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Equivalent reference points in Multiobjective Programming

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Abstract

In this paper, we concentrate on reference point based methods in multiobjective programming to demonstrate, as main contribution, that the solution to a multiobjective optimization problem stays unchanged if the reference point is changed to any point on a set defined by means of the original reference point, the nondominated objective solution and some parameters of the ASF. Concretely, this new set of 'equivalent reference points' is the convex linear combination of two straight lines, one containing the original reference point and the other a nondominated objective solution and where the slope of both straight lines is given by the inverses of the weights of the ASF. An illustrative example is used to show the results obtained and an empirical model (application with real data) allows us to point out possible implications.

Keywords: Multiobjective Programming, Reference Point, Achievement Scalarizing Function, interactive methods.

Mathematics Subject Classification (2010 database): 90C29, 90C90, 91B02.

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1. Introduction.

Multiple criteria decision making (MCDM) constitutes a formal approach to solve problems involving multiple conflicting criteria. We can identify two case scenarios: problems involving a finite set of alternatives, and problems described as the set of points satisfying a series of constraints (see, e.g., Stewart 1992; Vincke 1992). In this paper the focus is on the latter, i.e. on multiobjective programming, where several objective functions are optimized simultaneously. Because of the objective functions being typically conflicting, it is impossible to find a solution where all the objectives can reach their individual optima. As a consequence we can identify compromise solutions, which are the so-called Pareto optimal or nondominated solutions, where none of the objectives can achieve a better value without deteriorating at least one of the other objective values.

According to this last assumption, the task of solving multiobjective optimization problems is aimed at finding the best nondominated solution, i.e. the most preferred solution. In order to achieve this aim, a decision maker (DM), an expert in the domain of the problem in question, is usually needed to get additional information. In the so-called *a priori* methods the DM specifies desires and hopes before the solution process; alternatively, in the *a posteriori* methods, a representation of nondominated solutions is first generated and displayed to the DM, who is supposed to select the best of them as the final solution. The difficulty here is that it may be demanding for the DM to analyze many solutions and it may be computationally demanding for complicated real-life problems to generate many nondominated solutions. A way to overcome the above mentioned difficulties is to use interactive methods.

When applying interactive multiobjective optimization methods (see, e.g., Miettinen 1999 and references therein) a solution pattern is formed and iteratively repeated, and the DM takes part actively in the solution process by specifying and refining his preferences about the information, even changing his/her mind during the process. As a result, the DM can learn about the possibilities and limitations of the problem and about the interdependencies among the objective functions, taking part in an interactive solution process, so that the final solution can be expected to be more satisfactory than with alternative approaches.

Within these interactive methods it a common practice to use an achievement scalarizing function (ASF) to generate nondominated solutions. This kind of function was proposed by Wierzbicki (1980) and there is a bulk of literature related to it. This scheme is normally used in two types of preference information asked from the DM, namely reference point-based approaches (see Wierzbicki 1982; Nakayama and Sawaragi 1984; Korhonen and Laakso 1986; Buchanan 1997; Jaszkiewicz and Slowinski 1999; Luque *et al*. 2009a; Luque *et al*. 2009b; Nikulin *et al.* 2010, Luque *et al.* 2010), where the DM gives a reference value to each objective (these values constitute the so-called reference point), while in classification-based approaches (see Benayoun et al. 1971; Miettinen and Mäkelä 2006), the DM classifies the objectives into different classes (objectives to be improved, worsened, etc.).

For DMs, a reference point is a natural way of expressing desires in solutions in the way the DM would like to reach them for the objective functions. Once a reference point is given, the ASF is optimized to find a nondominated solution. The majority of the ASFs used in the reference point-based methods are extensions of the minimax distance (Tchebychev distance) which are able to support every Pareto optimal solution for any kind of multiobjective optimization problem.

Recently, Yang and Xu (2014) demonstrate that the minimax approach for reference point-based approaches is equivalent to input-oriented dual DEA models under certain conditions. This equivalence together with the previous work proposed in Yang, Xu and Yang (2012), allow them to propose a new interactive method for Hybrid Efficiency and Trade-off Analyses (HETA) taking into account the DM's preferences in an interactive way. A better understanding of HETA is then used to modify the efficiency measures, providing graphical and analytical procedures for performance analysis problems with many units. On the other hand, Deb et al. (2014) have also published recently the first part of the new NSGA-III based on the NSGA-II, the most cited evolutionary multiobjective optimization algorithm known to date. This new version uses an evenly distribution of reference points, which is modified taking into account the concentration of solutions. Departing from Luque *et al.* (2007) and previously in Kaliszewski (1994), we propose, as novelty, a set of new reference points called "equivalent reference points", which generates the same solution in the reference point-based approaches (minimax approach) . This new set of "equivalent reference points" is the convex linear combination of two straight lines: one containing the original reference point and the other the nondominated objective solution, where the slope of both straight lines is given by the inverses of the ASF weights. To the best of our knowledge, no formal proof of this fact has been developed in the literature to date; therefore this is the main contribution of this paper. Its practical applicability to real problems, including DEA models, is pointed out and commented. In addition, these results can be embedded in any evolutionary algorithm that uses reference points, like NSGA-III, and to improve its efficiency.

The rest of the paper is organized as follows. Section 2 introduces concepts and notations used as well as some achievement functions based on reference points. Then, theoretical results are presented in Section 3. Our ideas are illustrated with examples in Section 4, including both a theoretical and an empirical example (application with real data). Finally we conclude in Section 5 with a summary of the main results and we point out several research future lines.

2. Concepts and notation.

In this section, the basic definitions and notations are provided. Consider the following general multiobjective problem:

$$\begin{aligned} &\max f(\mathbf{x}) = \left(f_1(\mathbf{x}), \dots, f_k(\mathbf{x})\right) \\ &s.t.: \qquad \mathbf{x} \in X \end{aligned} \tag{1}$$

involving $k(\geq 2)$ conflicting objective functions $f_i: X \to \mathbb{R}$, which must be maximized simultaneously and where $\boldsymbol{x} = (x_1, \dots, x_n)^T$ are the *decision variables*. The $\boldsymbol{x}$ vector belongs to the feasible region $X \subset \mathbb{R}^n$, which is a nonempty compact set. The image of $X \subset \mathbb{R}^n$, $Z = f(X)$, is called feasible objective region, and $\boldsymbol{z} = f(\boldsymbol{x})$ an *objective vector* where $\boldsymbol{z} \in Z$ if $\boldsymbol{x} \in X$.

Multiobjective optimization generally lacks a feasible solution to simultaneously maximize all objective functions. Because of that another concept of optimality appears where none of the components can be improved without deteriorating at least one of the others: a decision vector $\boldsymbol{x}' \in X$ is called efficient or Pareto optimal solution of the problem (1) if there does not exist another $\boldsymbol{x} \in X$ such that$f_i(\boldsymbol{x}') \leq f_i(\boldsymbol{x})$ for all $i = 1, \dots, k$ and $f_j(\boldsymbol{x}') < f_j(\boldsymbol{x})$ for at least one index j. When this happens, $\boldsymbol{z}' = f(\boldsymbol{x}')$ is called nondominated objective vector. The efficient set is denoted by E and $f(E)$ is the nondominated objective set. A decision vector $\boldsymbol{x}' \in X$ is called weakly efficient or weakly Pareto optimal if there does not exist another $\boldsymbol{x} \in X$ such as $f_i\left(\boldsymbol{x}'\right) < f_i(\boldsymbol{x})$ for all $i = 1, \dots, k$. The corresponding objective vectors are called (weakly) nondominated objective vectors. It is important to remark that the set of efficient solutions is a subset of the weakly efficient solutions.

It is useful to know the bounds for the objective vectors in the nondominated objective set, since the set of nondominated objective vectors contains more than one vector. Specifically, upper bounds are given by the ideal values $\boldsymbol{z}^* = (z_1^*, \dots, z_k^*)$ where$z_i^* = \max_{x \in E} f_i(\boldsymbol{x}) = \max_{x \in X} f_i(\boldsymbol{x})$ for all $i = 1, \dots, k$. Conversely, lower bounds are set by the nadir vector $\boldsymbol{z^{nad}} = \left(z_1^{nad}, \dots, z_k^{nad}\right)$, where $z_i^{nad} = \min_{x \in E} f_i(\boldsymbol{x})$ for all $i = 1, \dots, k$. This vector is not easy to obtain and when estimated from the pay-off table the values achieved are not necessarily good approximations (for details, see e.g. Ehrgott and Tenfelde-Podehl 2003 and references therein). Recently, some approaches for a more reliable nadir vector generation were proposed in Deb and Miettinen (2010) and Deb *et al*. (2010). Both the ideal vector and the nadir vector are used frequently to normalize the objective functions.

Expressing preferences about the efficient solutions is commonly done by using a *reference point* $\boldsymbol{q} = (q_1, \dots, q_k)^T$, which consists of reference values for the objective functions. Considering a vector of weights for these values, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)^T$, the ASF is built and minimized over the feasible set. One of the most commonly used ASFs was proposed by Wierzbicki (1980):

$$s(\mathbf{q}, f(\mathbf{x}), \boldsymbol{\mu}) = \max_{i=1,\dots,k}\{\mu_i(q_i - f_i(\mathbf{x}))\} + \rho \sum_{i=1}^{k} \mu_i(q_i - f_i(\mathbf{x})) \quad (2)$$

which must be minimized in the feasible region:

$$\begin{aligned} \min\ & s(\mathbf{q}, f(\mathbf{x}), \boldsymbol{\mu}) \\ s.t.: \quad & \mathbf{x} \in X \end{aligned} \quad (3)$$

with $\mu_i > 0$ for all $i = 1, \dots, k$ and where $\rho > 0$ is a so-called augmentation coefficient.

Problem (3) produces nondominated solutions and it is demonstrated that any Pareto optimal solution can be found by solving (3) using the ideal objective vector as reference point (or any objective vector that dominates it, as an utopian vector), and varying the weight vector in the whole weight vector space (Wierzbicki 1980, Miettinen 1999). It is also demonstrated that fixing the weight vector and varying the reference point any Pareto optimal solution can be found by solving (3) (Miettinen 1999).

If the second term of the ASF is eliminated ($\rho = 0$), we obtain another ASF also very used in the literature:

$$s_0(\mathbf{q}, f(\mathbf{x}), \boldsymbol{\mu}) = \max_{i=1,\dots,k}\{\mu_i(q_i - f_i(\mathbf{x}))\} \quad (4)$$

which must also be minimized in the feasible region:

$$\begin{aligned} \min\ & s_0(\mathbf{q}, f(\mathbf{x}), \boldsymbol{\mu}) \\ s.t.: \quad & \mathbf{x} \in X \end{aligned} \quad (5)$$

In this case every solution of (5) is weakly Pareto optimal of (1) and it is Pareto optimal if it is unique.

To help the decision maker to provide the reference values $\boldsymbol{q} = (q_1, \dots, q_k)^T$, the ranges of the objective functions, $[z_i^{nad}, z_i^*]$ for all $i = 1, \dots, k$, can be very useful. Once the reference values are provided by the DM, if the (s)he does not like the solution or solutions generated by solving (3) (or (5)), we can make use of an interactive reference point-based method.

Any interactive method based on the reference point approach allows the decision maker to modify the reference values in order to try to get a suitable solution, but it is also possible to introduce preferential weights for some reference values in the ASF (Luque *et al.* 2009a, Luque et al. 2010). In this way, the results proved in this paper can be embedded in any interactive reference point-based method: at each iteration, the DM can know all or a part of the "equivalent reference points" obtained from the previous solution, before giving a new reference point (new reference values). For more details about interactive methods, see for example Miettinen 1999, Luque *et al.* 2011 or Ruiz *et al.* 2012.

3. Theoretical results.

The main purpose of this paper is to show that given a reference point, a weight vector and an efficient solution obtained by solving the problem (3), we can provide a new set of reference points, each of which yields the same efficient solution as the optimal solution of (3) Likewise, given a reference point, a weight vector and an weakly efficient solution obtained by solving the problem (5), we can provide a new set of reference points, each of which yields the same weakly efficient solution as the optimal solution of (5).

In Kaliszewski (1994) and also in Miettinen (1999) it is demonstrated that, given an efficient solution and fixed a reference point, a weight vector can be defined so that this efficient solution is an optimal solution of (3). Following this line, Luque *et al.* 2007 proved that, given a nondominated objective solution and a fixed weight vector, the points of the line passing through the nondominated objective solution and whose direction is given by the inverse of the weights, are "equivalent reference points", i.e., the efficient solution is an optimal solution of (3). This result was also proved by Luque et al. (2007) given a weakly nondominated objective solution.These results are presented at by the following theorems.

Theorem 1

Let $\mathbf{x}^*$be an optimal solution to the problem (3) where $\mu_i > 0$ for all $i = 1, \dots, k$and $\rho > 0$. Then, $\mathbf{x}^*$is also solution of (3) for the reference point $\hat{\boldsymbol{q}} = (\hat{q}_1, \dots, \hat{q}_k)^T$ defined by:

$$\hat{q}_i = f_i(\mathbf{x}^*) + \frac{1}{\mu_i} C \quad \forall i = 1, \dots, k \text{ with } C \in \mathbb{R} \tag{6}$$

where *C* is a parameter which can adopt any real number.

Proof

See Luque *et al.* 2007.

Theorem 2

Let $\mathbf{x}^*$be an optimal solution to the problem (5) where $\mu_i > 0$ for all $i = 1, \dots, k$. Then, $\mathbf{x}^*$is also solution of (5) for the reference point $\hat{\boldsymbol{q}} = (\hat{q}_1, \dots, \hat{q}_k)^T$ defined in (6) for any real value *C*.

Proof

See Luque *et al.* 2007.

Following the aim of this paper, we move now to provide formal proofs of the contribution of this research work. Firstly, instead of modifying the objective vector solution, we are going to modify the original reference point. Given a solution $\mathbf{x}^*$, a reference point $\boldsymbol{q} = (q_1, \dots, q_k)^T$and fixed a weight vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)^T$, the points of the line passing through the reference point, whose direction is given by the inverse of the weights, are reference points that generate the same solution when solving (3) or, respectively, (5). These new reference points can coincide with the ones proposed in Luque *et al.* (2007) when the minimum of (3) (or respectively (5)) is reached for all the components of the first term of the ASF, that is:

$$\mu_1\big(q_1 - f_1(\mathbf{x}^*)\big) = \mu_2\big(q_2 - f_2(\mathbf{x}^*)\big) = \cdots = \mu_k\big(q_k - f_k(\mathbf{x}^*)\big) \tag{7}$$

However, when the minimum is not reached for all the components (any equality of (7) is not verified), which is the most common, the reference points proposed by Luque *et al.* 2007 and the ones proposed in this first case are completely different (different straight lines).

Theorem 3

Given a reference point $\mathbf{q} = (q_1, \dots, q_k)^T$, let $\mathbf{x}^*$ be an optimal solution to the problem (3) where $\mu_i > 0$ for all $i = 1, \dots, k$ and $\rho > 0$. Then, $\mathbf{x}^*$ is also solution of (3) for the reference point $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_k)^T$ defined by:

$$\bar{q}_i = q_i + \frac{1}{\mu_i} C \quad \forall i = 1, \dots, k \text{ with } C \in \mathbb{R} \tag{8}$$

where *C* is a parameter which can adopt any real number.

Proof

Let us prove the Theorem by *reductio ad absurdum.* Assuming that $\mathbf{x}^*$ is not an optimal solution to (3) for the reference point $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_k)^T$. There exists $\mathbf{x}^1$ such that:

$$s(\bar{\mathbf{q}}, f(\mathbf{x}^1), \boldsymbol{\mu}) < s(\bar{\mathbf{q}}, f(\mathbf{x}^*), \boldsymbol{\mu}) \Rightarrow$$

$$\max_{i=1,\dots,k}\{\mu_i(\bar{q}_i - f_i(\mathbf{x}^1))\} + \rho \sum_{i=1}^{k} \mu_i(\bar{q}_i - f_i(\mathbf{x}^1)) <$$

$$< \max_{i=1,\dots,k}\{\mu_i(\bar{q}_i - f_i(\mathbf{x}^*))\} + \rho \sum_{i=1}^{k} \mu_i(\bar{q}_i - f_i(\mathbf{x}^*)) .$$

Substituting $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_k)^T$ by the expression (8) in the first term of the inequality we get:

$$\max_{i=1,\dots,k}\left\{\mu_i\left(q_i + \frac{1}{\mu_i}C - f_i(\mathbf{x}^1)\right)\right\} + \rho \sum_{i=1}^{k} \mu_i\left(q_i + \frac{1}{\mu_i}C - f_i(\mathbf{x}^1)\right) =$$

$$= \max_{i=1,\dots,k}\{C + \mu_i(q_i - f_i(\mathbf{x}^1))\} + \rho \sum_{i=1}^{k}\Big(C + \mu_i(q_i - f_i(\mathbf{x}^1))\Big) .$$

The maximum value of a constant plus an expression is equal to the constant plus the maximum of the expression, i.e.:

$$C + \max_{i=1,\dots,k}\{\mu_i(q_i - f_i(\mathbf{x}^1))\} + \rho \cdot k \cdot C + \rho \sum_{i=1}^{k}\Big(\mu_i(q_i - f_i(\mathbf{x}^1))\Big) =$$

$$= C(1 + \rho \cdot k) + \max_{i=1,\dots,k}\{\mu_i(q_i - f_i(\mathbf{x}^1))\} + \rho \sum_{i=1}^{k}\Big(\mu_i(q_i - f_i(\mathbf{x}^1))\Big)$$

and then

$$s(\bar{\mathbf{q}}, f(\mathbf{x}^1), \boldsymbol{\mu}) = C(1 + \rho \cdot k) + s(\mathbf{q}, f(\mathbf{x}^1), \boldsymbol{\mu}) .$$

In the same vein, we simplify the value of the ASF at $\mathbf{x}^*$ for the reference point $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_k)^T$ and get:

$$s(\bar{\mathbf{q}}, f(\mathbf{x}^*), \boldsymbol{\mu}) = C(1 + \rho \cdot k) + s(\mathbf{q}, f(\mathbf{x}^*), \boldsymbol{\mu})$$

This implies:

$$C(1 + \rho \cdot k) + s\left(\mathbf{q}, f(\mathbf{x}^1), \boldsymbol{\mu}\right) < C(1 + \rho \cdot k) + s(\mathbf{q}, f(\mathbf{x}^*), \boldsymbol{\mu}) \Rightarrow$$
$$\Rightarrow s\left(\mathbf{q}, f\left(\mathbf{x}^1\right), \boldsymbol{\mu}\right) < s(\mathbf{q}, f(\mathbf{x}^*), \boldsymbol{\mu}) .$$

which is a contradiction to $\mathbf{x}^*$ being an optimal solution to the problem (3) for the reference point $\mathbf{q} = (q_1, \dots, q_k)^T$. Therefore $\mathbf{x}^*$is an optimal solution to (3) for the reference point $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_k)^T$defined in (8).

□

Theorem 4

Given a reference point $\mathbf{q} = (q_1, \dots, q_k)^T$, let $\mathbf{x}^*$ be an optimal solution to the problem (5) where $\mu_i > 0$ for all $i = 1, \dots, k$. Then $\mathbf{x}^*$is also solution of (5) for the reference point $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_k)^T$ defined in (8) for any real value *C*.

Proof

The proof of the Theorem 2 is valid for $\rho = 0$ and therefore, this Theorem is proved.

□

Secondly, let us see the more general case. The convex linear combination of the set of reference points proposed in Luque *et al.* 2007, defined in (6), and the set of reference points given in the first case, defined in (8), constitute a new set of reference points that yields the same weakly efficient solution as optimal solution of (5), efficient if it is unique.

Theorem 5

Given a reference point $\mathbf{q} = (q_1, \dots, q_k)^T$, let $\mathbf{x}^*$ be an optimal solution to the problem (5) where $\mu_i > 0$ for all $i = 1, \dots, k$. Then, $\mathbf{x}^*$is also solution for the reference point $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_k)^T$ defined by:

$$\bar{q}_i = \lambda\, f_i(\mathbf{x}^*) + (1 - \lambda)\hat{q}_i \quad \forall i = 1, \dots, k \text{ with } \lambda \in [0, 1] \tag{9}$$

where

$$\hat{q}_i = q_i + \frac{1}{\mu_i}\hat{C} \ \ \forall i = 1, \dots, k \text{ with } \hat{C} \in \mathbb{R} \text{ is such that } \max_{i=1,\dots,k}\left\{\mu_i\left(\hat{q}_i - f_i(\mathbf{x}^*)\right)\right\} = 0 \tag{10}$$

Proof

The reference point $\hat{\mathbf{q}} = (\hat{q}_1, \dots, \hat{q}_k)^T$ can be calculated in the following way. Let $C_1, \dots, C_k$ given by

$$q_i + \frac{1}{\mu_i} C_i = f_i(\mathbf{x}^*) \Rightarrow C_i = \mu_i(f_i(\mathbf{x}^*) - q_i) \quad \forall i = 1, \dots, k$$

and if we set $\hat{C} = \min_{i=1,\dots,k} \{C_i\}$, then:

$$\hat{q}_i = q_i + \tfrac{1}{\mu_i} \hat{C} \le q_i + \tfrac{1}{\mu_i} C_i = f_i(\mathbf{x}^*) \quad \forall i = 1, \dots, k \text{ and } \exists\, j \text{such that } \hat{q}_j = f_j(\mathbf{x}^*)$$

which implies:

$$\max_{i=1,\dots,k} \{\hat{q}_i - f_i(\mathbf{x}^*)\} = 0 \Rightarrow \max_{i=1,\dots,k} \left\{\mu_i \left(\hat{q}_i - f_i(\mathbf{x}^*)\right)\right\} = 0$$

Let us prove this Theorem by *reductio ad absurdum.* Suppose that $\mathbf{x}^*$is not an optimal solution to (5) for the reference point $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_k)^T$. There exists $\mathbf{x}^1$ such that:

$$s_0(\bar{\mathbf{q}}, f(\mathbf{x}^1), \boldsymbol{\mu}) < s_0(\bar{\mathbf{q}}, f(\mathbf{x}^*), \boldsymbol{\mu}) \Rightarrow$$

$$\max_{i=1,\dots,k} \{\mu_i(\bar{q}_i - f_i(\mathbf{x}^1))\} < \max_{i=1,\dots,k} \{\mu_i(\bar{q}_i - f_i(\mathbf{x}^*))\}.$$

Substituting $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_k)^T$ by the expression (9) in the second term of the inequality:

$$\max_{i=1,\dots,k} \left\{\mu_i \left(\lambda\, f_i(\mathbf{x}^*) + (1-\lambda)\hat{q}_i - f_i(\mathbf{x}^*)\right)\right\} = \max_{i=1,\dots,k} \left\{\mu_i \left((1-\lambda)\left(\hat{q}_i - f_i(\mathbf{x}^*)\right)\right)\right\} =$$

$$= (1-\lambda) \max_{i=1,\dots,k} \left\{\mu_i \left(\hat{q}_i - f_i(\mathbf{x}^*)\right)\right\} = (1-\lambda) \cdot 0 = 0$$

Substituting $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_k)^T$ by the expression (9) in the first term of the inequality:

$$\max_{i=1,\dots,k} \left\{\mu_i \left(\lambda\, f_i(\mathbf{x}^*) + (1-\lambda)\hat{q}_i - f_i(\mathbf{x}^1)\right)\right\} = \max_{i=1,\dots,k} \left\{\mu_i \left(\hat{q}_i - f_i(\mathbf{x}^1) - \lambda\left(\hat{q}_i - f_i(\mathbf{x}^*)\right)\right)\right\}$$

Since $-\lambda\left(\hat{q}_i - f_i(\mathbf{x}^*)\right) \ge 0 \quad \forall i = 1, \dots, k$ given that $\hat{q}_i - f_i(\mathbf{x}^*) \le 0 \quad \forall i = 1, \dots, k$, we have:

$$\max_{i=1,\dots,k} \left\{\mu_i \left(\hat{q}_i - f_i(\mathbf{x}^1)\right)\right\} \le \max_{i=1,\dots,k} \left\{\mu_i \left(\hat{q}_i - f_i(\mathbf{x}^1) - \lambda\left(\hat{q}_i - f_i(\mathbf{x}^*)\right)\right)\right\} <$$

$$< \max_{i=1,\dots,k} \left\{\mu_i \left(\lambda f_i(\mathbf{x}^*) + (1-\lambda)\hat{q}_i - f_i(\mathbf{x}^*)\right)\right\} = 0 \;\Rightarrow$$

$$\max_{i=1,\dots,k} \left\{\mu_i \left(\hat{q}_i - f_i(\mathbf{x}^1)\right)\right\} < 0$$

Applying Theorem 4, $\mathbf{x}^*$is an optimal solution to the problem (5) for the reference point $\hat{\mathbf{q}} = (\hat{q}_1, \dots, \hat{q}_k)^T$. However:

$$\max_{i=1,\dots,k} \left\{\mu_i \left(\hat{q}_i - f_i(\mathbf{x}^1)\right)\right\} < 0 = \max_{i=1,\dots,k} \left\{\mu_i \left(\hat{q}_i - f_i(\mathbf{x}^*)\right)\right\} \Rightarrow$$

$$s_0(\hat{\mathbf{q}}, f(\mathbf{x}^1), \boldsymbol{\mu}) < s_0(\hat{\mathbf{q}}, f(\mathbf{x}^*), \boldsymbol{\mu})$$

which is a contradiction and therefore $\mathbf{x}^*$is an optimal solution to (5) for the reference point $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_k)^T$defined in (9) and (10).

□

Corollary 1

Given a reference point $\mathbf{q} = (q_1, \ldots, q_k)^T$, let $\mathbf{x}^*$ be an optimal solution to the problem (5) where $\mu_i > 0$ for all $i = 1, \ldots, k$. Then, $\mathbf{x}^*$is also solution for the reference point $\bar{\bar{\mathbf{q}}} = (\bar{\bar{q}}_1, \ldots, \bar{\bar{q}}_k)^T$ defined by:

$$\bar{\bar{q}}_i = \bar{q}_i + \frac{1}{\mu_i} C \quad \forall i = 1, \ldots, k \text{ with } C \in \mathbb{R} \tag{11}$$

with $\bar{\mathbf{q}} = (\bar{q}_1, \ldots, \bar{q}_k)^T$ defined in (9) and (10) for any $\lambda \in [0, 1]$.

Proof

It is trivial applying the Theorems 4 and 5.

In practice, the solution of the problem (5) usually matches with the solution of the problem (3) for a sufficiently small value of ρ, although it is not always true.

Therefore, we have been able to find, in the context of reference point approaches in MOP, a new set of reference points which generates the same solution.

4. Examples.

In the next two subsections we illustrate the implications of the Theorems shown in Section 3 using, firstly, a simple theoretical example and, secondly, an example based on real data coming from an international survey regarding, among other aspects, the satisfaction of European workers.

4.1. A simple theoretical example.

We illustrate here the implications of the Theorem with a nonlinear bi-objective optimization problem proposed in Luque *et al.* 2009:

$$\begin{aligned} &\min f_1(\mathbf{x}) = -4x_1 - x_2 \\ &\min \ f_2(\mathbf{x}) = x_1 - 2x_2 \\ &s.t.: 2x_1 + x_2 \leq 6, \\ &{x_1}^2 + {x_2}^2 \leq 9, \\ &x_1, x_2 \geq 0. \end{aligned} \tag{11}$$

The ideal and nadir values are given by the vectors $\mathbf{z}^* = (-12, -6)^{\mathrm{T}}$ and $\mathbf{z}^{\mathbf{nad}} = (-3, 3)^{\mathrm{T}}$, respectively. The corresponding feasible objective region is given by:

$$Z = \left\{ (z_1, z_2)^T \in \mathbb{R}^2 \middle| \begin{array}{c} \frac{5}{81}{\mathrm{z}_1}^2 + \frac{17}{81}{\mathrm{z}_2}^2 + \frac{4}{81}\mathrm{z}_1\mathrm{z}_2 \leq 9, \quad \frac{5}{9}\mathrm{z}_1 + \frac{2}{9}\mathrm{z}_2 \geq -6, \\ -\frac{2}{9}\mathrm{z}_1 + \frac{1}{9}\mathrm{z}_2 \geq 0, \quad \frac{1}{9}\mathrm{z}_1 + \frac{4}{9}\mathrm{z}_2 \leq 0 \end{array} \right\}$$

Let us consider the reference point $\mathbf{q} = (-1, -5)^T$ and the weight vector $\boldsymbol{\mu} = (1/6, 1/9)^T$. Solving (3), the solution obtained is $\mathbf{x}^* = (x_1^*, x_2^*) = (0, 3)^T$ and the values of the objective functions $f(\mathbf{x}^*) = (-3, -6)^T$. Applying Theorem (3) on this example:

$$\bar{\mathbf{q}} = (\bar{q}_1, \bar{q}_2)^T = (-1, -5)^T + C\left(\frac{1}{1/6}, \frac{1}{1/9}\right)^T = (-1 + 6C, -5 + 9C) \tag{12}$$

This means that, for any real value *C*, the solution of (3) considering $\bar{\mathbf{q}} = (\bar{q}_1, \bar{q}_2)$ as reference point is $\mathbf{x}^* = (x_1^*, x_2^*) = (0, 3)^T$ and the values of the objective functions $\mathbf{z}^* = (-3, -6)^T$. For example, let us consider two reference points of (12), $\bar{\mathbf{q}}^1 = (0, -7/2)^T$ obtained for $C = 1/6$ and $\bar{\mathbf{q}}^2 = (-5/3, -6)^T$ for $C = -1/9$.

If we analyze the value of the ASF in the optimal solution for the three reference points considered, it follows that the maximum value of the first part of the ASF (2) ($\max\limits_{i=1,\dots,k}\{\mu_i(/q_i - f_i(\mathbf{x}))\}$), which we call 'max term', is only reached for one of the components. Taking into account that our problem (6) is a minimization problem and, therefore, the term $(q_i - f_i(\mathbf{x}))$ must be substituted by $(f_i(\mathbf{x}) - q_i)$ in (2), then the 'max term' in the solution for the three reference points considered is given by:

$$\mathbf{q} = (-1, -5)^T \Rightarrow \max\left\{\frac{1}{6}(-3 - (-1)), \frac{1}{9}(-6 - (-5))\right\} = \max\left\{-\frac{1}{3}, -\frac{1}{9}\right\} = -\frac{1}{9}$$

$$\bar{\mathbf{q}}^1 = (0, -\frac{7}{2})^T \Rightarrow \max\left\{\frac{1}{6}(-3-0), \frac{1}{9}\left(-6-\left(-\frac{7}{2}\right)\right)\right\} = \max\left\{-\frac{1}{2}, -\frac{5}{18}\right\} = -\frac{5}{18}$$

$$\bar{\mathbf{q}}^2 = (-\frac{5}{3}, -6)^T \Rightarrow \max\left\{\frac{1}{6}\left(-3-\left(-\frac{5}{3}\right)\right), \frac{1}{9}(-6-(-6))\right\} = \max\left\{-\frac{2}{27}, 0\right\} = 0$$

This implies that the reference points obtained by (8) are different from the reference points proposed in Luque *et al.* (2007), defined in (6) and which are given by:

$$\hat{\mathbf{q}} = (\hat{q}_1, \hat{q}_2)^T = (-3, -6)^T + C\left(\frac{1}{1/6}, \frac{1}{1/9}\right)^T = (-3+6C, -6+9C) \tag{13}$$

On the other hand, we have considered $\rho = 0.001$ in the ASF (3). Given the reference point $\mathbf{q} = (-1, -5)^T$, the solution of (5) is the same one than the solution of (3). Thus, the reference points defined in Corollary 1 - starting from the Theorems 4 and 5- are given by:

$$\bar{\bar{\mathbf{q}}} = (\bar{\bar{q}}_1, \bar{\bar{q}}_2)^T = \lambda(-3, -6)^T + (1-\lambda)\left(-\frac{5}{3}, -6\right)^T + C\left(\frac{1}{1/6}, \frac{1}{1/9}\right)^T$$

$$\bar{\bar{\mathbf{q}}} = (\bar{\bar{q}}_1, \bar{\bar{q}}_2)^T = \left(-\frac{4}{3}\lambda - \frac{5}{3} + 6C, -6+9C\right) \tag{14}$$

Consequently the solution of (5) for these reference points is $\mathbf{x}^* = (0, 3)^T$ for any $\lambda \in [0, 1]$ and for any real value *C*. For example, if we consider $\lambda = 0.6$, we can obtain another line of "equivalent reference points" given in (14), where for $C = -1/9$, the reference point is given by $\bar{\bar{\mathbf{q}}} = (-9/5, -5)$. The graphical support to this example can be seen in Figure 1.

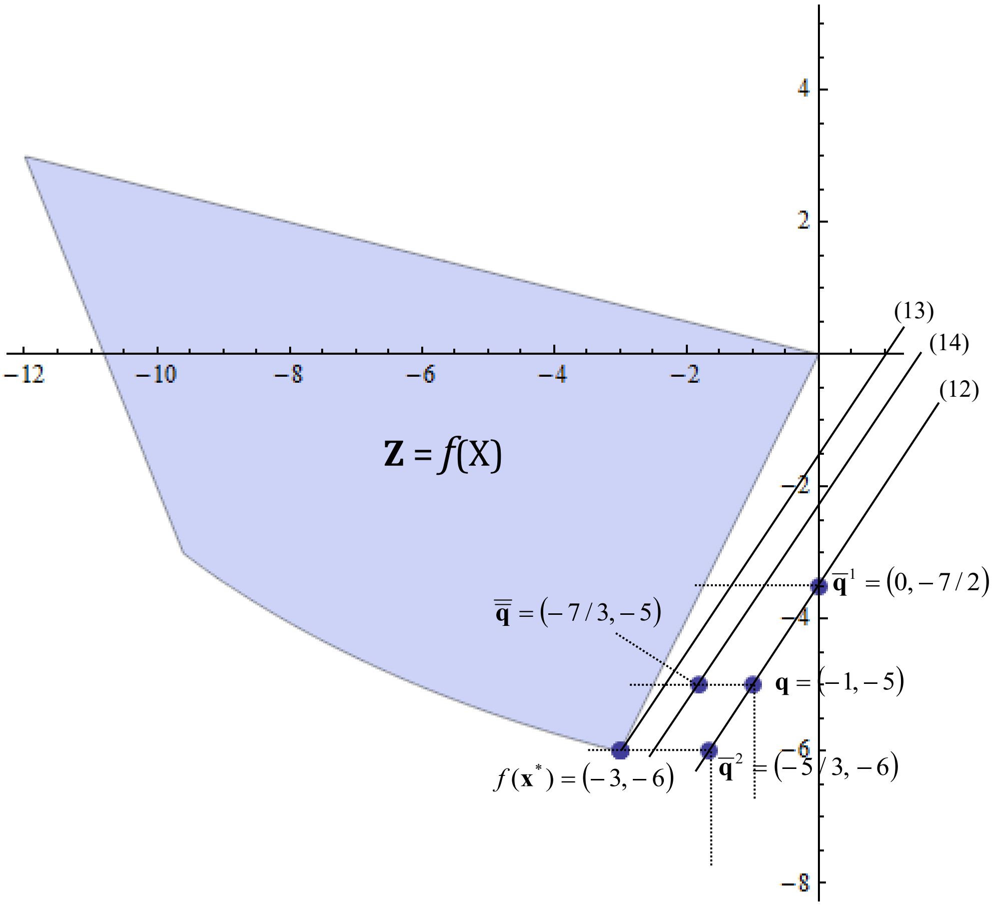

Fig. 1 Results obtained from the nonlinear bi-objective problem

4.2. Example with real data.

To complement the theoretical example shown above we include an additional example analyzed previously in Marcenaro *et al.* (2010), based on solving a problem with real data coming from the European Community Household Panel (ECHP), from1995 to 2001[1], which is the only wide European survey containing microdata on workers satisfaction with different work aspects, as well as on several individual personal and family characteristics. The idea behind this example is to determine the profile of the most satisfied worker in Spain[2], bearing in mind that there is some degree of conflict between different satisfaction aspects and, consequently, the multiobjective methodology is especially suitable for this model. We use as reference point, or target values for the satisfaction objective functions, the Danish workers' satisfaction levels, as they reported the highest average work satisfaction level within the 15 countries included in the survey.

[1] There is more up to date data from this survey but it does not contain data on satisfaction related variables.

[2] This example is based on a reelaboration of the estimates included in Marcenaro *et al.* (2010).

More precisely, regarding the information on satisfaction, workers in the ECHP were asked to evaluate seven different aspects of a job, on a scale from 1 to 6 where 1 is "not satisfied at all" and 6 is "fully satisfied". The job aspects presented were: earnings, job security, type of work, number of working hours, working times, working conditions/environment and distance from work/commuting[3].

On the ground of a sample of data contained in the above mentioned survey, an econometric analysis was carried out in order to find dependence relations of the satisfaction levels with respect to a set of personal and family characteristics of the individuals (the decision variables of our model have been listed in Table A1 –appendix-), as well as possible correlations among some data. We restricted the sample to those female workers working in the private sector aged between 25 and 65 (retirement age). Based on these results, the significant decision variables of the problem are identified, and the objective functions and constraints built. Finally, a reference point approach is used to solve the resulting multiobjective problem. After briefly presenting the econometric approach, we focus on the MOP problem to show the consequences of the Theorem presented in Section 3.

The level for each satisfaction target results from the combination of a set of individual and contextual features, unobservable factors and a random disturbance (ε). If individuals are indexed by "r", and job satisfaction aspects are indexed by "j", this model can be represented by the following set of equations:

$$ES_j(r) = \alpha^j + \beta_1^j ghw(r) + \beta_2^j edh(r) + \cdots + \beta_{35}^j y_7(r) + \varepsilon_j(r) \qquad (15)$$

$$r = 1, \dots, ns;\ j = 1, \dots, 7$$

Where *ns* is the number of observations, $ES_j(r)$ is a measure of the satisfaction category $j$of individual$r$,and $ghw(r), edh(r), \dots, year7(r)$, a group of explanatory variables; $\varepsilon_j(r)$ is a random disturbance; β^j a vector of slope coefficients and α^j a fixed but unknown population intercept.

From the seven equations represented by (15) we obtained the estimated coefficients on the key variables of interest (table not presented to conserve space), which are quite consistent with those found in the previous literature.

Based on those estimates we go a step forward presenting and solving the MOP problem. Enumerating all the variables by $x_1, x_2, \dots, x_{35}$, the multiobjective problem is the following:

$$\text{Max}\ \left(ES_1(\mathbf{x}), \dots, ES_7(\mathbf{x})\right) = \left(\sum_{i=1}^{35} \hat{\beta}_i^1 \cdot x_i + \hat{\alpha}^1, \dots, \sum_{i=1}^{35} \hat{\beta}_i^7 \cdot x_i + \hat{\alpha}^7\right) \qquad (16)$$

Subject to two sets of constraints[4]

[3] The precise wording of the questions was: How satisfied are you with your present job in terms of …?

[4] Particularly, we defined two set of constraints (not reported for the sake of brevity –available upon request-):

Thus, the model under study is a mixed integer linear multiobjective model with 7 objective functions, 20 constraints and several simple bounds and integrality constraints[5]. To solve this MOP problem by using the reference point approach, let us consider the equivalent differentiable formulation of (3) (see for example Miettinen 1999). In this way, we are able to solve it by using a single objective mixed integer linear optimization solver. Concretely, the resulting optimization problem was solved using the NAG library for C language (see NAG C 2000). Consequently, the reference point problem solved was:

$$\begin{aligned} &\min \alpha + \rho \textstyle\sum_{j=1}^{7} \left(q_j - \left(\sum_{i=1}^{35} \hat{\beta}_i^j . x_i + \hat{\alpha}^j \right) \right) \\ &s.t.: \qquad q_j - \left(\textstyle\sum_{i=1}^{35} \hat{\beta}_i^j . x_i + \hat{\alpha}^j \right) \leq \alpha \\ &\text{Constraints.} \end{aligned} \tag{17}$$

With regard to the solutions of the multiobjective model, as previously mentioned, we use as a reference point the Danish mean satisfaction levels (for female workers):

$$q_1 = 4.36 \;\; q_2 = 4.74 \;\; q_3 = 4.78 \;\; q_4 = 4.76 \;\; q_5 = 4.92 \;\; q_6 = 4.80 \;\; q_7 = 4.89$$

All the criteria are equally weighted. Note that, given that all the satisfaction levels are specified on a 1 – 6 scale, no normalization is needed in this formulation. Therefore, it is implicitly assumed that the achievements of all the reference levels are equally important for the DM. The solutions obtained are shown in Table 1, which provides a "taxonomy" of the most satisfied Spanish female workers.

Decision Variables			
Variable names	***Solution***	***Variable names***	***Solution***
Grosswage	7.98	***Unemploymentduration***	38
Highereducation	1	***Goodhealth***	1
Secondaryeducation	0	***Fairhealth***	0
Familyincome	9.5	***Regional unemploymentrate***	16.79
Age	51.61	***Mining and quarrying***	0
Seniority 3-4	0	***Utilities and construction***	0
Seniority 5-9	0	***Hotel sales***	1
Seniority 10-14	0	***Transport***	0
Seniority 15+	0	***Financeproperty***	0
Workinghours	0	***Otherindustry***	0
Permanentcontract	1	***Firmsize 5-19***	0
Supervisory	1	***Firmsize 20-99***	0
Intermediate	0	***Firmsize 100-499***	0
Married	1	***Firmsize 500+***	0
Children<6	1		

Satisfaction	**Value**	**Reference**
1	3.72	4.36
2	4.94	4.74

a) technical constraints, which ensure that certain binary variables do not take the value 1 simultaneously (e.g. $edh + eds \leq 1$).

b) some constraints to ensure that the profile of the worker that we are looking for is sufficiently realistic (e.g. defining bounds for the salary depending on the individuals' highest education level).

[5] The pay-off matrices obtained (displaying the values of the seven objective functions in each of the individual optima) proved that there was some degree of conflict between each pair of functions, meaning that the multiobjective analysis is appropriate.

3	5.16	4.78
4	4.48	4.76
5	4.28	4.92
6	5.04	4.80
7	4.86	4.89

Table 1. Solutions for the multiobjective model

Considering the reference point and nondominated objective vector of Table 1, we apply Theorem 3; we try some values for C to obtain new reference points. Specifically departing from the initial reference point we add 0.8 points to the mean Danish satisfaction levels (i.e. $C = 0.8$). Solving the same problem (17) with the new reference point $\bar{\mathbf{q}}^1$ shown in Table 2, the nondominated objective vector obtained is the same one. Alternatively, if we consider the value $C = -0.5$, i.e. we subtract 0.5 from the mean Danish satisfaction levels ($\bar{\mathbf{q}}^2$shown in Table 2) the results are also the same ones. Observe that these reference points are different again from the reference points proposed in Luque *et al.* (2007) and given in equation (6), due to the same reason mentioned in the previous example: the 'max term' is not reached for all components of the objective functions.

Once again, we have also considered $\rho = 0.001$ in the ASF (3) and the solution of (5) for the original reference point is the same one than the solution of (3). Table 2 reports the "equivalent reference points" given by (11), which depends on the equations (9) and (10) (Corollary 1, Theorems 4 and 5); we have considered the same values for C (0.8 and -0.5) and $\lambda = 0.6$.

Solution			***Theorems 1, 2 Luque et al. (2007)***		***Theorems 3, 4 New proposal***		***Theorem 5, Corollary 1 New proposal***	
			Equivalent Reference Points					
Satisf.	**Obj. $\mathbf{z}^*$**	$\mathbf{q}$	$\hat{\mathbf{q}}^1$	$\hat{\mathbf{q}}^2$	$\bar{\mathbf{q}}^1$	$\bar{\mathbf{q}}^2$	$\bar{\bar{\mathbf{q}}}^1$	$\bar{\bar{\mathbf{q}}}^2$
1	3.72	4.36	4.52	3.22	5.16	3.86	4.97	3.67
2	4.94	4.74	5.74	4.44	5.54	4.24	5.60	4.30
3	5.16	4.78	5.96	4.66	5.58	4.28	5.69	4.39
4	4.48	4.76	5.28	3.98	5.56	4.26	5.47	4.17
5	4.28	4.92	5.08	3.78	5.73	4.43	5.53	4.23
6	5.04	4.80	5.84	4.54	5.60	4.30	5.67	4.37
7	4.86	4.89	5.66	4.36	5.69	4.39	5.68	4.38

Table 2. Solutions for the multiobjective model (applied to real data) with alternative reference points

It is evident that, in case that the solution of (5) does not match with the solution of (3), the solution used for Corollary 1 and Theorem 5 is the solution obtained by solving (5). If so we want to compare these reference points with the rest of reference points (Theorems 1-4), we would use the solution of (5).

5. Conclusions.

In this paper, we propose a new set of reference points which allow generating the same Pareto optimal solution in reference point-based approaches. Starting from the result proposed in Luque *et al.* (2007), these new theorems provide a wider set of "equivalent reference points" for the same solution than the previous result.

Our main contribution in this work is therefore to perform a more exhaustive sensitivity analysis of the solution obtained in the reference point-based approaches, so that more different reference points produce the same Pareto optimal solution. This information has a practical advantage in any real solution process since that a sensibility analysis of the reference levels is now available. Many DMs, after solving a real model, make the following question: What will happen if we change this or these reference levels? In addition, in the interactive methods based on the reference point approach, given an iteration, the DM can have available all the "equivalent reference points" obtained from the previous solution, before giving a new reference point. The computational implementation can check if the new reference point is or not in the new set and can save any iteration.

An illustrative example is used to show the results obtained and a real model demonstrates the potential of our new methodological results. As it can be seen in the real model about the satisfaction of the workers in the Spanish labour market, certain modifications of the reference satisfaction levels produce the same Pareto optimal solution.

((OSCAR: VER SI SE PUEDE SACAR ALGUNA CONCLUSIÓN CON LOS NUEVOS PUNTOS DE REFERENCIA DE LAS SATISFACCIONES Y QUE NOS LLEVAN A LA MISMA SOLUCIÓN))

It is convenient to point out, when the optimal value of the ASF is reached for all the components of the first term, which can occurs in some cases, the new reference points coincide with previous results already published.

A future research line is to apply our results in the field of Economy of the Education through DEA models taking into account the publications aforementioned. Concretely, using the equivalence between the minimax approach in reference point-based approaches and DEA models, the method interactive could help the DM to provide new trade-offs by showing the new set of equivalent reference points. In addition to this, following the last papers published about evolutionary multiobjective optimization, many of them related to the reference point-based approaches, our work can be very useful. Concretely, the new NSGA-III published recently use an evenly distribution of reference points, which are adapted taking into account the concentration of solutions in the space of solutions. Our work can give information if the new reference points provide or don't new solutions. An analysis in depth of the possible implications in EMO algorithms carrying computational tests is also a future research line.

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Bibliography

Benayoun, R., Montgofier, J., Tergny, J., & Laritchev, O. (1971). Linear programming with multiple objective functions: Step method (STEM). *Mathematical Programming*, 1(3), 366-375.

Buchanan, J. T. (1997). A naïve approach for solving MCDM problems: The GUESS method. *Journal of the Operational Research Society*, 48, 202-206.

Deb, K., & Miettinen, K. (2010). Nadir point estimation using evolutionary approaches: better accuracy and computational speed through focused search. In M. Ehrgott, B. Naujoks, T. J. Stewart, & J. Wallenius (Eds.), *Multiple criteria decision making for sustainable energy and transportation systems, proceedings* (pp. 339–354). Berlin, Heidelberg: Springer.

Deb, K., Miettinen K., & Chaudhuri, S. (2010). Towards an estimation of nadir objective vector using a hybrid of evolutionary and local search approaches. *IEEE Transactions on Evolutionary Computation*, 14(6), 821–841.

Ehrgott, M., & Tenfelde-Podehl, D. (2003). Computation of ideal and nadir values and implications for their use in MCDM methods. *European Journal of Operational Research*, 151, 119–39.

Jaszkiewicz, A., & Slowinski, R. (1999). The 'light beam search' approach - an overview ofmethodology and applications. *European Journal of Operational Research*, 113(2), 300-314.

Kaliszewski, I. (1994). *Quantitative Pareto Analysis by Cone Separation Technique*. The Netherlands, Dordrecht: Kluwer Academic Publishers.

Korhonen, P., & Laakso, J. (1986). A visual interactive method for solving the multiplecriteria problem. *European Journal of Operational Research*, 24(2), 277-287.

Luque, M., Caballero, R., Molina, J., & Ruiz, F. (2007). Equivalent Information for Multiobjective Interactive Procedures. *Management Science*, 53, 125-134.

Luque, M., Miettinen, K., Eskelinen, P., & Ruiz, F. (2009a). Incorporating Preference Information in Interactive Reference Point Methods for Multiobjective Optimization. *OMEGA-International Journal of Management Science*, 37(2), 450-462.

Luque, M., Yang, J. B., & Wong, B. Y. H. (2009b). PROJECT method for multiobjective optimization based on the gradient projection and reference points. *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, 39(4), 864-879.

Luque, M., Ruiz, F., & Steuer, R. E. (2010). Modified Interactive Chebyshev Algorithm (MICA) for Convex Multiobjective Programming. *European Journal of Operational Research*, 204(3), 557-564.

Luque, M., Ruiz, F., & Miettinen, K. (2011). Global Formulation for Interactive Multiobjective Optimization. *OR-Spectrum*, 33(1), 27-48.

Luque, M., Miettinen, K., Ruiz, A. B., & Ruiz, F. (2012). A Two-Slope Achievement Scalarizing Function for Interactive Multiobjective Optimization. *Computers & Operations Research*, 39(7), 1673-1681.

Marcenaro, O., Luque, M., & Ruiz, F. (2010). An application of multiobjective programming to the study of worker's satisfaction in the Spanish Labour Market. *European Journal of Operational Research*, 203, 430-443.

Miettinen, K. (1999). *Nonlinear multiobjective optimization*. Boston: Kluwer Academic Publishers.

Miettinen, K., & Mäkelä, M. M. (2006). Synchronous approach in interactive multiobjective optimization. *European Journal of Operational Research*, 170(7-8), 909-922.

Miettinen, K., Mäkelä, M. M., & Kaario, K. (2006). Experiments with classification-based scalarizing functions in interactive multiobjective optimization. *European Journal of Operational Research*, 175, 931–47.

Nakayama, H., & Sawaragi, Y. (1984). Satisfacing trade-off method for multiobjective programming. In M. Grauer, & A. P. Wierzbicki (Eds.), *Interactive Decision Analysis* (pp. 113-122). Berlin: Springer-Verlag.

Nikulin, Y., Miettinen, K., &, Mäkelä, M. K. (2010). A new achievement scalarizing function based on parameterizationsin multiobjective optimization. *OR Spectrum,* 34(1), 69-87.

Ruiz, F., Luque, M., & Miettinen, K. (2012). Improving the Computational Efficiency in a Global Formulation (GLIDE) for Interactive Multiobjective Optimization. *Annals of Operations Research*, 197(1), 47-70.

Stewart, T. J. (1992). A critical survey on the status of multiple criteria decision making theory and practice. *Omega20*, 5(6), 569–86.

Vincke, P. (1992). *Multicriteria decision-aid*. Chichester: Wiley.

Wierzbicki, A. P. (1980). The use of reference objectives in multiobjective optimization. In G. Fandel, & T. Gal, (Eds.), *Multiple Criteria Decision Making, Theory and Applications* (pp 468-486). Berlin: Springer-Verlag.

Wierzbicki, A. P. (1982). A mathematical basis for satisficing decision making. *Mathematical Modelling*, 3, 391–405.

Wierzbicki, A. P. (1986). On the completeness and constructiveness of parametric characterizations to vector optimization problems. *OR Spectrum*, 8, 73–87.

Appendix

Name	*Notation*	*Variable*	*Type*	*Values*	*Description*
ghw	x_1	**Gross Wage**	Continuous	[0, ∞)	Gross Real Hourly wage (€)
		Education level:			Highest education level completed *(reference group: first level of secondary education or lower)*
edh	x_2	Higher education	Binary	0 or 1	Higher education
eds	x_3	Secondary education	Binary	0 or 1	Secondary (2nd level) education completed
nfi	x_4	**Family income**	Continuous	[0, ∞)	Net Equivalent family income (10^3 €)
age	x_5	**Age**	Continuous	[26, 64]	Age (years)
		Job seniority:			Seniority in the company *(reference group:0-2 years)*
j_{3-4}	x_6	Seniority 3-4	Binary	0 or 1	Seniority in the company (3-4 years)
j_{5-9}	x_7	Seniority 5-9	Binary	0 or 1	Seniority in the company (5-9 years)
j_{10-14}	x_8	Seniority 10-14	Binary	0 or 1	Seniority in the company (10-14 years)
j_{15+}	x_9	Seniority 15+	Binary	0 or 1	Seniority in the company (15- years)
mh40	x_{10}	**Working hours:**	Binary	0 or 1	Working more than 40 hours per week
per	x_{11}	**Permanent contract**	Binary	0 or 1	Type of contract signed: permanent *(reference group: non permanent; i.e. fixed term, etc.)*
		Occupational status:			Job status *(reference group: non supervisory/intermediate)*
sup	x_{12}	Supervisory	Binary	0 or 1	Supervisory status
int	x_{13}	Intermediate	Binary	0 or 1	Intermediate status
mar	x_{14}	**Married**	Binary	0 or 1	Civil State
ch6	x_{15}	**Children <6**	Binary	0 or 1	Having children under the age of 6 *(ref. group: over the age of 5)*
und	x_{16}	**Unemployment duration**	Integer	[0, 288]	Worker's unemployment duration previous to current job (months)
		Worker's Health:			General health status *(reference group: poor or very poor)*
ghe	x_{17}	Good health	Binary	0 or 1	Health status (Good)
fhe	x_{18}	Fair health	Binary	0 or 1	Health status (Fair)
rur	x_{19}	**Regional unemployment rate**	Continuous	[0, 100]	Regional unemployment rate
		Industry in current job:			Main activity in current job *(ref. group: Manufacturing)*
in1	x_{20}	Mining and quarrying	Binary	0 or 1	Industry (Mining and quarrying)
in3	x_{21}	Utilities and construction	Binary	0 or 1	Industry (Utilities and construction)
in4	x_{22}	Hotel sales	Binary	0 or 1	Industry (Sales, hotels and restaurants)
in5	x_{23}	Transport	Binary	0 or 1	Industry (Transport)
in6	x_{24}	Finance property	Binary	0 or 1	Industry (Finance property)
in7	x_{25}	Other industry	Binary	0 or 1	Industry (Other industry)
		Firm size:			Number of employees in current job*(ref. group: fewer than 5)*
fs_{5-19}	x_{26}	Firm size 5-19	Binary	0 or 1	Firm size (5-19 workers)
fs_{20-99}	x_{27}	Firm size 20-99	Binary	0 or 1	Firm size (20-99 workers)
$fs_{100-499}$	x_{28}	Firm size 100-499	Binary	0 or 1	Firm size (100-499 workers)
fs_{500+}	x_{29}	Firm size 500+	Binary	0 or 1	Firm size (>=500 workers)
		Year:			*(reference group: year 1995))*
year1	x_{30}	1995	Binary	0 or 1	1995
.	.	.	.	.	.
.	.	.	.	.	.
year7	x_{35}	2001	Binary	0 or 1	2001

Note: T*he duration of unemployment for those who have been out of the labor force is the sum of the duration of the first unemployment spell and the duration of the spell out of the labor force (excluding time spent in formal education).*

Table A1. Decision variables