WEIGHTED WEAK TYPE INEQUALITIES FOR MODIFIED HARDY OPERATORS AND GEOMETRIC MEANS OPERATORS IN DIMENSIONS ONE AND GREATER

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Abstract. We characterize the pairs of weights (u, v) such that the geometric mean operator G_1, defined for positive functions f on $(0, \infty)$ by

$$G_1 f(x) = \exp\left(\frac{1}{x}\int_0^x \log f\right),$$

verifies the weak type inequality

$$\left(\int_{\{x\in(0,\infty)/G_1f(x)>\lambda\}} u\right)^{\frac{1}{q}} \le \frac{C}{\lambda}\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}$$

in the case $0 < p \le q < \infty$.

Similar results are obtained for the n-dimensional geometric mean operator G_n defined by

$$G_n f(x_1, x_2, \ldots, x_n) = \exp\left(\frac{1}{x_1 x_2 \cdots x_n}\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} \log f\right).$$

1. Introduction and results

Let G_1 be the geometric mean operator defined for positive functions f on $(0, \infty)$ by

$$G_1 f(x) = \exp\left(\frac{1}{x}\int_0^x \log f\right).$$

P. Gurka and B. Opic [4] in the case $0 < p \le q < \infty$ and B. Opic and L. Pick [8] in the case $0 < q < p < \infty$ characterized the pairs of weights (u, v) such that the strong type inequality

$$\left(\int_0^\infty G_1 f(x)^q u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p v\right)^{\frac{1}{p}} \tag{1.1}$$

holds.

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More recently, L. E. Persson and V. D. Stepanov ([5], [6], [9]) have given a new characterization of the inequality (1.1) by applying a limiting process. This technique uses the fact that

$$G_1f(x)=\lim_{\alpha\to 0^+}\left(\frac{1}{x}\int_0^x f^\alpha\right)^{\frac{1}{\alpha}} \tag{1.2}$$

and needs to have a characterization of the weighted Hardy inequality

$$\left(\int_0^\infty\left(\frac{1}{x}\int_0^x f\right)^q u(x)dx\right)^{\frac{1}{q}}\leq C\left(\int_0^\infty f^pv\right)^{\frac{1}{p}}$$

provided with a suitable control of the best constant C.

A similar approach has been performed by A. Wedestig ([11], [12]) in relation to the characterization of the pairs of weights $(u(x,y),v(x,y))$ such that the inequality

$$\left(\int_0^\infty\int_0^\infty G_2f(x,y)^q u(x,y)dxdy\right)^{\frac{1}{q}}\leq C\left(\int_0^\infty\int_0^\infty f^pv\right)^{\frac{1}{p}}$$

holds, where $0<p\leq q<\infty$ and G_2 is the two-dimensional geometric mean operator defined by

$$G_2f(x,y)=\exp\left(\frac{1}{xy}\int_0^x\int_0^y\log f(s,t)dsdt\right).$$

The first purpose of this paper is to characterize the pairs of weights (u,v) such that the weak type inequality

$$\left(\int_{\{x\in(0,\infty):G_1f(x)>\lambda\}}u\right)^{\frac{1}{q}}\leq\frac{C}{\lambda}\left(\int_0^\infty f^pv\right)^{\frac{1}{p}} \tag{1.3}$$

holds for all $\lambda>0$ and all $f>0$ with a constant independent of f and λ, in the case $0<p\leq q<\infty$.

The inequality (1.3) can be written as

$$\|G_1f\|_{q,\infty;u}\leq C\|f\|_{p;v},$$

which expresses the boundedness of G_1 from $L^p(v)$ to $L^{q,\infty}(u)$. If $r>0$, the space $L^{r,\infty}(u)$ is defined by

$$L^{r,\infty}(u)=\{g:(0,\infty)\to\mathbf{R}:\|g\|_{r,\infty;u}<\infty\},$$

where

$$\|g\|_{r,\infty;u}=\sup_{\lambda>0}\lambda\left(\int_{\{x\in(0,\infty):|g(x)|>\lambda\}}u\right)^{\frac{1}{r}}.$$

It is well known that $\|\cdot\|_{r,\infty;u}$ is not a norm. If $r>1$, the space $L^{r,\infty}(u)$ is a Banach function space [2] with the norm

$$\|f\|^*_{r,\infty;u}=\sup_{t>0}t^{\frac{1}{r}}f^{**}_u(t),$$

where $f_u^{**}(t) = \frac{1}{t}\int_0^t f_u^*$ and f_u^*, the nonincreasing rearrangement of f with respect to the measure u, is defined by

$$f_u^*(t) = \inf\left\{ s > 0 : \int_{\{x:|f(x)|>s\}} u \leq t \right\}.$$

The relationship between $\|\cdot\|_{r,\infty;u}$ and $\|\cdot\|^*_{r,\infty;u}$ is given by the inequalities

$$\|f\|_{r,\infty;u} \leq \|f\|^*_{r,\infty;u} \leq r'\|f\|_{r,\infty;u}, \tag{1.4}$$

where $r' = \frac{r}{r-1}$ is the conjugate exponent of r.

Moreover, the associate space of $L^{r,\infty}(u)$ is $L^{r',1}(u)$, which consists of the functions f such that

$$\|f\|_{r',1;u} = \int_0^\infty \left(\int_{\{x\in(0,\infty):|f(x)|>t\}} u \right)^{\frac{1}{r'}} dt < \infty.$$

In order to characterize the inequality (1.3), we will perform a limiting process similar to Wedestig's one ([11], [12]), but conveniently adapted to the setting of the weak type inequalities. We will need a sort of weak type Minkowski's inequality which will allow us the permutation of the integral with the $L^{r,\infty}(u)$-norm, $r \geq 1$.

The weak type Minkowski's inequality is established in the following lemma.

Lemma 1. *Let $r \geq 1$ and let g, f, u be nonnegative measurable functions on $(0,\infty)$. Then,*

$$\left\| g(x)\int_0^x f \right\|_{r,\infty;u} \leq D_r \int_0^\infty f(t)\|\chi_{(t,\infty)}g\|_{r,\infty;u} dt,$$

where $D_r = r'$ if $r > 1$ and $D_1 = 4$.

It is worth noting that the above inequality asserts that if u is a positive function on $(0,\infty)$, then the pair of weights $(u(t), \|\chi_{(t,\infty)}g\|_{r,\infty;u})$ is always a good pair for the weighted weak type $(1,r)$ inequality for the modified Hardy operator $Tf(x) = g(x)\int_0^x f$ to hold.

Really, instead of Lemma 1, we will use the following lemma, which is an immediate consequence of Lemma 1.

Lemma 2. *Let $1 < p \leq q < \infty$ and let g, f, u be nonnegative functions on $(0,\infty)$. Then*

$$\left\| g(x)\left(\int_0^x f\right)^{\frac{1}{p}} \right\|^p_{q,\infty;u} \leq D_{p,q} \int_0^\infty f(t)\|\chi_{(t,\infty)}g\|^p_{q,\infty;u} dt,$$

where $D_{p,q} = \frac{q}{q-p}$ if $p < q$ and $D_{p,p} = 4$.

We will apply Lemma 2 in order to obtain a suitable characterization of the weighted weak type inequality for the modified Hardy operator. It is contained in the following theorem.

Theorem 1. *Let $1 < p \leq q < \infty$. Let u, v and h be nonnegative functions on $(0,\infty)$. Let $s_0 \in (1,p)$. The following statements are equivalent:*

(1) *There exists* $C > 0$ *such that the inequality*

$$\left\| h(x) \int_0^x f \right\|_{q,\infty;u} \leq C \|f\|_{p,v} \tag{1.5}$$

holds for all positive functions f.

(2) $B_{s_0}(p,q,u,v,h) < \infty$, *where*

$$B_{s_0}(p,q,u,v,h) = \sup_{t \in (0,\infty)} V(t)^{\frac{s_0-1}{p}} \|\chi_{(t,\infty)} h V^{\frac{p-s_0}{p}}\|_{q,\infty;u}$$

and

$$V(x) = \int_0^x v^{1-p'}.$$

Moreover, if C *is the best constant in (*1.5*), then*

$$B_{s_0}(p,q,u,v,h) \left(\frac{\left(\frac{p}{p-s_0}\right)^p}{\left(\frac{p}{p-s_0}\right)^p + \frac{1}{s_0-1}} \right)^{\frac{1}{p}} \leq C \leq \left(\frac{p-1}{p-s_0}\right)^{\frac{1}{p'}} C_{p,q} B_{s_0}(p,q,u,v,h),$$

where $C_{p,q} = \left(\dfrac{q}{q-p}\right)^{\frac{1}{p}}$ *if* $p < q$ *and* $C_{p,p} = 4^{\frac{1}{p}}$.

Observe that the theorem provides with a scale of characterizing conditions, one for each $s_0 \in (1,p)$.

The weighted weak type inequalities for modified Hardy operators has already been characterized by K. Andersen and B. Muckenhoupt [1], E. Ferreyra [3] and F. J. Martín-Reyes, P. Ortega and M. D. Sarrión [7]. However, the characterizations did not provide us with a suitable control of the best constant.

By applying (1.2) and Theorem 1, we obtain the result for G_1:

Theorem 2. *Let* $0 < p \leq q < \infty$. *Let* u *be a nonnegative function on* $(0,\infty)$ *and let* v *be a positive function on* $(0,\infty)$. *Let* $s_0 > 1$. *The following statements are equivalent:*

(1) *There exists* $C > 0$ *such that the inequality (*1.3*) holds for all positive functions* f *and all* $\lambda > 0$.

(2) $B_{\exp,s_0}(p,q,u,v) < \infty$, *where*

$$B_{\exp,s_0}(p,q,u,v) = \sup_{t \in (0,\infty)} t^{\frac{s_0-1}{p}} \|\chi_{(t,\infty)}(x) w(x) x^{\frac{-s_0}{p}}\|_{q,\infty;u}$$

and $w = G_1(v^{-\frac{1}{p}})$.

Moreover, if C *is the best constant in (*1.3*), then*

$$B_{\exp,s_0}(p,q,u,v) \left(1 + \frac{e^{-s_0}}{s_0-1}\right)^{-\frac{1}{p}} \leq C \leq e^{\frac{s_0-1}{p}} C_{p,q} B_{\exp,s_0}(p,q,u,v).$$

By applying Theorem 2 and performing tedious calculations, we can characterize the pairs of power weights for the weak type inequality (1.3) to hold.

Theorem 3. *Let $0 < p \le q < \infty$. If $u(x) = x^\alpha$ and $v(x) = x^\beta$, the inequality (1.3) holds for all $f > 0$ and all $\lambda > 0$ with a constant independent of f and λ if and only if $\frac{\alpha+1}{q} = \frac{\beta+1}{p}$.*

The techniques we apply for proving Theorems 1 and 2 also work in higher dimensions. Let G_n be the geometric mean operator defined for positive f on $(0,\infty)^n$ by

$$G_n f(x_1, x_2, \dots, x_n) = \exp\left(\frac{1}{x_1 x_2 \cdots x_n}\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n}\log f\right).$$

We will be able to characterize the pairs of weights (u, v) on $(0,\infty)^n$ such that the inequality

$$\left(\int_{\{(x_1,x_2,\dots,x_n)\in(0,\infty)^n : G_n f(x_1,x_2,\dots,x_n) > \lambda\}} u\right)^{\frac{1}{q}} \le \frac{C}{\lambda}\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty f^p v\right)^{\frac{1}{p}} \tag{1.6}$$

holds in the case $0 < p < q < \infty$.

We will perform a limiting process similar to the one used in Theorem 1. We will need a n- dimensional weak type Minkowski's inequality and the characterization of the weighted weak type inequality

$$\begin{aligned}&\left(\int_{\{(x_1,x_2,\dots,x_n)\in(0,\infty)^n : Tf(x_1,x_2,\dots,x_n) > \lambda\}} u\right)^{\frac{1}{q}}\\ &\le \frac{C}{\lambda}\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty f^p(x_1,x_2,\dots,x_n)\Pi_{i=1}^n v_i(x_i)\right)^{\frac{1}{p}}\end{aligned} \tag{1.7}$$

in the case $1 < p < q < \infty$, where T stands for the n-dimensional modified Hardy operator defined by

$$Tf(x_1, x_2, \dots, x_n) = h(x_1, x_2, \dots, x_n)\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} f.$$

The above mentioned n-dimensional weak type Minkowski's inequality reads as follows:

Lemma 3. *Let $1 < p < q < \infty$ and let g, f, u be nonnegative functions on $(0,\infty)^n$. Then*

$$\begin{aligned}&\left\| g(x_1,x_2,\dots,x_n)\left(\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} f\right)^{\frac{1}{p}}\right\|_{q,\infty;u}^p\\ &\le D_{p,q}\int_0^\infty\int_0^\infty\cdots\int_0^\infty f(t_1,t_2,\dots,t_n)\|\chi_{\Pi_{i=1}^n(t_i,\infty)}g\|_{q,\infty;u}^p dt_1 dt_2 \dots dt_n.\end{aligned}$$

The inequality (1.7) has independent interest. E. Sawyer characterized in a celebrated paper [10] the two-dimensional weighted weak type inequality

$$\left(\int_{\left\{(x,y)\in(0,\infty)^2 : \int_0^x\int_0^y f > \lambda\right\}} u\right)^{\frac{1}{q}} \le \frac{C}{\lambda}\left(\int_0^\infty\int_0^\infty f^p v\right)^{\frac{1}{p}}$$

in the case $1 < p \leq q < \infty$. The characterization involved two independent conditions. Moreover, it is not clear that the techniques used by Sawyer could be applied in dimensions greater than two.

Our next Theorem characterizes the inequality (1.7). Observe that the weight in the starting space is of product type: $v(x_1, x_2, \dots, x_n) = v_1(x_1)v_2(x_2)\cdots v_n(x_n)$. For such a particular case, our Theorem improves Sawyer's result in a triple sense: it is a n-dimensional result, it refers to modified Hardy operators and there is only one characterizing condition.

The theorem reads as follows:

Theorem 4. *Let $1 < p < q < \infty$. Let u be a nonnegative function on $(0,\infty)^n$ and let $v_1, v_2, \cdots, v_n$ be nonnegative functions on $(0,\infty)$. Let $s_1, s_2, \dots, s_n \in (1, p)$. The following statements are equivalent:*

1. *There exists $C > 0$ such that the inequality (1.7) holds for all positive functions f and all $\lambda > 0$.*
2. *$B_{s_1,s_2,\dots,s_n}(p,q,u,v_1,v_2,\dots,v_n,h) < \infty$, where*

$$B_{s_1,s_2,\dots,s_n}(p,q,u,v_1,v_2,\dots,v_n,h) = \sup_{(t_1,t_2,\dots,t_n)\in(0,\infty)^n} \Pi_{i=1}^n V_i(t_i)^{\frac{s_i-1}{p}} \|\chi_{\Pi_{i=1}^n(t_i,\infty)} h \Pi_{i=1}^n V_i(x_i)^{\frac{p-s_i}{p}}\|_{q,\infty;u}$$

and

$$V_i(x_i) = \int_0^{x_i} v_i^{1-p'}.$$

Moreover, if C is the best constant in (1.7), then

$$B_{s_1,s_2,\dots,s_n} \prod_{i=1}^n \left(\frac{\left(\frac{p}{p-s_i}\right)^p}{\left(\frac{p}{p-s_i}\right)^p + \frac{1}{s_i-1}} \right)^{\frac{1}{p}} \leq C \leq \prod_{i=1}^n \left(\frac{p-1}{p-s_i}\right)^{\frac{1}{p'}} C_{p,q} B_{s_1,s_2,\dots,s_n}.$$

From Theorem 4 and the fact that

$$G_n f(x) = \lim_{\alpha\to 0^+} \left(\frac{1}{x_1 x_2 \cdots x_n} \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} f^\alpha \right)^{\frac{1}{\alpha}}, \tag{1.8}$$

we can prove the following Theorem.

Theorem 5. *Let $0 < p < q < \infty$. Let u be a nonnegative function on $(0,\infty)^n$ and let v be a positive function on $(0,\infty)^n$. Let $s_1, s_2, \dots, s_n > 1$. The following statements are equivalent:*

1. *There exists $C > 0$ such that the inequality (1.6) holds for all positive functions f and all $\lambda > 0$.*
2. *$B_{\exp,s_1,s_2,\dots,s_n}(p,q,u,v) < \infty$, where*

$$B_{\exp,s_1,s_2,\dots,s_n} = \sup_{t_1,t_2,\dots,t_n\in(0,\infty)^n} \Pi_{i=1}^n t_i^{\frac{s_i-1}{p}} \|\chi_{\Pi_{i=1}^n(t_i,\infty)} w \Pi_{i=1}^n x_i^{\frac{-s_i}{p}}\|_{q,\infty;u}$$

and $w = G_n(v^{-\frac{1}{p}})$.

Moreover, if C is the best constant in (1.6), then

$$B_{\exp,s_1,s_2,\dots,s_n}\prod_{i=1}^{n}\left(1+\frac{e^{-s_i}}{s_i-1}\right)^{-\frac{1}{p}} \le C \le \Pi_{i=1}^{n} e^{\frac{s_i-1}{p}} C_{p,q} B_{\exp,s_1,s_2,\dots,s_n}.$$

The paper is organized as follows. Section 2 is devoted to the proof of the lemmas. The sections 3, 4, 5 and 6 contain the proofs of Theorems 1, 2, 4 and 5, respectively.

2. Proofs of the Lemmas

We only prove Lemmas 1 and 2. The proof of Lemma 3 is similar.

Proof of Lemma 1. Suppose first that $r > 1$. By the definition of the associate space and Fubini's Theorem,

$$\left\| g(x)\int_0^x f \right\|^*_{r,\infty;u} = \sup_{\|h\|_{r',1;u}\le 1}\int_0^\infty g(x)\left(\int_0^x f(t)dt\right)h(x)u(x)dx$$

$$= \sup_{\|h\|_{r',1;u}\le 1}\int_0^\infty f(t)\left(\int_t^\infty g(x)h(x)u(x)dx\right)dt \le \int_0^\infty f(t)\|\chi_{(t,\infty)}g\|^*_{r,\infty;u}dt.$$

Then, (1.4) yields

$$\left\| g(x)\int_0^x f \right\|_{r,\infty;u} \le r'\int_0^\infty f(t)\|\chi_{(t,\infty)}g\|_{r,\infty;u}dt.$$

The above argument does not work in the case $r = 1$. The result in this case is a consequence of a theorem due to F. J. Martín-Reyes, P. Ortega and M. D. Sarrión. They showed in [7] that the couples of weights (u, v) such that the weak type inequality

$$\left\| g(x)\int_0^x f \right\|_{1,\infty;u} \le C\int_0^\infty fv \tag{2.1}$$

holds for all positive f are those that satisfy

$$J = \sup_{0<t<\infty}\|g\chi_{(t,\infty)}\|_{1,\infty;u}\operatorname*{ess\,sup}_{x\in(0,t)} v^{-1}(x) < \infty. \tag{2.2}$$

Moreover, they proved that the best constant C in the inequality (2.1) verifies $C \le 4J$.

It is immediate that the couple $(u(t), \|\chi_{(t,\infty)}g\|_{1,\infty;u})$ satisfies the condition (2.2) with $J \le 1$. Therefore

$$\left\| g(x)\int_0^x f \right\|_{1,\infty;u} \le 4\int_0^\infty f(t)\|\chi_{(t,\infty)}g\|_{1,\infty;u}dt,$$

as we wished to prove. □

Proof of Lemma 2. By the definition of $\|\cdot\|_{q,\infty;u}$ and Lemma 1 with $r=\frac{q}{p}$, we have

$$\begin{aligned}\left\|g(x)\left(\int_0^x f\right)^{\frac{1}{p}}\right\|_{q,\infty;u}^p &= \sup_{\lambda>0}\lambda^p\left(\int_{\left\{x\in(0,\infty):g(x)^p\int_0^x f>\lambda^p\right\}}u\right)^{\frac{p}{q}}\\ &= \left\|g(x)^p\int_0^x f\right\|_{\frac{q}{p},\infty;u}\le D_{p,q}\int_0^\infty f(t)\|\chi_{(t,\infty)}g^p\|_{\frac{q}{p},\infty;u}dt\\ &= D_{p,q}\int_0^\infty f(t)\|\chi_{(t,\infty)}g\|_{q,\infty;u}^p dt.\end{aligned}$$

□

3. Proof of Theorem 1

(1) ⇒ (2) Suppose that (1.5) holds. It is equivalent to

$$\left(\int_{\left\{x\in(0,\infty):h(x)\int_0^x g^{\frac{1}{p}}v^{-\frac{1}{p}}>\lambda\right\}}u\right)^{\frac{1}{q}}\le\frac{C}{\lambda}\left(\int_0^\infty g\right)^{\frac{1}{p}}. \tag{3.1}$$

Let $t\in(0,\infty)$. Let g be the function defined by

$$g(x)=\left(\frac{p}{p-s_0}\right)^p V(t)^{-s_0}v(x)^{1-p'}\chi_{(0,t)}(x)+V(x)^{-s_0}v(x)^{1-p'}\chi_{(t,\infty)}(x).$$

For all $x\in(t,\infty)$,

$$\begin{aligned}h(x)\int_0^x g^{\frac{1}{p}}v^{-\frac{1}{p}} &= h(x)\int_0^t g^{\frac{1}{p}}v^{-\frac{1}{p}}+h(x)\int_t^x g^{\frac{1}{p}}v^{-\frac{1}{p}}\\ &= h(x)\frac{p}{p-s_0}V(t)^{-\frac{s_0}{p}}\int_0^t v^{1-p'}+h(x)\int_t^x V^{-\frac{s_0}{p}}(z)v^{1-p'}(z)dz\\ &= \frac{p}{p-s_0}h(x)V(x)^{\frac{p-s_0}{p}}.\end{aligned}$$

On the other hand, the integral on the right-hand side of (3.1) can be estimated as follows:

$$\begin{aligned}\int_0^\infty g &= \left(\frac{p}{p-s_0}\right)^p V(t)^{-s_0}\int_0^t v^{1-p'}+\int_t^\infty V^{-s_0}(z)v(z)^{1-p'}(z)dz\\ &\le\left[\left(\frac{p}{p-s_0}\right)^p+\frac{1}{s_0-1}\right]V(t)^{1-s_0}.\end{aligned}$$

Hence, (3.1) implies

$$\left(\int_{\left\{x\in(t,\infty):\frac{p}{p-s_0}h(x)V(x)^{\frac{p-s_0}{p}}>\lambda\right\}}u\right)^{\frac{1}{q}}\le\frac{C}{\lambda}\left[\left(\frac{p}{p-s_0}\right)^p+\frac{1}{s_0-1}\right]^{\frac{1}{p}}V(t)^{\frac{1-s_0}{p}}$$

for all $\lambda>0$.

By the definition of $\|\cdot\|_{q,\infty;u}$, the above inequality gives

$$\left\|\chi_{(t,\infty)}(x)h(x)V(x)^{\frac{p-s_0}{p}}\right\|_{q,\infty;u}V(t)^{\frac{s_0-1}{p}}\left(\frac{p}{p-s_0}\right)\left[\left(\frac{p}{p-s_0}\right)^p+\frac{1}{s_0-1}\right]^{-\frac{1}{p}}\le C,$$

that is,

$$B_{s_0}(p,q,u,v,h)\left(\frac{p}{p-s_0}\right)\left[\left(\frac{p}{p-s_0}\right)^p+\frac{1}{s_0-1}\right]^{-\frac{1}{p}}\leq C.$$

(2) $\Rightarrow$ (1) We are going to prove the inequality (3.1). Let $g>0$, $\lambda>0$ and $O_\lambda=\{x\in(0,\infty):h(x)\int_0^x g^{\frac{1}{p}}v^{-\frac{1}{p}}>\lambda\}$.

For all $x\in O_\lambda$, by applying Hölder's inequality with exponents p and p', we have

$$\begin{aligned}\lambda\ &< h(x)\int_0^x g^{\frac{1}{p}}v^{-\frac{1}{p}}=h(x)\int_0^x g(t)^{\frac{1}{p}}V(t)^{\frac{s_0-1}{p}}V(t)^{\frac{1-s_0}{p}}v(t)^{-\frac{1}{p}}dt\\ &\leq h(x)\left(\int_0^x g(t)V(t)^{s_0-1}dt\right)^{\frac{1}{p}}\left(\int_0^x V(t)^{\frac{(1-s_0)p'}{p}}v^{1-p'}(t)dt\right)^{\frac{1}{p'}}\\ &=h(x)\left(\int_0^x gV^{s_0-1}\right)^{\frac{1}{p}}V(x)^{\frac{p-s_0}{p}}\left(\frac{p-1}{p-s_0}\right)^{\frac{1}{p'}}.\end{aligned}$$

This implies that

$$O_\lambda\subset\left\{x\in(0,\infty):h(x)\left(\int_0^x gV^{s_0-1}\right)^{\frac{1}{p}}V(x)^{\frac{p-s_0}{p}}\left(\frac{p-1}{p-s_0}\right)^{\frac{1}{p'}}>\lambda\right\}.$$

Then, the above inclusion and Lemma 2 give

$$\begin{aligned}\lambda\left(\int_{O_\lambda}u\right)^{\frac{1}{q}}\ &\leq\lambda\left(\int_{\left\{x\in(0,\infty):h(x)\left(\int_0^x gV^{s_0-1}\right)^{\frac{1}{p}}V(x)^{\frac{p-s_0}{p}}\left(\frac{p-1}{p-s_0}\right)^{\frac{1}{p'}}>\lambda\right\}}u\right)^{\frac{1}{q}}\\ &\leq\left(\frac{p-1}{p-s_0}\right)^{\frac{1}{p'}}\left\|h(x)\left(\int_0^x gV^{s_0-1}\right)^{\frac{1}{p}}V(x)^{\frac{p-s_0}{p}}\right\|_{q,\infty;u}\\ &\leq\left(\frac{p-1}{p-s_0}\right)^{\frac{1}{p'}}C_{p,q}\left(\int_0^\infty g(t)V(t)^{s_0-1}\|\chi_{(t,\infty)}hV^{\frac{p-s_0}{p}}\|_{q,\infty;u}^p dt\right)^{\frac{1}{p}}\\ &\leq C_{p,q}B_{s_0}(p,q,u,v,h)\left(\frac{p-1}{p-s_0}\right)^{\frac{1}{p'}}\left(\int_0^\infty g(t)dt\right)^{\frac{1}{p}},\end{aligned}$$

and we are done.

$\square$

4. Proof of Theorem 2

(1) $\Rightarrow$ (2) Suppose that (1.3) holds. It is equivalent to

$$\left(\int_{\{x\in(0,\infty):w(x)G_1f(x)>\lambda\}}u\right)^{\frac{1}{q}}\leq\frac{C}{\lambda}\left(\int_0^\infty f^p\right)^{\frac{1}{p}}. \tag{4.1}$$

Let $t>0$ and $f(x)=\frac{1}{t}\chi_{(0,t)}(x)+e^{-\frac{s_0}{p}}t^{\frac{s_0}{p}-1}x^{-\frac{s_0}{p}}\chi_{(t,\infty)}(x)$.

The function f verifies

$$\int_0^\infty f^p=t^{1-p}\left(1+\frac{e^{-s_0}}{s_0-1}\right)$$

and if $x \in (t, \infty)$, then

$$G_1 f(x) = t^{\frac{s_0}{p}-1} x^{-\frac{s_0}{p}}.$$

By (4.1) we have

$$\left(\int_{\{x \in (t,\infty): w(x) t^{\frac{s_0}{p}-1} x^{-\frac{s_0}{p}} > \lambda\}} u \right)^{\frac{1}{q}} \leq \frac{C}{\lambda} t^{\frac{1-p}{p}} \left(1 + \frac{e^{-s_0}}{s_0 - 1} \right)^{\frac{1}{p}}$$

for all $\lambda > 0$, i. e. ,

$$t^{\frac{s_0-1}{p}} \| \chi_{(t,\infty)}(x) w(x) x^{\frac{-s_0}{p}} \|_{q,\infty;u} \left(1 + \frac{e^{-s_0}}{s_0 - 1} \right)^{-\frac{1}{p}} \leq C.$$

Then

$$B_{\exp,s_0}(p,q,u,v) \left(1 + \frac{e^{-s_0}}{s_0 - 1} \right)^{-\frac{1}{p}} \leq C.$$

(2) ⇒ (1) Suppose that $B_{\exp,s_0}(p,q,u,v) < \infty$. There exists α_0 such that $s_0 < \frac{p}{\alpha}$ for all $\alpha \in (0, \alpha_0)$. Then

$$B_{s_0}\left(\frac{p}{\alpha}, \frac{q}{\alpha}, u, 1, \frac{w(x)^\alpha}{x} \right) = \sup_{t \in (0,\infty)} t^{\frac{(s_0-1)\alpha}{p}} \| \chi_{(t,\infty)}(x) w(x)^\alpha x^{\frac{-s_0 \alpha}{p}} \|_{\frac{q}{\alpha},\infty;u}$$

$$= \sup_{t \in (0,\infty)} t^{\frac{(s_0-1)\alpha}{p}} \| \chi_{(t,\infty)}(x) w(x) x^{\frac{-s_0}{p}} \|^{\alpha}_{q,\infty;u} = B_{\exp,s_0}(p,q,u,v)^\alpha < \infty.$$

Applying Theorem 1, we have that for every $\alpha \in (0, \alpha_0)$ there exists $C_\alpha > 0$ such that

$$\left(\int_{\{x \in (0,\infty): \frac{w(x)^\alpha}{x} \int_0^x f > \lambda\}} u \right)^{\frac{\alpha}{q}} \leq \frac{C_\alpha}{\lambda} \left(\int_0^\infty f^{\frac{p}{\alpha}} \right)^{\frac{\alpha}{p}}, \tag{4.2}$$

where

$$C_\alpha^{\frac{1}{\alpha}} \leq B_{\exp,s_0}(p,q,u,v) \left(\frac{p-\alpha}{p-\alpha s_0} \right)^{\frac{p-\alpha}{p\alpha}} C^{\frac{1}{\alpha}}_{\frac{p}{\alpha},\frac{q}{\alpha}}. \tag{4.3}$$

The inequality (4.2) is equivalent to

$$\left(\int_{\{x \in (0,\infty): w(x) \left(\frac{1}{x} \int_0^x f^\alpha \right)^{\frac{1}{\alpha}} > \lambda\}} u \right)^{\frac{1}{q}} \leq \frac{C_\alpha^{\frac{1}{\alpha}}}{\lambda} \left(\int_0^\infty f^p \right)^{\frac{1}{p}}. \tag{4.4}$$

By (1.2) and dominated convergence, the left-hand side of the inequality (4.4) tends to the left-hand side of the inequality (4.1) when α tends to 0.

On the other hand, (4.3) implies that the upper limit of $C_\alpha^{\frac{1}{\alpha}}$ when α tends to 0 is bounded above by $e^{\frac{s_0-1}{p}} C_{p,q} B_{\exp,s_0}(p,q,u,v) < \infty$. Then, taking upper limit when α tends to 0 in the inequality (4.4), we obtain the inequality (4.1) with $C \leq e^{\frac{s_0-1}{p}} C_{p,q} B_{\exp,s_0}(p,q,u,v)$.

□

5. Proof of Theorem 4

(1) $\Rightarrow$ (2) Suppose that (1.7) holds. It is equivalent to

$$\left(\int_{\left\{x\in(0,\infty)^n : h(x)\int_0^{x1}\int_0^{x2}\cdots\int_0^{xn} g^{\frac{1}{p}}\Pi_{i=1}^n v_i^{-\frac{1}{p}}>\lambda\right\}} u\right)^{\frac{1}{q}} \leq \frac{C}{\lambda}\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty g\right)^{\frac{1}{p}}. \qquad (5.1)$$

Let $(t_1,t_2,\ldots,t_n)\in(0,\infty)^n$. Let g be the function defined by

$$g=\prod_{i=1}^n\left(\left(\frac{p}{p-s_i}\right)^p V_i(t_i)^{-s_i}v_i(x_i)^{1-p'}\chi_{(0,t_i)}(x_i)+V_i(x_i)^{-s_i}v_i(x_i)^{1-p'}\chi_{(t_i,\infty)}(x_i)\right).$$

For all $(x_1,x_2,\ldots,x_n)\in\Pi_{i=1}^n(t_i,\infty)$,

$$h(x_1,x_2,\ldots,x_n)\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} g^{\frac{1}{p}}\Pi_{i=1}^n v_i^{-\frac{1}{p}} = h(x_1,x_2,\ldots,x_n)\prod_{i=1}^n\frac{p}{p-s_i}V_i(x_i)^{\frac{p-s_i}{p}}.$$

On the other hand,

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty g\leq\prod_{i=1}^n\left[\left(\frac{p}{p-s_i}\right)^p+\frac{1}{s_i-1}\right]V_i(t_i)^{1-s_i}.$$

Hence, (5.1) implies

$$\left\|h\prod_{i=1}^n\frac{p}{p-s_i}\chi_{(t_i,\infty)}(x_i)V_i(x_i)^{\frac{p-s_i}{p}}\right\|_{q,\infty;u}\leq C\prod_{i=1}^n\left[\left(\frac{p}{p-s_i}\right)^p+\frac{1}{s_i-1}\right]^{\frac{1}{p}}V_i(t_i)^{\frac{1-s_i}{p}},$$

which means

$$B_{s_1,s_2,\ldots,s_n}\prod_{i=1}^n\left(\frac{\left(\frac{p}{p-s_i}\right)^p}{\left(\frac{p}{p-s_i}\right)^p+\frac{1}{s_i-1}}\right)^{\frac{1}{p}}\leq C.$$

(2) $\Rightarrow$ (1) Let us prove the inequality (5.1). Let $g>0$, $\lambda>0$ and $O_\lambda=\{x\in(0,\infty)^n : h(x)\int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_n} g^{\frac{1}{p}}\Pi_{i=1}^n v_i^{-\frac{1}{p}}>\lambda\}$.

If $x\in O_\lambda$, Hölder's inequality gives

$$\lambda<h(x)\left(\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} g\Pi_{i=1}^n V_i^{s_i-1}\right)^{\frac{1}{p}}\prod_{i=1}^n V_i(x_i)^{\frac{p-s_i}{p}}\left(\frac{p-1}{p-s_i}\right)^{\frac{1}{p'}}.$$

Then, by the n-dimensional weak type Minkowski's inequality (Lemma 3) and the definition of $B_{s_1,s_2,\cdots,s_n}$ we have

$$\lambda\left(\int_{O_\lambda}u\right)^{\frac{1}{q}}\leq\left\|h(x)\left(\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} g\Pi_{i=1}^n V_i^{s_i-1}\right)^{\frac{1}{p}}\prod_{i=1}^n V_i(x_i)^{\frac{p-s_i}{p}}\left(\frac{p-1}{p-s_i}\right)^{\frac{1}{p'}}\right\|_{q,\infty;u}$$

$$\leq C_{p,q}\prod_{i=1}^n\left(\frac{p-1}{p-s_i}\right)^{\frac{1}{p'}}$$

$$\times\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty g(t_1,t_2,\dots,t_n)\Pi_{i=1}^n V_i(t_i)^{s_i-1}\|h\Pi_{i=1}^n\chi_{(t_i,\infty)}V_i(x_i)^{\frac{p-s_i}{p}}\|^p_{q,\infty;u}dt_1dt_2\cdots dt_n\right)^{\frac{1}{p}}$$

$$\le C_{p,q}B_{s_1,s_2,\dots,s_n}\prod_{i=1}^n\left(\frac{p-1}{p-s_i}\right)^{\frac{1}{p'}}\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty g\right)^{\frac{1}{p}}.$$

□

6. Proof of Theorem 5

(1) ⇒ (2) Suppose that (1.6) holds. It is equivalent to

$$\left(\int_{\{x\in(0,\infty)^n:w(x)G_nf(x)>\lambda\}}u\right)^{\frac{1}{q}}\le\frac{C}{\lambda}\left(\int_0^\infty\int_0^\infty\cdots\int_0^\infty f^p\right)^{\frac{1}{p}}. \tag{6.1}$$

Let $(t_1,t_2,\dots,t_n)\in(0,\infty)^n$ and let f be the function defined by

$$f(x)=\prod_{i=1}^n\left(\frac{1}{t_i}\chi_{(0,t_i)}(x_i)+e^{-\frac{s_i}{p}}t_i^{\frac{s_i}{p}-1}x_i^{-\frac{s_i}{p}}\chi_{(t_i,\infty)}(x_i)\right).$$

If $x\in\Pi_{i=1}^n(t_i,\infty)$, then $G_nf(x)=\Pi_{i=1}^n t_i^{\frac{s_i}{p}-1}x_i^{-\frac{s_i}{p}}$.
On the other hand,

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty f^p=\prod_{i=1}^n t_i^{1-p}\left(1+\frac{e^{-s_i}}{s_i-1}\right).$$

Substituting the above identities in (6.1), we have

$$\left(\int_{\{x\in(0,\infty)^n:w(x)\Pi_{i=1}^n t_i^{\frac{s_i}{p}-1}x_i^{-\frac{s_i}{p}}>\lambda\}}u\right)^{\frac{1}{q}}\le\frac{C}{\lambda}\prod_{i=1}^n t_i^{\frac{1-p}{p}}\left(1+\frac{e^{-s_i}}{s_i-1}\right)^{\frac{1}{p}}$$

for all $\lambda>0$, i. e. ,

$$\Pi_{i=1}^n t_i^{\frac{s_i-1}{p}}\|w\Pi_{i=1}^n\chi_{(t_i,\infty)}x_i^{\frac{-s_i}{p}}\|_{q,\infty;u}\prod_{i=1}^n\left(1+\frac{e^{-s_i}}{s_i-1}\right)^{-\frac{1}{p}}\le C.$$

Then

$$B_{\exp,s_1,s_2,\dots,s_n}(p,q,u,v)\prod_{i=1}^n\left(1+\frac{e^{-s_i}}{s_i-1}\right)^{\frac{1}{p}}\le C.$$

(2) ⇒ (1) Suppose that $B_{\exp,s_1,s_2,\dots,s_n}(p,q,u,v)<\infty$. There exists $\alpha_0>0$ such that $s_i<\frac{p}{\alpha}$ for all i and all $\alpha\in(0,\alpha_0)$. Then

$$\begin{aligned}&B_{s_1,s_2,\dots,s_n}\left(\frac{p}{\alpha},\frac{q}{\alpha},u,1,1,\dots,1,\frac{w(x_1,x_2,\dots,x_n)^\alpha}{x_1x_2\cdots x_n}\right)\\&=\sup_{(t_1,t_2,\dots,t_n)\in(0,\infty)^n}\Pi_{i=1}^n t_i^{\frac{(s_i-1)\alpha}{p}}\|\chi_{\Pi_{i=1}^n(t_i,\infty)}w^\alpha\Pi_{i=1}^n x_i^{\frac{-s_i\alpha}{p}}\|_{\frac{q}{\alpha},\infty;u}\\&=\sup_{(t_1,t_2,\dots,t_n)\in(0,\infty)^n}\Pi_{i=1}^n t_i^{\frac{(s_i-1)\alpha}{p}}\|\chi_{\prod_{i=1}^n(t_i,\infty)}w\Pi_{i=1}^n x_i^{\frac{-s_i}{p}}\|^\alpha_{q,\infty;u}\\&=B_{\exp,s_1,s_2,\dots,s_n}(p,q,u,v)^\alpha<\infty.\end{aligned}$$

By Theorem 4, we have that for every $\alpha \in (0, \alpha_0)$ there exists $C_\alpha > 0$ such that

$$\left(\int_{\left\{x \in (0,\infty)^n : \frac{w(x)^\alpha}{x_1 x_2 \cdots x_n} \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} f > \lambda \right\}} u \right)^{\frac{\alpha}{q}} \leq \frac{C_\alpha}{\lambda} \left(\int_0^\infty \int_0^\infty \cdots \int_0^\infty f^{\frac{p}{\alpha}} \right)^{\frac{\alpha}{p}}, \tag{6.2}$$

where

$$C_\alpha^{\frac{1}{\alpha}} \leq B_{\exp, s_1, s_2, \ldots, s_n}(p, q, u, v) \prod_{i=1}^{n} \left(\frac{p - \alpha}{p - \alpha s_i} \right)^{\frac{p-\alpha}{p\alpha}} C^{\frac{1}{\alpha}}_{\frac{p}{\alpha}, \frac{q}{\alpha}}. \tag{6.3}$$

The inequality (6.2) is equivalent to

$$\left(\int_{\left\{x \in (0,\infty)^n : w(x) \left(\frac{1}{x_1 x_2 \cdots x_n} \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} f^\alpha \right)^{\frac{1}{\alpha}} > \lambda \right\}} u \right)^{\frac{1}{q}} \leq \frac{C_\alpha^{\frac{1}{\alpha}}}{\lambda} \left(\int_0^\infty \int_0^\infty \cdots \int_0^\infty f^p \right)^{\frac{1}{p}}. \tag{6.4}$$

Then, taking upper limit when α tends to 0 in the inequality (6.4), we obtain the inequality (6.2) with $C \leq C_{p,q} B_{\exp, s_1, s_2, \ldots, s_n} \prod_{i=1}^{n} e^{\frac{s_i - 1}{p}}$.

□

REFERENCES

[1] K. F. Andersen and B. Muckenhoupt, *Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions*, Studia Math. 72 (1982), 9-26.

[2] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press (1988).

[3] E.V. Ferreyra, *Weighted Lorentz norm inequalities for integral operators*, Studia Math. 96 (1990), 125-134.

[4] P. Gurka and B. Opic, *Weighted inequalities for geometric means*, Proc. Amer. Math. Soc. 120(3) (1994), 771-779.

[5] A. Kufner, L. Maligranda and L. E. Persson, *The Hardy inequality-About its history and some related results*, Book manuscript, Department of Mathematics, Lulea University of Technology, 2006.

[6] A. Kufner and L. E. Persson, *Weighted inequalities of Hardy Type*, World Scientific, 2003.

[7] F. J. Martín-Reyes, P. Ortega Salvador and M. D. Sarrión Gavilán, *Boundedness of operators of Hardy type in $\Lambda^{p,q}$-spaces and weighted mixed inequalities for singular integral operators*, Proc. Royal Soc. Edinburgh, 127A (1997), 157-170.

[8] B. Opic and L. Pick, *On the geometric mean operator*, J. Math. Anal. Appl. 183(3) (1994), 652-662.

[9] L. E. Persson and V. D. Stepanov, *Weighted integral inequalities with the geometric mean operator*, J. Inequal. Appl. 7(5) (2002), 727-746.

[10] E. Sawyer, *Weighted inequalities for the two-dimensional Hardy operator*, Studia Math. 82 (1985), 1-16.

[11] A. Wedestig, *Weighted inequalities of Hardy-type and their limiting inequalities*, PhD thesis, Department of Mathematics, Lulea University of Technology, 2003.

[12] A. Wedestig, *Weighted inequalities for the Sawyer two-dimensional Hardy operator and its limiting geometric mean operator*, J. Inequal. Appl. 2005:4 (2005), 387-394.

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