Article

MMS

Mathematics and Mechanics of Solids
2016, Vol. 21(5) 552–561

DOI: 10.1177/1081286514530753
mms.sagepub.com
SAGE

Stress concentration in an elastic square plate with a full-strength hole

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Received 27 January 2014; accepted 14 March 2014

Abstract
This paper addresses the problem of plate bending for a square weakened with a full-strength hole including the origin of coordinates. The vertices of the square lie on the coordinate axes and its neighbourhoods are cut by the same smooth full-strength arcs which are symmetric with respect to the coordinate axes. Rigid bars are attached to each component of the broken line of the outer boundary of the plate. The plate bends under the action of concentrated moments applied to the middle points of the bars. The unknown part of the boundary is free from external forces. Using the methods of complex analysis, the unknown part of the boundary is found under the condition that the tangential–normal moment takes a constant value. Numerical analysis is performed and the corresponding graphs are constructed.

Keywords
Plate bending, full-strength, elasticity, mixed problem, stress state, complex variable theory

1. Introduction

The problems of stress concentration control on the boundary of the hole of an elastic body are, from the applied viewpoint, very important and mathematically rather complicated. Their solvability provides the optimal stress distribution by selecting the appropriate hole boundary.

The tangential–normal stresses, in the case of plane elasticity theory, or the tangential–normal moments, in the case of bending of thin plates, can at some points reach values that cause destruction of plates or the formation of plastic zones near the hole. Proceeding from this, the following problem was posed: given a load applied to the plate's boundary, choosing a hole shape such that at its boundary the maximum value of the

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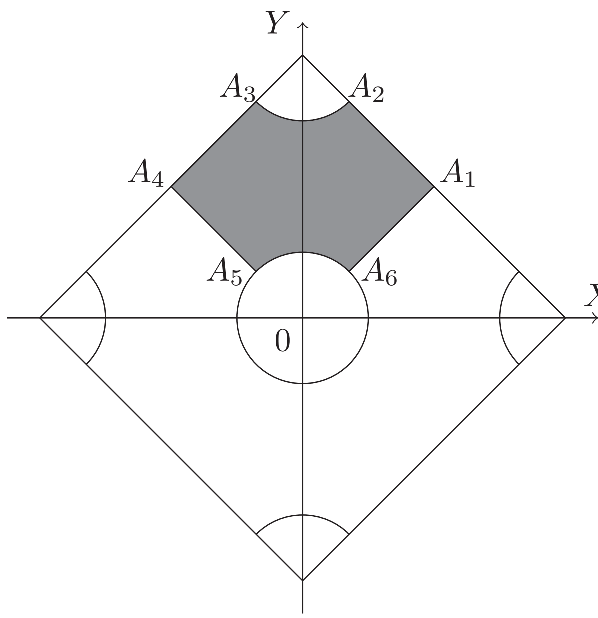

Figure 1. Statement of the problem.

tangential–normal stress or the tangential–normal moment is minimal in comparison with the all other holes is required. It is proved that such a condition is valid provided that tangential–normal stress and tangential–normal moments are constant at that hole. As is known, such holes are called full-strength ones.

Boundary-value problems of the plane theory of elasticity and plate bending for infinite plates weakened by unknown full-strength holes with normal stresses acting on their boundaries and forces applied at infinity were analysed in [1–6].

Boundary-value problems for a finite doubly connected domain with a part of its boundary being unknown full-strength and the other part being a polygonal line are solved in [7, 8].

The axis-symmetric and cycle-symmetric problems of the plane theory of elasticity and plate bending with partially unknown boundaries are studied in [9–19].

In this article the cycle problem of bending for a square weakened with a full-strength hole is considered. The vertices of the square lie on the coordinate axes and its neighbourhoods are cut by the same smooth full-strength arcs which are symmetric with respect to the coordinate axes. Rigid bars are attached to the linear parts of the boundary. The plate bends by moments applied to the middle points of the bars. The unknown part of the boundary is free from external forces. Using methods of complex analysis, the analytical image of Kolosov–Muskhelishvili's complex potentials (characterizing an elastic equilibrium of the body) is constructed and the unknown parts of its boundary are determined under the condition that the tangential–normal moment on that takes a constant value.

2. Statement of the problem

Let a middle surface of a homogeneous isotropic elastic plate on the complex plane $z = x + iy$ occupy a doubly connected domain, representing a square weakened by an unknown hole. Assume that the point $z = 0$ is located inside of the unknown hole. The vertices of the square lie on the coordinate axes and its neighbourhoods are cut by the same unknown arcs which are symmetric with respect to the coordinate axes (Figure 1).

The side length of the square is assumed to be equal to $2a$. Rigid bars are attached to each component of the broken line of the outer boundary of the plate. The plate bends under the action of concentrated moments M applied to the middle points of the bars. It is evident that the normals to the middle surface drawn at the points of rectilinear segments of the boundary turn through one and the same constant angle. Unknown parts of the boundary are free from outer actions.

Consider the following problem.

Problem: Find the plate deflection $w(x,y)$ and the unknown parts of its boundary such that the tangential–normal moment on that plate takes a constant value $M_s = M_0 = \text{const}$.

According to the approximate theory of the plate bending, the deflection of the middle surface $w(x,y)$ in the case under consideration is a biharmonic function. Since the angle of rotation of the middle surface normal $\partial w/\partial n$ is constant on the linear part of the outer boundary, it is not difficult to see that the torque $H_{ns} = 0$ and shearing force $N_n = 0$. Thus on the entire boundary $H_{ns} = N_n = 0$. On the known part of the plate boundary $\partial w/\partial n = \text{const}$ and on the unknown part of the boundary the normal bending moment $M_n = 0$ and the tangential–normal moment $M_s = \text{const}$.

Since the problem is symmetric with respect to segments connecting the middle points of the opposite square sides, and to the coordinate axes OX and OY, it is easy to show that on the segments $[A_6A_1]$, $[A_4A_5]$,

$$\frac{\partial w}{\partial n} = 0, \qquad H_{ns} = N_n = 0.$$

To study the above-formulated problem, it is sufficient to consider a curvilinear hexagon $A_1A_2A_3A_4A_5A_6$ which is denoted by S. Let us introduce the notation $\Gamma_1 = A_1A_2$, $\Gamma_2 = A_3A_4$, $\Gamma_3 = A_4A_5$, $\Gamma_4 = A_6A_1$, $\Gamma = \cup_{j=1}^{4}\Gamma_j$, $\gamma_1 = A_2A_3$, $\gamma_2 = A_5A_6$, $\gamma = \cup_{j=1}^{2}\gamma_j$. The arc abscissa of the point $t \in \Gamma \cup \gamma$ counted from the point A_1 is denoted by s. By M_1, M_2, M_3, M_4 we denote the principal bending moments applied to Γ_1, Γ_2, Γ_3, Γ_4 respectively. Due to the equilibrium of the cut-out part of the plate and $A_1A_2 \| A_4A_5$, $A_3A_4 \| A_6A_1$, we have

$$M_1 = \int_{\Gamma_1} M_n\, ds = M_3 = \int_{\Gamma_3} M_n\, ds = \frac{M}{2}, \qquad M_4 = \int_{\Gamma_4} M_n\, ds = M_2 = \int_{\Gamma_2} M_n\, ds = \frac{M}{2}. \tag{1}$$

The boundary conditions of the problem can be written in the form

$$\frac{\partial w}{\partial n} = \begin{cases} d_1 = \text{const}, & t \in \Gamma_1 \cup \Gamma_2 \\ 0 & t \in \Gamma_3 \cup \Gamma_4 \end{cases} \tag{2}$$

$$M_s = \text{const}, \qquad H_{ns} = M_n = N_n = 0, \quad t \in \gamma, \tag{3}$$

$$H_{ns} = N_n = 0, \qquad t \in \Gamma. \tag{4}$$

3. Problem solution

As is known, the biharmonic in S function $w(x,y)$ can be presented by the Goursat formula

$$w(x,y) = \text{Re}\left[\bar{z}\varphi(z) + \chi(z)\right], \tag{5}$$

where $\varphi(t)$ and $\chi(t)$ are holomorphic functions in S. We can replace formula (5) by a formula more convenient for our purpose:

$$\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}, \qquad z \in S, \tag{6}$$

where $\psi(z) = \chi'(z)$.

It is known that M_n, N_n, H_{ns} and M_s are expressed in terms of the functions $\varphi(z)$ and $\psi(z)$ by formulas analogous to those of Kolosov–Muskhelishvili (see [20, 21]):

$$\frac{1}{D(1-\nu)} \int_{z_0}^{z} \left[M_n + i\int_{s_0}^{s} \left(N_n + \frac{\partial H_{ns}}{\partial s}\, ds \right) \right] dz = l\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}, \qquad z \in S \tag{7}$$

$$(M_n + M_s) = M_x + M_y = -4D(1+\nu)\text{Re}\varphi'(z), \qquad z \in S \tag{8}$$

$$N_x + iN_y = -4D\varphi''(z), \qquad z \in S \tag{9}$$

where

$$l = -\frac{3+\nu}{1-\nu},$$

ν is the Poisson coefficient and D is the cylindrical rigidity.

Taking into account formulas (6), (7) and (8), the boundary conditions of problem (2)–(4) can be presented in the form

$$\operatorname{Re}\left[e^{-i\alpha(t)}\left(\varphi(t)+t\overline{\varphi(t)}+\overline{\psi(t)}\right)\right]=\frac{\partial w}{\partial n},\qquad t\in\Gamma,\tag{10}$$

$$\operatorname{Re}\left[e^{-\alpha(t)}\left(l\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}\right)\right]=C(t),\qquad t\in\Gamma,\tag{11}$$

$$l\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=B(t),\qquad t\in\gamma\tag{12}$$

$$\operatorname{Re}\varphi'(t)=-\frac{M_0}{4D(1+\nu)},\qquad t\in\gamma\tag{13}$$

where

$$B(t)=\frac{i}{D(1-\nu)}\left(\int_{A_1}^{t}M_n(t_0)e^{i\alpha(t_0)}\,ds_0-\frac{M}{2}e^{\frac{\pi}{4}i}\right),\qquad C(t)=\operatorname{Re}\left(e^{-i\alpha(t)}B(t)\right),$$

where $\alpha(t)$ is the angle size, made by the outward normal n and the axis OX:

$$\begin{gathered}\alpha(t)=\alpha_k,\quad t\in\Gamma_k,\quad k=1,2,3,4,\\ \alpha(t)=\alpha_k=\frac{\pi}{4}+\frac{\pi}{2}(k-1).\end{gathered}\tag{14}$$

Taking into account (1), (3) and (14), one gets

$$B(t)=\begin{cases}0, & t\in\gamma_1\\ -\frac{Me^{\frac{\pi}{4}}(1+i)}{2D(1+\nu)}, & t\in\gamma_2\end{cases},\qquad C(t)=\begin{cases}0, & t\in\Gamma_1\cup\Gamma_2\\ \frac{M}{2D(1-\nu)}, & t\in\Gamma_3\cup\Gamma_4\end{cases}.\tag{15}$$

Subtracting (10) from (11) and differentiating with respect to the arc abscissa s and taking into account that $C(t)$, $\alpha(t)$ and $\partial w/\partial n$ are piecewise-constant on Γ, we obtain

$$\operatorname{Im}\varphi'(t)=0,\qquad t\in\Gamma.\tag{16}$$

In what follows, the functions $\varphi(z)$ will be assumed to be continuous in the closed domain S, while functions $\varphi'(z)$ and $\psi(z)$ are continuously extendable on the boundary of S, with the exclusion of the points A_2, A_3, A_5, A_6 in the neighbourhood of which the following estimate is admitted:

$$|\varphi'(z)|<M|z-A_k|^{-\delta_k},\quad |\psi(z)|<M|z-A_k|^{-\delta_k},\qquad 0\le\delta_k<1,\quad k=2,3,5,6.\tag{17}$$

Equalities (13) and (16) are, in fact, the Keldysh–Sedov problem for the domain S:

$$\begin{aligned}&\operatorname{Re}\varphi'(t)=-\frac{M_0}{4D(1+\nu)},\qquad t\in\gamma\\ &\operatorname{Im}\varphi'(t)=0,\qquad t\in\Gamma.\end{aligned}\tag{18}$$

Problem (18) has a unique solution:

$$\varphi'(z)=\frac{-M_0}{4D(1+\nu)}.\tag{19}$$

Hence one obtains

$$\varphi(z)=-\frac{M_0}{4D(1+\nu)}z\tag{20}$$

where the constant was neglected because it is not necessary for the construction of the solution to the posed problem. It is generally used for the determination of some additional conditions for obtaining analytical solutions, which is not relevant in this case.

Substituting the values $B(t)$, $C(t)$ and $\varphi(t)$ into the boundary conditions (10) to (12), one gets the following problem:

$$\mathrm{Re}\left[e^{-i\alpha(t)}\left(\frac{K}{2}t+\overline{\psi(t)}\right)\right]=\begin{cases}0, & t\in\Gamma_1\cup\Gamma_2\\ \frac{P}{2}, & t\in\Gamma_3\cup\Gamma4\end{cases} \tag{21}$$

$$e^{-\frac{\pi}{4}i}\left(\frac{K}{2}t+\overline{\psi(t)}\right)=\begin{cases}0, & t\in\gamma_1\\ -(1+i)\frac{P}{2}, & t\in\gamma_2\end{cases} \tag{22}$$

where

$$K=\frac{M_s}{D(1-\nu)},\quad P=\frac{M}{D(1-\nu)}. \tag{23}$$

Let $t\in\Gamma_j, j=1,2,3,4$. Then $t-A_j=i|t-A_j|e^{i\alpha_j}$. Hence we obtain

$$\mathrm{Re}\left(te^{-i\alpha(t)}\right)=\mathrm{Re}\left(e^{-i\alpha(t)}A(t)\right),$$

where $A(t)$ is a piecewise-constant function, $A(t)=A_j$, $t\in\Gamma_j, j=1,2,3,4$.

Thus if $t\in\Gamma_j, j=1,2,3,4$, then

$$\mathrm{Re}\left(te^{-i\alpha_j}\right)=\begin{cases}a, & j=1,2\\ 0, & j=3,4\end{cases}. \tag{24}$$

Taking into account (14), the conditions (21), (22) and (24) can be written as follows:

$$\mathrm{Re}\left[\frac{K}{2}e^{-\frac{\pi}{4}i}t+e^{\frac{\pi}{4}i}\psi(t)\right]=\begin{cases}0, & t\in\Gamma_1\cup\gamma_1\\ -\frac{P}{2}, & t\in\Gamma_3\cup\Gamma_2\end{cases}, \tag{25}$$

$$\mathrm{Im}\left[\frac{K}{2}e^{-\frac{\pi}{4}i}t-e^{\frac{\pi}{4}i}\psi(t)\right]=\begin{cases}0, & t\in\Gamma_2\cup\gamma_1\\ -\frac{P}{2}, & t\in\Gamma_4\cup\Gamma_2\end{cases}, \tag{26}$$

$$\mathrm{Re}\left[e^{-\frac{\pi}{4}i}t\right]=\begin{cases}a, & t\in\Gamma_1\\ 0, & t\in\Gamma_3\end{cases}, \tag{27}$$

$$\mathrm{Im}\left[e^{-\frac{\pi}{4}i}t\right]=\begin{cases}a, & t\in\Gamma_2\\ 0, & t\in\Gamma_4\end{cases}. \tag{28}$$

Combining equations (28) and (26) in the following way: K (28) – (26), we obtain

$$\mathrm{Im}\left[\frac{K}{2}e^{-\frac{\pi}{4}i}t+\psi(t)e^{\frac{\pi}{4}i}\right]=\begin{cases}Ka, & t\in\Gamma_2\\ \frac{P}{2}, & t\in\Gamma_4\end{cases}. \tag{29}$$

Analogously, from (25) and (27) we obtain

$$\mathrm{Re}\left[\frac{K}{2}e^{-\frac{\pi}{4}i}t-\psi(t)e^{\frac{\pi}{4}i}\right]=\begin{cases}aK, & t\in\Gamma_1\\ \frac{P}{2}, & t\in\Gamma_3\end{cases}. \tag{30}$$

Let $z=\omega(\zeta)$, $\zeta=\xi+i\eta$, conformally map the domain S onto the upper half-plane $\mathrm{Im}\zeta>0$. By a_j we denote the image of the point $A_j, j=1,\ldots,6$. Assume that $a_6=1$, $a_5=-1$, and the middle point of the arc A_2A_3 turns into $\zeta=\infty$. Since the domain is symmetric with respect to the axis OY, $-a_2=a_3$ and $-a_1=a_4$.

Denote

$$\Phi(\zeta)=\frac{K}{2}e^{-\frac{\pi}{4}i}\omega(\zeta)+e^{\frac{\pi}{4}i}\psi(\omega(\zeta)), \tag{31}$$

$$\Psi(\zeta)=\frac{K}{2}e^{-\frac{\pi}{4}i}\omega(\zeta)-e^{\frac{\pi}{4}i}\psi(\omega(\zeta)). \tag{32}$$

Taking into account (31) and (32), the boundary conditions (25), (26), (29) and (30) take the form

$$\Phi(\xi)+\overline{\Phi(\xi)}=\begin{cases}0, & \xi\in(-\infty,a_2)\cup(a_1,\infty)\\ -P, & \xi\in(-a_1,1)\end{cases}, \qquad \Phi(\xi)-\overline{\Phi(\xi)}=\begin{cases}2aKi, & \xi\in(-a_2,-a_1),\\ Pi, & \xi\in(1,a_1)\end{cases}, \tag{33}$$

$$\Psi(\xi)+\overline{\Psi(\xi)}=\begin{cases}2aK, & \xi\in(a_1,a_2),\\ P, & \xi\in(-a_1,-1)\end{cases}, \qquad \Psi(\xi)-\overline{\Psi(\xi)}=\begin{cases}0, & \xi\in(-\infty,-a_1)\cup(a_2,\infty),\\ -Pi, & \xi\in(-1,a_1)\end{cases}. \tag{34}$$

The solution to the boundary-value problem (33) and (34) is given by the Keldysh–Sedov formula [22]:

$$\Phi(\zeta)=\frac{X_1(\zeta)}{2\pi i}\left[\int_{-a_2}^{-a_1}\frac{2aKi\,d\xi}{X_1(\xi)(\xi-\zeta)}-\int_{-a_1}^{1}\frac{P\,d\xi}{X_1(\xi)(\xi-\zeta)}+\int_{1}^{a_1}\frac{Pi\,d\xi}{X_1(\xi)(\xi-\zeta)}-C\right], \tag{35}$$

$$\Psi(\zeta)=\frac{X_2(\zeta)}{2\pi i}\left[\int_{-a_1}^{-1}\frac{P\,d\xi}{X_2(\xi)(\xi-\zeta)}-\int_{-1}^{a_1}\frac{P_i\,d\xi}{X_2(\xi)(\xi-\zeta)}+\int_{a_1}^{a_2}\frac{2aK\,d\xi}{X_2(\xi)(\xi-\zeta)}+C_i\right], \qquad \mathrm{Im}\zeta>0, \tag{36}$$

where

$$X_1(\zeta)=\sqrt{\frac{(\zeta+a_2)(\zeta-1)}{(\zeta+a_1)(\zeta-a_1)}}, \qquad \mathrm{Im}\zeta>0, \tag{37}$$

$$X_2(\zeta)=\sqrt{\frac{(\zeta-a_2)(\zeta+1)}{(\zeta+a_1)(\zeta-a_1)}}, \qquad \mathrm{Im}\zeta>0. \tag{38}$$

By $X_1(\zeta)$ and $X_2(\zeta)$ we mean the branch of the holomorphic function in the upper half plane which at infinity takes the value

$$X_1(\infty)=X_2(\infty)=1.$$

It easy to show that

$$X_1(\xi)=\begin{cases}|X_1(\xi)|, & \xi\in(-\infty,-a_2)\cup(-a_1,1)\cup(a_1,\infty),\\ -i|X_1(\xi)|, & \xi\in(-a_2,-a_1)\cup(1,a_1)\end{cases}, \tag{39}$$

$$X_2(\xi)=\begin{cases}|X_2(\xi)|, & \xi\in(-\infty,-a_1)\cup(-1,a_1)\cup(a_2,\infty),\\ i|X_2(\xi)|, & \xi\in(-a_1,-1)\cup(a_1,a_2)\end{cases}, \tag{40}$$

$$|X_1(\xi)|=|X_2(-\xi)|. \tag{41}$$

Taking into account (39) and (40), the formulas (35) and (36) can be written as follows:

$$\Phi(\zeta)=\frac{X_1(\zeta)i}{2\pi}\left[\int_{-a_2}^{-a_1}\frac{2aK\,d\xi}{|X_1(\xi)|(\xi-\zeta)}+\int_{-a_1}^{a_1}\frac{P\,d\xi}{|X_1(\xi)|(\xi-\zeta)}+C\right], \qquad \mathrm{Im}\zeta>0, \tag{42}$$

$$\Psi(\zeta)=-\frac{X_2(\zeta)}{2\pi}\left[\int_{-a_2}^{-a_1}\frac{2aK\,d\xi}{|X_2(\xi)|(\xi-\zeta)}+\int_{-a_1}^{a_1}\frac{P\,d\xi}{|X_2(\xi)|(\xi-\zeta)}-C\right], \qquad \mathrm{Im}\zeta>0. \tag{43}$$

4. Definition of deflection

Since all mechanical values such as deflection, bending moments, etc. are defined by complex potentials $\varphi(z)$, $\psi(z)$, solutions can be obtained in a straightforward way.

To obtain a numerical solution to the plate deflection $w(x,y)=\mathrm{Re}[\bar{z}\varphi(z)+\chi(z)]$, where $\varphi(z)$ and $\psi(z)$ are holomorphic functions in S ($\psi(z)=\chi'(z)$, that is, $\chi(z)=\int\psi(z)\,dz+C$, $C=\text{const}$), we consider a conformal mapping function $z=\omega(\zeta)$, $\zeta=\xi+i\eta$, which maps the domain S onto the upper half-plane $\mathrm{Im}\zeta>0$. By virtue of (20) and (31) the obtained solutions $\varphi(z)$ and $\psi(z)$ are defined as functions of variable ζ:

$$\varphi(z)=\varphi(\omega(\zeta))=-\frac{M_0}{4D(\nu+1)}\omega(\zeta),$$

$$\psi(z)=\psi(\omega(\zeta))=\left(\Phi(\zeta)-\frac{K}{2}e^{-\frac{\pi}{4}i}\omega(\zeta)\right)e^{-\frac{\pi}{4}i}.$$

We calculate these functions $\varphi(z) = \varphi(\omega(\zeta))$, $\psi(z) = \psi(\omega(\zeta))$ at the points ζ. In the half-plane, if we choose these points ζ sufficiently close to each other, then the points z (where $z = \omega(\zeta)$) of the considered domain S corresponding to them will also be close to each other. Therefore we can calculate values of deflection bending moments of the considered body at any point with high accuracy, since mapping functions $z = \omega(\zeta)$, $\varphi(z) = \varphi(\omega(\zeta))$ and $\psi(z) = \psi(\omega(\zeta))$ are analytical.

Taking into account (20), (31) and $\bar{z} \cdot z = |z|^2$, where $z = \omega(\zeta)$, the plate deflection has the form

$$w(x, y) = \mathrm{Re}\left[-\frac{-M_0|\omega(\zeta)|^2}{4D(\nu+1)} + \chi(\omega(\zeta))\right],$$

where $x = \mathrm{Re}z = \mathrm{Re}(\omega(\zeta))$, $y = \mathrm{Im}z = \mathrm{Im}(\omega(\zeta))$ and

$$\chi(\omega(\zeta)) = \int \psi(\omega(\zeta))\omega'(\zeta)\,d\zeta + C_1 = \int \left(\Phi(\zeta) - \frac{K}{2}e^{-\frac{\pi}{4}i}\omega(\zeta)\right) e^{-\frac{\pi}{4}i}\omega'(\zeta)\,d\zeta + C_1,$$

where $\Phi(\zeta)$, $\omega(\zeta)$ and K are defined by (42), (49) and (47) correspondingly. Thus, one obtains

$$w(x, y) = \mathrm{Re}\left[-\frac{M_0|\omega(\zeta)|^2}{4D(\nu+1)} + \int \left(\Phi(\zeta) - \frac{K}{2}e^{-\frac{\pi}{4}i}\omega(\zeta)\right) e^{-\frac{\pi}{4}i}\omega'(\zeta)\,d\zeta + C_1\right]$$

where $C_1 = \mathrm{const}$.

5. Construction of unknown parts of the boundary

Since the functions $X_1(\zeta)$ and $X_2(\zeta)$ have singularities of order 0.5 at the points $\xi = \pm a_1$, for the functions $\Phi(\zeta)$ and $\Psi(\zeta)$ to be bounded near the points a_1 and $-a_1$ it is necessary and sufficient for the following conditions to be fulfilled:

$$2aK\int_{-a_2}^{-a_1} \frac{d\xi}{|X_1(\xi)|(\xi \pm a_1)} + P\int_{-a_1}^{a_1} \frac{d\xi}{|X_1(\xi)|(\xi \pm a_1)} + C = 0, \tag{44}$$

$$2aK\int_{a_1}^{a_2} \frac{d\xi}{|X_2(\xi)|(\xi \pm a_1)} + P\int_{-a_1}^{a_1} \frac{d\xi}{|X_2(\xi)|(\xi \pm a_1)} - C = 0. \tag{45}$$

If we replace ξ by $-\xi$ in condition (45) and take into account condition (41), we will obtain the condition which coincides with condition (44). This condition is in fact the system of two equations involving unknown parameters a_1, a_2, C and K:

$$\begin{cases} 2aK\int_{-a_2}^{-a_1} \frac{d\xi}{|X_1(\xi)|(\xi - a_1)} + P\int_{-a_1}^{a_1} \frac{d\xi}{|X_1(\xi)|(\xi - a_1)} + C = 0, \\ 2aK\int_{-a_2}^{-a_1} \frac{d\xi}{|X_1(\xi)|(\xi + a_1)} + P\int_{-a_1}^{a_1} \frac{d\xi}{|X_1(\xi)|(\xi + a_1)} + C = 0. \end{cases} \tag{46}$$

Solving the system with respect to K and C we obtain

$$K = \frac{P}{2a}\frac{\int_{-a_1}^{a_1} \frac{d\xi}{|X_1(\xi)|(a_1^2 - \xi^2)}}{\int_{-a_2}^{-a_1} \frac{d\xi}{|X_1(\xi)|(\xi^2 - a_1^2)}}, \tag{47}$$

$$C = \int_{-a_2}^{-a_1} \frac{2aK\,d\xi}{|X_1(\xi)|(a_1 - \xi)} + P\int_{-a_1}^{a_1} \frac{d\xi}{|X_1(\xi)|(a_1 - \xi)}. \tag{48}$$

Conditions (47) and (48) are valid for arbitrary parameters a_1, a_2, $1 < a_1 < a_2 < \infty$, that is, conditions (44) and (45) are valid for these conditions.

Hence we conclude that boundary problems (33) and (34) have infinite solutions, which are given by formulas (42) and (43). From equalities (31) and (32) we obtain

$$\omega(\zeta) = \frac{\Phi(\zeta) + \Psi(\zeta)}{K}e^{\frac{\pi}{4}i}, \qquad \mathrm{Im}\zeta > 0. \tag{49}$$

Passing in formulas (42) and (43) to the limit $\zeta \to \xi_0 \in (-1, 1)$ and using the Sokhotski–Plemelj formulas, we obtain

$$\Phi(\xi_0) = i\Phi_0(\xi_0) - \frac{P}{2}, \qquad \Psi(\xi_0) = \Psi_0(\xi_0) - \frac{P}{2}i, \tag{50}$$

where

$$\Phi_0(\xi_0) = \frac{X_1(\xi_0)}{2\pi}\left[\int_{-a_2}^{-a_1} \frac{2aK\,d\xi}{|X_1(\xi)|(\xi - \xi_0)} + \int_{-a_1}^{a_1} \frac{P\,d\xi}{|X_1(\xi_0)|(\xi - \xi_0)} + C\right], \qquad \xi_0 \in (-1, 1), \tag{51}$$

$$\Psi_0(\xi_0) = -\frac{X_2(\xi_0)}{2\pi}\left[\int_{a_1}^{a_2} \frac{2aK\,d\xi}{|X_2(\xi)|(\xi - \xi_0)} + \int_{-a_1}^{a_1} \frac{P\,d\xi}{|X_2(\xi)|(\xi - \xi_0)} - C\right], \qquad \xi_0 \in (-1, 1). \tag{52}$$

The integrals

$$\int_{-1}^{1} \frac{P\,d\xi}{|X_1(\xi)|(\xi - \xi_0)}$$

and

$$\int_{-1}^{1} \frac{P\,d\xi}{|X_2(\xi)|(\xi - \xi_0)}$$

included in equalities (51) and (52) are singular and exist in the Cauchy principal value sense.

Replacing ξ_0 by $-\xi_0$ and ξ by $-\xi$ in equality (51), and using (41), we get

$$\Phi_0(-\xi_0) = \Psi_0(\xi_0), \qquad \xi_0 \in (-1, 1). \tag{53}$$

The equation of arc γ_2 is given by formula

$$t = \omega(\xi_0) = \frac{\Phi(\xi_0) + \Psi_0(\xi_0)}{K}, \qquad \xi_0 \in (-1, 1). \tag{54}$$

Inserting in the last formula the values $\Phi(\xi_0)$ and $\Psi(\xi_0)$ defined by equality (50) and taking into account (53), we find

$$t = \omega(\xi_0) = \frac{\Phi(-\xi_0) - \Phi_0(\xi_0)}{2K}\sqrt{2} + i\frac{\Phi(\xi_0) + \Phi_0(-\xi_0) - P}{2K}\sqrt{2}, \qquad \xi_0 \in (-1, 1). \tag{55}$$

Hence, it can be easily seen that the arc γ_2 is symmetric with respect to the OY-axis. Analogously, the equation of arc γ_1 is given by the formula

$$t = \omega(\xi_0) = \frac{\Phi(-\xi_0) - \Phi_0(\xi_0)}{2K}\sqrt{2} + i\frac{\Phi(\xi_0) + \Phi_0(-\xi_0)}{2K}\sqrt{2}, \qquad \xi_0 \in (-\infty, a_2) \cup (a_2, +\infty).$$

$$\Phi_0(\xi_0) = \frac{X_1(\xi_0)}{2\pi}\left[\int_{-a_2}^{-a_1} \frac{2aK\,d\xi}{|X_1(\xi)|(\xi - \xi_0)} + \int_{-a_1}^{a_1} \frac{P\,d\xi}{|X_1(\xi)|(\xi - \xi_0)} + C\right], \qquad \xi_0 \in (-\infty, -a_2) \cup (a_2, +\infty). \tag{56}$$

Taking into account equality (9), from equality (20) we obtain that the shearing force $N_n = 0$, in other words, all the conditions on the boundary are fulfilled.

The graph for the arc γ_2 is constructed by $\omega_1(\xi_0) = \omega(\xi_0)$ for $\xi_0 \in (-1, 1)$, and graphs for the arc γ_1 by $\omega_2(\xi_0) = \omega(\xi_0)$ for $\xi_0 \in (a_2, +\infty)$ and $\omega_3(\xi_0) = \omega(\xi_0)$ for $\xi_0 \in (-\infty, -a_2)$. Since the problem is cyclically symmetric, the remaining parts of the full-strength contour are constructed by rotating the graph $\omega(\xi_0)$ by value $\pi/2$. Values K and C are defined from formulas (47) and (48) for given P.

Let the width of the plate be $h = 0.1$, the Poisson coefficient be $\nu = 1/3$ and Young's modulus be $E = 5 \times 10^5$. Then the cylindrical rigidity will be

$$D = \frac{Eh^3}{12(1 - \nu^2)} = 46.875.$$

In Figure 2, the graphs of a full-strength boundary for different parameter selections are constructed.

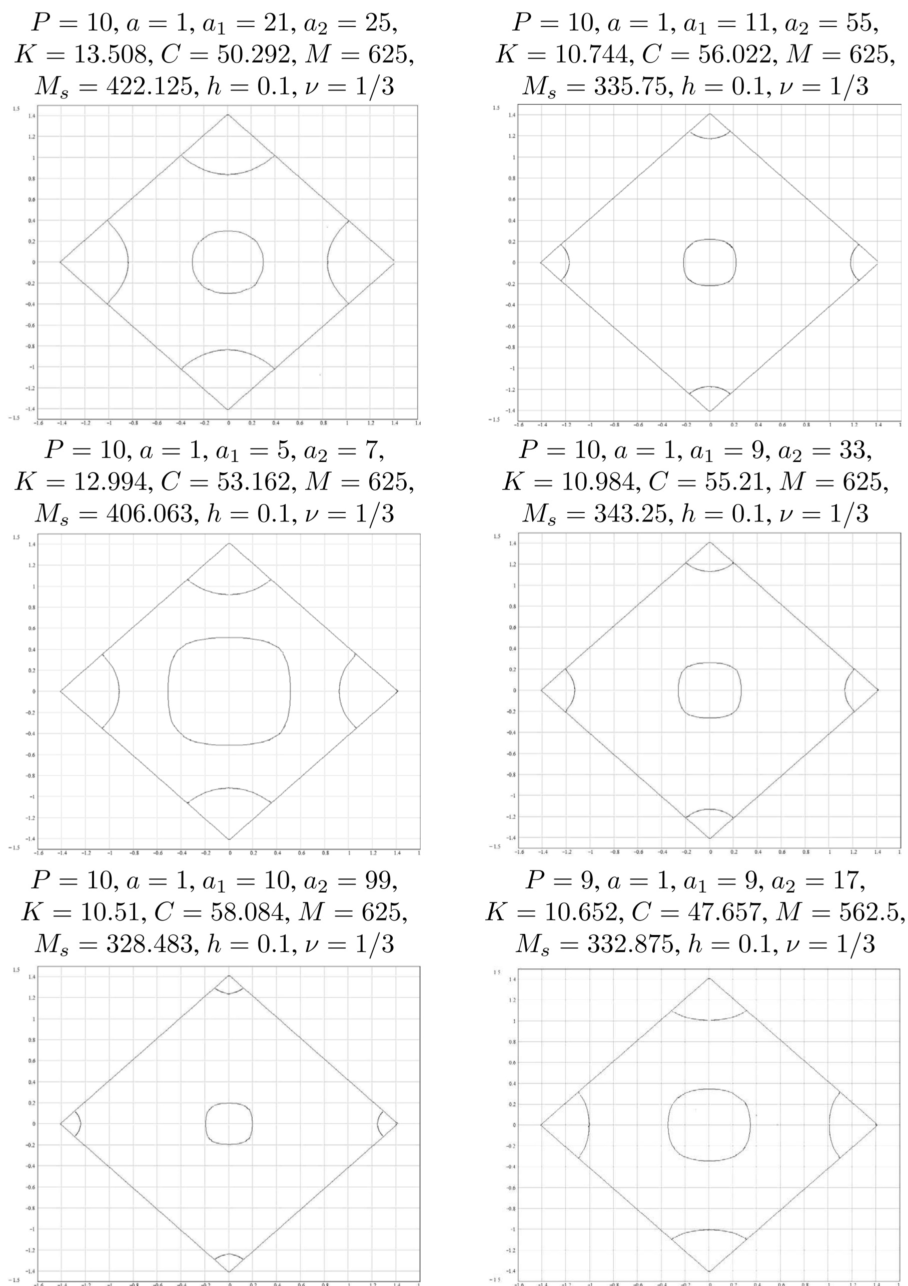

Figure 2. Graphs of a full-strength boundary for different parameter selections.

6. Conclusion

The most effective methods for studying the problem posed here turned out to be those of the theory of analytical functions of complex variables. Kolosov–Muskhelishvili formulas are used for investigating and the solution is written in quadratures. The unknown full-strength parts of the plate boundary are constructed. This problem is both of mechanics and of geometry since we search the shape of a hole in a plate and the conformal mapping function is used to define it.

The applications of elastic plates weakened by full-strength holes are of great interest in several areas of mechanical construction (building practice, mechanical engineering, shipbuilding, aircraft construction, etc).

Cherepanov [3, 4] proved that the tangential–normal stresses of a plate weakened by a hole with a full-strength contour is less than 40% of the maximum tangential–normal stress of a plate weakened by a circular hole.

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

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