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The number of t-norms on some special lattices

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Abstract

We estimate the number of triangular norms on some classes of finite lattices. One of them is obtained from two chains by identifying their zero elements, unit elements and an atom. Another one is the set of the dual lattices of the previous one. The obtained formulas involve the number of triangular norms on the corresponding chains. We derive several properties of a triangular norm for this kind of lattices, that enable us to obtain better estimates. Moreover, we obtain the number of t-norms in another class of lattices, which includes the so-called Chinese lantern. Finally, we estimate the number of Archimedean t-norms and divisible t-norms on these lattices.

Keywords: lattice, triangular norm, number of triangular norms, divisible t-norm, Archimedean t-norm, chain.

1. Introduction

The first definition of a t-norm was given by Menger in 1942 (see [11]). The most significant application for t-norms can be found in fuzzy logic, where they represent a mathematical model for the conjunction. In classical models of fuzzy logic, the domain of t-norms is usually the unit interval. A detailed analysis of t-norms and their properties can be found in [9]. If we restrict the set of possible

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truth values to some set of linguistic expressions (for instance, “no, perhaps no, perhaps yes, yes”), we work with a finite chain as a lattice (see [4]). If we also assume the existence of incomparable values, we need more general lattices, for instance, horizontal sums of finite chains.

One object of interest of different authors is to find how many operators defined on bounded lattices are t-norms, or at least to estimate the number of them. If the bounded lattice is not finite, the number of t-norms is infinite. For this reason, we focus on finite lattices. More information concerning t-norms on finite lattices can be found in [1].

B. De Baets and R. Mesiar wrote a seminal paper in this context. They provided the number of t-norms on discrete chains up to a length of fourteen by computational methods (see [3]).

The horizontal sum of discrete chains is another kind of finite lattice which appears in some studies of theoretical physics (see [14]). The number of t-norms in this kind of lattices was found by S. Saminger-Platz, E.P. Klement and R. Mesiar in [17]. Methods of extending t-norms from sublattices to lattices can be found in [12, 13]. So far, there are no similar results for more general lattices.

We will estimate the number of t-norms for two classes of finite lattices. The structure of the first class is given by a horizontal sum by adding a special element called atom (see Figure 2 in Section 3). The second part is devoted to the study of t-norms on the dual lattices class (see Figure 5 in Section 3).

A relevant class of t-norms is the class of Archimedean t-norms. An Archimedean t-norm is a t-norm which does not have non-trivial idempotent elements (see [3, 19, 16]). We estimate the number of Archimedean t-norms on the lattice families mentioned before.

Divisible t-norms are another important class that is studied in this paper. Divisibility is the proper equivalent to the continuity of t-norms on the unit interval in the framework of discrete t-norms (see [9]). Divisible t-norms on chains have been classified in [10]. More details about divisible t-norms can be found in [3, 7].

2. Preliminaries

We introduce the notions which will be used along the paper.

Definition 2.1 ([2])**.** *Let* $(L, \wedge, \vee)$ *be a lattice. We say that* L *is a bounded lattice if there are* $0, 1 \in L$ *such that* $0 \vee x = x$ *and* $1 \wedge x = x$ *for all* $x \in L$*.*

The crucial object of our research will be triangular norms.

Definition 2.2 ([5])**.** *Let* L *be bounded a lattice. A function* $T : L \times L \longrightarrow L$ *is called triangular norm (a t-norm) on* L *if it satisfies the following axioms:*

1. $T(x, y) = T(y, x)$ *for all* $x, y \in L$*.*
2. $T(x, 1) = x$ *for all* $x \in L$*.*
3. *If* $x \leq y$*, then* $T(x, z) \leq T(y, z)$ *for all* $x, y, z \in L$*.*
4. $T(x, T(y, z)) = T(T(x, y), z)$ *for all* $x, y, z \in L$*.*

Definition 2.3 ([2])**.** *Let* $(L, \wedge, \vee)$ *be a lattice and* S *a subset of* L*. We say that* S *is a sublattice if for all* $a, b \in S$*,* $a \wedge b \in S$ *and* $a \vee b \in S$*.*

The definition of a horizontal sum was given by S. Saminger in [15] for bounded posets. We present that notion for bounded lattices.

Definition 2.4 ([15])**.** *Let* I *be a finite index set and let* $(L_i, \leq_i, 0, 1)_{i \in I}$ *be bounded lattices with* $L_i \cap L_j = \{0, 1\}$ *whenever* $i \neq j$*. We say that* $(L, \leq, 0, 1)$ *is a horizontal sum of* $(L_i, \leq_i, 0, 1)_{i \in I}$ *if* $L = (\cup L_i, \leq, 0, 1)$*, where* $x \leq y$ *if and only if there is* $i \in I$ *such that* $\{x, y\} \subset L_i$ *and* $x \leq_i y$*.*

Example 2.5. *A significant example of a horizontal sum is the non-distributive lattice* $\mathrm{MO2} = \{0, a, b, a', b', 1\}$ *of Figure 1. This lattice is well-known as "Chinese lantern" (see [6]).*

Each t-norm defined in a horizontal sum is represented as an extension of its restrictions to the summands as we can see in the following result.

Lemma 2.6 ([17])**.** *Let* L *be a bounded lattice which is a horizontal sum of some non-empty family* $\{L_i\}_{k \in I}$ *of bounded lattices. Then a binary operation*

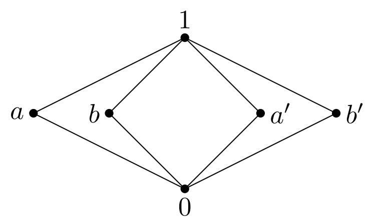


Figure 1: MO2

$T : L \times L \longrightarrow L$ is a t-norm on L if and only if T is given by

$$T(x,y) = \begin{cases} T|_{L_i^2}(x,y) & \text{if} \quad x,y \in L_i \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The lattice MO2 is the horizontal sum of four chains of three elements and there are two t-norms defined on each chain of three elements. Therefore, the number of t-norms defined on MO2 is 16.

We remind that given two elements a and b of a lattice L, the closed interval $[a,b]$ in L is defined by $[a,b] := \{x \in L \mid a \le x \le b\}$ and the open interval $]a,b[$ of L is defined by $]a,b[:= \{x \in L \mid a < x < b\}$.

Definition 2.7 ([15])**.** *Consider a bounded lattice $(L, \le, 0, 1)$ and a linearly ordered index set $I \neq \emptyset$. Further, let $(]a_i, b_i[)_{i \in I}$ be a family of pairwise disjoint subintervals of L and $(T^{[a_i,b_i]})_{i \in I}$ a family of t-norms on the corresponding intervals $([a_i,b_i])_{i\in I}$. Then the ordinal sum $T = (\langle a_i, b_i, T^{[a_i,b_i]}\rangle)_{i \in I} : L^2 \longrightarrow L$ is given by*

$$T(x,y) = \begin{cases} T^{[a_i,b_i]}(x,y) & \text{if} \quad x,y \in [a_i,b_i] \\ x \wedge y & \text{otherwise.} \end{cases} \tag{2}$$

The ordinal sum is not always a t-norm. We present the following characterization.

Proposition 2.8 ([15])**.** *Consider a bounded lattice $(L, \le, 0, 1)$ and a linearly ordered index set $I \neq \emptyset$. Then the following are equivalent:*

i) The ordinal sum T as defined by Equation (2) is a t-norm for arbitrary *families of pairwise disjoint subintervals* $(]a_i, b_i[)_{i\in I}$ *and for arbitrary t-norms* $T^{[a_i,b_i]}$ *on the corresponding* $[a_i, b_i]$.

ii) *L is a horizontal sum of chains.*

Before the following definition we recall a recursive notation. Given a bounded lattice L and $x \in L$,

$$x_T^{(1)} = x \text{ and } x_T^{(n)} = T(x_T^{(n-1)}, x) \text{ for } n \geq 2.$$

Definition 2.9 ([18]). *Let* L *be a bounded lattice and* $T : L \times L \longrightarrow L$ *a t-norm on* L. *The t-norm* T *is called Archimedean if for each* $x, y \in L \setminus \{0, 1\}$ *there is* $n \in \mathbb{N}$ *such that* $x_T^{(n)} < y$.

We will use the following characterization of Archimedean t-norms, which is valid for finite lattices.

Proposition 2.10. *Let* L *be a finite lattice and* $T : L \times L \longrightarrow L$ *a t-norm. The following facts are equivalent.*

1. T *is an Archimedean t-norm.*

2. T *has only two idempotents:* 0 *and* 1.

Proof. Suppose that T is an Archimedean t-norm and x is an idempotent element, $0 \neq x \neq 1$. Then $x_T^{(n)} = x$ for all $n \in \mathbb{N}$, but this is a contradiction as T is an Archimedean t-norm. Conversely, suppose that T has only two idempotents which are 0 and 1. Suppose that there are $x, y \in L \setminus \{0, 1\}$ satisfying $x_T^{(n)} \geq y$ for all $n \in \mathbb{N}$. Since L is a finite lattice and T is non-decreasing, there is $k \in \mathbb{N}$ such that $x_T^{(k)} = x_T^{(l)}$ for each $l \geq k$. Take $z = x_T^{(k)}$, by hypothesis, we have $1 > x \geq z \geq y > 0$. Then,

$$T(z, z) = T(x_T^{(k)}, x_T^{(k)}) = x_T^{(2k)} = x_T^{(k)} = z$$

that is, z is an idempotent element and $0 \neq z \neq 1$. □

Definition 2.11 ([8]). *Let* L *be a bounded lattice and* $T : L \times L \longrightarrow L$ *a t-norm on* L. *The t-norm* T *is called divisible if for all* $x, y \in L$ *satisfying* $x \leq y$, *there is* $z \in L$ *such that* $T(y, z) = x$.

Denote by P_n the number of t-norms on the chain

$$C = \{0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1\}.$$

That means that P_n is the number of t-norms on a chain with $n+2$ elements. We can see in Table 1 the values of $P_1, P_2, \ldots, P_{12}$ obtained computationally in [3].

n	P_n	n	P_n
1	2	7	13775
2	6	8	86417
3	22	9	590489
4	94	10	4446029
5	451	11	37869449
6	2386	12	382549464

Table 1: Number of t-norms on chains.

3. Main results

Definition 3.1 ([2]). *Let L be a bounded lattice and $T : L \times L \longrightarrow L$ on L. Then an element $\alpha \in L$ is called atom if $\alpha > 0$ and there is not $x \in L$ such that $0 < x < \alpha$.*

We introduce now one of the families of finite lattices that we are going to study.

Let us consider two finite chains:

$$\begin{aligned} C_1 &= \{0 < \alpha < \beta_1 < \cdots < \beta_p < 1\} \\ C_2 &= \{0 < \alpha < \gamma_1 < \cdots < \gamma_q < 1\} \end{aligned}$$

where p, q are natural numbers. Let us identify the zero and unit elements in these chains, as well as the atom α.

We are going to study the number of t-norms on $L = \{0, \alpha, \beta_1, ..., \beta_p, \gamma_1, ..., \gamma_q, 1\}$.

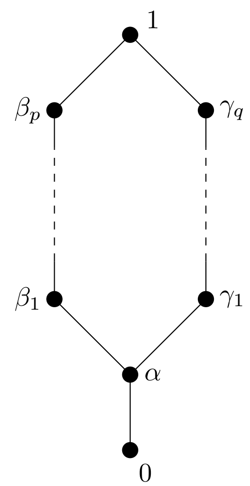


Figure 2: The lattice L.

We will focus on estimating the number of t-norms on L, assuming that the numbers of t-norms on C_1, C_2 are known. First we will find an upper bound for the number of t-norms on L. After that, we will find some more conditions for different classes of t-norms on L.

Theorem 3.2. *Let L be the lattice of Figure 2. Then the number of t-norms on L is bounded from above by the number*

$$P_p \cdot P_q + \binom{p+q}{p}(P_{p+1} - P_p)(P_{q+1} - P_q) \tag{3}$$

Proof. Let T be a t-norm on L. There are two possible cases:

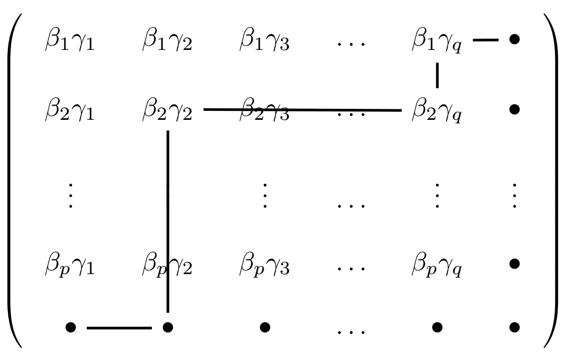
1. $T(\alpha, \alpha) = \alpha$.
2. $T(\alpha, \alpha) = 0$.

The total number of t-norms on L will be the sum of numbers of t-norms for both cases.

Suppose that $T(\alpha, \alpha) = \alpha$. Because of the monotonicity we have $T(\beta_i, \gamma_j) = \alpha$ for arbitrary $i \in \{1, 2, \ldots, p\}, j \in \{1, 2, \ldots, q\}$. Thus T is completely described by its values on the chains $C_1 \setminus \{0\}$ and $C_2 \setminus \{0\}$ and each pair of t-norms on these chains uniquely defines a t-norm on L satisfying $T(\alpha, \alpha) = \alpha$. Therefore, there is a bijection between the set of all t-norms on L satisfying $T(\alpha, \alpha) = \alpha$ and the set of all t-norms on the horizontal sum of $C_1 \setminus \{0\}$ and $C_2 \setminus \{0\}$. By Lemma 2.6, the number of t-norms on L satisfying $T(\alpha, \alpha) = \alpha$ is $P_p \cdot P_q$, what is the first term in the estimate.

Suppose now that $T(\alpha, \alpha) = 0$. For the elements of both chains we use the notation from the Figure 2. Then the restrictions $T\big|_{C_1^2}$ and $T\big|_{C_2^2}$ are obviously t-norms. The numbers of all t-norms on C_1 and C_2 are P_{p+1} and P_{q+1} respectively, but our restrictions do not satisfy $T(\alpha, \alpha) = \alpha$, so we have to subtract the number of t-norms for which $T(\alpha, \alpha) = \alpha$ holds, what is P_p and P_q. Thus the number of all pairs of these t-norms is $(P_{p+1} - P_p)(P_{q+1} - P_q)$. However, contrary to the case when $T(\alpha, \alpha) = \alpha$, there may be many ways for the choice of values for $T(\beta_i, \gamma_j)$. Since $T(\beta_i, \gamma_j) \leq \beta_i \wedge \gamma_j = \alpha$, we have $T(\beta_i, \gamma_j) \in \{0, \alpha\}$.

From the monotony property it follows that $T(\beta_i, \gamma_j) = \alpha$ implies $T(\beta_r, \gamma_s) = \alpha$ for all $r \geq i, s \geq j$. This means that each t-norm on L satisfying $T(\alpha, \alpha) = 0$ is given by a path in the following matrix, where $T(\beta_i, \gamma_j) = \alpha$ for all pairs on the path and below it, while the value at the remaining pairs is zero. The mentioned monotony means that only turns to the right and up are allowed. So the number of pairs of t-norms has to be multiplied by the number of all possible paths.



For example, the path in the image represents the case when $\{T(\beta_2, \gamma_2) = T(\beta_1, \gamma_q)\} = \alpha$ as well as all the values at pairs on and below this line, while $T(\beta_i, \gamma_j) = 0$ at all the pairs above it. The path joining only the bullets (without a turn to the right) corresponds to the case when the zero value is attained at all pairs.

The number of such paths is $\binom{p+q}{p}$ as the length of the path is $p+q$ and p vertical steps have to be chosen. Therefore the number of the pairs of t-norms in the estimate is multiplied by this coefficient. □

Example 3.3. *Consider the finite lattice of Figure* 3*. The estimate of formula* (3) *is*

$$P_2 \cdot P_1 + \binom{3}{2}(P_3 - P_2)(P_2 - P_1) = 204.$$

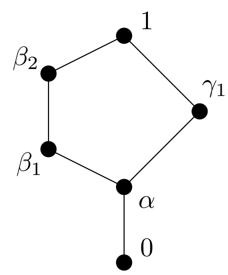


Figure 3: Lattice of Figure 2 for $p = 2$ and $q = 1$.

In order to obtain formula (3) as upper bound of the number of t-norms on the lattice L of Figure 2, we took into account only the monotonicity, commutativity and the property of the neutral element, but not associativity of the t-norm. Thus the number of t-norms on L is much smaller than the upper bound obtained.

Next, we present some additional properties of t-norms on L which are derived from associativity and can help to obtain better estimates of the number of t-norms.

As for the case $T(\alpha, \alpha) = \alpha$ we have an exact number of t-norms on L, let us assume that $T(\alpha, \alpha) = 0$.

For a fixed t-norm, for which $T(\alpha, \alpha) = 0$ denote $M = \{(\beta_i, \gamma_j); T(\beta_i, \gamma_j) = \alpha\}$. On this set we have an order $(\beta_i, \gamma_j) \leq (\beta_r, \gamma_s)$ if and only if $i \leq r$ and $j \leq s$. The following proposition deals with minimal elements in M. Note that these are exactly the pairs at which the path from the previous proof turns to the right, starting from the lower most to the left bullet.

Proposition 3.4. *Let L be the lattice of Figure 2 and $T : L \times L \longrightarrow L$ a t-norm such that $T(\alpha, \alpha) = 0$. If (β_m, γ_n) is a minimal element of the set M, then $T(\alpha, \beta_i) = 0$ and $T(\alpha, \gamma_j) = 0$ for all $i < m$ and for all $j < n$.*

Proof. Let (β_m, γ_n) be a minimal element of M, $i < m$ and $j < n$. By commutativity and associativity we have

$$T(\alpha, \beta_i) = T(T(\beta_m, \gamma_n), \beta_i) = T(\beta_m, T(\beta_i, \gamma_n)) = T(\beta_m, 0) = 0.$$

$$T(\alpha, \gamma_j) = T(T(\gamma_n, \beta_m), \gamma_j) = T(\gamma_n, T(\beta_m, \gamma_j)) = T(\gamma_n, 0) = 0.$$

□

Proposition 3.5. *Let L be the lattice of Figure 2 and $T : L \times L \longrightarrow L$ a t-norm such that $T(\alpha, \alpha) = 0$. For at least one of the restrictions $T\big|_{C_1^2}$ and $T\big|_{C_2^2}$ the value of T at all pairs (α, δ) where $\delta \in L \setminus \{1\}$ is zero.*

Proof. If $T(\beta_p, \gamma_q) = 0$ for all $\beta_p < 1, \gamma_q < 1$, then from monotonicity

$$T(\alpha, \beta_i) = T(\alpha, \gamma_i) \leq T(\beta_p, \gamma_q) = 0.$$

Suppose that there exist β_p, γ_q for which $T(\beta_p, \gamma_q) = \alpha$. Then

$$T(T(\alpha, \beta_p), \gamma_q) = T(\alpha, T(\beta_p, \gamma_q)) = T(\alpha, \alpha) = 0.$$

This means that either $T(\alpha, \beta_p) = 0$, or, in the opposite case $T(\alpha, \beta_p) = \alpha$, but then $T(\alpha, \gamma_q) = 0$. The statement of the proposition follows immediately from the monotonicity. □

In the sequel, we present several propositions that we obtain from the previous results.

Corollary 3.6. *Let L be the lattice of Figure 2. Then, the number of Archimedean t-norms is bounded from above by*

$$\binom{p+q}{q}(P_{p+1} - P_p)(P_{q+1} - P_q).$$

Proof. This is a consequence of Theorem 3.2, taking into account that the case $T(\alpha, \alpha) = \alpha$ cannot be considered due to Proposition 2.10. □

Some additional properties for Archimedean t-norms on L are the following:

Proposition 3.7. *Let L be the lattice of Figure 2 and T an Archimedean t-norm on L. Then, $T(\alpha, x) = 0$ for all $x \in L \setminus \{1\}$.*

Proof. Since T is Archimedean, Proposition 2.10 states $T(\alpha, \alpha) = 0$. Suppose there is $y \in L \setminus \{1\}$ such that $T(\alpha, y) = \alpha$. Then, $y = \beta_i$ or $y = \gamma_j$ for some i or j. It is sufficient to check the case $y = \beta_i$. Taking $z = \min\{\beta_k \mid T(\beta_k, \alpha) = \alpha\}$, we have

$$T(T(z, z), \alpha) = T(z, T(z, \alpha)) = T(z, \alpha) = \alpha.$$

Since $T(\alpha, \alpha) = 0$, $T(z, z) > \alpha$. Then,

$$T(z, z) \in \{\beta_k \mid T(\beta_k, \alpha) = \alpha\}.$$

This implies $T(z, z) \geq z$. By monotonicity, $T(z, z) \leq z$. Hence, $T(z, z) = z$. This is a contradiction since T is an Archimedean t-norm. □

Proposition 3.8. *Let L be the lattice of Figure 2 and T an Archimedean t-norm on L. Then,*

$$T(\beta_i, \gamma_q) = 0 \qquad \text{for all } \beta_i \leq T(\beta_p, \beta_p)$$

and

$$T(\beta_p, \gamma_j) = 0 \qquad \text{for all } \gamma_j \leq T(\gamma_q, \gamma_q).$$

Proof. Let us take β_i satisfying $\beta_i \leq T(\beta_p, \beta_p)$, then

$$T(\beta_i, \gamma_q) \leq T(T(\beta_p, \beta_p), \gamma_q) = T(\beta_p, T(\beta_p, \gamma_q)) \leq T(\beta_p, \alpha) = 0.$$

Now, let us take γ_j satisfying $\gamma_j \leq T(\gamma_q, \gamma_q)$, then

$$T(\beta_p, \gamma_j) \leq T(\beta_p, T(\gamma_q, \gamma_q)) = T(T(\beta_p, \gamma_q), \gamma_q) \leq T(\alpha, \gamma_q) = 0.$$

□

The following two results involve the number of divisible t-norms on the lattice L.

Lemma 3.9 ([10])**.** *Let C be the chain $\{0 < 1 < 2 < ... < n-1 < n\}$. Then there is a bijection between the set of all divisible t-norms on C and the power set $\wp(C \setminus \{0, n\})$. Thus there are 2^{n-1} divisible t-norms on C.*

Proposition 3.10. *Let L be the lattice of Figure 2. Then, the number of divisible t-norms is bounded from above by*

$$2^p \cdot 2^q + \binom{p+q}{q}(P_{p+1} - P_p)(P_{q+1} - P_q).$$

Proof. By Theorem 3.2, the number of divisible t-norms is bounded from above by Equation (3). In that formula, the first term, $P_p \cdot P_q$, is the number of t-norms T on L satisfying $T(\alpha, \alpha) = \alpha$ and the second term, $\binom{p+q}{p}(P_{p+1} - P_p)(P_{q+1} - P_q)$, is an upper bound of the number of t-norms T on L satisfying $T(\alpha, \alpha) = 0$. In this proof, we focus on t-norms satisfying the first condition and we prove that the number of divisible t-norms is then $2^p \cdot 2^q$.

Consider the set

$$M = \{T : L \times L \longrightarrow L \mid T(\alpha, \alpha) = \alpha, T \text{ is a divisible t-norm}\}$$

and for each $i \in \{1, 2\}$

$$A_i = \{T : C_i \times C_i \longrightarrow C_i \mid T(\alpha, \alpha) = \alpha, T \text{ is a divisible t-norm}\}.$$

Note that by monotonicity, if $T \in A_i$, then $T(\alpha, x) = \alpha$ for all $x \in C_i \setminus \{0\}$. Let us define the following map

$$\begin{array}{rccc} \phi : & A_1 \times A_2 & \longrightarrow & M \\ & (T_1, T_2) & \mapsto & \phi(T_1, T_2) \end{array}$$

where $\phi(T_1, T_2)(x, y) = \begin{cases} T_i(x, y) & \text{if} \quad x, y \in C_i \\ \alpha & \text{otherwise.} \end{cases}$

Let us see that $\phi(T_1, T_2)$ is a divisible t-norm, that is, $\phi(T_1, T_2) \in M$.

Note that $\phi(T_1, T_2)$ is exactly the ordinal sum described in Equation (2) on the lattice $L \setminus \{0\}$. Since $L \setminus \{0\}$ is a horizontal sum, by Proposition 2.8 we see that $\phi(T_1, T_2)$ is a t-norm.

We will check that $\phi(T_1, T_2)$ is divisible. Take $x, y \in L$. If $x = 0$ or $x = 1$, the condition of the divisibility is clear. If $x = \alpha$, taking $z = \alpha$, we have $T(y, z) = x$. Suppose $x \notin \{0, \alpha, 1\}$, then $x \in C_1 \setminus C_2$ or $x \in C_2 \setminus C_1$. Without loss of generality, suppose $x \in C_1 \setminus C_2$, then $x = \beta_i$ for some $i \in \{1, ..., p\}$. Because of the structure of L and since $x \leq y$, we have $y \in C_1$. Since T_1 is a divisible t-norm on C_1, there is an element $z \in C_1$ such that $T_1(y, z) = x$, or equivalently, $\phi(T_1, T_2)(y, z) = x$. We conclude that $\phi(T_1, T_2) \in M$.

Let us check that ϕ is injective. Suppose that there are $(T_1, T_2), (T_1', T_2') \in A_1 \times A_2$ satisfying $\phi(T_1, T_2)(x, y) = \phi(T_1', T_2')(x, y)$ for all $x, y \in L$. Taking $x, y \in C_i$ where $i \in \{1, 2\}$, we have the following equality

$$T_i(x, y) = \phi(T_1, T_2)(x, y) = \phi(T_1', T_2')(x, y) = T_i'(x, y).$$

That means $T_i = T_i'$ for each $i \in \{1, 2\}$. Thus ϕ is injective and

$$|A_1| \cdot |A_2| \leq |M|.$$

Finally, we will check that ϕ is onto and consequently $|A_1| \cap |A_2| = |M|$. If $T \in M$, then T is a divisible t-norm and $T(\alpha, \alpha) = \alpha$. By monotonicity, $T(\alpha, x) = \alpha$ for all $x \in L \setminus \{0\}$. Consider $T_1 = T|_{C_1^2}$ and $T_2 = T|_{C_2^2}$. Let us prove that they are divisible t-norms and satisfy $T_1(\alpha, \alpha) = T_2(\alpha, \alpha) = \alpha$.

For T_1, take $x, y \in C_1$ satisfying $x \leq y$. If $x = 0$ or $x = 1$, the divisibility holds obviously. If $x = \alpha$, taking $z = \alpha$, we have $T(y, z) = x$. Otherwise, since T is a divisible t-norm on L, there is $z \in L$ such that $T(y, z) = x$. This implies that $z \geq x$ and since $x = \beta_i$ for some $i \in \{1, ..., p\}$, we have $z \in C_1$. Note that $T_1(\alpha, \alpha) = \alpha$ is clear.

For T_2 the procedure is the same.

Then, $T_1 \in A_1$ and $T_2 \in A_2$ and $\phi(T_1, T_2) = T$. We conclude that ϕ is onto.

Therefore,

$$|M| = |A_1||A_2|.$$

Since C_1 and C_2 are chains with $p+3$ and $q+3$ elements, by the Lemma 3.9, the number of divisible t-norms on C_1 and C_2 is 2^{p+1} and 2^{q+1} respectively. By definition of A_1 and A_2, we have

$$|A_1| = 2^p \qquad \text{and} \qquad |A_2| = 2^q.$$

Therefore,

$$2^p \cdot 2^q = |A_1||A_2| = |M|$$

is the number of divisible t-norms T satisfying $T(\alpha, \alpha) = 0$. We conclude that

$$2^p \cdot 2^q + \binom{p+q}{q}(P_{p+1} - P_p)(P_{q+1} - P_q)$$

is a upper estimate of the number of divisible t-norms on L. □

In the sequel, we will deal with t-norms on a lattice of Figure 4.

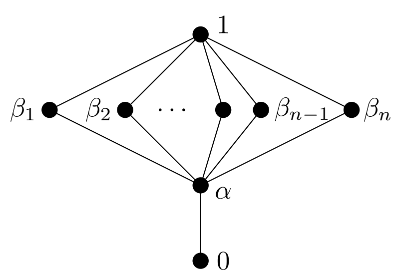


Figure 4: The lattice L.

Proposition 3.11. *Let L be the lattice from the Figure 4. There are exactly*

$$2^n + \sum_{k=0}^{n} \binom{n}{k} \cdot 2^{\frac{(n-k)(n-k+1)}{2}} + n$$

t-norms on L.

Proof. Given a t-norm T on L, we can distinguish two cases: $T(\alpha, \alpha) = 0$ and $T(\alpha, \alpha) = \alpha$.

If $T(\alpha, \alpha) = \alpha$, by monotonicity $T(\beta_i, \beta_j) = \alpha$ for all $i \neq j$. Therefore, the number of t-norms on L satisfying $T(\alpha, \alpha) = \alpha$ is equal to the number of t-norms on the horizontal sum $L \setminus \{0\}$, where α is the zero element. By Lemma 2.6, we have 2^n t-norms.

If $T(\alpha, \alpha) = 0$. Denote by A the following set:

$$A = \{\beta_i \in L \mid T(\alpha, \beta_i) = \alpha\}.$$

We will show that for any t-norm T on L the set A is either empty or a singleton. Suppose this is not the case and there are two different $\beta_i, \beta_j \in A$. From the monotonicity property of the t-norm we have $T(\beta_i, \beta_j) \geq T(\beta_i, \alpha) = \alpha$ and hence $T(\beta_i, \beta_j) = \alpha$. Then

$$T(\alpha, T(\beta_i, \beta_j)) = T(\alpha, \alpha) = 0 \neq \alpha = T(\alpha, \beta_j) = T(T(\alpha, \beta_i), \beta_j)$$

what is in contradiction to the associativity of T.

Suppose that the set A is empty, i.e. $T(\alpha, \beta_i) = 0$ for all $i = 1, 2, \ldots, n$. We will show that if for some $p \in \{1, 2, \ldots, n\}$ there is $T(\beta_p, \beta_p) = \beta_p$, then $T(\beta_p, \beta_i) = 0$ for all $i \in \{1, 2, \ldots, n\}, i \neq p$. This follows from the equalities

$$T(\beta_p, \beta_i) = T(T(\beta_p, \beta_p), \beta_i) = T(\beta_p, T(\beta_p, \beta_i)) = 0$$

where the last equality follows from the fact that $T(\beta_p, \beta_i) \in \{0, \alpha\}$. Thus, for those β_p for which $T(\beta_p, \beta_p) = \beta_p$ we already know the values $T(\beta_p, \gamma)$ for all $\gamma \in L$. For the remaining β_i, β_j (not necessarily distinct) there is $T(\beta_i, \beta_j) \in \{0, \alpha\}$. It is easy to see that if we choose either $T(\beta_i, \beta_j) = T(\beta_j, \beta_i) = 0$ or $T(\beta_i, \beta_j) = T(\beta_j, \beta_i) = \alpha$, in both cases T will be a t-norm.

We will estimate the number of such t-norms. There may exist k elements β_p for which $T(\beta_p, \beta_p) = \beta_p$, where $k \in \{0, 1, \ldots, n\}$. For each such k there are $\binom{n}{k}$ possibilities to choose such elements. For each such choice of k elements we have $n - k$ remaining elements and for them (accounting the symmetry of a t-norm) we have $2^{n-k}, 2^{n-k-1}, \ldots, 2^1, 2^0$ possibilities of choosing α as the value of $T(\beta_i, \beta_j)$, while the remaining values will be zeros, what is altogether $2^{n-k} 2^{n-k-1} \ldots 2^1 2^0 = 2^{1+2+\cdots+(n-k)} = 2^{\frac{(n-k)(n-k+1)}{2}}$ possibilities. Summing over k we have $\sum_{k=0}^{n} \binom{n}{k} 2^{\frac{(n-k)(n-k+1)}{2}}$ t-norms.

Now we will deal with the case when there is exactly one element (for simplicity let it be β_1) for which $T(\alpha, \beta_1) = \alpha$. For all $i = 2, 3, \ldots, n$ from the monotonicity we have $T(\beta_1, \beta_i) \geq T(\beta_1, \alpha) = \alpha$, hence $T(\beta_1, \beta_i) = \alpha$.

For $i > 1$ from the equality

$$T(T(\beta_1, \beta_1), \beta_i) = T(\beta_1, T(\beta_1, \beta_i)) = T(\beta_1, \alpha) = \alpha$$

we obtain $T(\beta_1, \beta_1) = \beta_1$, as in case $T(\beta_1, \beta_1) = \alpha$ we would get $T(\alpha, \beta_i) = \alpha$, which is not possible. Thus we have all the values $T(\beta_1, \gamma)$ for $\gamma \in L$.

Now let $i, j > 1$, not necessarily distinct. Then from

$$T(\beta_1, T(\beta_i, \beta_j)) = T(T(\beta_1, \beta_i), \beta_j) = T(\alpha, \beta_j) = 0$$

we have $T(\beta_i, \beta_j) = 0$. Hence there is no choice for the values of T except the choice of the element denoted here as β_1 which can be chosen in n ways.

Finally, we conclude that the number of t-norms on L is

$$2^n + \sum_{k=0}^{n} \binom{n}{k} \cdot 2^{\frac{(n-k)(n-k+1)}{2}} + n$$

□

If $n = 1$, the lattice is a chain with 4 elements and the number of t-norms is $2^1 + \binom{1}{0}2^1 + \binom{1}{1}2^0 + 1 = 6$, which is the same value that appears in Table 1 given originally in [3].

Corollary 3.12. *Let L be the lattice from Figure 4. There are*

$$2^{\frac{n(n+1)}{2}}$$

Archimedean t-norms on L.

Proof. According to the proof of Proposition 3.11, there are

$$2^n + \sum_{k=0}^{n} \binom{n}{k} \cdot 2^{\frac{(n-k)(n-k+1)}{2}} + n$$

t-norms on L. Moreover, there are 2^n t-norms T such that $T(\alpha, \alpha) = \alpha$. Clearly, they are not Archimedean t-norms.

Given a t-norm T, consider the set

$$A = \{\beta_i \in L \mid T(\alpha, \beta_i) = \alpha\}.$$

We have that $|A| \in \{0, 1\}$. Also, we have the following assertions:

1. If $|A| = 1$, we have n t-norms on L and each of them satisfy $T(\beta_p, \beta_p) = \beta_p$ for some $p \in \{1, ..., n\}$. Then, they are not Archimedean t-norms.
2. If $|A| = 0$, we have $\sum_{k=0}^{n} \binom{n}{k} 2^{\frac{(n-k)(n-k+1)}{2}}$ t-norms on L.

Therefore, we must check what t-norms, which satisfy $|A| = 0$, are Archimedean t-norms. This only happens when $k = 0$, and the number of Archimedean t-norms is

$$2^{\frac{n(n+1)}{2}}$$

□

Lemma 3.13. *Let L be the lattice from Figure 4. Then the following statements are equivalent.*

(i) *T is a divisible t-norm on L.*

(ii) *For each β_i, there is β_j such that $T(\beta_i, \beta_j) = \alpha$.*

Proof. $(i) \Rightarrow (ii)$ Take $x = \alpha$ and $y = \beta_i$. Since $x \leq y$ and T is divisible, there is $z \in L$ such that $T(y, z) = x$. Note that z cannot be 0 or 1. If $z = \alpha$, by monotonicity, $T(y, \beta_j) = x$ for all $\beta_j \neq \beta_i$.

$(ii) \Rightarrow (i)$ Take $x, y \in L$ satisfying $x \leq y$. If $x = 0$ or $x = 1$, the divisibility is clear. If $x = \beta_i$ for some $i \in \{1, ..., n\}$, then $y \in \{\beta_i, 1\}$. Both cases are trivial taking $z = 1$ and $z = x$ respectively. Finally, if $x = \alpha$, then $y \in \{\alpha, \beta_1, ..., \beta_n, 1\}$. The cases $y = \alpha$ and $y = 1$ are trivial taking $z = 1$ and $z = x$ respectively. The case $y = \beta_i$ for some $i \in \{1, ..., n\}$ holds by hypothesis. □

Lemma 3.14. *Let $S_1, \cdots, S_n$ be finite sets. The principle of inclusion–exclusion states that*

$$|\bigcup_{i=1}^{n} S_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < ... < i_k \leq n} |S_{i_1} \cap ... \cap S_{i_k}| \quad (4)$$

i.e.,

$$|\bigcup_{i=1}^{n} S_i| = \sum_{i=1}^{n} |S_i| - \sum_{1\le i<j\le n} |S_i \cap S_j| + \sum_{1\le i<j<k\le n} |S_i \cap S_j \cap S_k| - \cdots + (-1)^{n+1} |S_1 \cap S_2 \cap \cdots \cap S_n|.$$

Corollary 3.15. *Let L be the lattice from Figure 4. There are*

$$2^n + \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cdot 2^{\frac{(n-k)(n-k+1)}{2}} + n$$

divisible t-norms on L.

Proof. We will count the divisible t-norms on L using Lemma 3.13, which states that the following facts are equivalent:

(i) T is a divisible t-norm on L.

(ii) for each β_i, there is β_j such that $T(\beta_i, \beta_j) = \alpha$.

If $T(\alpha, \alpha) = \alpha$, by monotonicity we have that $T(\beta_i, \beta_j) = \alpha$ for all $i, j \in \{1, \ldots, n\}$. The number of t-norms on L satisfying $T(\alpha, \alpha) = \alpha$ is equal to the number of t-norms on the horizontal sum $L \setminus \{0\}$, that is, 2^n t-norms. Since (ii) holds, we have that every t-norm T on L satisfying $T(\alpha, \alpha) = \alpha$ is divisible. Therefore, there are 2^n divisible t-norms.

If $T(\alpha, \alpha) = 0$, let us consider the set

$$A = \{\beta_i \in L \mid T(\alpha, \beta_i) = \alpha\}.$$

As in the proof of Proposition 3.11, we obtain that $|A| \in \{0, 1\}$.

If $|A| = 1$, there is β_k such that $T(\beta_k, \alpha) = \alpha$. By monotonicity, $T(\beta_k, \beta_l) = \alpha$ for all $l \neq k$. Then, for each β_i, there is β_j such that $T(\beta_i, \beta_j) = \alpha$. Since (ii) holds, we have n divisible t-norms in this case.

If $|A| = 0$, suppose that there is β_p such that $T(\beta_p, \beta_p) = \beta_p$.

Since

$$T(\beta_p, \beta_j) = T((T\beta_p, \beta_p), \beta_j) = T(\beta_p, T(\beta_p, \beta_j)) \le T(\beta_p, \alpha) = 0$$

we have $T(\beta_p, \beta_j) = 0$ for all $j \neq p$. Therefore, T does not satisfy condition (ii). Hence, there are not any divisible t-norms in this case.

In the other case, suppose that $T(\beta_p,\beta_p)<\beta_p$ for all $p\in\{1,...,n\}$. Then,

$$T\in B:=\{T \text{ is a t-norm on } L \mid T(\alpha,\alpha)=0,\ T(\beta_p,\beta_p)<\beta_p \text{ for all } p\in\{1,...,n\}\}.$$

As in the proof of Proposition 3.11, we obtain that the number of t-norms satisfying $T(\alpha,\alpha)=0$ and $T(\beta_p,\beta_p)<\beta_p$ for all $p\in\{1,...,n\}$ is $2^{\frac{n(n+1)}{2}}$ because $T(\beta_i,\beta_j)\in\{0,\alpha\}$ and we have $\frac{n(n+1)}{2}$ values to assign. Hence, $|B|=2^{\frac{n(n+1)}{2}}$.

Fix $i\in\{1,...,n\}$ and consider the following set

$$B_i=\{T\in B \mid T(\beta_i,\beta_j)=0 \text{ for all } j\neq i\}.$$

Note that if $T\in B_i$, then $T(\alpha,\beta_i)=T(\alpha,\alpha)=0$. Since T does not satisfy condition (ii), we have that $\bigcup B_i$ is the set formed by all non-divisible t-norms such that $T(\alpha,\alpha)=0$ and $T(\beta_p,\beta_p)<\beta_p$ for all $p\in\{1,...,n\}$.

Therefore, the number of divisible t-norms satisfying $T(\alpha,\alpha)=0$ and $T(\beta_p,\beta_p)<\beta_p$ for all $p\in\{1,...,n\}$ is

$$|B|-|\bigcup B_i|.$$

By the inclusion-exclusion principle (Lemma 3.14), we have

$$|\bigcup_{i=1}^{n} B_i|=\sum_{k=1}^{n}(-1)^{k+1}\sum_{1\leq i_1<...<i_k\leq n}|B_{i_1}\cap...\cap B_{i_k}|.$$

Moreover, by definition of the sets B_i, we know that $|B_{i_1}\cap...\cap B_{i_k}|=|B_1\cap...\cap B_k|$. Here, we have $T(\beta_i,\beta_j)=0$ for all $i\leq k$ and for all $j\in\{1,...,n\}$. The values for $T(\beta_i,\beta_j)$ for each $i,j>k$ are not assigned. Taking into account the commutativity, we have $\frac{(n-k)(n-k+1)}{2}$ values which can be 0 or α. Therefore, $2^{\frac{(n-k)(n-k+1)}{2}}$ t-norms belong to $B_1\cap...\cap B_k$. This implies that

$$\sum_{1\leq i_1<...<i_k\leq n}|B_{i_1}\cap...\cap B_{i_k}|=\binom{n}{k}\cdot 2^{\frac{(n-k)(n-k+1)}{2}}$$

and

$$\sum_{k=1}^{n}(-1)^{k+1}\sum_{1\leq i_1<...<i_k\leq n}|B_{i_1}\cap...\cap B_{i_k}|=\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\cdot 2^{\frac{(n-k)(n-k+1)}{2}}$$

Hence, the number of divisible t-norms satisfying $T(\alpha,\alpha)=0$ and $T(\beta_p,\beta_p)<\beta_p$ for all $p\in\{1,...,n\}$ is

$$|B|-|\bigcup B_i|=2^{\frac{n(n+1)}{2}}-\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\cdot 2^{\frac{(n-k)(n-k+1)}{2}}=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\cdot 2^{\frac{(n-k)(n-k+1)}{2}}$$

□

Now, we study the number of t-norms on $L = \{0, \beta_1, ..., \beta_p, \gamma_1, ..., \gamma_q, \alpha, 1\}$, given in Figure 5.

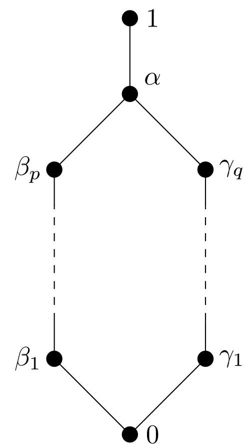


Figure 5: The lattice L.

Let us consider the chains

$$C_1 = \{0 < \beta_1 < \cdots < \beta_p < \alpha < 1\}$$
$$C_2 = \{0 < \gamma_1 < \cdots < \gamma_q < \alpha < 1\}$$

Lemma 3.16. *Given a t-norm T on L, $T(\beta_i, \gamma_j) = 0$ for $i \leq p$ and for $j \leq q$.*

Proof. The statement is straightforward because $T(\beta_i, \gamma_j) \leq \beta_i \wedge \gamma_j = 0$. □

Lemma 3.17. *Given a t-norm T on L, the following statements hold:*

1. *If $T(\alpha, \alpha) = \gamma_k$ for some $k \in \{1, ..., q\}$, then $T(\beta_i, \beta_j) = 0$ for all $i, j \in \{1, ..., p\}$.*
2. *If $T(\alpha, \alpha) = \beta_k$ for some $k \in \{1, ..., p\}$, then $T(\gamma_i, \gamma_j) = 0$ for all $i, j \in \{1, ..., q\}$.*

Proof. We will prove the first case; the second one is analogous. Suppose $T(\alpha, \alpha) = \gamma_k$ for some $k \in \{1, ..., q\}$. Since $T(\beta_i, \beta_j) \leq T(\alpha, \alpha) = \gamma_k$ and

$T(\beta_i, \beta_j) \leq \beta_i \wedge \beta_j$, we conclude $T(\beta_i, \beta_j) \leq \beta_i \wedge \beta_j \wedge \gamma_k = 0$ for all $i, j \in \{1, .., p\}$. □

Let us consider the sets

$$A_i = \{T : C_i \times C_i \longrightarrow C_i \mid T(\alpha, \alpha) = \alpha, \text{ T is a t-norm}\}$$

$$B_i = \{T : C_i \times C_i \longrightarrow C_i \mid T(\alpha, \alpha) < \alpha, \text{ T is a t-norm}\}$$

for each $i \in \{1, 2\}$.

Since the elements of the sets A_1 and A_2 are t-norms satisfying that the biggest non-trivial element is idempotent, we have that $|A_1| \leq P_{p+1}$ and $|A_2| \leq P_{q+1}$.

Now, the following result determine the number of t-norms on L.

Lemma 3.18. *Let L be the lattice of Figure 5. The number of t-norms on L is*

$$|A_1||A_2| + P_{p+1} + P_{q+1} - |A_1| - |A_2| - 1.$$

Proof. Consider the set M defined by

$$M = \{T : L \times L \longrightarrow L \mid \text{ T is a t-norm and } T(\alpha, \alpha) = \alpha\}$$

and for each $i \in \{1, 2\}$ put

$$K_i = \{T : L \times L \longrightarrow L \mid \text{ T is a t-norm and } T(\alpha, \alpha) \in C_i \setminus \{\alpha\}\}.$$

The cardinality of the set $M \cup (K_1 \cup K_2)$ is the number of t-norms on L. Since $M \cap (K_1 \cup K_2) = \emptyset$, we must calculate the formula

$$|M| + |K_1| + |K_2| - |K_1 \cap K_2|.$$

First of all, note that the only t-norm in $K_1 \cap K_2$ is the drastic t-norm. Hence, $|K_1 \cap K_2| = 1$.

Let us prove that $|K_i| = |B_i|$ for each $i \in \{1, 2\}$. We define the following function

$$\begin{array}{rccc} \phi_i : & B_i & \longrightarrow & K_i \\ & T & \mapsto & \phi_i(T) \end{array}$$

where $\phi_i(T)(x,y) = \begin{cases} T(x,y) & \text{if} \quad x,y \in C_i \\ 0 & \text{otherwise.} \end{cases}$

Given a t-norm $T \in B_i$, $\phi_i(T)$ is a binary operator which satisfies associativity, commutativity, monotonicity and it has neutral element 1, then ϕ_i is well-defined. Moreover, by Lemma 3.16 and Lemma 3.17, ϕ_i are injective functions for each $i \in \{1,2\}$. Now, take $i \in \{1,2\}$, given a t-norm $T \in K_i$, we see that $T|_{C_i^2} : C_i \times C_i \longrightarrow C_i$ is a t-norm. Since $T|_{C_i^2}(x,y) \neq \alpha$, we have $T|_{C_i^2} \in B_i$. Since $\phi(T|_{C_i^2}) = T$, we conclude that ϕ is onto.

Then, $|K_1| = |B_1|$ and $|K_2| = B_2|$. Note that

$$|B_1| = P_{p+1} - |A_1| \text{ and } |B_2| = P_{q+1} - |A_2|.$$

Now, we prove that $|M| = |A_1||A_2|$. Again, we define the following function,

$$\begin{array}{rccc} \phi : & A_1 \times A_2 & \longrightarrow & M \\ & (T_1, T_2) & \mapsto & \phi(T_1, T_2) \end{array}$$

where $\phi(T_1,T_2)(x,y) = \begin{cases} T_1(x,y) & \text{if} \quad x,y \in C_1 \\ T_2(x,y) & \text{if} \quad x,y \in C_2 \\ 0 & \text{otherwise.} \end{cases}$

Note that $\phi(T_1,T_2)$ is the horizontal sum on $L \backslash \{1\}$ described in Equation (2). By Lemma 2.8, $\phi(T_1,T_2)$ is a t-norm. Moreover, we know that as $T_1(\alpha,\alpha) = \alpha = T_2(\alpha,\alpha)$, then $\phi(T_1,T_2) \in M$. The map ϕ is injective by construction, hence $|M| \geq |A_1||A_2|$.

We prove that ϕ is onto. Take $T \in M$, we have $T|_{C_i^2}$ is a t-norm on C_i for each i and by definition of M, we have that $T|_{C_i^2}(\alpha,\alpha) = \alpha$. Then, $T|_{C_i^2} \in A_i$ for each $i \in \{1,2\}$. By Lemma 3.16, $T(\beta_i,\gamma_j) = 0$. Therefore, $\phi(T|_{C_1}, T|_{C_2}) = T$.

We conclude that the number of t-norms on L is

$$|A_1||A_2| + P_{p+1} + P_{q+1} - |A_1| - |A_2| - 1.$$

$\square$

We present in Table 2 the number $|A_1|$ of t-norms T on the chain $C = \{0, \beta_1, \beta_2, \ldots, \beta_p, \alpha, 1\}$ satisfying that $T(\alpha,\alpha) = \alpha$. This result has been obtained computationally until $p = 8$. It is worth noticing that the cardinal of C is $p + 3$.

p	$\|A_1\|$
1	3
2	11
3	48
4	236
5	1278
6	7502
7	47303
8	318885

Table 2: Number of t-norms on chains satisfying that the biggest non-trivial element is idempotent.

Example 3.19. *If $p = 3$ and $q = 5$, the number of t-norms on the lattice given in Figure 6 is*

$$|A_1|\cdot|A_2|+P_4+P_6-|A_1|-|A_2|-1 = 48\cdot1278+94+2386-48-1278-1 = 62497.$$

Theorem 3.20. *Let L be the lattice of Figure 5. Then the number of t-norms on L is bounded from above by the number*

$$P_{p+1} \cdot P_{q+1} + P_{p+1} + P_{q+1} - P_p - P_q - 1.$$

For Archimedean t-norms, we have $T(\alpha, \alpha) \neq \alpha$. Then, if we remove the terms $|A_1|$ and $|A_2|$ from the formula in Lemma 3.18, we obtain the following corollary.

Corollary 3.21. *Let L be the lattice of Figure 5. Then the number of Archimedean t-norms on L is bounded from above by the number*

$$P_{p+1} + P_{q+1} - 1.$$

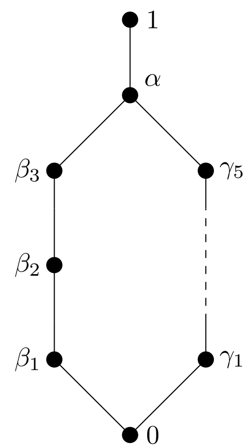


Figure 6: The lattice L when $p = 3$ and $q = 5$.

In the following result, we show the exact number of divisible t-norms on L.

Proposition 3.22. *Let L be the lattice of Figure 5. Then the number of divisible t-norms on L is*

$$2^{p+q}.$$

Proof. Consider the following sets

$$N = \{T : L^2 \longrightarrow L \mid T(\alpha, \alpha) < \alpha \text{ and T is a divisible t-norm}\}$$

$$M = \{T : L^2 \longrightarrow L \mid T(\alpha, \alpha) = \alpha \text{ and T is a divisible t-norm}\}.$$

First of all, we see that $N = \emptyset$. Suppose that $T \in N$, we enumerate three cases.

1. $T(\alpha, \alpha) = 0$. Since $\beta_1 \leq \alpha$, there is $z \in L$ which satisfies $T(\alpha, z) = \beta_1$. Since $T(\alpha, 1) = \alpha$, we have $z \neq 1$, and $T(\alpha, z) \leq T(\alpha, \alpha) = 0$. But $T(\alpha, z) = \beta_1 \neq 0$.
2. $T(\alpha, \alpha) = \beta_i$ for some $i \in \{1, ..., p\}$. If $T(\alpha, \alpha) = \beta_i$, since $\gamma_1 \leq \alpha$, there is $z \in L$ which satisfies $T(\alpha, z) = \gamma_1$. Again, $z \neq 1$, and $T(\alpha, z) \leq T(\alpha, \alpha) = \beta_i$. Therefore,

$$T(\alpha, z) \in \{0, \beta_1, \beta_2, ..., \beta_i\}.$$

But $T(\alpha, z) = \gamma_1$.

3. $T(\alpha, \alpha) = \gamma_j$ for some $j \in \{1, ..., q\}$. The proof is similar to case 2.

Now, we are going to find the cardinality of M. For each $i \in \{1, 2\}$, we define the set A_i as follows:

$$A_i = \{T : C_i \times C_i \longrightarrow C_i \mid T(\alpha, \alpha) = \alpha, \text{ T is a divisible t-norm}\}.$$

The relationship between the sets A_i and M is established by the following function:

$$\begin{array}{rccc} \phi: & A_1 \times A_2 & \longrightarrow & M \\ & (T_1, T_2) & \mapsto & \phi(T_1, T_2) \end{array}$$

where $\phi(T_1, T_2)(x, y) = \begin{cases} T_1(x, y) & \text{if} \quad x, y \in C_1 \\ T_2(x, y) & \text{if} \quad x, y \in C_2 \\ 0 & \text{otherwise.} \end{cases}$

As we have proved in Lemma 3.18, the operator $\phi(T_1, T_2)$ is a t-norm on L because $\phi(T_1, T_2)$ is an ordinal sum defined on the horizontal sum $L \setminus \{1\}$ given in Equation (2). Moreover, $T_1(\alpha, \alpha) = \alpha = T_2(\alpha, \alpha)$ for all $(T_1, T_2) \in A_1 \times A_2$. Let us discuss the divisibility. Take $x, y \in L$ satisfying $x \leq y$. Since $x \in C_i$ for some $i \in \{1, 2\}$ and $x \leq y$, we have $y \in C_i$ due to the lattice structure (see Figure 4). Since T_i is divisible on C_i, there is $z \in C_i$ such that $T_i(y, z) = x$. Hence $\phi(T_1, T_2)(y, z) = T_i(y, z) = x$. Therefore, ϕ is a well-defined function. Moreover, it is injective.

Now, let us prove that ϕ is onto. Take $T \in M$, we see that $T|_{C_i^2}$ is a t-norm on C_i for each i and by definition of M, we have that $T|_{C_i^2}(\alpha, \alpha) = \alpha$. Then, $T|_{C_i^2} \in A_i$ for each $i \in \{1, 2\}$. By Lemma 3.17, $T(\beta_i, \gamma_j) = 0$. Therefore, $\phi(T|_{C_1}, T|_{C_2}) = T$.

Since ϕ is one-to-one and onto, we have that the cardinality of M is the cardinality of the Cartesian product $A_1 \times A_2$. Since the cardinality of the Cartesian product is the product of their cardinalities, we conclude that

$$|M| = |A_1| \cdot |A_2|.$$

By Lemma 3.9 and the definitions of A_1 and A_2, we have that

$$|M| = |A_1| \cdot |A_2| = 2^{p+q}.$$

Then, the number of divisible t-norms on L is

$$2^{p+q}$$

□

We can extend Lemma 3.18 from two chains to n chains. Consider the chains

$$C_i = \{0 < \beta_i^1 < \beta_i^2 < ... < \beta_i^{p_i} < \alpha < 1\}$$

where p_i is an integer number for each $i \in \{1, 2, ..., n\}$. We define the lattice L_n as follows

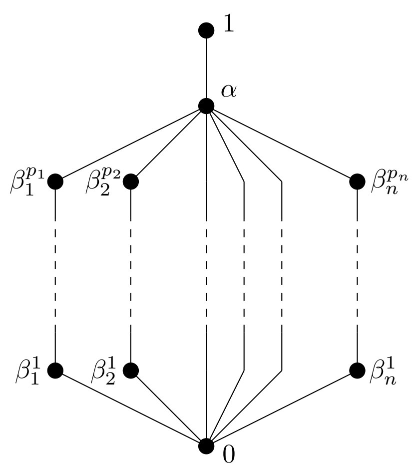


Figure 7: The lattice L_n.

Theorem 3.23. *Let L_n be the lattice of Figure 7. Then, the number of t-norms on L_n is*

$$\prod_{i=1}^{n} |A_i| + \sum_{i=1}^{n} P_{p_i+1} - \sum_{i=1}^{n} |A_i| + 1 - n$$

where $A_i = \{T : C_i \times C_i \longrightarrow C_i \mid T(\alpha, \alpha) = \alpha)\}$.

Proof. Let $\bar{L} = \{T \mid T \text{ is a t-norm on } L\}$. Consider the set M defined by:

$$M = \{T : L \times L \longrightarrow L \mid T(\alpha, \alpha) = \alpha\}$$

and for each $i \in \{1, \cdots, n\}$,

$$K_i = \{T : L \times L \longrightarrow L \mid T(\alpha, \alpha) \in C_i\}.$$

By these definitions, we see that the cardinality of the set $M \cup (\cup_i^n K_i)$ is the number of t-norms on L. Since $M \cap (\cup_i^n K_i) = \emptyset$,

$$|M \cup (\cup_{i=1}^n K_i) = |M| + |\cup_{i=1}^n K_i|.$$

Using the Equation (4) from Principle of inclusion–exclusion (Lemma 3.14), we must calculate

$$|M| + |\cup_{i=1}^n K_i| = |M| + \sum_{k=1}^{n} (-1)^{k+1} (\sum_{1 \leq i_1 < \ldots < i_k \leq n} |K_{i_1} \cap \ldots \cap K_{i_k}|)$$

or equivalently,

$$|M| + \sum_{i=1}^{n} |K_i| + \sum_{k=2}^{n} (-1)^{k+1} (\sum_{1 \leq i_1 < \ldots < i_k \leq n} |K_{i_1} \cap \ldots \cap K_{i_k}|). \tag{5}$$

As in the proof of Lemma 3.18, we have that

$$|M| = \prod_{i=1}^{n} |A_i|$$

and

$$\sum_{i=1}^{n} |K_i| = \sum_{i=1}^{n} P_{p_i+1} - \sum_{i=1}^{n} |A_i|.$$

Now, we study the last term in Equation (5). The proof will be completed by showing that term is equal to $1-n$. Given K_i and K_j with $i \neq j$, note that the $K_i \cap K_j = T_D$, where T_D is the drastic t-norm. Then,

$$\sum_{k=2}^{n} (-1)^{k+1} (\sum_{1 \leq i_1 < \ldots < i_k \leq n} |K_{i_1} \cap \ldots \cap K_{i_k}|) = \sum_{k=2}^{n} (-1)^{k+1} (\sum_{1 \leq i_1 < \ldots < i_k \leq n} 1) = \sum_{k=2}^{n} (-1)^{k+1} \binom{n}{k}.$$

Since

$$0 = -(1-1)^n = -\sum_{i=0}^{n} \binom{n}{i} 1^{n-i} (-1)^i = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1} = -\binom{n}{0} + \binom{n}{1} + \sum_{i=2}^{n} \binom{n}{i} (-1)^{i+1}$$

we have $\sum_{i=2}^{n} \binom{n}{i} (-1)^{i+1} = 1-n$. Therefore, the last term is equal to $1-n$. □

If $n = 1$, the lattice is a chain with $p+3$ elements and the number of t-norms is P_{p+1}. This number appears in Table 1 for $p = \{0, 1, \ldots, 11\}$.

Corollary 3.24. *Let L_n be the lattice of Figure 7. Then,*

1. *The number of t-norms on L_n is bounded from above by*
$$\prod_{i=1}^{n} P_{p_1+1} + \sum_{i=1}^{n} P_{p_i+1} - \sum_{i=1}^{n} P_{p_i} + 1 - n$$
2. *The number of Archimedean t-norms on L_n is bounded from above by*
$$\sum_{i=1}^{n} P_{p_i+1} + 1 - n$$
3. *The number of divisible t-norms on L_n is*
$$\prod_{i=1}^{n} 2^{p_i}$$

Conclusion

We have estimated the t-norms defined on four families of bounded lattices. For one of them, we have provided the exact number of t-norms that can be defined on each lattice of the family. For another one, we have given an upper bound. For the other two families, we have expressed the number of t-norms in terms of the number of t-norms defined on chains.

In each of these four families, we have done the same for Archimedean and divisible norms. In the case of the Archimedean norms, we have obtained the exact number in one of the families and given upper bounds in the rest of the families. In the case of divisible norms, we have obtained an upper bound in one of them and the exact number in the rest of the families.

This study extends the results given by other authors in discrete chains (see [3, 10]).

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