Inc}(B_u^c, A_u^c)$ for all $u \in \mathcal{U}$. As a consequence:

$$\begin{aligned}\widehat{M}_{Inc}(A,B) &= \frac{\sum_{u\in\mathcal{U}} M_{Inc}(A_u, B_u)}{Card(\mathcal{U})} \\ &= \frac{\sum_{u\in\mathcal{U}} M_{Inc}(B_u^c, A_u^c)}{Card(\mathcal{U})} \\ &= \widehat{M}_{Inc}(B^c, A^c).\end{aligned}$$

□

In the following example we illustrate the differences between the measures of inclusion M_{Inc} and $\widehat{M}_{Inc}$.

Example 5. *Let us consider the universe $\mathcal{U} = \{u_1, u_2, u_3\}$ and the fuzzy sets A, B and C given by the following table:*

$\mathcal{U}$	u_1	u_2	u_3
A	1	0.5	1
B	0.1	0.2	0.4
C	0.1	0.6	1

The reader can check that $Inc(A,B)$ coincides with $Inc(A,C)$ and:

$$Inc(A,B) = Inc(A,C) = \begin{cases} x & \text{if } 0 \leq x \leq 0.1 \\ 0.1 & \text{otherwise.} \end{cases}$$

Therefore, we have the measures of inclusion $M_{Inc}(A,B) = M_{Inc}(A,C) = 0.19$.

For the respective single measures of inclusion for u_1, u_2 and u_3 we have:

$$Inc(A_{u_1}, B_{u_1}) = \begin{cases} x & \textit{if } 0 \leq x \leq 0.1 \\ 0.1 & \textit{otherwise.} \end{cases}$$

$$Inc(A_{u_2}, B_{u_2}) = \begin{cases} x & \textit{if } 0 \leq x \leq 0.2 \\ 0.2 & \textit{if } 0.2 < x \leq 0.5 \\ x & \textit{otherwise.} \end{cases}$$

$$Inc(A_{u_3}, B_{u_3}) = \begin{cases} x & \textit{if } 0 \leq x \leq 0.4 \\ 0.4 & \textit{otherwise.} \end{cases}$$

$$Inc(A_{u_1}, C_{u_1}) = \begin{cases} x & \textit{if } 0 \leq x \leq 0.1 \\ 0.1 & \textit{otherwise.} \end{cases}$$

and $Inc(A_{u_2}, C_{u_2}) = Inc(A_{u_3}, C_{u_3}) = id$. *As a result, we have the following single measures of inclusion:*

$$M_{Inc}(A_{u_1}, B_{u_1}) = 0.19 \qquad M_{Inc}(A_{u_2}, B_{u_2}) = 0.91$$
$$M_{Inc}(A_{u_3}, B_{u_3}) = 0.64 \qquad M_{Inc}(A_{u_1}, C_{u_1}) = 0.19$$
$$M_{Inc}(A_{u_2}, C_{u_2}) = 1 \qquad M_{Inc}(A_{u_3}, C_{u_3}) = 1$$

Therefore, the mean measures of inclusion are $\widehat{M}_{Inc}(A, B) = 0.58$ *and* $\widehat{M}_{Inc}(A, C) = 0.73$.

Note that although the measures of inclusion coincide with respect to M_{Inc}*, they do not coincide with respect to the measure* $\widehat{M}_{Inc}$*. The reason is that the measure of inclusion related to* u_1 *collapses the others in* M_{Inc}*; so the other truth-values have not effect. However, in the measure of inclusion* $\widehat{M}_{Inc}$ *all the truth-values are considered, and then, the single inclusions related to* u_2 *and* u_3 *are also taken into account.*

4.2. A φ-index of entropy from the φ-index of inclusion

In this last section we present some measures of entropy induced by the φ-index of inclusion. It is well known that measures of entropy of a fuzzy set A can be defined by measuring the inclusion of $A \cap A^c$ into $A \cup A^c$ by using a Young measure of inclusion [2, 30]. This construction of measures of entropy can be extended by means of the use of arbitrary negation operators to generalize the complement. Specifically, given a negation operator n, i.e., a decreasing mapping $n\colon [0,1] \to [0,1]$ such that $n(0) = 1$ and $n(1) = 0$, we can define the n-complement of a fuzzy set A as $n(A)(u) = n(A(u))$ for all $u \in \mathcal{U}$, and with it, the following measure of entropy.

Definition 13. *Let A be a fuzzy set defined on a finite universe, and let n be a strictly decreasing negation such that $n(0.5) = 0.5$. The measure of fuzzy entropy E induced by the φ-index of inclusion and the negation n is given by:*

$$E(A) = \widehat{M}_{Inc}(A \cup n(A), A \cap n(A))$$

where $\widehat{M}_{Inc}$ is the mean measure of inclusion given in Definition 12.

The following result shows that, effectively, Definition 13 introduces a measure of fuzzy entropy in the sense of De Luca & Termini [10].

Theorem 9. *The fuzzy entropy measure E induced by the φ-index of inclusion and a strictly decreasing negation n such that $n(0.5) = 0.5$, is a De Luca-Termini measure of fuzzy entropy, i.e.*

(E1) *$E(A) = 0$ if and only if $A(u) \in \{0, 1\}$ for all $u \in \mathcal{U}$.*

(E2) *$E(A)$ reaches the maximum value 1 if and only if $A(u) = 0.5$ for all $u \in \mathcal{U}$.*

(E3) *$E(A) \leq E(B)$ if A is a refinement of B.*

Proof. The proof is similar to the one in [2] and [5] but for strictly decreasing negation n such that $n(0.5) = 0.5$.

(E1) Let A be a fuzzy set. Note firstly that, since n is a strictly decreasing negation such that $n(0.5) = 0.5$, necessarily $(A \cup n(A))(u) = \max\{A(u), n(A)(u)\} \geq 0.5$ for all $u \in \mathcal{U}$. Then, by using Theorem 8 and axiom (Y2) we have that $E(A) = 0$ if and only if $(A \cup n(A))(u) = \max\{A(u), n(A)(u)\} = 1$ for all $u \in \mathcal{U}$. Consequently, either $A(u) = 1$ or $n(A)(u) = 1$ for all $u \in \mathcal{U}$; note that the latter is equivalent to say that $A(u) = 0$.

(E2. Note that the maximum possible value of the measure of entropy E is 1 since, firstly, the greatest possible value of $\widehat{M}_{Inc}$ is 1 and secondly, such value is obtained for the fuzzy set $1/2$ that assigns to each element in the universe the value 0.5, since

$$M_{Inc}\left((1/2)_u \cup n\big((1/2)_u\big), (1/2)_u \cap n\big((1/2)_u\big)\right) = 1 \quad \text{for all } u \in \mathcal{U}.$$

Then, by definition of E, by Theorem 8, axiom (Y1) and that n is a strictly decreasing negation such that $n(0.5) = 0.5$ we have that

$$\begin{aligned} E(A) = 1 \quad & \text{if and only if} \quad M_{Inc}(A_u \cup n(A_u), A_u \cap n(A_u)) = 1 \text{ for all } u \in \mathcal{U} \\ & \text{if and only if} \quad A_u \cup n(A_u) \leq A_u \cap n(A_u) \text{ for all } u \in \mathcal{U} \\ & \text{if and only if} \quad A_u \cup n(A_u) = A_u \cap n(A_u) \text{ for all } u \in \mathcal{U} \\ & \text{if and only if} \quad A(u) = 0.5 \text{ for all } u \in \mathcal{U}. \end{aligned}$$

(E3) Let A and B be two fuzzy sets such that A is a refinement of B. Consider $u \in \mathcal{U}$ and let us assume firstly that $B(u) \leq 0.5$. Then, since A is a refinement of B we have that $A(u) \leq B(u)$. Since n is decreasing, we have $A(u) \leq B(u) \leq 0.5 \leq n(B(u) \leq n(A(u))$ and, then, the following chain of inequalities:

$$\min\{A(u), n(A(u))\} \leq \min\{B(u), n(B(u))\} \leq \max\{B(u), n(B(u))\} \leq \max\{A(u), n(A(u))\}$$

On the other hand, if $B(u) \geq 0.5$, since A is a refinement of B we have that $B(u) \leq A(u)$. Since n is decreasing and $n(0.5) = 0.5$, we have $n(A(u)) \leq n(B(u)) \leq 0.5 \leq B(u) \leq A(u)$ and then, the chain of inequalities

$$\min\{A(u), n(A(u))\} \leq \min\{B(u), n(B(u))\} \leq \max\{B(u), n(B(u))\} \leq \max\{A(u), n(A(u))\}$$

Therefore, in cases we have the same chain of inequalities which, by Theorem 8, and axioms (Y3) and (Y4), leads us to:

$$E(A) = \widehat{M}_{Inc}(A \cup n(A), A \cap n(A)) \leq \widehat{M}_{Inc}(B \cup n(B), B \cap n(B)) = E(B)$$

as we wanted to prove. □

At this point, it is worth mentioning that, in order to define a measure of fuzzy entropy, Definition 13 requires the use of the mean measure of inclusion $\widehat{M}_{Inc}$ and that, if we substitute it by M_{Inc} we do not obtain an entropy measure, as the following example shows.

Example 6. *Let us consider the universe $\mathcal{U} = \{u_1, u_2\}$, the fuzzy set given by $A(u_1) = 1$ and $A(u_2) = 0.5$, and the standard negation $n(x) = 1 - x$. Then, the formula*

$$M_{Inc}(A \cup n(A), A \cap n(A))$$

does not satisfies axiom $(E2)$ *of a measure of entropy. Specifically, we have that*

$$A \cup n(A)(u_1) = \max\{1,0\} = 1 \qquad A \cup n(A)(u_2) = \max\{0.5, 0.5\} = 0.5$$

$$A \cap n(A)(u_1) = \min\{1,0\} = 0 \qquad A \cap n(A)(u_2) = \min\{0.5, 0.5\} = 0.5$$

By applying Theorem 2, item 2, we have that $Inc(A \cup n(A), A \cap n(A)) = \bot$ *and therefore* $M_{Inc}(A \cup n(A), A \cap n(A)) = 0$. *However,* $A(u_2) = 0.5 \notin \{0,1\}$ *and then,* $M_{Inc}(A \cup n(A), A \cap n(A))$ *does not satisfy axiom (E1).*

Hereafter, we will focus on the structure of the measure of fuzzy entropy E induced by the φ-index of inclusion and an involutive negation n. For the sake of presentation, we refer to E simply as the measure of fuzzy entropy induced by an involutive negation n. In the following lemma, we rewrite the measure E directly in terms of the measure of inclusion M_{Inc}.

Lemma 4. *Let* A *be a fuzzy set defined on a finite universe, and let* n *be a strictly decreasing negation* n *such that* $n(0.5) = 0.5$, *then*

$$E(A) = \frac{\sum_{u \in \mathcal{U}} M_{Inc}\big(A_u \cup n(A_u), A_u \cap n(A_u)\big)}{Card(\mathcal{U})}$$

Proof. Straightforward. □

Thanks to the previous lemma and to the fact that the computation of M_{Inc} is very simple for singleton universes, the measure of fuzzy entropy induced by negations have associated easy formulas as the following proposition shows.

Proposition 8. *Let* A *be a fuzzy set defined on a finite universe and let* n *be a strictly decreasing negation such that* $n(0.5) = 0.5$, *then*

$$E(A) = \frac{\sum_{u \in \mathcal{U}} 1 - \big(A(u) - n(A(u))\big)^2}{Card(\mathcal{U})}$$

Proof. Let A be a fuzzy set defined on a finite universe, let n be a strictly decreasing negation such that $n(0.5) = 0.5$ and let $u \in \mathcal{U}$. Then,

$$M_{Inc}\big(A_u \cup n(A_u), A_u \cap n(A_u)\big) = \begin{cases} M_{Inc}\big(A_u, n(A_u)\big) & \text{if } A_u \leq 0.5 \\ M_{Inc}\big(n(A_u), A_u\big) & \text{if } A_u > 0.5 \end{cases}.$$

Let us assume that $A_u \leq 0.5$, then $Inc(A_u, n(A_u))$ is easily obtainable by following the steps in Example 1. The analytical formula so obtained is:

$$Inc(A_u, n(A_u))(x) = \begin{cases} x & \text{if } x \leq A_u(u) \\ A_u(u) & \text{if } A_u(u) \leq x \leq n(A_u(u)) \\ x & \text{if } x \geq n(A_u(u)) \end{cases} .$$

Then, after a chain of easy computation we obtain that

$$M_{Inc} = \int_0^1 Inc(A_u, n(A_u))(x)\ dx = 1 - \big(A(u) - n(A(u))\big)^2.$$

On the other hand, and by following a similar procedure we have that, if $A_u \geq 0.5$, then $M_{Inc} = \int_0^1 Inc(n(A_u), A_u)(x)\ dx = 1 - \big(A(u) - n(A(u))\big)^2$ as well. As a result, by applying Lemma 4 we have:

$$E(A) = \frac{\sum_{u\in\mathcal{U}} M_{Inc}\big(A_u \cup n(A_u), A_u \cap n(A_u)\big)}{Card(\mathcal{U})} = \frac{\sum_{u\in\mathcal{U}} 1 - \big(A(u) - n(A(u))\big)^2}{Card(\mathcal{U})}.$$

□

Example 7. *If we consider the standard negation $n(x) = 1 - x$ then its induced measure of fuzzy entropy is given by:*

$$E_{1-x}(A) = \frac{4\sum_{u\in\mathcal{U}} A(u) - A(u)^2}{Card(\mathcal{U})}$$

If we consider the negation

$$n(x) = \begin{cases} x + \sqrt{1-2x} & \text{if } x \leq 0.5 \\ x - \sqrt{2x-1} & \text{if } x > 0.5 \end{cases}$$

it is not difficult to check that the fuzzy entropy induced by n coincides with Yager's measure of entropy [29]:

$$E_Y(A) = \frac{\sum_{u\in\mathcal{U}} 1 - |2A(u) - 1|}{Card(\mathcal{U})}$$

Finally, if we consider the following negation, where we asume by convention that $0Ln(0) = 0$,

$$n(x) = \begin{cases} x + \sqrt{1 - \frac{x\ln(x) + (1-x)\ln(1-x)}{(1.5\ln(0.5))}} & \text{if } x \leq 0.5 \\ x - \sqrt{1 - \frac{x\ln(x) + (1-x)\ln(1-x)}{(1.5\ln(0.5))}} & \text{if } x > 0.5 \end{cases}$$

we obtain the normalized De Luca & Termini measure of entropy based on Shannon's entropy:

$$E_Y(A) = \frac{2}{3\ln(0.5) * Card(\mathcal{U})} \sum_{u\in\mathcal{U}} A(u)\ln(A(u)) + (1 - A(u))\ln(1 - A(u))$$

From the previous result and examples, it is evident the existence of relationships between the measures of fuzzy entropies induced by negations n and the Knopfmacher's family of measures of entropies provided by the following constructive technique[2].

Definition 14 ([17]). *Let A be a fuzzy set defined on a finite universe and let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be a real function satisfying the following properties:*

i) $\sigma(0) = \sigma(1) = 0$.

ii) $\sigma(x) = \sigma(1-x)$ *and*

iii) σ *is strictly increasing in* $[0, 0.5]$

the Knopfmacher's measure of entropy induced by σ *is given by*

$$d(A) = \frac{\sum_{u \in \mathcal{U}} \sigma(A(u))}{Card(\mathcal{U})}.$$

It is convenient to recall that the construction of measures of fuzzy entropies given above is equivalent to the technique proposed by Capocelli & De Luca in [6] and that both techniques were published in parallel. The following result shows that any Knopfmacher's measure of entropy satisfies all De Luca & Termini axioms given in Definition 1.

Theorem 10 ([17]). *Any Knopfmacher's entropy measure d is a De Luca & Termini measure of fuzzy entropy and, moreover, satisfies the following two extra properties for any pair of fuzzy sets A and B:*

i) $d(A \cup B) + d(A \cap B) = d(A) + d(B)$

ii) $d(A^c) = d(A)$.

The following theorem shows that the class of measures of fuzzy entropy induced by the φ-index of inclusion extend the class of Knopfmacher's measures of fuzzy entropy that get the value 1 as a maximum.

Theorem 11. *Given a Knopfmacher's measure of fuzzy entropy d such that $d\left(\frac{1}{2}\right) = 1$, there exists a negation n such that d is the measure of fuzzy entropy induced by n.*

[2]The definition has been rewritten in the context of finite universes and to make it compatible with the notation used in this paper. The original definition is defined on a more general context by assuming measure spaces and σ-algebras.

Proof. Let d be a Knopfmacher's measure of entropy, then there exists a mapping $\sigma\colon \mathbb{R} \to \mathbb{R}$ satisfying

i) $\sigma(0) = \sigma(1) = 0$,

ii) $\sigma(x) = \sigma(1-x)$ and

iii) σ is strictly increasing in $[0, 0.5]$.

and such that

$$d(A) = \frac{\sum_{u\in\mathcal{U}} \sigma(A(u))}{Card(\mathcal{U})}.$$

Note that, since $d\left(\frac{1}{2}\right) = 1$, we have that

$$d\left(\frac{1}{2}\right) = \frac{\sum_{u\in\mathcal{U}} \sigma(0.5)}{Card(\mathcal{U})} = 1 \quad\iff\quad \sigma(0.5) = 1.$$

Let us prove that the measure of fuzzy entropy induced by the negation

$$n(x) = \begin{cases} x + \sqrt{1-\sigma(x)} & \text{if } x \leq 0.5 \\ x - \sqrt{1-\sigma(x)} & \text{if } 0.5 < x \end{cases}.$$

coincides with d. But first, it is necessary to prove that n is actually a negation operator such that $n(0.5) = 0.5$.

- $n(0) = 0 + \sqrt{1-\sigma(0)} = 1$.
- $n(1) = 1 - \sqrt{1-\sigma(1)} = 1 - 1 = 0$.
- n is decreasing. For $x \in [0, 0.5]$, since $\sigma(x)$ is increasing in $[0, 0.5]$, we have that $n(x) = x + \sqrt{1-\sigma(x)}$ is decreasing in $[0, 0.5]$. On the other hand, $x \in (0.5, 1]$, since $\sigma(x)$ is increasing in $[0, 0.5]$ and $\sigma(1-x) = \sigma(x)$, then $\sigma(x)$ is decreasing in $x \in (0.5, 1]$. Then, we have that $n(x) = x - \sqrt{1-\sigma(x)}$ is decreasing in $[0.5, 1]$. As a result, n is decreasing.
- $n(0.5) = 0.5 + \sqrt{1-\sigma(0.5)} = 0.5 + \sqrt{1-1} = 0.5$.

Let A be a fuzzy set and let us show now that $d(A) = E(A)$. Note that if we prove the equality $1 - \big(A(u) - n(A(u))\big)^2 = \sigma(A(u))$ for all $u \in \mathcal{U}$, then by Proposition 8 the proof ends, since

$$E(A) = \frac{\sum_{u\in\mathcal{U}} 1 - \big(A(u) - n(A(u))\big)^2}{Card(\mathcal{U})} = \frac{\sum_{u\in\mathcal{U}} \sigma(A(u))}{Card(\mathcal{U})} = d(A)$$

Let us prove such an equality for all $u \in \mathcal{U}$. Let $u \in \mathcal{U}$ and let us assume firstly that $A(u) \leq 0.5$. Then,

$$1-\big(A(u)-n(A(u))\big)^2 = 1-\Big(A(u) - \big(A(u) + \sqrt{1-\sigma(A(u))}\big)\Big)^2 = 1-\Big(\sqrt{1-\sigma(A(u))}\Big)^2 = \sigma(A(u)).$$

On the other hand, if $A(u) > 0.5$ then

$$1-\big(A(u)-n(A(u))\big)^2 = 1-\Big(A(u) - \big(A(u) - \sqrt{1-\sigma(A(u))}\big)\Big)^2 = 1-\Big(-\sqrt{1-\sigma(A(u))}\Big)^2 = \sigma(A(u)).$$

As a consequence, $1 - \big(A(u) - n(A(u))\big)^2 = \sigma(A(u))$ for all $u \in \mathcal{U}$ as we wanted to prove. □

In the following example we show the existence of measures of entropy induced by negations that are not Knopfmacher's measures of fuzzy entropy.

Example 8. *Let us consider a singleton universe* $\mathcal{U} = \{u_0\}$ *and the measure of entropy induced by the negation* $n(x) = 1 - x^2$. *By Proposition 8, we have that for any fuzzy set* A *on* $\mathcal{U}$

$$E(A) = \frac{\sum_{u\in\mathcal{U}} 1 - \big(A(u) - n(A(u))\big)^2}{Card(\mathcal{U})} = 1 - \big(A(u_0) - 1 + A(u_0)^2)\big)^2.$$

Let us assume that there exists a Knopfmacher's measure of entropy d *such that* $E(A) = d(A)$ *for any fuzzy set* A *defined on* $\mathcal{U}$. *Then there exists* $\sigma\colon \mathbb{R} \to \mathbb{R}$ *satisfying the three items of Definition 14 and, since* $\mathcal{U}$ *is a singleton, we have that*

$$E(A) = d(A) = \frac{\sum_{u\in\mathcal{U}} \sigma(A(u))}{Card(\mathcal{U})} = \sigma(A(u_0)).$$

In other words, for any fuzzy set A *we have the equality:*

$$1 - \big(A(u_0) - 1 + A(u_0)^2)\big)^2 = \sigma(A(u_0)).$$

Let us consider the fuzzy sets given by $A_1(u_0) = 0.25$ *and* $A_2(u_0) = 0.75$. *Then, by the previous formula we have that*

$$\sigma(0.25) = 0.527 \quad \textit{and} \quad \sigma(0.75) = 0.902$$

which contradicts item iii) of Definition 14, since $0.527 = \sigma(0.25) \neq \sigma(1-0.25) = \sigma(0.75) = 0.902$.

Finally we relate the measures of entropy induced by negations with those that satisfy the Knopfmacher's measure of entropy, in addition to De Luca & Termini axioms as stated in Theorem 10. The following result shows that the measures of entropy induced by negations satisfy the relation between union and intersection.

Proposition 9. *Let A be a fuzzy set defined on a finite universe, let n be a strictly decreasing negation such that $n(0.5) = 0.5$ and let E be the measure of entropy induced by n. Then:*

$$E(A \cup B) + E(A \cap B) = E(A) + E(B)$$

Proof. Let $u \in \mathcal{U}$ and let us assume without loss of generality that $A(u) \leq B(u)$. Then $(A \cup B)(u) = B(u)$ and $(A \cap B)(u) = A(u)$; as a result:

$$\big((A\cup B)(u) - n(A\cup B(u))\big)^2 + \big((A\cap B)(u) - n(A\cap B(u))\big)^2 = \big(A(u) - n(A(u))\big)^2 + \big(B(u) - n(B(u))\big)^2$$

Therefore, by applying Proposition 8 we have:

$$\begin{aligned} E(A \cup B) + E(A \cap B) = \\ &= \frac{\sum_{u\in\mathcal{U}} 1 - \big((A \cup B)(u) - n(A \cup B(u))\big)^2}{Card(\mathcal{U})} + \frac{\sum_{u\in\mathcal{U}} 1 - \big((A \cap B)(u) - n(A \cap B(u))\big)^2}{Card(\mathcal{U})} \\ &= \frac{\sum_{u\in\mathcal{U}} 1 - \big(B(u) - n(B(u))\big)^2}{Card(\mathcal{U})} + \frac{\sum_{u\in\mathcal{U}} 1 - \big(A(u) - n(A(u))\big)^2}{Card(\mathcal{U})} = E(A) + E(B) \end{aligned}$$

□

The last result shows that, if the negation satisfies certain property related to symmetry, then the measures of entropy induced by negations also satisfy the relation with the complement of fuzzy sets.

Proposition 10. *Let n be a strictly decreasing negation such that $n(0.5) = 0.5$ and $n(1 - x) = 1 - n(x)$, and let E be the measure of entropy induced by n. Then, for any fuzzy set A defined on a finite universe, we have that $E(A^c) = E(A)$.*

Proof. Let A be a fuzzy set, then by Proposition 8 we have

$$\begin{aligned} E(A^c) &= \frac{\sum_{u\in\mathcal{U}} 1 - \big(1 - A(u) - n(1 - A(u))\big)^2}{Card(\mathcal{U})} \\ &= \frac{\sum_{u\in\mathcal{U}} 1 - \big(1 - A(u) - 1 + n(A(u))\big)^2}{Card(\mathcal{U})} = \frac{\sum_{u\in\mathcal{U}} 1 - \big(A(u) - n(A(u))\big)^2}{Card(\mathcal{U})} = E(A^c). \end{aligned}$$

□

5. Conclusions

In this paper we have continued the comparison started in [20] between the notion of φ-index of inclusion and the four main axiomatic approaches in the literature dealing with inclusion between fuzzy sets; namely, Kitainik [16], Sinha & Dougherty [25], Young [30], and Fan, Xie & Pei [12]. To perform such a comparison, we have defined two different measures of inclusion that satisfy the axioms proposed by Fan, Xie & Pei [12] and that are in accordance, respectively, with the approaches given, firstly, by Kitainik and Sinha & Dougherty and, secondly, by the approach of Young. Concerning the results obtained from the first measure of inclusion, we obtain an important conclusion: the φ-index of inclusion actually gathers the idea underlying the axiomatic approaches of Kitainik and Sinha & Dougherty. This conclusion is based also on the previous work given in [20], where it is shown that the only axiom that is it not satisfied directly by the φ-index of inclusion is the one concerning the complement of fuzzy sets. The reason of such a discrepancy is because the relationship between the φ-index of inclusion and the complement of fuzzy sets is given in terms of adjoint pairs, which is a property that cannot be rewritten directly in terms of values in the unit interval $[0, 1]$. The measure of inclusion proposed in this paper shows that the inherent amount of inclusion of A in B and B^c in A^c is the same when the inclusion is represented by means of the φ-index of inclusion.

Concerning the second measure of inclusion, due to the satisfiability of the axioms proposed by Young and the relationship of those axioms with measures of fuzzy entropy, we have proposed the use of the φ-index of inclusion as a base to define measures of fuzzy entropy according to the axioms proposed by De Luca & Termini [10]. We have shown that the formula of this measure of entropy can be simplified conveniently and that it covers the constructive measures of entropy given by Knopfmacher [17] and Capocelli & De Luca [6]; which includes the Yager measure of fuzzy entropy [29] and the normalised version of the De Luca measure of entropy based on Shannon entropy.

The intrinsic differences between the measures of inclusion based on Young and Sinha & Dougherty open up two different and plausible future research lines. On the one hand, Young measures are closely related to entropy measures and have been applied to decision making [7], whereas Kitainik and Sinha & Dougherty are related to fuzzy implications and to knowledge-based systems [4].

Moreover, note that one of the goals of defining the measures provided in this paper is to motivate the use of the φ-index of inclusion as a good representation of the inclusion between fuzzy sets. In other words, the functional feature of the φ-index of inclusion is of great interest for frameworks where functional operators, similarities and inclusion are combined. One of those frameworks is Fuzzy Logic Programming [26], where the inference is based on IF-THEN rules associated to monotonic functions. We believe that the monotonic functions used in fuzzy logic programming can be related to the φ-index of inclusion and, as a result, it would be possible to define inference procedures based on the φ-index of inclusion.

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