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A non markovian retrial queueing system[☆]

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ABSTRACT

This paper studies a discrete-time queueing system in which a customer who enters in the system with a server occupied may choose to go to the retrial group or to begin its service displacing to the retrial group the customer that was in the server. If the server is idle the arriving customer begins its service immediately.

Retrial and service times are general, and the only repeated customer that can go directly to the server is the one situated at the first place of the retrial group.

A complete study of the model has been curried out noting specially the analysis of a recursive algorithm implemented by Theorems 2 and 3 and the study of the waiting time of a customer in the queue and in the system. Finally numerical examples are given.

1. Introduction

Researches on retrial queues with non-geometrical retrial times is motivated in computers and telecommunication networks, where retrial times are non convenient to be treated as geometrical.

Research on queueing theory allow models, structural insights, problem solutions and algorithms to many application areas. Due to its practical applicability to communications technology, production and manufacturing more and more complex systems requires a complex elaborated models, techniques and algorithm. Discrete-time systems are more convenient to treat problems that with its continuous counterpart, that is:

1. the basic unit describes a numbering scheme in which there are only two possible values and
2. the occurrence of simultaneous events.

For a further reading we recommend the following cites [1–4], and the references therein.

There are situations in which is convenient to take into account displacement of tasks, customers, etc. This procedure is named as *synchronised or triggered motion*, see [5,6] and for inverse order discipline we refer to [7]–[8]. If in this sort of actions are considered the elimination of tasks then enters in consideration what its called usually in the queueing literature as negative customers see for example [5] and [9]–[10].

In this work it is analysed a retrial queueing system with general retrial times in which customers may interrupt a service displacing to the orbit the customer that was in the server. A complete study of the model has been carried out including the waiting time of a customer in the orbit and in the system.

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The remainder of this paper is structured as follows. The following section gives a description of the queueing model. In Section 3 the study of the Markov chain is carried out. The orbit and system size distributions are obtained together with several performance measures of the system. In Section 4, the recursive computation of the steady-state distributions of the orbit and system size is investigated. Section 5 is devoted to introduce the busy period of an auxiliary system that will be useful to study the customer's delay. In Section 6 the distributions of the sojourn time of a customer in the server, the orbit and in the system are obtained. In Section 7, numerical examples to illustrate the impact of the parameters on several performance characteristics are provided.

2. The mathematical model

We are going to study a discrete-time queueing system such that the axis that represents the times is divided into consecutive intervals of the same length, called *slots*. This time axis will be numbered and we suppose that the arrival, departure and retrials happened in the border of this intervals.

Let as denote by a the probability that a customer enters in the system in a slot. If the server is free the customers gets service immediately but if the server is occupied, the arriving customer has two options: displace the one at service to a pool of customers called *orbit* with probability θ or leave the service area to the *orbit* with probability $\bar{\theta}$ in accordance with a *First in First out* (FIFO discipline).

It will be assumed that only the customer situated in the first place of the retrial group is allowed for access to the server. Let $\{a_i\}_{i=0}^{\infty}$ be the distribution of the inter-retrial times with generating function (GF) $A(x) = \sum_{i=0}^{\infty} a_i x^i$ and $\{s_i\}_{i=1}^{\infty}$ the distribution of the service times with GF $S(x) = \sum_{i=1}^{\infty} s_i x^i$.

3. The Markov chain

At time m^+, the system can be expressed by the process

$$Y_m = (C_m, \xi_{0,m}, \xi_{i,m}, N_m)$$

where C_m can take two values: cero or one depending if the server is idle or occupied respectively and N_m is the size of the orbit. If $C_m = 0$ and $N_m > 0$, $\xi_{0,m}$ denotes the remaining retrial time. If $C_m = 1$, $\xi_{1,m}$ corresponds to the remaining service time of the customer being served.

The state space of our system is:

$$\{(0,0); (0,i,k) : i \geq 1, k \geq 1; (1,i,k) : i \geq 1, k \geq 0\}.$$

Our goal is to find the following probabilities:

$$\begin{aligned}
\pi_{0,0} &= \lim_{m\to\infty} P[C_m = 0, N_m = 0], \\
\pi_{0,i,k} &= \lim_{m\to\infty} P[C_m = 0, \xi_{0,m} = i, N_m = k], \ \ i \geq 1, k \geq 1, \\
\pi_{1,i,k} &= \lim_{m\to\infty} P[C_m = 1, \xi_{0,m} = i, N_m = k], \ \ i \geq 1, k \geq 0,
\end{aligned}$$

of the Markov chain $\{Y_m, m \in \mathbb{N}\}$.

The system of equilibrium equations (SEE) is given by

$$\pi_{0,0} = \bar{a}\pi_{0,0} + \bar{a}\pi_{1,1,0} \iff a\pi_{0,0} = \bar{a}\pi_{1,1,0} \tag{1}$$

$$\pi_{0,i,k} = \bar{a}\pi_{0,i+1,k} + \bar{a}a_i\pi_{1,1,k}, \ \ i \geq 1, \ k \geq 1 \tag{2}$$

$$\begin{aligned}
\pi_{1,i,k} = {} & \delta_{0,k} a s_i \pi_{0,0} + \bar{a} s_i \pi_{0,1,k+1} + (1-\delta_{0,k}) a s_i \sum_{j=1}^{\infty} \pi_{0,j,k} \\
& + a s_i \pi_{1,1,k} + \bar{a} a_0 s_i \pi_{1,1,k+1} \\
& + (1-\delta_{0,k}) a\bar{\theta} \pi_{1,i+1,k-1} + \bar{a}\pi_{1,i+1,k} \\
& + (1-\delta_{0,k}) a\theta s_i \sum_{j=2}^{\infty} \pi_{1,j,k-1}, \ \ i \geq 1, \ k \geq 0,
\end{aligned} \tag{3}$$

with $\bar{a} = 1 - \bar{a}$.

The normalisation relation is

$$\pi_{0,0} + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \pi_{0,i,k} + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \pi_{1,i,k} = 1.$$

Lets introduce the following generating functions to solve (1)–(3):

$$\varphi_0(x,z) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{0,i,k}x^i z^k$$

$$\varphi_1(x,z) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i,k}x^i z^k$$

$$\varphi_{0,i}(z) = \sum_{k=1}^{\infty}\pi_{0,i,k}z^k,\ \ i \geq 1$$

$$\varphi_{1,i}(z) = \sum_{k=0}^{\infty}\pi_{1,i,k}z^k,\ \ i \geq 1.$$

From (1)–(3) we readily obtain:

$$\varphi_{0,i}(z) = \bar{a}\varphi_{0,i+1}(z) + \bar{a}a_i\varphi_{1,1}(z) - aa_i\pi_{0,0} \tag{4}$$

$$\begin{aligned}\varphi_{1,i}(z) = {}& (\bar{a}+a\bar{\theta}z)\varphi_{1,i+1}(z) + \frac{\bar{a}a_0 + az(1-\theta z)}{z}s_i\varphi_{1,1}(z)\\ &+\frac{\bar{a}}{z}s_i\varphi_{0,1}(z) + as_i\varphi_0(1,z) + a\theta z s_i\varphi_1(1,z) + \frac{z-a_0}{z}as_i\pi_{0,0}.\end{aligned} \tag{5}$$

and from (4) and (5):

$$\frac{x-\bar{a}}{x}\varphi_0(x,z) = \bar{a}[A(x)-a_0]\varphi_{1,1}(z) - \bar{a}\varphi_{0,1}(z) - a[A(x)-a_0]\pi_{0,0} \tag{6}$$

$$\begin{aligned}z\frac{x-(\bar{a}+a\bar{\theta}z)}{x}\varphi_1(x,z) = {}& [(\bar{a}a_0 + az(1-\theta z))S(x) - z(\bar{a}+a\bar{\theta}z)]\varphi_{1,1}(z)\\ &+\bar{a}S(x)\varphi_{0,1}(z) + azS(x)\varphi_0(1,z)\\ &+a\theta z^2 S(x)\varphi_1(1,z) + (z-a_0)aS(x)\pi_{0,0}.\end{aligned} \tag{7}$$

For $x = 1$ in (6) implies:

$$a\varphi_0(1,z) = \bar{a}(1-a_0)\varphi_{1,1}(z) - \bar{a}\varphi_{0,1}(z) - a(1-a_0)\pi_{0,0}. \tag{8}$$

and substituting the above equation into (7) yields

$$\begin{aligned}z\frac{x-(\bar{a}+a\bar{\theta}z)}{x}\varphi_1(x,z) = {}& \Big[\Big(\bar{a}a_0(1-z) + z(1-a\theta z)\Big)S(x) -\\ &-z(\bar{a}+a\bar{\theta}z)\Big]\varphi_{1,1}(z) + \bar{a}(1-z)S(x)\varphi_{0,1}(z) +\\ &+a\theta z^2 S(x)\varphi_1(1,z) - aa_0(1-z)S(x)\pi_{0,0}.\end{aligned} \tag{9}$$

For $x = 1$ in (9) gives

$$az\varphi_1(1,z) = (\bar{a}a_0 + az)\varphi_{1,1}(z) + \bar{a}\varphi_{0,1}(z) - aa_0\pi_{0,0}. \tag{10}$$

by substituting (10) into (9) we get

$$\begin{aligned}z\frac{x-(\bar{a}+a\bar{\theta}z)}{x}\varphi_1(x,z) = {}& [(\bar{a}a_0(1-\bar{\theta}z) + z)S(x) - z(\bar{a}+a\bar{\theta}z)]\varphi_{1,1}(z) +\\ &+\bar{a}(1-\bar{\theta}z)S(x)\varphi_{0,1}(z) - aa_0(1-\bar{\theta}z)S(x)\pi_{0,0}.\end{aligned} \tag{11}$$

By choosing $x = \bar{a}$ and $x = \bar{a} + a\bar{\theta}z$ in (6) and (11) respectively, we obtain

$$a[A(\bar{a}) - a_0]\pi_{0,0} = \bar{a}[A(\bar{a}) - a_0]\varphi_{1,1}(z) - \bar{a}\varphi_{0,1}(z), \tag{12}$$

$$\begin{aligned}aa_0(1-\bar{\theta}z)S(\bar{a}+a\bar{\theta}z)\pi_{0,0} = {}& \Big[(\bar{a}a_0(1-\bar{\theta}z) + z)S(\bar{a}+a\bar{\theta}z) -\\ &-z(\bar{a}+a\bar{\theta}z)\Big]\varphi_{1,1}(z) + \bar{a}(1-\bar{\theta}z)S(\bar{a}+a\bar{\theta}z)\varphi_{0,1}(z),\end{aligned} \tag{13}$$

From (12) and (13) we get:

$$\varphi_{1,1}(z) = \frac{aA(\bar{a})(1-\bar{\theta}z)S(\bar{a}+a\bar{\theta}z)}{D(z)}\pi_{0,0} \tag{14}$$

$$\varphi_{0,1}(z) = \frac{a[A(\bar{a}) - (a_0)]z[(\bar{a} + a\bar{\theta}z) - S(\bar{a} + a\bar{\theta}z)]}{D(z)} \frac{\pi_{0,0}}{\bar{a}} \tag{15}$$

where

$$D(z) = \bar{a}A(\bar{a})(1 - \bar{\theta}z)S(\bar{a} + a\bar{\theta}z) - z[(\bar{a} + a\bar{\theta}z) - S(\bar{a} + a\bar{\theta}z)].$$

Finally, substituting (14) and (15) into (6) and (11), we get

$$\varphi_0(x, z) = \frac{A(x) - A(\bar{a})}{x - \bar{a}} \cdot \frac{xaz[(\bar{a} + a\bar{\theta}z) - S(\bar{a} + a\bar{\theta}z)]}{D(z)} \pi_{0,0} \tag{16}$$

$$\varphi_1(x, z) = \frac{S(x) - S(\bar{a} + a\bar{\theta}z)}{x - (\bar{a} + a\bar{\theta}z)} \cdot \frac{xa(1 - \bar{\theta}z)(\bar{a} + a\bar{\theta}z)}{D(z)} A(\bar{a})\pi_{0,0} \tag{17}$$

Making use of the normalising relation, that can be written as $\pi_{0,0} + \varphi_0(1, 1) + \varphi_1(1, 1) = 1$, we get:

$$\pi_{0,0} = \frac{\bar{a}\theta A(\bar{a})S(\bar{a} + a\bar{\theta}) - [(\bar{a} + a\bar{\theta}) - S(\bar{a} + a\bar{\theta})]}{\theta S(\bar{a} + a\bar{\theta})A(\bar{a})} \tag{18}$$

since $\pi_{0,0} > 0$, it results that the inequality

$$\bar{a}\theta A(\bar{a})S(\bar{a} + a\bar{\theta}) - [(\bar{a} + a\bar{\theta}) - S(\bar{a} + a\bar{\theta})] > 0, \tag{19}$$

is a necessary condition for the ergodicity of the model, then we have:

Theorem 1.
If (19) is fulfilled the generating functions of the stationary distribution of the chain are given by

$$\varphi_0(x, z) = \frac{A(x) - A(\bar{a})}{x - \bar{a}} \cdot \frac{xaz[(\bar{a} + a\bar{\theta}z) - S(\bar{a} + a\bar{\theta}z)]}{D(z)} \pi_{0,0}$$

$$\varphi_1(x, z) = \frac{S(x) - S(\bar{a} + a\bar{\theta}z)}{x - (\bar{a} + a\bar{\theta}z)} \cdot \frac{xa(1 - \bar{\theta}z)(\bar{a} + a\bar{\theta}z)}{D(z)} A(\bar{a})\pi_{0,0}$$

where

$$D(z) = \bar{a}A(\bar{a})(1 - \bar{\theta}z)S(\bar{a} + a\bar{\theta}z) - z[(\bar{a} + a\bar{\theta}z) - S(\bar{a} + a\bar{\theta}z)].$$

and

$$\pi_{0,0} = \frac{\bar{a}\theta A(\bar{a})S(\bar{a} + a\bar{\theta}) - [(\bar{a} + a\bar{\theta}) - S(\bar{a} + a\bar{\theta})]}{\theta S(\bar{a} + a\bar{\theta})A(\bar{a})}$$

Corollary 1 (*1*)**.** *the probability generating function of the orbit size (i.e., of the variable N) is given by*

$$\psi(z) = \pi_{0,0} + \varphi_0(1, z) + \varphi_1(1, z) = $$
$$= \frac{\theta z S(\bar{a} + a\bar{\theta}z) + (1 - z)(\bar{a} + a\bar{\theta}z)}{D(z)} A(\bar{a})\pi_{0,0}$$

(2) The probability generating function of the system size (i.e., of the variable L) is given by

$$\Phi(z) = \pi_{0,0} + \varphi_0(1, z) + z\varphi_1(1, z) = $$
$$= \frac{(\bar{a} + az)(1 - \bar{\theta}z)S(\bar{a} + a\bar{\theta}z)}{D(z)} A(\bar{a})\pi_{0,0}.$$

Corollary 2 (*1*)**.** *The expected value of the orbit size is*

$$E[N] = \psi'(1) = \frac{\triangle}{\nabla}$$

with

$$\triangle = \theta S(\bar{a} + a\bar{\theta})\Big[a\bar{\theta} - \bar{a}A(\bar{a})[(\bar{a} + a\bar{\theta}) - S(\bar{a} + a\bar{\theta})]\Big] + (\bar{a} + a\bar{\theta})\Big[(\bar{a} + a\bar{\theta}) - S(\bar{a} + a\bar{\theta})\Big] -$$
$$- a\theta\bar{\theta}(\bar{a} + a\bar{\theta})S'(\bar{a} + a\bar{\theta})$$
$$\nabla = \theta S(\bar{a} + a\bar{\theta})\Big[\bar{a}\theta A(\bar{a})S(\bar{a} + a\bar{\theta}) - [(\bar{a} + a\bar{\theta}) - S(\bar{a} + a\bar{\theta})]\Big]$$

(2) The expected value of the system size is

$$E[L] = \Phi'(1) = E[N] + \varphi_1(1, 1),$$

where

$$\varphi_1(1,1)=\frac{(\bar a+a\bar\theta)[1-S(\bar a+a\bar\theta)]}{\theta S(\bar a+a\bar\theta)}.$$

- Interesting Cases

1. If $a_0=1$ we have

$$\Phi(z)=\frac{(\bar a+az)(1-\bar\theta z)S(\bar a+a\bar\theta z)}{(\bar a+az+\bar a\theta z)S(\bar a+a\bar\theta z)-z(\bar a+a\bar\theta z)}\pi_0$$

where

$$\pi_0=\frac{(1+\bar a\theta)S(\bar a+a\bar\theta)-(\bar a+a\bar\theta)}{\theta S(\bar a+a\bar\theta)}.$$

See [11].

2. When $\theta=0$,

$$\Phi(z)=\frac{(1-z)(\bar a+az)S(\bar a+az)}{\bar aA(\bar a)(1-z)S(\bar a+az)-z[(\bar a+az)-S(\bar a+az)]}A(\bar a)\pi^*_{0,0}$$

where

$$\pi^*_{0,0}=\lim_{\theta\to 0}\pi_{0,0}=\frac{a+\bar aA(\bar a)-\rho}{A(\bar a)},\ \text{with } \rho=aS'(1).$$

See [12].

4. Calculation of the steady-state probabilities of the system and orbit size

In this section we present recursive formulae for calculating the steady-state probability of the system and orbit size.

Theorem 2. *The steady-state distribution of the system size is*

$$\Phi_0=P[L=0]=\pi_{0,0} \tag{20}$$

$$\Phi_k=P[L=k]=\frac{\sum_{n=0}^{k-1}[b_{k-n}-\bar aA(\bar a)c_{k-n}]\Phi_n+[\bar ac_k+ac_{k-1}]A(\bar a)\pi_{0,0}}{A(\bar a)S(\bar a)},\ k\ge 1 \tag{21}$$

where

$$b_n=\sum_{i=n}^{\infty}\binom{i-1}{n-1}S_{i+1}\bar a^{\,i-n}a(a\bar\theta)^{n-1},\ n\ge 1$$

$$c_n=\sum_{i=n}^{\infty}\binom{i}{n}s_{i+1}\bar a^{\,i-n}(a\bar\theta)^n,\ n\ge 1$$

$$c_0=\frac{S(\bar a)}{\bar a}.$$

Proof. The GF $\Phi(z)$ of the system size verifies

$$\Phi(z)G(z)=(\bar a+az)\frac{S(\bar a+a\bar\theta z)}{\bar a+a\bar\theta z}A(\bar a)\pi_{0,0} \tag{22}$$

where

$$G(z)=\bar aA(\bar a)\frac{S(\bar a+a\bar\theta z)}{\bar a+a\bar\theta z}-\frac{z}{1-\bar\theta z}\left[1-\frac{S(\bar a+a\bar\theta z)}{\bar a+a\bar\theta z}\right]=\sum_{n=0}^{\infty}g_nz^n.$$

Using Newton's binomial, [[13], theorem 3], we have

$$G(z)=A(\bar a)S(\bar a)-\sum_{n=1}^{\infty}[b_n-\bar aA(\bar a)c_n]z^n.$$

The right hand side of (22) can be expressed as

$$\left[S(\bar{a}) + \sum_{n=1}^{\infty}(\bar{a}c_n + ac_{n-1})z^n\right]A(\bar{a})\pi_{0,0}$$

and from (22) by comparing the coefficients of z^k we get

$$\begin{aligned} \Phi_0 g_0 &= A(\bar{a})S(\bar{a})\pi_{0,0} \\ \sum_{n=0}^{k}\Phi_n g_{k-n} &= (\bar{a}c_k + ac_{k-1})A(\bar{a})\pi_{0,0}, \ k \geq 1, \end{aligned}$$

and as a consequence the formulae (20) and (21) are obtained. □

Theorem 3. *The steady- state distribution of the orbit size is*

$$\psi_0 = P[N=0] = \frac{\pi_{0,0}}{S(\bar{a})} \tag{23}$$

$$\psi_k = P[N=k] = \tag{24}$$

$$\frac{\sum_{n=0}^{k-1}[b_{k-n} - \bar{a}A(\bar{a})c_{k-n}]\Psi_n + [b_{k+1} - b_k + c_k]A(\bar{a})\pi_{0,0}}{A(\bar{a})S(\bar{a})}, \ k \geq 1 \tag{25}$$

Proof. The GF $\psi(z)$ of the orbit verifies

$$\begin{aligned} \psi(z)G(z) &= \left(\frac{S(\bar{a}+a\bar{\theta}z)}{\bar{a}+a\bar{\theta}z} + \frac{1-z}{1-\bar{\theta}z}\left[1 - \frac{S(\bar{a}+a\bar{\theta}z)}{\bar{a}+a\bar{\theta}z}\right]\right)A(\bar{a})\pi_{0,0} \\ &= \sum_{n=0}^{\infty} l_n z^n \end{aligned} \tag{26}$$

with

$$\begin{aligned} l_0 &= A(\bar{a})\pi_{0,0} \\ l_n &= [b_{n+1} - b_n + c_n]A(\bar{a})\pi_{0,0}, \ n \geq 1. \end{aligned}$$

equating the coefficients of z^k in (26) we get

$$\begin{aligned} \psi_0 g_0 &= A(\bar{a})\pi_{0,0} \\ \sum_{n=0}^{k}\psi_n g_{k-n} &= [b_{k+1} - b_k + c_k]\,A(\bar{a})\pi_{0,0}, \ k \geq 1. \end{aligned}$$

Taking into account the expression of the sequence $\{g_n\}_{n=0}^{\infty}$, the formulae (23) and (24) are obtained. □

5. Busy period

In this paragraph we are going to analyse the busy period (BP) of a system whose difference from the one under study consists in that a customer arrives with probability $a\theta$ and goes to the server displacing to the first place of the orbit the customer that was receiving service. For this system, that will be of interest in the study of the waiting tomes, we find its main characteristics.

Let h_k, $k \geq 0$ be the probability that the length of the BP is k slots. Then:

$$\begin{aligned} h_0 &= 0 \\ h_k &= (1-a\theta)^k s_k + \sum_{i=1}^{k}(1-a\theta)^{i-1}a\theta s_i h_{k-i} + \\ &\quad + \sum_{i=1}^{k}(1-a\theta)^{i-1}S_{i+1}a\theta\sum_{j=0}^{k-i}h_j\sum_{l=0}^{k-i-j}w_l h_{k-i-j-l}, \ k \geq 1 \end{aligned}$$

where w_l, $l \geq 0$, is the probability that the time spent by the customer situated in the first place of the retrial group since the ending of a BP till the moment that this customer begins its service is l slots.

Explanation of the above formula:

- The term $(1-a\theta)^k s_k$:
  The arriving customer chooses a service time of k slots, with probability s_k, and in this k slots no customer arrives to the system, with probability $(1-a\theta)^k$.
- The term $\sum_{i=1}^{k}(1-a\theta)^{i-1}a\theta s_i h_{k-i}$:
  The arriving customer chooses a service times of i slots, $i=1,\dots,k$, with probability s_i, no customer arrives in the first $i-1$ slots, with probability $(1-a\theta)^{i-1}$ and in the slot i a customer arrives, with probability $a\theta$, opening a BP of length $k-i$ slots, with probability h_{k-i}.
- The term $\sum_{i=1}^{k}(1-a\theta)^{i-1}S_{i+1}a\theta\sum_{j=0}^{k-i}h_j\sum_{l=0}^{k-i-j}w_l h_{k-i-j-l},\ k\geq 1$
  The arriving customer chooses a service time that lasts not less than $i+1$ slots, $i=1,\dots,k$, with probability S_{i+1}. In the first $i-1$ slots no customer arrives, with probability $(1-a\theta)^{i-1}$ and in the slot i a customer arrives, with probability $a\theta$, initiating a BP of length j slots, $j=0,\dots,k-i$.
  Once the BP is finished the customer that was expelled to the head of the orbit for the arrival of the new customer will remain there l slots, $l=0,\dots,k-i-j$, with probability w_l, till the beginning of its service and then will open a BP of length of $k-i-j-l$ slots.

The generating function $h(x)=\sum_{k=0}^{\infty}h_k x^k$ is given by

$$h(x)=S[(1-a\theta)x]+\frac{a\theta}{1-a\theta}S[(1-a\theta)x]h(x)+ \\ +\frac{a\theta}{1-a\theta}\frac{(1-a\theta)x-S[(1-a\theta)x]}{1-(1-a\theta)x}h^2(x)w(x),$$

where $w(x)$ is the generating function of $\{w_l,\ l\geq 0\}$. This formula can be expressed as

$$a\theta[(1-a\theta)x-S[(1-a\theta)x]]w(x)h^2(x)+ \\ +[a\theta[1-(1-a\theta)x]S[(1-a\theta)x]-(1-a\theta)[1-(1-a\theta)x]]h(x)+ \\ +(1-a\theta)[1-(1-a\theta)x]S[(1-a\theta)x]=0. \tag{27}$$

In order to find the generating function $w(x)=\sum_{k=0}^{\infty}w_k x^k$, let us note that the probabilities w_k, $k\geq 0$, satisfy the following recursive formulae:

$$w_0=a_0 \\ w_k=(1-a\theta)^k a_k+(1-\delta_{1,k})a\theta\sum_{l=1}^{k-1}(1-a\theta)^{l-1}A_l\sum_{i=1}^{k-l}h_i w_{k-l-i},\ k\geq 1 \tag{28}$$

where $A_l=\sum_{i=l}^{\infty}a_i$.

Let us explain the formulae (28). Consider the slot in which a BP ends, say the slot 0, then the customer at the first place of the orbit will begin immediately its service with probability a_0 (let us note that no customer arrives in the slot 0 since in this slot a BP has finished). Now, we consider the equation for $k\geq 1$. The customer at the head of the orbit will wait there k, $k\geq 1$, slots since ending a BP till the beginning of its service if:

1. In the first k slots no customers arrive and in the slot k a retrial occurs (with probability $(1-a\theta)^k a_k$), or
2. before the slot l, $1\leq l\leq k-1$, no customer arrives and no retrial occurs (with probability $(1-a\theta)^{l-1}A_l$), and in the slot l a new customer arrives opening a BP of length i ($i=1,\dots,k-l$), and ended this BP the customer placed at the head of the orbit will wait there till the beginning of its service $k-l-i$ slots (all with probability $a\theta h_i w_{k-l-i}$).

The GF $w(x)$ has the form

$$w(x)=\frac{[1-(1-a\theta)x]A[(1-a\theta)x]}{1-x\Big[1-a\theta+a\theta h(x)[1-A[(1-a\theta)x]]\Big]} \tag{29}$$

and its expected value is given by

$$\bar{w}=w'(1)=\frac{(1+a\theta\bar{h})[1-A(1-a\theta)]}{a\theta A(1-a\theta)}, \tag{30}$$

where $\bar{h}=h'(1)$.

Taking into account (29), the formula (27) can be expressed as

$$\alpha(x)h^2(x)+\beta(x)h(x)+\gamma(x)=0 \tag{31}$$

with

$$\alpha(x) = a\theta\Big[\big[(1-a\theta)x - S[(1-a\theta)x]\big]A[(1-a\theta)x]\Big] + \\ + x\big[(1-a\theta) - a\theta S[(1-a\theta)x][1-A[(1-a\theta)x]]\big]$$

$$\beta(x) = -\Big[\,\big[(1-a\theta) - a\theta S[(1-a\theta)x]\big][1-(1-a\theta)x] + \\ + a\theta x(1-a\theta)S[(1-a\theta)x][1-A[(1-a\theta)x]]\,\Big]$$

$$\gamma(x) = (1-a\theta)[1-(1-a\theta)x]S[(1-a\theta)x]$$

Then, the GF $h = h(x)$ verifies:

$$f(h) = 0 \tag{32}$$

where

$$f(h) = \alpha(x)h^2 + \beta(x)h + \gamma(x)$$

For $0 < x < 1$ we have

$$a(x) > 0$$
$$f(0) = \gamma(x) > 0$$

and

$$f(1) < (1-a\theta)(x-1)[1-S[(1-a\theta)x]] < 0.$$

The above inequalities show that for $x,\ 0 < x < 1$, Eq. (31) has two solutions $h(x)$ and $h^*(x)$ verifying $0 < h(x) < 1 < h^*(x)$ with

$$h(x) = \frac{-\beta(x) - \Big[\beta(x)^2 - 4\alpha(x)\gamma(x)\Big]^{1/2}}{2\alpha(x)}$$

$$h^*(x) = \frac{-\beta(x) + \Big[\beta(x)^2 - 4\alpha(x)\gamma(x)\Big]^{1/2}}{2\alpha(x)}$$

For $x = 1$ is $f(1) = 0$ which means that at least one of the two solutions $h(x)$ or $h^*(x)$ takes the value 1 for $x = 1$. The inequality $h^*(x) > 1$ takes place iff

$$[\beta(1)^2 - 4\alpha(1)\gamma(1)]^{1/2} > 2\alpha(1) + \beta(1) \\ = 1 - a\theta - S(1-a\theta) - (1-a\theta)A(1-a\theta)S(1-a\theta).$$

If (19) is satisfied, the right hand side of this formula is negative and consequently $h^*(1) > 1$ and $h(1) = 1$. Therefore, the GF of the BP is $h(x)$.

The expected value of the BP is

$$\bar{h} = h'(1) = \frac{a\theta A(1-a\theta)S(1-a\theta) + 1 - a\theta - S(1-a\theta)}{a\theta\Big[(1-a\theta)A(1-a\theta)S(1-a\theta) - [1-a\theta-S(1-a\theta)]\Big]}.$$

where (30) has been used.

It will be useful to study the GF $h(x;m)$ of the BP that starts with a customer whose remaining service time consists in m slots. The GF $h(x;m)$ has the following expression

$$h(x;m) = \frac{[(1-a\theta)x]^m}{1-a\theta}[1-a\theta+a\theta h(x)] + \\ + x\frac{1-[(1-a\theta)x]^{m-1}}{1-(1-a\theta)x}a\theta w(x)h^2(x).$$

The former expression can be put in the following way

$$h(x;m) = \frac{1}{(1-a\theta)[1-(1-a\theta)x]}\cdot \\ \cdot\Big[[(1-a\theta)x]^m([1-(1-a\theta)x][1-a\theta+a\theta h(x)] - a\theta w(x)h^2(x)) + \\ + a\theta(1-a\theta)xw(x)h^2(x)\Big] \tag{33}$$

6. Sojourn times

6.1. Sojourn time of a customer in the server

Let b_k, $k \geq 0$ be the probability that the spent time of a customer in the server (taking into account possible interruptions) is k slots. Then

$$b_0 = 0$$
$$b_k = (\bar{a} + a\bar{\theta})^{k-1} s_k + \sum_{i=1}^{k} (\bar{a} + a\bar{\theta})^{i-1} a\theta S_{i+1} b_{k-i}, \ k \geq 1.$$

The GF $b(x) = \sum_{k=0}^{\infty} b_k x^k$ has the following form

$$b(x) = \frac{1}{\bar{a} + a\bar{\theta}} S[(\bar{a} + a\bar{\theta})x] + \frac{a\theta}{\bar{a} + a\bar{\theta}} \frac{(\bar{a} + a\bar{\theta})x - S((\bar{a} + a\bar{\theta})x)}{1 - (\bar{a} + a\bar{\theta})x} b(x)$$

That can be written as

$$b(x) = \frac{[1 - (\bar{a} + a\bar{\theta})x] S[(\bar{a} + a\bar{\theta})x]}{a\theta S[(\bar{a} + a\bar{\theta})x] + (1 - x)(\bar{a} + a\bar{\theta})}, \tag{34}$$

and the corresponding expected time is

$$\bar{b} = b'(1) = \frac{(\bar{a} + a\bar{\theta})[1 - S(\bar{a} + a\bar{\theta})]}{a\theta S(\bar{a} + a\bar{\theta})}.$$

Remark. Condition (19), necessary for the stability of the model, can be expressed in the following way

$$a[\bar{b} - 1] < \bar{a} A(\bar{a}),$$

where $a[\bar{b} - 1]$ is the mean number of customers who enter in a service period and $\bar{a}A(\bar{a})$ is the mean number of the customers in the orbit who goes to the server at the moment in which a service begins. Accordingly, if the condition (19) is verified, then the system is stable and therefore this condition is also sufficient.

6.2. Elapsed time of a customer in the system from the beginning of the service till its departure

We denote by g_k, $k \geq 0$ the probability that the time spent by a customer from the beginning o its service till its departure consists in k slots. Therefore:

$$g_0 = 0$$
$$g_k = (\bar{a} + a\bar{\theta})^{k-1} s_k + a\theta \sum_{i=1}^{k} (\bar{a} + a\bar{\theta})^{i-1} S_{i+1} \sum_{l=0}^{k-i} h_l \sum_{j=0}^{k-i-l} w_j g_{k-l-i-j}, \ k \geq 1.$$

The GF $g(x) = \sum_{k=0}^{\infty} g_k x^k$ is

$$g(x) = \frac{1}{\bar{a} + a\bar{\theta}} S[(\bar{a} + a\bar{\theta})x] + \frac{a\theta}{\bar{a} + a\bar{\theta}} \frac{(\bar{a} + a\bar{\theta})x - S[(\bar{a} + a\bar{\theta})x]}{1 - (\bar{a} + a\bar{\theta})x} h(x) w(x) g(x),$$

which implies

$$g(x) = \frac{[1 - (\bar{a} + a\bar{\theta})x] S[(\bar{a} + a\bar{\theta})x]}{(\bar{a} + a\bar{\theta})[1 - (\bar{a} + a\bar{\theta})x] - a\theta h(x) w(x)[(\bar{a} + a\bar{\theta})x - S[(\bar{a} + a\bar{\theta})x]]}.$$

and its expected value is

$$\bar{g} = g'(1) = \frac{(\bar{a} + a\bar{\theta})[1 - S(\bar{a} + a\bar{\theta})] + a\theta[\bar{h} + \bar{w}][(\bar{a} + a\bar{\theta}) - S[(\bar{a} + a\bar{\theta})]]}{a\theta S(\bar{a} + a\bar{\theta})}.$$

6.3. Sojourn time of a customer in the orbit

The stationary distribution of the sojourn time of a customer in the orbit till the beginning of its service has the following GF

$$W(x) = \pi_{0,0} + \sum_{k=0}^{\infty} \pi_{1,1,k} + \varphi_0(1,1) + \theta \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \pi_{1,i+1,k} + \\ + \bar{\theta} w(x) \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \pi_{1,i+1,k} h(x;i) \Big(h(x)w(x)\Big)^k, \tag{35}$$

The explanation of the above formula rises as follows:

An arriving customer begins immediately its service and in consequence spends zero slots in the orbit till the beginning of its service with probability $\pi_{0,0}+\sum_{k=0}^{\infty}\pi_{1,1,k}+\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{0,i,k}+\theta\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i+1,k}$, and with probability $\bar{\theta}\pi_{1,i+1,k}$, $i \geq 1$, $k \geq 0$, goes to the orbit finding k other customers before him, and a customer in the server with a remaining service time of i slots. Then the customer placed at the head of the orbit will wait there till the beginning of its service a period of time, counted since the arrival of the new customer, with GF $h(x,i)w(x)$. Once this customer begins its service a BP is opened with GF $h(x)$, and when this BP ends, the customer that was situated in the second place of the orbit at the moment of the arrival of the new customer will wait in the orbit till the beginning of its service a period of time with GF $w(x)$.

Continuing in this way results de term $\bar{\theta}w(x)\pi_{1,i+1,k}h(x,i)(h(x)w(x))^k$, and summing over i and k the formula is obtained.

From , the above expression has the following form

$$W(x) = \pi_{0,0} + \bar{\theta}\varphi_{1,1} + \varphi_0(1,1) + \theta\varphi_1(1,1) + \\ + \bar{\theta}w(x)\Big[F_1(x)\Big[\frac{1}{(1-a\theta)x}\varphi_1[(1-a\theta)x, h(x)w(x)] - \\ - \varphi_{1,1}(h(x)w(x))\Big] + F_2(x)\Big[\varphi_1(1, h(x)w(x))\Big] - \\ - \varphi_{1,1}(h(x)w(x))\Big] \tag{36}$$

where

$$F_1(x) = \frac{[1-(1-a\theta)x][1-a\theta+a\theta h(x)] - a\theta w(x)h^2(x)}{(1-a\theta)[1-(1-a\theta)x]},\ F_1(1)=0$$

$$F_2(x) = \frac{a\theta x w(x) h^2(x)}{1-(1-a\theta)x},\ F_2(1)=0$$

The corresponding mean value is given by

$$\bar{W} = W'(1) = \bar{\theta}\ \Big[\ \bar{w}[\varphi_1(1,1)-\varphi_{1,1}(1)] + F_1'(1)\Big[\frac{1}{1-a\theta}\varphi_1(1-a\theta,1) - \varphi_1(1,1)\Big] + \\ + F_2'(1)[\varphi_1(1,1)-\varphi_{1,1}(1)] + \varphi_1'\big(1, h(x)w(x)\big)|_{x=1} - \varphi_{1,1}'(h(x)w(x))|_{x=1}\ \Big]$$

where

$$\varphi_1(1,1)-\varphi_{1,1}(1) = \frac{(\bar{a}+a\bar{\theta}) - S(\bar{a}+a\bar{\theta})}{\theta S(\bar{a}+a\bar{\theta})}$$

$$\frac{1}{1-a\theta}\varphi_1(1-a\theta,1) - \varphi_{1,1}(1) = \frac{(1-a\theta)S'(1-a\theta) - S(1-a\theta)}{S(1-a\theta)}a$$

$$\varphi_1'\big(1, h(x)w(x)\big)|_{x=1} - \varphi_{1,1}'(h(x)w(x))|_{x=1} = \frac{a\bar{\theta}[1-S'(\bar{a}+a\bar{\theta})]D(1) - [(\bar{a}+a\bar{\theta}) - S(\bar{a}+a\theta)]}{\theta S(\bar{a}+a\bar{\theta})D(1)}\ \cdot \\ \cdot\ (\bar{h}+\bar{w})$$

$$F_1'(1) = \frac{a\theta\bar{h}(a\theta-2) - (1-a\theta) - a\theta\bar{w}}{a\theta(1-a\theta)}$$

$$F_2'(1) = \frac{1+\bar{w}+2\bar{h}}{a\theta}$$

$$D(1) = \bar{a}\theta A(\bar{a})S(\bar{a}+a\bar{\theta}) - [(\bar{a}+a\bar{\theta}) - S(\bar{a}+a\bar{\theta})]$$

The expected total time that a customer remains in the retrial group is

$$\bar{W}_T = \bar{W} + \bar{g} - \bar{b}.$$

Table 1
$a = 0.3,\ r = 0.6.$

	$\theta = 0$	$\theta = 0.3$	$\theta = 0.6$	$\theta = 0.9$	$\theta = 1$
ψ_0	0.5408159	0.4530159	0.3459426	0.2124681	0, 1603491
ψ_1	0.2483318	0.225491	0.1855382	0.1221474	0, 0942461
ψ_2	0.1140299	0.1325335	0.0139059	0.1032021	0, 0836674
ψ_3	0.0523606	0.0778972	0.095204	0.0871952	0.0742761
ψ_4	0.0240431	0.0457844	0.0681971	0.073671	0.0659390
ψ_5	0.0110402	0.0, 02691	0.0488513	0.0622444	0.0585377
ψ_{10}	0.0002253	0.00042917	0.00063927	0.00069058	0.0322775

Table 2
$a = 0.3,\ r = 0.6.$

	$\theta = 0$	$\theta = 0.3$	$\theta = 0.6$	$\theta = 0.9$	$\theta = 1$
Φ_0	0.2649998	0.2219778	0.1695119	0.1041094	0.0785711
Φ_1	0.348826	0.2921949	0.2231325	0.1370416	0.1034248
Φ_2	0.2088483	0.2002802	0.1722893	0.1178999	0.0918158
Φ_3	0.0958991	0.1177157	0.1234154	0.0996134	0.0815099
Φ_4	0.0440355	0.069188	0.0884057	0.0841631	0.0723608
Φ_5	0.0202203	0.0406653	0.0633273	0.0711092	0.064238
Φ_{10}	0.00041278	0.0006485	0, 00082870	0.00078893	0.035420

6.4. Sojourn time of a customer in the system

The GF $V(x)$ of the time spent by a customer in the system is

$$V(x) = W(x)g(x)$$

with expected value:

$$\bar{v} = v'(1) = \bar{W} + \bar{g}.$$

If the service times are of geometrical type the performance characteristics of the system: $\pi_{0,0}, E[N], E[L], \bar{b}$, etc, are independent of the parameter θ, which is due to the memoryless property of the geometrical distribution.

7. Numerical results

In this paragraph we present various numerical results to exhibit the effect of several characteristic of the system.

In the graphics and tables considered it is supposed that the service times take exactly two slots and that the retrial times are governed by a geometrical distribution with generating function $A(x) = \frac{1-r}{1-rx}$.

In Fig. 1, $\pi_{0,0}$ is plotted against θ. It have been presented three curves which corresponds to $r = 0, 0.4, 0.6$ respectively. As is to be expected the value of $\pi_{0,0}$ decreases with increasing values of θ and r.

In Fig. 2, $E[N]$ is plotted against the parameters θ for $r = 0, 0.4, 0.6$. As intuition tells us $E[N]$ increases with increasing values of θ and r.

Let us note that in this paper is relevant the implementation yielded by Theorems 2 and 3. The formulae (20), (21), (23) and (24) have been implemented in Tables 1 and 2.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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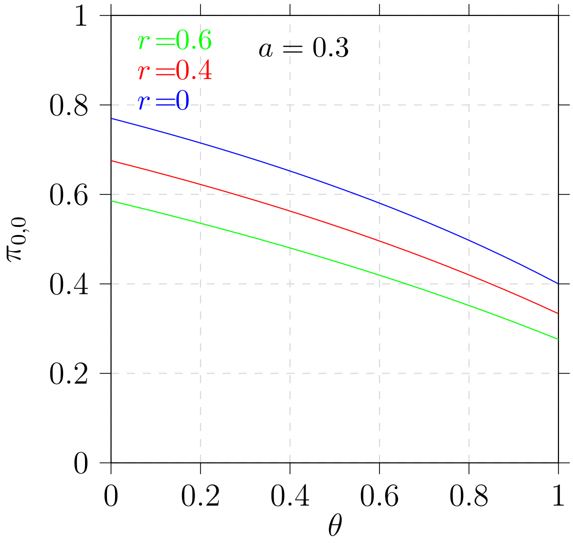

Fig. 1. $\pi_{0,0}$ against θ.

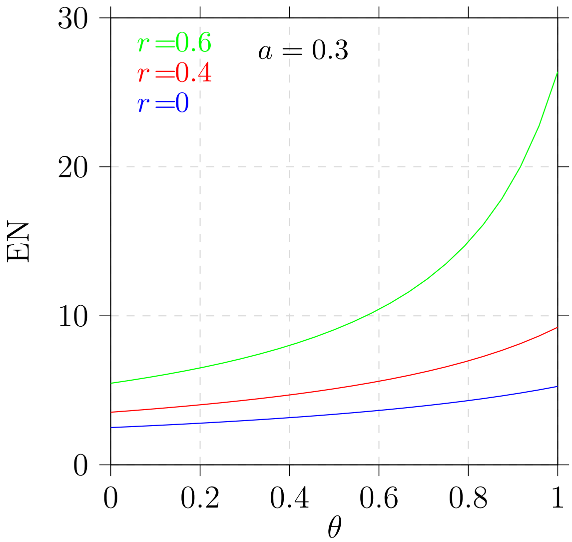

Fig. 2. $E[N]$ against θ.

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