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A discrete-time queue with service time adjustments and general retrial times

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ABSTRACT

This paper examines a discrete-time retrial queueing system where incoming customers can either choose a last-come, first-served (LCFS) discipline or enter an orbit. It accounts for the possibility of varying service times, which follow an arbitrary distribution, and the retrial times are also governed by an arbitrary distribution. The underlying Markov chain of the system has been analyzed, leading to the derivation of the generating function for the number of customers in both the orbit and the overall system, along with their expected values. The paper also establishes the stochastic decomposition law and, as an application, provides bounds for the difference between the steady-state distributions of the system in question and its standard equivalent. Recursive formulas for determining the steady-state distribution of customers in the orbit and the system are presented. The paper derives the distribution of the time a customer spends at the server and, consequently, the distribution of service times subject to possible variations. A detailed analysis of the time a customer spends in the orbit is also conducted. Finally, numerical examples are included to demonstrate how key parameters impact various system characteristics, with the main contributions of the research summarized in the conclusion.

1. Introduction

Queueing systems play a vital role in modeling and analyzing system performance. These systems are encountered frequently in everyday life, and queueing theory was developed to create models that predict the behavior of systems under random demand. Due to the practical and valuable applications of discrete-time retrial queues, researchers continue to focus on analyzing these models.

Discrete-time models are particularly well-suited for many scenarios. However, they present greater technical challenges compared to continuous-time systems due to two key factors: (i) the basic unit of time is a binary code, and (ii) the possibility of simultaneous events occurring. The regulation of simultaneous events, such as arrivals and departures, in queueing systems has been thoroughly investigated with regard to their steady-state behavior, as explored in [1]. Further discussions and applications of discrete-time queues are available in the works of [1–4], among others.

The inclusion of retrials significantly complicates the analysis due to the spatial heterogeneity of the underlying processes, a direct result of the retrial mechanism. In fact, detailed analytical results are available for specific retrial queueing models with particular assumptions, such as the distribution of retrial times, the number of servers, and customer homogeneity. However, for many other models, performance evaluations rely on numerically tractable approximations, as shown in studies like [5–7].

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Service interruption in queues was first examined by [8], where they explored an $M/M/1$ pre-emptive two-priority queueing system with exponentially distributed service interruptions. Following this initial work, numerous studies have expanded on such models, including contributions by [9–12], and for a comprehensive review of queues with service interruptions, see [13].

In [14] it was studied the stationary distribution of the sojourn time of a customer in the orbit when the retrial times were ruled by a geometrical law, in our present work these results are extended to systems with general retrial times, which adds a new research contribution to the field of retrial queues.

Changes in the repair and vacation times were considered, for the first time, in [15,16], in the present work the study of changes in the service times turns out to be more complex due to the queueing activities that take place during a service time.

The consideration of changes in the service times is a quite realistic one. The service times can be shortened or enlarged due to different causes such as staff shortage, the state of the machinery, the current speed of the communications, and for applications to the field of home automation we refer to [17].

The structure of the paper is as follows. In the next section, we provide a detailed description of the queueing system model under consideration. Section 3 focuses on analyzing the associated Markov chain and derives the distribution of customers both in the orbit and in the overall system. In addition, key performance metrics for the system are presented. Section 4 introduces the stochastic decomposition property for the steady-state system size and provides upper and lower bounds on the distance between the steady-state distributions of the given queueing model and its corresponding standard version. In Section 5, we develop a recursive algorithm to compute the steady-state probabilities for the number of customers in both the orbit and the system. Section 6 explores the busy period of an auxiliary system, which is used to analyze customer delay. In Section 7, we determine the distribution of a customer's sojourn time at the server, in the orbit, and in the entire system. Section 8 provides numerical examples to demonstrate the impact of the parameter governing changes in service times on various performance metrics. Finally, the paper concludes with a section summarizing the key contributions of the research.

2. The mathematical model

Consider a discrete-time queueing system where the time axis is divided into equal intervals, referred to as slots. The time is marked as $0, 1, \ldots, m, \ldots$, and it is assumed that all queue-related events (such as arrivals, departures, and retrials) occur near the slot boundaries. At epoch m, departures happen in the interval (m^-, m), while arrivals and retrials occur sequentially in the interval (m, m^+). Specifically, external arrivals take place just after the slot boundary, followed by retrials, and departures occur immediately before the slot boundary.

Customers arrive according to a geometric process with arrival probability a, meaning a is the likelihood that an arrival will occur within a slot. If the arriving customer finds the server idle, service begins immediately. However, if the server is busy, with probability θ, the incoming customer displaces out of the system the current one and begins immediately its service and with complementary probability $\overline{\theta}$, the new customer joins an "orbit" of blocked customers, which follows a First-Come-First-Served (FCFS) discipline. Only the customer at the front of the orbit is eligible to attempt accessing the server. The retrial times follow a general distribution $\{a_i\}_{i=0}^{\infty}$, with a generating function $A(x) = \sum_{i=0}^{\infty} a_i x^i$, where a_i is the probability that the customer at the head of the orbit makes a retry in i slots. Service times are independent and follow a distribution $\{s_i\}_{i=1}^{\infty}$, with a generating function $S(x) = \sum_{i=1}^{\infty} s_i x^i$ where s_i represents the probability that a service lasts i slots. The probability that service lasts at least k slots is denoted as $S_k = \sum_{i=k}^{\infty} s_i$

Service times can change dynamically. In each slot, there is a probability v that the remaining service time of the customer currently being served will change and with probability s_i, $i \geq 1$, the remaining service time will become equal to i independently of the elapsed service time, and with complementary probability $\overline{v}$, no change occurs in the remaining service time.

3. The Markov chain

Considering the instant time immediately after the slot m, the state of the system can be described by the process

$$\mathcal{Y}_m = \left(C_m, \xi_{0,m}, \xi_{1,m}, \mathcal{N}_m \right)$$

where C_m denotes the state of the server 0 or 1 according whether the server is free or busy respectively and N_m is the number of repeated customers. If $C_m = 0$ and $\mathcal{N}_m > 0$, $\xi_{0,m}$ denotes the remaining retrial time. If $C_m = 1$, then $\xi_{1,m}$ represents the remaining service time of the customer currently being served.

It can be shown that $\{\mathcal{Y}_m, m \in \mathbb{N}\}$ is the Markov chain of our queueing system, whose space states is

$$\{(0,0), (0,i,k) : i,k \geq 1, (1,i,k); i \geq 1, k \geq 0\}.$$

Our first task is to find the stationary distribution

$$\begin{aligned}
\pi_{0,0} &= \lim_{m\to\infty} P[C_m = 0, \mathcal{N}_m = 0] \\
\pi_{0,i,k} &= \lim_{m\to\infty} P[C_m = 0, \xi_{0,m} = i, \mathcal{N}_m = k], \ i,k \geq 1 \\
\pi_{1,i,k} &= \lim_{m\to\infty} P[C_m = 1, \xi_{1,m} = i, \mathcal{N}_m = k], \ i \geq 1, \ k \geq 0
\end{aligned}$$

of the Markov chain $\{\mathcal{Y}_m, m \in \mathbb{N}\}$.

The Kolmogorov equations for the stationary distribution of the system are given by

$$\pi_{0,0} = \bar{a}\pi_{0,0} + \bar{a}\pi_{1,1,0} \tag{1}$$

$$\pi_{0,i,k} = \bar{a}\pi_{0,i+1,k} + \bar{a}a_i\pi_{1,1,k}, \; i,k \geq 1 \tag{2}$$

$$\begin{aligned}\pi_{1,i,k} &= \delta_{0,k}as_i\pi_{0,0} + \bar{a}s_i\pi_{0,1,k+1} + (1-\delta_{0,k})as_i\sum_{j=1}^{\infty}\pi_{0,j,k} \\ &+ as_i\pi_{1,1,k} + \bar{a}a_0s_i\pi_{1,1,k+1} + (1-\delta_{0,k})a\bar{\theta}\bar{v}\pi_{1,i+1,k-1} \\ &+ \bar{a}\bar{v}\pi_{1,i+1,k} + (1-\delta_{0,k})a\bar{\theta}vs_i\sum_{j=2}^{\infty}\pi_{1,j,k-1} \\ &+ (\bar{a}v + a\theta)s_i\sum_{j=2}^{\infty}\pi_{1,j,k}, \; i\geq 1, \; k\geq 0\end{aligned} \tag{3}$$

where $\bar{a} = 1-a$ and $\delta_{a,b}$ is the Kronecker's delta.

The normalization condition is

$$\pi_{0,0} + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{0,i,k} + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i,k} = 1$$

In order to solve the previous equations we introduce the following generating functions

$$\begin{aligned}\varphi_0(x,z) &= \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{0,i,k}x^iz^k \\ \varphi_1(x,z) &= \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i,k}x^iz^k\end{aligned}$$

and the auxiliary generating functions

$$\begin{aligned}\varphi_{0,i}(z) &= \sum_{k=1}^{\infty}\pi_{0,i,k}z^k, \; i\geq 1 \\ \varphi_{1,i}(z) &= \sum_{k=0}^{\infty}\pi_{1,i,k}z^k, \; i\geq 1\end{aligned}$$

Multiplying Eqs. (2) and (3) by z^k, summing over k, and taking into account (1), these equations become

$$\varphi_{0,i}(z) = \bar{a}\varphi_{0,i+1}(z) + \bar{a}a_i\varphi_{1,1}(z) - aa_i\pi_{0,0} \tag{4}$$

$$\begin{aligned}\varphi_{1,i}(z) &= (\bar{a}+a\bar{\theta}z)\bar{v}\varphi_{1,i+1}(z) + \frac{\bar{a}}{z}s_i\varphi_{0,1}(z) + as_i\varphi_0(1,z) \\ &+ \frac{\bar{a}a_0 + a\bar{\theta}z - (\bar{a}+a\bar{\theta}z)vz}{z}s_i\varphi_{1,1}(z) \\ &+ [a\theta + (\bar{a}+a\bar{\theta}z)v]s_i\varphi_1(1,z) + \frac{z-a_0}{z}as_i\pi_{0,0}\end{aligned} \tag{5}$$

Multiplying (4) and (5) by x^i and summing over i gives

$$\begin{aligned}\frac{x-\bar{a}}{x}\varphi_0(x,z) &= \bar{a}[A(x)-a_0]\varphi_{1,1}(z) \\ &- \bar{a}\varphi_{0,1}(z) - a[A(x)-a_0]\pi_{0,0}\end{aligned} \tag{6}$$

$$\begin{aligned}z\frac{x-(\bar{a}+a\bar{\theta}z)\bar{v}}{x}\varphi_1(x,z) &= \Big[\,[\bar{a}a_0 + a\bar{\theta}z - (\bar{a}+a\bar{\theta}z)vz]S(x) \\ &- (\bar{a}+a\bar{\theta}z)\bar{v}z\,\Big]\,\varphi_{1,1}(z) + \bar{a}S(x)\varphi_{0,1}(z) \\ &+ azS(x)\varphi_0(1,z) \\ &+ [a\theta + (\bar{a}+a\bar{\theta}z)v]zS(x)\varphi_1(1,z) \\ &+ a(z-a_0)S(x)\pi_{0,0}\end{aligned} \tag{7}$$

Putting $x = 1$ in (6), we have

$$a\varphi_0(1,z) = \bar{a}(1-a_0)\varphi_{1,1}(z) - \bar{a}\varphi_{0,1}(z) - a(1-a_0)\pi_{0,0} \tag{8}$$

and substituting the above equation into (7) yields

$$\begin{aligned}z\frac{x-(\bar{a}+a\bar{\theta}z)\bar{v}}{x}\varphi_1(x,z) &= \Big[\,\big(\bar{a}a_0(1-z) + (\bar{a}+a\bar{\theta})z - (\bar{a}+a\bar{\theta}z)vz\big)S(x) \\ &- (\bar{a}+a\bar{\theta}z)\bar{v}z\,\Big]\,\varphi_{1,1}(z) \\ &+ \bar{a}(1-z)S(x)\varphi_{0,1}(z)\end{aligned}$$

$$+ \; [a\theta + (\bar{a} + a\bar{\theta}z)\nu]zS(x)\varphi_1(1,z) - a_0(1-z)aS(x)\pi_{0,0} \tag{9}$$

setting $x = 1$ in Eq. (9) gives

$$a\bar{\theta}z\varphi_1(1,z) = (\bar{a}a_0 + a\bar{\theta}z)\varphi_{1,1}(z) + \bar{a}\varphi_{0,1}(z) - aa_0\pi_{0,0} \tag{10}$$

Substituting (10) into (9), we get:

$$\begin{aligned} a\bar{\theta}z\frac{x-(\bar{a}+a\bar{\theta}z)\bar{\nu}}{x}\varphi_1(x,z) = & \Big[\big(a\bar{\theta}z + \bar{a}a_0[1-(\bar{a}+a\bar{\theta}z)\bar{\nu}]\big)S(x) \\ & - \bar{a}\bar{\theta}z(\bar{a}+a\bar{\theta}z)\bar{\nu} \Big] \varphi_{1,1}(x) \\ & + \bar{a}[1-(\bar{a}+a\bar{\theta}z)\bar{\nu}]S(x)\varphi_{0,1}(z) \\ & - a_0[1-(\bar{a}+a\bar{\theta}z)\bar{\nu}]aS(x)\pi_{0,0} \end{aligned} \tag{11}$$

Setting $x = \bar{a}$ and $x = (\bar{a} + a\bar{\theta}z)\bar{\nu}$ in (6) and (11) respectively, we obtain

$$a[A(\bar{a}) - a_0]\pi_{0,0} = \bar{a}[A(\bar{a}) - a_0]\varphi_{1,1}(z) - \bar{a}\varphi_{0,1}(z) \tag{12}$$

$$\begin{aligned} a_0[1-(\bar{a}+a\bar{\theta}z)\bar{\nu}]aS[(\bar{a}+a\bar{\theta}z)\bar{\nu}]\pi_{0,0} = & \Big[(a\bar{\theta}z + \bar{a}a_0[1-(\bar{a}+a\bar{\theta}z)\bar{\nu}]) \\ & \cdot \; S[(\bar{a}+a\bar{\theta}z)\bar{\nu}] \\ & - a\bar{\theta}z(\bar{a}+a\bar{\theta}z)\bar{\nu} \Big] \varphi_{1,1}(z) \\ & + \bar{a}[1-(\bar{a}+a\bar{\theta}z)\bar{\nu}] \\ & \cdot \; S[(\bar{a}+a\bar{\theta}z)\bar{\nu}]\varphi_{0,1}(z) \end{aligned} \tag{13}$$

From (12) and (13), we find the generating functions $\varphi_{0,1}(z)$ and $\varphi_{1,1}(z)$:

$$\varphi_{0,1}(z) = \frac{a^2\bar{\theta}z[A(\bar{a}) - a_0] \Big[(\bar{a}+a\bar{\theta}z)\bar{\nu} - S[(\bar{a}+a\bar{\theta}z)\bar{\nu}] \Big]}{D(z)} \frac{\pi_{0,0}}{\bar{a}} \tag{14}$$

$$\varphi_{1,1}(z) = \frac{aA(\bar{a})[1-(\bar{a}+a\bar{\theta}z)\bar{\nu}]S[(\bar{a}+a\bar{\theta}z)\bar{\nu}]}{D(z)}\pi_{0,0} \tag{15}$$

where

$$\begin{aligned} D(z) = & \; \bar{a}A(\bar{a})[1-(\bar{a}+a\bar{\theta}z)\bar{\nu}]S[(\bar{a}+a\bar{\theta}z)\bar{\nu}] - a\bar{\theta}z \Big[(\bar{a}+a\bar{\theta}z)\bar{\nu} \\ & - S[(\bar{a}+a\bar{\theta}z)\bar{\nu}] \Big] \end{aligned}$$

By substituting (14), (15) into (6) and (11), we finally have

$$\varphi_0(x,z) = \frac{A(x) - A(\bar{a})}{x - \bar{a}} \frac{xa^2\bar{\theta}z \Big[(\bar{a}+a\bar{\theta}z)\bar{\nu} - S[(\bar{a}+a\bar{\theta}z)\bar{\nu}] \Big]}{D(z)}\pi_{0,0} \tag{16}$$

$$\varphi_1(x,z) = \frac{S(x) - S[(\bar{a}+a\bar{\theta}z)\bar{\nu}]}{x - (\bar{a}+a\bar{\theta}z)\bar{\nu}} \frac{xaA(\bar{a})[1-(\bar{a}+a\bar{\theta}z)\bar{\nu}](\bar{a}+a\bar{\theta}z)\bar{\nu}}{D(z)}\pi_{0,0} \tag{17}$$

From the normalization condition we find the unknown constant $\pi_{0,0}$:

$$\pi_{0,0} = \frac{\bar{a}A(\bar{a})[1-(\bar{a}+a\bar{\theta})\bar{\nu}]S[(\bar{a}+a\bar{\theta})\bar{\nu}] - a\bar{\theta} \Big[(\bar{a}+a\bar{\theta})\bar{\nu} - S[(\bar{a}+a\bar{\theta})\bar{\nu}] \Big]}{\Big[\nu S[(\bar{a}+a\bar{\theta})\bar{\nu}] + a\theta\bar{\nu} \Big] (\bar{a}+a\bar{\theta})A(\bar{a})}$$

Since $\pi_{0,0} > 0$, the inequality

$$a\bar{\theta} \Big[(\bar{a}+a\bar{\theta})\bar{\nu} - S[(\bar{a}+a\bar{\theta})\bar{\nu}] \Big] < \bar{a}A(\bar{a})[1-(\bar{a}+a\bar{\theta})\bar{\nu}]S[(\bar{a}+a\bar{\theta})\bar{\nu}] \tag{18}$$

is a necessary condition for the stability of the system.

Let us note that if $\nu = 1$, i.e. the case in which a change in the service times occur in each slot, the probability $\pi^*_{0,0}$ of an empty system is given by

$$\pi^*_{0,0} = \lim_{\nu\to 1} \pi_{0,0} = \frac{[\bar{a}A(\bar{a}) + a\bar{\theta}]s_1 - a\bar{\theta}}{[(\bar{a}+a\bar{\theta})s_1 + a\theta]A(\bar{a})}$$

Therefore, $\pi^*_{0,0}$ has the same value for all the service time distributions $\{s^*_i\}_{i=1}^{\infty}$ with $s^*_1 = s_1$, and if $s_1 = 0$, the system, in this particular case, becomes unstable, as expected. Moreover the system is stable only if $s_1 \in \Big(\frac{a\bar{\theta}}{\bar{a}A(\bar{a})+a\bar{\theta}}, 1 \Big]$.

Let us summarize the previous results:

Theorem 1. *If condition (18) is fulfilled, the stationary distribution of the Markov chain $\{Y_m, m \in \mathbb{N}\}$ has the following generating functions*

$$\varphi_0(x,z) = \frac{A(x) - A(\bar{a})}{x - \bar{a}} \frac{xa^2\bar{\theta}z \Big[(\bar{a}+a\bar{\theta}z)\bar{\nu} - S[(\bar{a}+a\bar{\theta}z)\bar{\nu}] \Big]}{D(z)}\pi_{0,0}$$

$$\varphi_1(x,z) = \frac{S(x) - S[(\bar{a}+a\bar{\theta}z)\bar{v}]}{x-(\bar{a}+a\bar{\theta}z)\bar{v}} \frac{xaA(\bar{a})[1-(\bar{a}+a\bar{\theta}z)\bar{v}](\bar{a}+a\bar{\theta}z)\bar{v}}{D(z)}\pi_{0,0}$$

where

$$D(z) = \bar{a}A(\bar{a})[1-(\bar{a}+a\bar{\theta}z)\bar{v}]S[(\bar{a}+a\bar{\theta}z)\bar{v}] - a\bar{\theta}z\left[(\bar{a}+a\bar{\theta}z)\bar{v} - S[(\bar{a}+a\bar{\theta}z)\bar{v}]\right]$$

and

$$\pi_{0,0} = \frac{\bar{a}A(\bar{a})[1-(\bar{a}+a\bar{\theta})\bar{v}]S[(\bar{a}+a\bar{\theta})\bar{v}] - a\bar{\theta}\left[(\bar{a}+a\bar{\theta})\bar{v} - S[(\bar{a}+a\bar{\theta})\bar{v}]\right]}{\left[vS[(\bar{a}+a\bar{\theta})\bar{v}] + a\theta\bar{v}\right](\bar{a}+a\bar{\theta})A(\bar{a})}$$

Corollary 1.

(1) The probability generating function of the number of customers in the orbit (i.e., of the variable N) is given by

$$\Psi(z) = \pi_{0,0} + \varphi_0(1,z) + \varphi_1(1,z) = \frac{(\bar{a}+a\bar{\theta}z)A(\bar{a})\left[a\bar{v}(1-\bar{\theta}z) + vS[(\bar{a}+a\bar{\theta}z)\bar{v}]\right]}{D(z)}\pi_{0,0} \quad (19)$$

(2) The generating function for the number of customers in the system (i.e., the variable L) is expressed as

$$\Phi(z) = \pi_{0,0} + \varphi_0(1,z) + z\varphi_1(1,z) = \frac{1}{D(z)} \cdot \left[(\bar{a}+a\bar{\theta}z)A(\bar{a})\left[a\theta z\bar{v} + [v+a(1-z)\bar{v}]S[(\bar{a}+a\bar{\theta}z)\bar{v}]\right]\right]\pi_{0,0} \quad (20)$$

Corollary 2.

(1) The average number of customers in the orbit can be expressed as

$$E[N] = \Psi'(1) = \frac{1}{D(1)(vS^* + a\theta\bar{v})(\bar{a}+a\bar{\theta})} \times \left[a[(\bar{a}+a\bar{\theta})\bar{v}]^2\left[(\bar{a}+a\bar{\theta})\bar{v}(\bar{a}v - a\bar{\theta}\bar{v}) + \bar{a}vA(\bar{a}) - avS^*\right]S^* - (\bar{a}+a\bar{\theta})\bar{v}[1-(\bar{a}+a\bar{\theta})\bar{v}]\left[\bar{a}A(\bar{a}) + a\bar{v}(\bar{\theta}+a\theta A(\bar{a}))S'[(\bar{a}+a\theta)\bar{v}]\right]\right]$$

where

$$D(1) = \bar{a}A(\bar{a}[1-(\bar{a}+a\bar{\theta})\bar{v}]S^* - a\bar{\theta}[(\bar{a}+a\bar{\theta})\bar{v} - S^*])$$

and $S^* = S[(\bar{a}+a\bar{\theta})\bar{v}]$

(2) The mean number of customers in the system is given by

$$E[L] = \Phi'(1) = E[N] + \varphi_1(1,1)$$

where

$$\varphi_1(1,1) = \frac{a\bar{v}(1-S[(\bar{a}+a\bar{\theta})\bar{v}])}{vS[(\bar{a}+a\bar{\theta})\bar{v}] + a\theta\bar{v}}$$

4. Stochastic decomposition

This section focuses on the analysis of the stochastic decomposition property of the system size distribution. The concept of stochastic decomposition in queueing systems was first introduced by [18] in the continuous-time case. They established that, in steady-state, the number of customers in a vacation system is distributed as the sum of two independent random variables: one representing the number of customers in a standard $M/G/1$ system and the other representing the number of customers in the vacation system when the server is on vacation. This result was later extended by [19].

In the context of discrete-time queueing systems, this property has been explored by [20,21]. For discrete-time $Geo/G/1$ retrial queues, the stochastic decomposition has also been investigated, see [22–25].

The system considered in this paper exhibits the stochastic decomposition property since it can be modeled as a queueing system with server vacations. In this model, the server enters a vacation whenever a service is completed, and there are no new arrivals or retrials. The duration of these vacations depends on the arrival process and the inter-retrial times. A vacation ends when the server either serves a customer from the orbit or an external customer arrives. Given these assumptions, the stochastic decomposition for this model is expressed as follows:

$$\Phi(z) = \underbrace{\frac{a\theta z\bar{v} + [v+a(1-z)\bar{v}]S[(\bar{a}+a\bar{\theta}z)\bar{v}]}{[1-\bar{a}\bar{v}]S(\bar{a}+a\bar{\theta}z)\bar{v} - a\bar{\theta}z\bar{v}} \frac{(1-\bar{a}\bar{v})S[(\bar{a}+a\bar{\theta}z)\bar{v}] - a\bar{\theta}\bar{v}}{a\theta\bar{v} + vS[(\bar{a}+a\bar{\theta})\bar{v}]}}_{*}$$

$$\times \underbrace{\frac{\pi_{0,0}+\varphi_0(1,z)}{\pi_{0,0}+\varphi_0(1,1)}}_{**}$$

In this equation, the first two terms (∗) represent the probability generating function of the number of customers in the $Geo/G/1/\infty$ variant of our system. The third term (∗∗) represents the probability generating function of the number of customers in the orbit, conditioned on the server being idle. Therefore, the stochastic decomposition law for our queueing system implies that the total number of customers in the system can be modeled as the sum of two independent random variables: one representing the total number of customers in the standard system, and the other representing the number of customers in the orbit, conditioned on the server being idle.

Theorem 2. *The total number of customers in the system (L) can be described as the sum of two independent random variables: the first corresponds to the total number of customers in an equivalent $Geo/G/1/\infty$ system (L_0), and the second accounts for the number of retrial customers when the server is idle (M). Thus, we have $L = L_0 + M$.*

As a practical application of the stochastic decomposition property, we provide a method to quantify the closeness of the steady-state distributions for the $Geo/G/1/\infty$ system and our queueing system. The significance of the following bounds is to establish upper and lower estimates for the difference between these two distributions.

Theorem 3. *The following inequalities hold*

$$2\frac{[1-\bar{a}\bar{v}]S[(\bar{a}+a\bar{\theta})\bar{v}]-a\bar{\theta}\bar{v}}{vS[(\bar{a}+a\bar{\theta})\bar{v}]+a\theta\bar{v}}\Big[1-\frac{\big(vS[(\bar{a}+a\bar{\theta})\bar{v}]+a\theta\bar{v}\big)\pi_{0,0}}{[1-\bar{a}\bar{v}]S[(\bar{a}+a\bar{\theta})\bar{v}]-a\bar{\theta}\bar{v}}\Big]$$
$$\le \sum_{j=0}^{\infty}\mid P[L=j]-P[L_0=j]\mid \le 2\Big[1-\frac{\big(vS[(\bar{a}+a\bar{\theta})\bar{v}]+a\theta\bar{v}\big)\pi_{0,0}}{[1-\bar{a}\bar{v}]S[(\bar{a}+a\bar{\theta})\bar{v}]-a\bar{\theta}\bar{v}}\Big]$$

The proof of the previous theorem follows the methodology outlined in [26] and is therefore omitted here.

As anticipated, the distance between the distributions of the variables L and L_0, given by $\sum_{j=0}^{\infty}\mid P[L=j]-P[L_0=j]\mid$, diminishes as a_0 approaches 1. Thus, for values of a_0 near 1, our queueing system can be closely approximated by the corresponding model with an infinite buffer.

5. Calculation of steady-state probabilities

In this section recursive formulae of some of the main stationary distributions of the system are given

Theorem 4. *The steady-state distribution of the orbit size is given by the following recursive formulae*

$$\psi_0 = P[N=0] = \frac{a\bar{v}+vS(\bar{a}\bar{v})}{(1-\bar{a}\bar{v})S(\bar{a}\bar{v})}\pi_{0,0} \tag{21}$$

$$\psi_k = P[N=k] = \frac{\sum_{n=0}^{k-1}[b_{k-n}-\bar{a}\bar{v}A(\bar{a})c_{k-n}]\psi_n - vd_kA(\bar{a})\pi_{0,0}}{A(\bar{a})S(\bar{a}\bar{v})}, k\ge 1 \tag{22}$$

where

$$b_n = \sum_{i=n}^{\infty}\binom{i-1}{n-1}S_{i+1}(\bar{a}\bar{v})^{i-n}(a\bar{\theta}\bar{v})^n,\ n\ge 1$$
$$c_n = \sum_{i=n}^{\infty}\binom{i}{n}s_{i+1}(\bar{a}\bar{v})^{i-n}(a\bar{\theta}\bar{v})^n,\ n\ge 1$$
$$d_n = \sum_{i=n}^{\infty}\binom{i}{n}S_{i+1}(\bar{a}\bar{v})^{i-n}(a\bar{\theta}\bar{v})^n,\ n\ge 1$$

Proof. Let us observe that the generating function $\psi(z)$ of the number of customers in the orbit can be written in the form

$$\psi(z) = \frac{1-\frac{1-S[(\bar{a}+a\bar{\theta}z)\bar{v}]}{1-(\bar{a}+a\bar{\theta}z)\bar{v}}v}{G(z)}A(\bar{a})\pi_{0,0}$$

where

$$G(z) = \bar{a}\bar{v}A(\bar{a})\frac{S[(\bar{a}+a\bar{\theta}z)\bar{v}]}{(\bar{a}+a\bar{\theta}z)\bar{v}} - \frac{a\bar{\theta}\bar{v}z}{1-[(\bar{a}+a\bar{\theta}z)\bar{v}]}$$
$$\cdot\Big[1-\frac{S[(\bar{a}+a\bar{\theta}z)\bar{v}]}{(\bar{a}+a\bar{\theta}z)\bar{v}}\Big] = \sum_{n=0}^{\infty}g_nz^n$$

The coefficients g_n, $n \geq 0$, can be obtained using the properties of the generating functions and Newton's binomial (see [22], theorem 3) getting

$$G(z) = A(\bar{a})S(\bar{a}\bar{v}) - \sum_{n=1}^{\infty}[b_n - \bar{a}\bar{v}A(\bar{a})c_n]z^n$$

Since

$$\psi(z)G(z) = \left[1 - v\frac{1 - S[(\bar{a} + a\bar{\theta}z)\bar{v}]}{1 - (\bar{a} + a\bar{\theta}z)\bar{v}}\right] A(\bar{a})\pi_{0,0} \tag{23}$$

and taking into account that the development in power series of the right hand side of the above quality is given by

$$\left[\frac{a\bar{v} + vS(\bar{a}\bar{v})}{1 - \bar{a}\bar{v}} - v\sum_{n=1}^{\infty} d_n z^n\right] A(\bar{a})\pi_{0,0}$$

we obtain, after equating the coefficients of z^k on both sides of (23), the following formulae

$$\begin{aligned} \psi_0 g_0 &= \frac{a\bar{v} + vS(\bar{a}\bar{v})}{1 - \bar{a}\bar{v}} A(\bar{a})\pi_{0,0} \\ \sum_{n=0}^{k} \psi_n g_{k-n} &= -v d_k A(\bar{a})\pi_{0,0} \end{aligned}$$

with $g_0 = A(\bar{a})S(\bar{a}\bar{v})$, and consequently the expressions (21) and (22) are obtained. □

Theorem 5. *The steady-state system size distribution can be determined using the following recursive equations*

$$\phi_0 = P[L = 0] = \pi_{0,0} \tag{24}$$

$$\phi_k = P[L = k] = \frac{\sum_{n=0}^{k-1} \left[\, b_{k-n} - \bar{a}\bar{v}A(\bar{a})c_n \,\right] \phi_n + [a\theta\bar{v}\alpha_k + \beta_k]A(\bar{a})\pi_{0,0}}{A(\bar{a})S(\bar{a}\bar{v})}, \ k \geq 1 \tag{25}$$

where

$$\begin{aligned} \alpha_n &= \sum_{i=n}^{\infty} \binom{i-1}{n-1} S_i(\bar{a}\bar{v})^{i-n}(a\bar{\theta}\bar{v})^{n-1}, \ n \geq 1 \\ \beta_n &= \sum_{i=n}^{\infty} \binom{i}{n} s_i(\bar{a}\bar{v})^{i-n}(a\bar{\theta}\bar{v})^{n}, \ n \geq 1 \end{aligned}$$

Proof. The generating function $\phi(z)$ of the number of customers in the system satisfies the following relation

$$\phi(z)G(z) = \left(a\theta\bar{v}z\frac{1 - S[(\bar{a} + a\bar{\theta}z)\bar{v}]}{1 - (\bar{a} + a\bar{\theta}z)\bar{v}} + S[(\bar{a} + a\bar{\theta}z)\bar{v}]\right) A(\bar{a})\pi_{0,0} \tag{26}$$

where $G(z)$ and its expression in power series have been given in the proof of the previous theorem.

The development in power series of the right hand side of the above formula is given by

$$\left(S(\bar{a}\bar{v}) + \sum_{n=1}^{\infty}[a\theta\bar{v}\alpha_n + \beta_n]z^n\right) A(\bar{a})\pi_{0,0}$$

then after comparing the coefficients of z^k on both sides of (26) we get

$$\begin{aligned} \phi_0 g_0 &= S(\bar{a}\bar{v})A(\bar{a})\pi_{0,0} \\ \sum_{n=0}^{k} \phi_n g_{k-n} &= [a\theta\bar{v}\alpha_k + \beta_k]A(\bar{a})\pi_{0,0} \end{aligned}$$

with $g_0 = A(\bar{a})S(\bar{a}\bar{v})$, and from the above formulae, (24) and (25) are readily obtained. □

6. Busy period

A busy period refers to the time interval that begins when a customer arrives to find the system empty and continues until the system becomes empty again after the completion of service.

In this section, we examine an auxiliary system that differs from the original system in that the arrival probability is $a\tau$, and any arriving customer immediately takes over the server, displacing the customer currently being served, if present. For this auxiliary system, we will determine the distribution of the busy period, which will help in analyzing customer delay in the original system. Let h_k, $k \geq 0$, represent the probability that the busy period lasts exactly k time slots. We then have the following:

$$h_0 = 0$$

$$\begin{aligned} h_k &= [(1-a\theta)\bar{v}]^{k-1}(1-a\theta)s_k + \sum_{i=1}^{k}[(1-a\theta)\bar{v}]^{i-1}s_i a\theta h_{k-i} + \\ &+ \sum_{i=1}^{k}[(1-a\theta)\bar{v}]^{i-1}S_{i+1}a\theta h_{k-i} \\ &+ \sum_{i=1}^{k}[(1-a\theta)\bar{v}]^{i-1}(1-a\theta)vS_{i+1}h_{k-i}, \quad k \geq 1 \end{aligned}$$

A recursive procedure of the above formula can lead to obtain numerically the distribution $\{h_k, k \geq 0\}$, but in order to find the moments of the distribution we will use the GF $h(x) = \sum_{k=0}^{\infty} h_k x^k$, that is given by

$$\begin{aligned} h(x) &= \frac{1}{\bar{v}}S[(1-a\theta)\bar{v}x] + \frac{a\theta}{(1-a\theta)\bar{v}}S[(1-a\theta)\bar{v}x]h(x) + \\ &+ \frac{a\theta}{(1-a\theta)\bar{v}}\frac{(1-a\theta)\bar{v}x - S[(1-a\theta)\bar{v}x]}{1-(1-a\theta)\bar{v}x}h(x) + \\ &+ \frac{v}{\bar{v}}\frac{(1-a\theta)\bar{v}x - S[(1-a\theta)\bar{v}x]}{1-(1-a\theta)\bar{v}x}h(x) \end{aligned}$$

that is

$$h(x) = \frac{[1-(1-a\theta)\bar{v}x]S[(1-a\theta)\bar{v}x]}{\bar{v}(1-x) + (v + a\theta\bar{v}x)S[(1-a\theta)\bar{v}x]}$$

The mean length of a busy period is given by

$$\bar{h} = h'(1) = \frac{\bar{v}[1 - S[(1-a\theta)\bar{v}]]}{(v + a\theta\bar{v})S[(1-a\theta)\bar{v}]}$$

To determine the generating function of the time a customer spends in the orbit, we need the generating function $h(x;i)$ for the distribution of the busy period that begins with a customer in service, with exactly i slots remaining until the service is completed. This generating function is given by the following expression:

$$\begin{aligned} h(x;i) &= \frac{[(1-a\theta)\bar{v}x]^i}{(1-a\theta)\bar{v}}\left[1 - a\theta + a\theta h(x)\right] + x\frac{1-[(1-a\theta)\bar{v}x]^{i-1}}{1-(1-a\theta)\bar{v}x} \\ &\cdot \left[a\theta h(x) + (1-a\theta)vh(x)\right], \ i \geq 1 \end{aligned}$$

The above formula can be explained as follows:

If, after the first $i-1$ slots, no new customers have arrived and no changes in the service times have occurred (with probability $[(1-a\theta)\bar{v}]^{i-1}$), the busy period ends with probability $1-a\theta$. However, if in the ith slot a new customer arrives (with probability $a\theta$), a new busy period begins with the generating function $h(x)$.

Alternatively, if after $k-1$ slots, where $k = 1, \ldots, i-1$, no customers have arrived and no service time changes have occurred (with probability $[(1-a\theta)\bar{v}]^{k-1}$), and a new customer arrives in the kth slot (with probability $a\theta$), a new busy period starts with the generating function $h(x)$. If no customer arrives, but the service time changes (with probability $(1-a\theta)v$), a busy period also begins with the generating function $h(x)$. By summing over all k from 1 to $i-1$, the formula for $h(x;i)$ is derived. Let us note that the expression of $h(x;i)$ can be written in the following form

$$\begin{aligned} h(x;i) &= \frac{[(1-a\theta)\bar{v}x]^i}{\bar{v}[1-(1-a\theta)\bar{v}x]}\left[1 - vh(x) - \bar{v}x(1 - a\theta + a\theta h(x))\right] \\ &+ \frac{[1-(1-a\theta)\bar{v}]xh(x)}{1-(1-a\theta)\bar{v}x} \end{aligned}$$

7. Sojourn times

7.1. Sojourn time of a customer in the server

In this section, we derive the distribution of the time a customer spends being served. Let b_k denote the probability that a customers time in the server lasts exactly k slots. Thus, the following holds:

$$\begin{aligned} b_0 &= 0 \\ b_k &= [(\bar{a} + a\bar{\theta})\bar{v}]^{k-1}s_k + \sum_{i=1}^{k}[(\bar{a} + a\bar{\theta})\bar{v}]^{i-1}(\bar{a} + a\bar{\theta})vS_{i+1}b_{k-i} \\ &+ [(\bar{a} + a\bar{\theta})\bar{v}]^{k-1}a\theta S_{k+1}, \quad k \geq 1 \end{aligned}$$

The corresponding GF $b(x) = \sum_{k=0}^{\infty} b_k x^k$ is given by

$$b(x) = \frac{[1-(\bar{a} + a\bar{\theta})\bar{v}x]S[(\bar{a} + a\bar{\theta})\bar{v}x] + a\theta\left[(\bar{a} + a\bar{\theta})\bar{v}x - S[(\bar{a} + a\bar{\theta})\bar{v}x]\right]}{(\bar{a} + a\bar{\theta})\left[\bar{v}[1-(\bar{a} + a\bar{\theta})\bar{v}x] - v\left[(\bar{a} + a\bar{\theta})\bar{v}x - S[(\bar{a} + a\bar{\theta})\bar{v}x]\right]\right]}$$

The mean sojourn time that a customer spends in the server is given by

$$\bar{b} = b'(1) = \frac{\bar{v}[1 - S[(\bar{a} + a\bar{\theta})\bar{v}]]}{vS[(\bar{a} + a\bar{\theta})\bar{v}] + a\theta\bar{v}}$$

If $v = 1$, that is, the case in which in each slot a change in the service times occurs, the mean time that a customer spends in the server is given by

$$\bar{b} = \frac{1}{(\bar{a} + a\bar{\theta})s_1 + a\theta}$$

Remark 1. If $\theta = 0$, that is, the case in which there is not service interruptions, we obtain the GF $S^*(x)$ of the service time distribution subjected to possible changes:

$$S^*(x) = \frac{(1 - \bar{v}x)S(\bar{v}x)}{\bar{v}(1 - x) + vS(\bar{v}x)}$$

with mean

$$(S^*)'(1) = \frac{\bar{v}[1 - S(\bar{v})]}{vS(\bar{v})}$$

Let us note that if $v = 0$, the case in which there is not changes in the service times, $S^*(x)$ and $(S^*)'(1)$ coincide with $S(x)$ and $S'(1)$ respectively.

If $v = 1$, that is, in each slot a change in the service times occurs, the mean service time is given by

$$(S^*)'(1) = \frac{1}{s_1}$$

Therefore, the mean increases when s_1 decreases, and becomes infinite when $s_1 = 0$, as expected. Thus, if $\theta = 0$ and $v = 1$, the mean service time depends only of the coefficient of x of the original service time distribution $S(x)$.

Remark 2. Let us observe that condition (18), necessary for the stability of the system, implies the inequality:

$$a\bar{\theta}[\bar{b} - 1] < \bar{a}A(\bar{a})$$

The first term represents the average number of new customers arriving to the orbit during a service interval, while the second term represents the expected number of retrial customers that begin service when a new service cycle starts. Therefore, condition (18) is not only necessary but also sufficient to ensure the system's stability.

7.2. Sojourn time of a customer in the orbit

The GF of the stationary distribution of the sojourn time of a customer in the orbit is given by

$$w(x) = \pi_{0,0} + \varphi_0(1,1) + \theta\varphi_1(1,1) + \bar{\theta}\omega(x)\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i,k}h(x;i)\big(h(x)\omega(x)\big)^k \tag{27}$$

The above formula can be interpreted as follows:

A newly arriving customer will spend 0 slots in the orbit with probability $\pi_{0,0}+\varphi_0(1,1)+\tau\varphi_1(1,1)$. Alternatively, with probability $\bar{\theta}\pi_{1,i,k}$ (where $i \geq 1, k \geq 0$), the customer enters the orbit and finds k other customers ahead, along with a customer in service who has i slots remaining. The customer at the head of the orbit will then wait a period of time, starting from the arrival of the new customer, with a generating function (GF) $h(x;i), \omega(x)$, where $\omega(x)$ represents the GF of the elapsed time between the end of the busy period $h(x;i)$ and the start of the new customer's service. Once the customer begins service, a busy period is initiated with GF $h(x)$, and when this period ends, the customer in the second position in the orbit (at the time of the new customer's arrival) will wait until their service begins, also with GF $\omega(x)$. Following this pattern results in the term $\pi_{1,i,k}h(x;i)\big(h(x)\omega(x)\big)^k$, and summing over i and k, the formula (27) is derived.

By using the generating functions from Sections 3 and 7, the formula (27) takes the following form:

$$\begin{aligned} w(x) \;=\; & \pi_{0,0} + \varphi_0(1,1) + \theta\varphi_1(1,1) + \frac{\bar{\theta}\omega(x)}{\bar{v}[1 - (1 - a\theta)\bar{v}x]} \\ & \cdot \Big[[1 - \bar{v}h(x) - \bar{v}(1 - a\theta + a\theta h(x))]\varphi_1\big((1 - a\theta)\bar{v}x, h(x)\omega(x)\big) \\ & + \bar{v}xh(x)[v + a\theta\bar{v}]\varphi_1\big(1, h(x)\omega(x)\big) \Big] \end{aligned}$$

Now, we will derive the expression for the generating function (GF) $\omega(x)$. Let ω_k represent the probability that the customer at the front of the orbit spends exactly k slots, for $k \geq 0$, in the orbit between the end of a busy period (BP) and the start of their service. This gives us the following:

$$\begin{aligned} \omega_0 &= a_0 \\ \omega_k &= \bar{a}^k a_k + (1 - \delta_{1,k})a\sum_{l=1}^{k-1}\bar{a}^{l-1}A_l\sum_{i=1}^{k-l}h_i\omega_{k-l-i},\; k \geq 1 \end{aligned} \tag{28}$$

where $A_l = \sum_{i=l}^{\infty} a_i$ is the probability that until the slot l no retrial has taken place.

The GF of the distribution $\{\omega_k, k \geq 0\}$ is given by

$$\omega(x) = \frac{(1-\bar{a}x)A(\bar{a}x)}{1 - x[\bar{a} + ah(x)[1 - A(\bar{a}x)]]}$$

and the corresponding mean time is given by

$$\bar{\omega} = \omega'(1) = \frac{(1 + a\bar{h})[1 - A(\bar{a})]}{aA(\bar{a})}$$

Let us note that if $a_0 = 1$, that is, the case in which the orbit becomes a queue, $\bar{\omega} = 0$ and $\omega(x) = 1$.

Let us clarify formula (28). Consider the moment when a busy period (BP) ends, say at slot 0. The customer at the front of the orbit will begin service immediately with probability a_0 (note that no customer arrives in slot 0 as the BP just finished). Now, for $k \geq 1$, the customer at the head of the orbit will wait exactly k slots after the BP ends before starting their service if:

(a) No customers arrive during the first k slots, and a retrial occurs in slot k (with probability α^k, a_k), or
(b) Before slot l, where $1 \leq l \leq k-1$, no customer arrives and no retrial happens (with probability $\alpha^{l-1} A_l$). In slot l, a new customer arrives and starts a new BP of length i ($i = 1, \ldots, k-l$). After the BP ends, the customer at the front of the orbit waits an additional $k-l-i$ slots until service begins (with total probability a, h_i, ω_{k-l-i}).

The mean sojourn time that a customer spends in the orbit is given by

$$\begin{aligned}\bar{w} &= w'(1) = \frac{\bar{\theta}}{\bar{v}(v + a\theta\bar{v})} \Big[[(v + a\theta\bar{v})\bar{\omega} + (1 - a\theta)\bar{v}]\varphi_1(1,1) \\ &+ (v + a\theta\bar{v})\varphi_1'\big(1, h(x)\omega(x)\big)|_{x=1} - [1 + (v + a\theta\bar{v})\bar{h}]\varphi_1\big((1 - a\theta)\bar{v}, 1\big) \Big]\end{aligned}$$

where

$$\varphi_1[(1 - a\theta)\bar{v}, 1] = S'[(\bar{a} + a\bar{\theta})\bar{v}]\frac{a\bar{v}}{vS[(\bar{a} + a\bar{\theta})\bar{v}] + a\theta\bar{v}}$$

and

$$\varphi_1(1, h(x)\omega(x))'|_{x=1} = \frac{\triangle}{\nabla} a\bar{\theta}\bar{v}(\bar{h} + \bar{w})$$

where

$$\begin{aligned}\triangle &= (1 - S^*) \Big[[\bar{a}A(\bar{a})(1 + a\theta\bar{v}) - (1 - \bar{a}\bar{v})]S^* + \bar{v}(\bar{a} + a\bar{\theta})(1 + a\theta) \Big] \\ &- \bar{v}[1 - (\bar{a} + a\bar{\theta})\bar{v}][a\bar{\theta} + \bar{a}A(\bar{a})]S'[(\bar{a} + a\bar{\theta})\bar{v}] \\ \nabla &= \Big[\bar{a}A(\bar{a})[1 - (\bar{a} + a\bar{\theta})\bar{v}]S^* - a\bar{\theta}[(\bar{a} + a\bar{\theta})\bar{v} - S^*] \Big] (vS^* + a\theta\bar{v})\end{aligned}$$

and $S^* = S[(\bar{a} + a\bar{\theta})\bar{v}]$.

7.3. *Sojourn time of a customer in the system*

The GF $\upsilon(x)$ of the stationary distribution of the sojourn time of a customer in the system is given by

$$\upsilon(x) = w(x) \cdot b(x)$$

and its corresponding mean is given by

$$\bar{\upsilon} = \upsilon'(1) = \bar{w} + \bar{b}$$

8. Numerical results

This section is devoted to illustrate the effect of the most significant parameters of the model on several performance characteristics. The following graphics and tables corroborate what the analytical results and the intuition tell as.

It has been supposed that the retrial times are governed by a geometrical distribution with generating functions $A(x) = \frac{1-r}{1-rx}$.

In Tables 1 and 2, and in the Figs. 1(a), 2(a) and 3(a) it has been supposed that the service times take exactly two slots, and for the Figs. 1(b), 2(b) and Fig. 3(b) it has been chosen a distribution with GF $S(x) = s_1\, x + (1 - s_1)\, x^2$.

With respect to Tables 1 and 2 let us note that an important feature of this work is the recursion scheme provided by Theorems 4 and 5. The formulae (21), (22), (24) and (25) een implemented in Tables 1 and 2 for $a = 0.3$, $r = 0.6$ and $\theta = 0$.

Fig. 1 (a) depicts the behavior of $\pi_{0,0}$ against the parameter v. As was expected $\pi_{0,0}$ is a decreasing function of v, and increases with an increasing parameter θ. Let us note that the graphics of $\pi_{0,0}$ show a similar behavior for any service time distribution $\{s_i\}_{i=1}^{\infty}$ with $s_1 = 0$.

In Fig. 1 (b) the probability that the system is empty is plotted against the parameter s_1. As is to be expected $\pi_{0,0}$ increases with increasing values of s_1 and decreasing values of v. Let us observe that $\pi_{0,0}$ is nearly independent of the values of the parameter v when s_1 approaches to 1, which is also true for any general distribution of the service times.

Table 1

	$\nu = 0$	$\nu = 0.3$	$\nu = 0.6$	$\nu = 0.9$
ψ_0	0.9540814	0.9263036	0.8472105	0.1673068
ψ_1	0.0438095	0.0672208	0.1254262	0.1291483
ψ_2	0.0459182	0.0839647	0.179081	0.8448998
ψ_{10}	$1.8856 \cdot 10^{-11}$	$1.9778 \cdot 10^{-9}$	$7.4086 \cdot 10^{-7}$	0.0396939
ψ_{20}	$7.8579 \cdot 10^{-25}$	$3.4447 \cdot 10^{-20}$	$2.5132 \cdot 10^{-14}$	0.0073582
ψ_{30}	$3.2746 \cdot 10^{-38}$	$5.9995 \cdot 10^{-31}$	$8.5256 \cdot 10^{-22}$	0.0013640

Table 2

	$\nu = 0$	$\nu = 0.3$	$\nu = 0.6$	$\nu = 0.9$
ϕ_0	0.467499	0.402304	0.286298	0.022157
ϕ_1	0.499461	0.540119	0.580631	0.151253
ϕ_2	0.031520	0.052739	0.109131	0.128103
ϕ_{10}	$6.4345 \cdot 10^{-9}$	$2.2009 \cdot 10^{-7}$	$2.0103 \cdot 10^{-5}$	0.055153
ϕ_{20}	$2.6813 \cdot 10^{-22}$	$3.8329 \cdot 10^{-18}$	$6.8185 \cdot 10^{-13}$	0.010224
ϕ_{30}	$1.1173 \cdot 10^{-35}$	$6.6751 \cdot 10^{-29}$	$2.3131 \cdot 10^{-20}$	0.001895

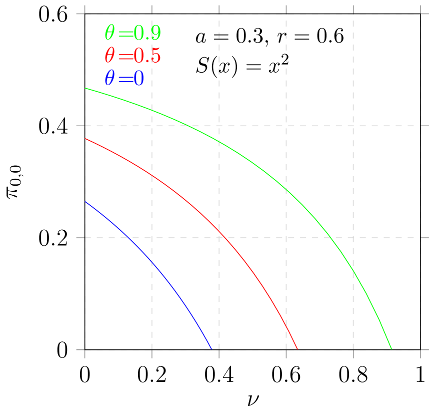

Fig. 1(a).

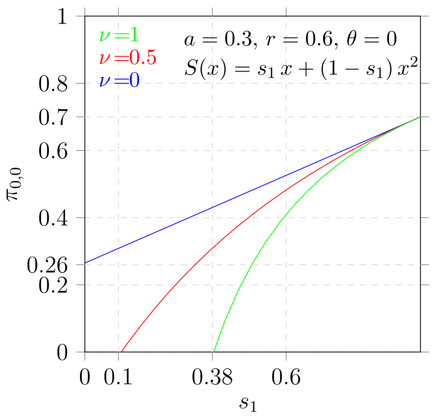

Fig. 1(b).

Fig. 2 (a) and (b) depicts the behavior of the mean number of repeated customers against the parameters ν and s_1 respectively. In Fig. 2 (a), we present three curves corresponding to $\theta = 0, 0.5, 0.9$. As is to be expected, $E[N]$ increases with increasing values of the parameter ν and decreases for increasing values of the parameter θ. The curves of Fig. 2 (b) shows that $E[N]$ is a decreasing function of s_1, and that for values of s_1 next to 1, $E[N]$ is nearly independent of ν.

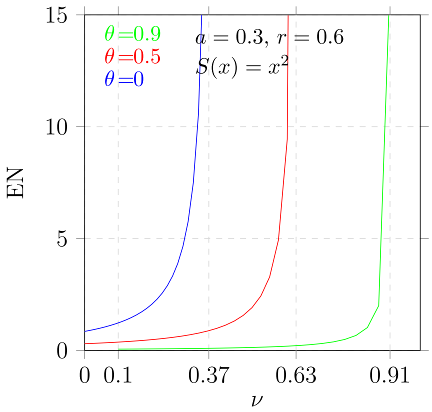

Fig. 2(a).

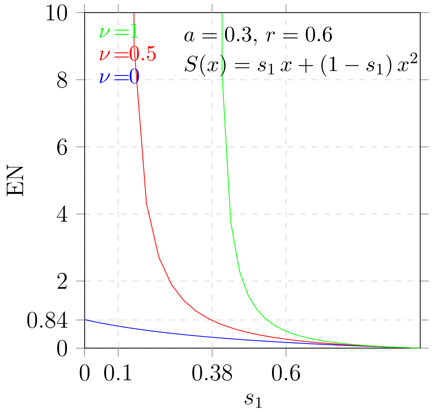

Fig. 2(b).

Fig. 3(a) and (b) show the behavior of the mean sojourn time $\bar{b}$ that a customer spends in the server against the parameter ν and s_1 respectively. As it is expected $\bar{b}$ increases as a function of ν and decreases as a function of s_1. In agreement with the intuitive expectation, when s_1 approaches to 1, the different values that the parameter ν can take affects very little to the value of $\bar{b}$, with is also true for any general distribution of the service time.

9. Conclusions and findings

This paper investigates a discrete-time retrial queue with general retrial times. Upon arrival, customers have two options: if the server is occupied, they can either displace the customer currently being served, or enter an orbit queue following a First-Come-First-Served (FCFS) discipline. Changes in service times have also been incorporated into the model. A thorough analysis of the system has been conducted, producing generating functions for the key distributions associated with the model. Specifically, we have derived the generating functions for both the number of customers in the orbit and in the entire system, along with their respective mean values. Additionally, the generating function of the time spent by a customer at the server has been obtained, which in turn allowed us to derive the generating function of the service times when subject to potential changes. To determine customer delays, the busy period of an auxiliary system has been introduced. Furthermore, we utilized the stochastic decomposition law to establish bounds for the closeness between the steady-state distributions of the studied system and those of its standard version.

One of the significant contributions of this research is a detailed and novel analysis of the time spent by a customer in the orbit. As far as the author is aware, this aspect has not been previously studied in such depth for systems with general retrial times.

Another notable contribution of this work is the realistic consideration of variable service times in discrete-time models. The effect of the parameter ν, which controls potential changes in service times, on the main performance measures of the system was investigated in the section on numerical examples. The results show that this effect strongly depends on the initial distribution $\{s_i\}_{i=1}^{\infty}$ of service times, and particularly on the parameter s_1.

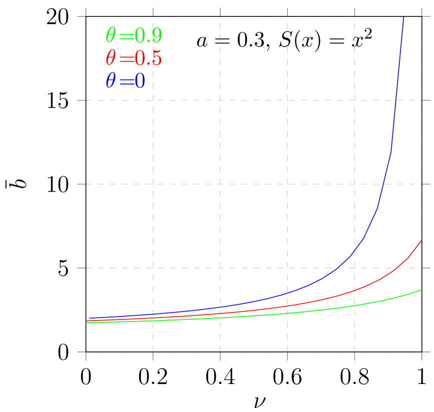

Fig. 3(a).

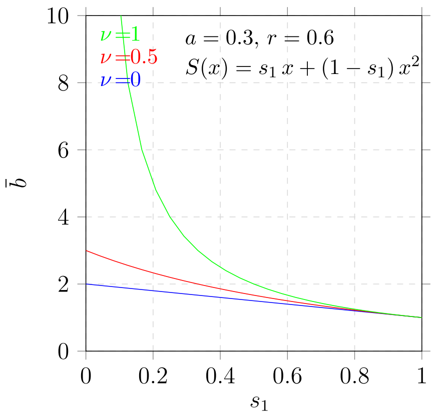

Fig. 3(b).

Moreover, the recursive scheme introduced in Theorems 4 and 5 and its implementation in Tables 1 and 2 also represent an important feature of this work.

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Data availability

No data was used for the research described in the article.

References

[1] J. Hunter, Mathematical Techniques of Applied Probability, Academic Press, 1983.
[2] H. Bruneel, B. Kim, Discrete-Time Models for Communication Systems Including ATM, Kluwer Academic Publishers, 1993.
[3] M.E. Woodward, Communication and Computer Networks: Modelling with Discrete-Time Queues, IEEE Computer Society, 1994.
[4] A. Alfa, Queueing Theory for Telecommunications 1: Discrete Time Modelling of a Single Node System, Springer, 2010.
[5] J. Artalejo, M. Lopez-Herrero, A simulation study of a discrete-time multiserver retrial queue with finite population, J. Statist. Plann. Inference 137 (8) (2007) 2536–2542.
[6] J. Artalejo, A. Economou, A. Gomez-Corral, Algorithm analysis of the $Geo/Geo/c$ retrial queue, European J. Oper. Res. 189 (2008) 1042–1056.
[7] F. Yang, J. Liu, Simulation-based transfer function for transient analysis of general queueing systems, European J. Oper. Res. 223 (2012) 150–165.
[8] H. White, L. Christie, Queuing with preemptive priorities or with breakdown, Oper. Res. 6 (1) (1958) 79–95.
[9] A. Krishnamoorthy, B. Gopakumar, V. Viswanath Narayanan, A retrial queue with server interruptions, resumption and restart of service, Oper. Res. Int. J. 12 (2012) 133–149.
[10] A. Krishnamoorthy, P. Pramod, S. Chakravarthy, A note on characterizing service interruptions with phase-type distribution, J. Stoch. Anal. Appl. 31 (4) (2013) 671–683.
[11] I. Atencia, I. Fortes, S. Sánchez, Discrete-time queueing system with expulsions, Commun. Comput. Inf. Sci. 356 (2013) 20–25.

[12] D. Fiems, B. Steyaert, H. Bruneel, Randomly interrupted $GI/G/1$ queues: service strategies and stability issues, Ann. Oper. Res. 112 (2002) 171–183.
[13] A. Krishnamoorthy, P. Pramod, S. Chakravarthy, A survey on queues with interruptions, TOP 22 (2014) 290–320.
[14] I. Atencia, A $Geo/G/1$ retrial queueing system with priority services, European J. Oper. Res. 256 (2017) 178–186.
[15] I. Atencia, A discrete-time system queueing system with server breakdowns and changes in the repair times, Ann. Oper. Res. 235 (2015) 37–49.
[16] I. Atencia, A discrete-time queueing system with changes in the vacations times, Int. J. Appl. Math. Comput. Sci. 26 (2) (2016) 379–390.
[17] F. Guzmán-Navarro, S. Merino-Córdoba, Gestión de la energía y gestión técnica de edificios, first ed., Autor-Editor, 2015.
[18] S. Fuhrmann, R. Cooper, Stochastic decomposition in the $M/G/1$ queue with generalized vacations, Oper. Res. 33 (5) (1985) 1117–1129.
[19] J. Shanthikumar, On stochastic decomposition in $M/G/1$ type queues with generalized server vacations, Oper. Res. 36 (1988) 566–569.
[20] H. Takagi, Queueing analysis: A foundation of performance evaluation, in: Discrete-Time Systems, North-Holland, 1993.
[21] N. Tian, Z. Zhang, The discrete-time $GI/Geo/1$ queue with multiple vacations, Queueing Syst. 40 (2002) 283–294.
[22] T. Yang, H. Li, On the steady-state queue size distribution of the discrete-time $Geo/G/1$ queue with repeated customers, Queueing Syst. 21 (1995) 199–215.
[23] J. Artalejo, I. Atencia, P. Moreno, A discrete-time $Geo^{[X]}/G/1$ retrial queue with control of admission, Appl. Math. Model. 29 (2005) 1100–1120.
[24] I. Atencia, P. Moreno, Discrete-time $Geo^{[X]}/G/1$ retrial queue with Bernoulli feedback, Comput. Math. Appl. 47 (2004) 1273–1294.
[25] I. Atencia, P. Moreno, The discrete-time $Geo/Geo/1$ queue with negative customers and disasters, Comput. Oper. Res. 31 (9) (2004) 1537–1548.
[26] J. Artalejo, G. Falin, Stochastic decomposition for retrial queues, Top 2 (1994) 329–342.