WEIGHTED INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATORS WITH KERNEL SATISFYING HÖRMANDER TYPE CONDITIONS

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Abstract. In this paper we study inequalities with weights for fractional integrals T_α given by convolution with a kernel K_α which is supposed to satisfy some size condition and a fractional Hörmander type condition. As it is done for singular integrals, the conditions on the kernel have been generalized from the scale of Lebesgue spaces to that of Orlicz spaces. Our fractional integrals include as particular cases the classical fractional integral I_α, fractional integrals associated to an homogeneous function and fractional integrals given by a Fourier multiplier.

1. Introduction

Suppose that T is a convolution integral operator with kernel K which satisfies some regularity condition and suppose that we know of some behavior on T with respect to Lebesgue measure. Sometimes, if one wants to know the behavior of T when we change the measure, i.e., when we consider the measure $w(x)dx$ where w is a weight, i.e., $0 \leq w \in L^1_{loc}(\mathbb{R}^n)$, we get an inequality of the type

$$\int |Tf|^p w \leq C \int (\mathcal{M}_T f)^p w \,, \tag{1.1}$$

for all $0 < p < \infty$ and $w \in A_\infty$, where $\mathcal{M}_T$ is a maximal operator related to the operator T which is normally easier to deal with. In general, $\mathcal{M}_T$ is strongly related with the kernel K and it will be bigger as much rough will be the kernel.

For T a Calderón-Zygmund singular integral operator (i.e., $K \in H^*_\infty$, see the definition in section ??) inequality (1.1) holds with $\mathcal{M}_T = M$, where M is the Hardy-Littlewood maximal function (see [9]). If T is a singular integral operator with less regular kernel as in [20], then inequality (1.1) holds with $\mathcal{M}_T = M_r$, where $M_r f = [M(|f|^r)]^{1/r}$ for some $1 \leq r < \infty$ (see [34]). The value of the exponent r is determined by the smoothness of the kernel, namely, the kernel satisfies an $L^{r'}$-Hörmander condition (see the precise definition in section ??). In [23], the L^r-Hörmander condition is generalized to the scale of the Orlicz spaces. For a Young function A, the L^A-Hörmander condition is introduced in that paper (for $A(t) = t^r$ we get the L^r-Hörmander condition) and it is proved that if the kernel satisfies the

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L^A-Hörmander condition, then inequality (1.1) holds with $\mathcal{M}_T = M_{\overline{A}}$, where $\overline{A}$ is the complementary function of A and $M_{\overline{A}}$ is the Orlicz maximal operator associated to $\overline{A}$ (see the definition in section 2). The corresponding inequality (1.1) for commutators (with symbol $b \in BMO$) of Calderón-Zygmund singular integrals appears in [28]. For commutators of generalized singular integrals, associated to a kernel satisfying a Hörmander type condition given by a Young function $\mathcal{A}$, the corresponding inequality (1.1) appears in [21].

In 1974, Muckenhoupt and Wheeden [25] proved inequality (1.1) for T the classical Riesz potential I_α and $\mathcal{M}_T$ the fractional maximal function M_α, defined for $0 < \alpha < n$ and locally integrable function f by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \quad \text{and} \quad M_\alpha f(x) = \sup_{x\in B} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)| dy\,.$$

There are fractional integrals with less regular kernel than the Riesz transform. In [19], Kurtz stated a fractional L^r-Hörmander type condition and he applied it to study the boundedness with weights of fractional integrals given by a multiplier (see Section 5 for definitions). Other generalization of fractional integrals are those whose kernel is associated to an homogeneous function Ω. Suppose that Ω is homogeneous of degree zero and $\Omega \in L^1(S^{n-1})$, where S^{n-1} denotes the unit sphere on $\mathbb{R}^n$. Define the fractional integral associated to Ω by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^{n-\alpha}} f(x-y)\, dy.$$

In [14] inequalities with weights were establish for this operator, when $\Omega \in L^s(S^{n-1})$, $s > 1$, which generalized the corresponding inequalities for I_α given by Muckenhoupt and Wheeden in [25]. In a more general context and with an aditional condition in Ω, that is, Ω satisfying the $L^s(S^{n-1})$-Dini smoothness condition, Segovia and Torrea [35], studied the good weights for this operator and its commutators, using extrapolation theorems. The $L^s(S^{n-1})$-Dini smoothness condition in Ω provides a fractional L^s-Hörmander type condition on the kernel $K_\alpha(x) = \frac{\Omega(x/|x|)}{|x|^{n-\alpha}}$.

In this paper we study operators T_α, $0 < \alpha < n$, which includes as particular cases the operators $T_{\Omega,\alpha}$ and the fractional integrals associated to a multipliers as in [19]. We shall obtain inequalities of the type (1.1) for these operators when its kernels K_α satisfy a size condition and a fractional L^A-Hörmander condition (we will denote it by $H_{\alpha,\mathcal{A}}$). We would like to point out that no boundedness of the operator T_α is used to derive (1.1). Moreover, if we know some boundedness of the operator T_α and K_α satisfies a size condition and a suitable fractional $L^{\mathcal{A}}$-Hörmander condition we shall prove (1.1) for the commutators of T_α.

These results will allow us to obtain, for general operators T_α and its commutators, two-weight inequalities of the type

$$\int |Tf|^p w \le C \int |f|^p \mathcal{M}_T w\,, \tag{1.2}$$

for $1 < p < \infty$ and with no assumptions on the weight w. The operators $\mathcal{M}_T$ are again suitable maximal operators related with T and not necessarily the same for inequalities (1.1) and (1.2).

There is a great amount of works that deal with inequalities of the type (1.2). When

T is a Calderón-Zygmund operator (with kernel $K \in H^*_\infty$), inequality (1.2) holds with $\mathcal{M}_T = M^{[p]+1}$, where $[p]$ is the integer part of p and, for $k \in \mathbb{N}$, M^k denote the Hardy-Littlewood maximal function iterated k times (see [27]). The corresponding result for commutators (with symbol $b \in BMO$) of Calderón-Zygmund singular integrals was proved in [28]. For T a singular integral associated to a kernel K satisfying a general Hörmander's condition given by a Young function A, the corresponding results, that include as particular cases those of C. Pérez, has been proved in [21] When T is the Riesz fractional integral I_α, then inequality (1.2) holds with $\mathcal{M}_T = M_{\alpha p}(M^{[p]})$ (this result is also due to C. Pérez, see [31]). For commutators (with symbol $b \in BMO$) of I_α see [3] and [4].

The paper is organized as follows. Section 2 contains preliminaries and definitions that are needed to state the results. In Section 3 we estate the conditions on the kernels that we are going to deal with. We introduce the classes $H_{\alpha,\mathcal{A},k}$ for $\mathcal{A}$ a Young function. These classes appeared in [21] in the case $\alpha = 0$ and in [23] in the case $\alpha = 0$ and $k = 0$. Section 4 is dedicated to study the Coifman type inequality (1.1), for generalized fractional integrals and their commutators. In Section 5 we give some applications: we obtain weighted norm inequalities for fractional integrals associated to a homogeneous function, or a multiplier. Section 6 is devoted to state a strong type two-weight norm inequality, obtained from the Coifman type inequality and to apply it to fractional integrals and their commutators.

2. PRELIMINARIES

A function $\mathcal{A} : [0,\infty) \to [0,\infty)$ is said to be a Young function if it is continuous, convex, increasing and satisfies $\mathcal{A}(0) = 0$ and $\mathcal{A}(t) \to \infty$ as $t \to \infty$.

Given a Young function $\mathcal{A}$, define the $\mathcal{A}$-mean Luxemburg norm of a function f on a ball (or a cube) B by

$$||f||_{\mathcal{A},B} = \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \mathcal{A}\left(\frac{|f|}{\lambda}\right) \le 1\right\}. \tag{2.1}$$

It is well known that if $\mathcal{A}(t) \le C\mathcal{B}(t)$ for all $t \ge t_0$ then $||f||_{\mathcal{A},B} \le C||f||_{\mathcal{B},B}$, for all balls B and functions f. Thus, the behavior of $\mathcal{A}(t)$ for $t \le t_0$ is not important. If $\mathcal{A} \approx \mathcal{B}$, i.e., there are constants $t_0, c_1, c_2 > 0$ such that $c_1\mathcal{A}(t) \le \mathcal{B}(t) \le c_2\mathcal{A}(t)$ for $t \ge t_0$, then $||f||_{\mathcal{A},B} \approx ||f||_{\mathcal{B},B}$.
Each Young function $\mathcal{A}$ has an associated complementary Young function $\overline{\mathcal{A}}$ satisfying

$$t \le \mathcal{A}^{-1}(t)\overline{\mathcal{A}}^{-1}(t) \le 2t, \qquad t > 0.$$

There is a generalization of Hölder's inequality

$$\frac{1}{|B|}\int_B |fg| \le ||f||_{\mathcal{A},B}||g||_{\overline{\mathcal{A}},B}, \tag{2.2}$$

and even another one that will be used later (see [26]): If $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are Young functions and

$$\mathcal{A}^{-1}(t)\mathcal{B}^{-1}(t) \le \mathcal{C}^{-1}(t)$$

then

$$\|fg\|_{\mathcal{C},B} \le 2\|f\|_{\mathcal{A},B}\|g\|_{\mathcal{B},B}. \tag{2.3}$$

When $\mathcal{A}(t)=t$ set $\overline{\mathcal{A}}(t)=0$ if $0\le t\le 1$ and $\overline{\mathcal{A}}(t)=\infty$ otherwise. Observe that $\overline{\mathcal{A}}$ is not a Young function, still $L^{\overline{\mathcal{A}}}$ can be identified with L^∞. Also, writing $\overline{\mathcal{A}}^{-1}(t)\equiv 1$, the previous Hölder inequalities make sense if one of the functions is $\mathcal{A}$ or $\overline{\mathcal{A}}$.

For a complete account on Young functions and Orlicz spaces see [33] and [26].

For each locally integrable function f and $0\le\alpha<n$, the fractional maximal operator associated to the Young function $\mathcal{A}$ is defined as

$$M_{\alpha,\mathcal{A}}f(x)=\sup_{B\ni x}|B|^{\alpha/n}||f||_{\mathcal{A},B}.$$

For $\alpha=0$, write $M_{\mathcal{A}}$ instead of $M_{0,\mathcal{A}}$. When $\mathcal{A}(t)=t^r$, $r>1$, then we write $M_{\alpha,\mathcal{A}}=M_{\alpha,r}$ and if $r=1$ we simply write $M_{\alpha,\mathcal{A}}=M_\alpha$ which is the classical fractional maximal operator. For $\alpha=0$ and $\mathcal{A}(t)=t$, then $M_{0,\mathcal{A}}=M$ is the Hardy-Littlewood maximal operator.

For $1<p<\infty$, a Young function $\mathcal{A}$ is said to belong to B_p if there exists $c>0$ such that $\int_c^\infty \frac{\mathcal{A}(t)}{t^p}\frac{dt}{t}<\infty$. This condition appears first in [30] and it was shown that $\mathcal{A}\in B_p$ if and only if $M_{\mathcal{A}}$ is bounded on $L^p(dx)$.

3. THE CONDITIONS IN THE KERNELS

In this paper, we shall consider fractional convolution operators of the type $T_\alpha f=K_\alpha * f$, $0<\alpha<n$, where the kernels K_α are supposed to satisfy conditions that ensure certain control on their smoothness and their size.

In [19] a sort of fractional Hörmander condition appears in the scale of L^r spaces. As mentioned in the introduction, the L^r-Hörmander condition for singular integrals has been generalized to the scale of Orlicz spaces (see [23]). In this way, the same can be done for fractional integrals.

From now on, we adopt the following convection: $|x|\sim s$ will stand for the set $\{s<|x|\le 2\,s\}$ and $\|f\|_{\mathcal{A},|x|\sim s}$ will stand for $\|f\,\chi_{\{|x|\sim s\}}\|_{\mathcal{A},B(0,2\,s)}$.

Definition 3.1. *Let $\mathcal{A}$ be a Young function and let $0\le\alpha<n$. The kernel K_α is said to satisfy the $H_{\alpha,\mathcal{A}}$ Hörmander condition, we write $K_\alpha\in H_{\alpha,\mathcal{A}}$, if there exist $c\ge 1$ and $C>0$ such that for any $y\in\mathbb{R}^n$ and $R>c\,|y|$,*

$$\sum_{m=1}^{\infty}(2^m\,R)^{n-\alpha}\,\|K_\alpha(\cdot-y)-K_\alpha(\cdot)\|_{\mathcal{A},|x|\sim 2^m\,R}\le C.$$

If $\mathcal{A}$ gives rise to L^r, $1\le r\le\infty$, then simply write $H_{\alpha,r}$ instead of $H_{\alpha,\mathcal{A}}$.

Definition 3.2. *The kernel K_α is said to satisfy the $H^*_{\alpha,\infty}$ condition if there exist $c\ge 1$ and $C>0$ such that*

$$|K_\alpha(x-y)-K_\alpha(x)|\le C\frac{|y|}{|x|^{n+1-\alpha}},\qquad |x|>c|y|\,.$$

When $\alpha=0$, then $H_{0,\mathcal{A}}=H_{\mathcal{A}}$ as defined in [23]. If $\mathcal{A}$ is a Young function, then there exists $t_0>0$ such that $t\le C\,\mathcal{A}(t)$ for all $t\ge t_0$, and therefore $H_{\alpha,\mathcal{A}}\subset H_{\alpha,1}$. Also, it is easy to see that $H^*_{\alpha,\infty}\subset H_{\alpha,\infty}\subset H_{\alpha,\mathcal{A}}$ and $H_{\alpha,r}\subset H_{\alpha,s}$, for $1\le s<r\le\infty$.

Definition 3.3. *Let $\mathcal{A}$ be a Young function and let $0\le\alpha<n$. The kernel K_α is said to satisfy the $S_{\alpha,\mathcal{A}}$ condition, denote it by saying $K_\alpha\in S_{\alpha,\mathcal{A}}$, if there exists a constant $C>0$ such that*

$$\|K_\alpha\|_{\mathcal{A},|x|\sim s}\le Cs^{\alpha-n}.$$

When $\alpha = 0$, simply write $S_{0,\mathcal{A}} = S_{\mathcal{A}}$, and when $\mathcal{A}(t) = t$, write $S_{\alpha,\mathcal{A}} = S_\alpha$. In this case the condition is $\int_{|x|\sim s} |K_\alpha(x)|dx \le Cs^\alpha$. Clearly $S_{\alpha,\mathcal{A}} \subset S_\alpha$.

Next, we define new classes of kernels depending on a Young function $\mathcal{A}$ and some exponent $k \ge 0$, which will be related with the order of the commutator, (when $k = 0$, $H_{\alpha,\mathcal{A},0}$ coincides with the class $H_{\alpha,\mathcal{A}}$):

Definition 3.4. *Let $\mathcal{A}$ be a Young function and $k \in \mathbb{N}$. We say that the kernel K satisfies the $H_{\alpha,\mathcal{A},k}$ Hörmander condition, we write $K \in H_{\alpha,\mathcal{A},k}$, if there exist $c \ge 1$ and $C > 0$ (depending on $\mathcal{A}$ and k) such that for all $y \in \mathbb{R}^n$ and $R > c\,|y|$*

$$\sum_{m=1}^{\infty} (2^m\, R)^{n-\alpha}\, m^k\, \|K_\alpha(\cdot - y) - K_\alpha(\cdot)\|_{\mathcal{A},|x|\sim 2^m\, R} \le C.$$

As before, if $\mathcal{A}$ gives rise to L^r, $1 \le r \le \infty$, simply write $H_{\alpha,r,k}$ instead of $H_{\alpha,\mathcal{A},k}$. Observe also that $H_{\alpha,\mathcal{A},k+1} \subset H_{\alpha,\mathcal{A},k}$.

The next proposition shows the relation between the conditions $S_{\alpha,\mathcal{A}}$ and $H_{\alpha,\mathcal{A},k}$ with the corresponding ones for $\alpha = 0$.

Proposition 3.5. *If $K_\alpha(x) = |x|^\alpha K(x)$ with $K \in H_{\mathcal{A},k} \cap S_{\mathcal{A}}$ then $K_\alpha \in H_{\alpha,\mathcal{A},k} \cap S_{\alpha,\mathcal{A}}$.*

Proof. It is clear that $K_\alpha \in S_{\alpha,\mathcal{A}}$ is equivalent to $K \in S_{\mathcal{A}}$. To prove that $K_\alpha \in H_{\alpha,\mathcal{A},k}$ let $|x| \sim s$ and $|y| < s/2$ then $s/2 < |x - y| < 5s/2$ and therefore, by the mean value theorem,

$$\begin{aligned}|K_\alpha(x - y) - K_\alpha(x)| &\le |x - y|^\alpha |K(x - y) - K(x)| + |K(x)|\big||x - y|^\alpha - |x|^\alpha\big| \\ &\le Cs^\alpha \left[|K(x - y) - K(x)| + \frac{|y|}{s}|K(x)|\right].\end{aligned}$$

Let $R > 0$ and $s = 2^m R$. Then, for $|y| < R$ and $|x| \sim 2^m R$, we have

$$|K_\alpha(x - y) - K_\alpha(x)| \le C(2^m R)^\alpha \left[|K(x - y) - K(x)| + 2^{-m}|K(x)|\right].$$

Therefore, since $K \in H_{\mathcal{A},k} \cap S_{\mathcal{A}}$,

$$\begin{aligned}&\sum_{m=1}^{\infty} (2^m R)^{n-\alpha}\, m^k \|K_\alpha(\cdot - y) - K_\alpha(\cdot)\|_{\mathcal{A},|x|\sim 2^m R} \\ &\qquad \le C \sum_{m=1}^{\infty} (2^m R)^n\, m^k \|K(\cdot - y) - K(\cdot)\|_{\mathcal{A},|x|\sim 2^m R} \\ &\qquad\quad + C \sum_{m=1}^{\infty} 2^{-m}\, m^k (2^m R)^n \|K\|_{\mathcal{A},|x|\sim 2^m R} \le C\,.\end{aligned}$$

□

4. The Coifman type inequalities

We first state and prove the result for the operator $T_\alpha f = K_\alpha * f$.

Theorem 4.1. *Let $\mathcal{A}$ be a Young function and let $T_\alpha f = K_\alpha * f$, with $K_\alpha \in H_{\alpha,\mathcal{A}} \cap S_\alpha$. Then for any $0 < p < \infty$ and any $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T_\alpha f(x)|^p \, w(x)\, dx \le C \int_{\mathbb{R}^n} M_{\alpha,\overline{\mathcal{A}}} f(x)^p w(x)\, dx\,, \qquad f \in L^\infty_c\,, \tag{4.1}$$

whenever the left-hand side is finite.

Proof. The proof follows standard procedures. For simplicity we may assume that $c = 1$ in the condition $H_{\alpha,\mathcal{A}}$. First we shall prove that if $w \in A_\infty$, $0 < p < \infty$, $0 < \delta < \min\{1, p\}$ and $f \in L^\infty_c$, then

$$M^\sharp_\delta(T_\alpha f)(x) \le C M_{\alpha,\overline{\mathcal{A}}} f(x), \tag{4.2}$$

where $M^\sharp_\delta f = (M^\sharp |f|^\delta)^{1/\delta}$ with

$$M^\sharp f(x) = \sup_{x\in B} \inf_{a\in\mathbb{R}} \frac{1}{|B|} \int_B |f(y) - a|\, dy.$$

Finally, we shall check that it is possible to apply the Fefferman-Stein inequality, see [16], which states that for all $0 < p < \infty$ and $w \in A_\infty$,

$$||Mf||_{L^p(w)} \le C ||M^\sharp f||_{L^p(w)}, \tag{4.3}$$

for all functions such that the left-hand side is finite.

First, let us prove (4.2). Fix $x \in \mathbb{R}^n$, and a ball $B = B(x_B, R)$ containing x. For $\widetilde{B} = B(x_B, 2R)$, set $f_1 = f\,\chi_{\widetilde{B}}$ and $f_2 = f - f_1$. Choose $a = |T_\alpha f_2(x_B)|^\delta$. Then, by Jensen's inequality,

$$\begin{aligned}\left(\frac{1}{|B|}\int_B ||T_\alpha f|^\delta(y) - a|dy\right)^{1/\delta} &\le \frac{1}{|B|}\int_B |T_\alpha f(y) - T_\alpha f_2(x_B)|dy \\ &\le \frac{1}{|B|}\int_B |T_\alpha f_1(y)|dy + \frac{1}{|B|}\int_B |T_\alpha f_2(y) - T_\alpha f_2(x_B)|dy \\ &\le I + II.\end{aligned}$$

Since $K_\alpha \in S_\alpha$ we have

$$\begin{aligned} I &\le \frac{1}{|B|}\int_B \left(\int_{\widetilde{B}} |K_\alpha(y-z) f(z)|dz\right) dy \\ &= \frac{1}{|B|}\int_{\widetilde{B}} |f(z)| \left(\int_B |K_\alpha(y-z)|dy\right) dz \\ &\le \frac{1}{|B|}\int_{\widetilde{B}} |f(z)| \left(\int_{|y-z|\le 3R} |K_\alpha(y-z)|dy\right) dz \\ &\le C\frac{(3R)^\alpha}{|B|}\int_{\widetilde{B}} |f(z)|dz \le C M_\alpha f(x).\end{aligned}$$

Notice that $M_\alpha f(x) \leq CM_{\alpha,\overline{\mathcal{A}}}f(x)$. To estimate II observe that using the generalized Hölder's inequality for $\mathcal{A}$ and $\overline{\mathcal{A}}$ and the fact that $K_\alpha \in H_{\alpha,\mathcal{A}}$, we get

$$
\begin{aligned}
&|T_\alpha f_2(y) - T_\alpha f_2(x_B)| = \left| \int_{\mathbb{R}^n \setminus \widetilde{B}} K_\alpha(y-z)f(z)dz - \int_{\mathbb{R}^n - \widetilde{B}} K_\alpha(x_B - z)f(z)dz \right| \\
&\leq \int_{|z-x_B|>2R} |f(z)||K_\alpha(y-z) - K_\alpha(x_B - z)|dz \\
&\leq \sum_{m=1}^{\infty} \int_{2^mR<|z-x_B|\leq 2^{m+1}R} |f(z)||K_\alpha(y-z) - K_\alpha(x_B - z)|dz \\
&\leq C\sum_{m=1}^{\infty} (2^mR)^n ||f||_{\overline{\mathcal{A}},|z-x_B|\sim 2^mR} ||K_\alpha(y-.) - K_\alpha(x_B - .)||_{\mathcal{A},|z-x_B|\sim 2^mR} \\
&= C\sum_{m=1}^{\infty} (2^mR)^\alpha ||f||_{\overline{\mathcal{A}},|z-x_B|\sim 2^mR} (2^mR)^{n-\alpha} ||K_\alpha(y-.) - K_\alpha(x_B - .)||_{\mathcal{A},|z-x_B|\sim 2^mR} \\
&\leq CM_{\alpha,\overline{\mathcal{A}}}f(x),
\end{aligned}
$$

so its integral average over B with respect to the y-variable gives the same bound for II.

The reasoning to complete the proof follows the same arguments given in [21]. Here we include them for the sake of completeness. By the extrapolation results obtained in [11], the inequality (4.1) will hold for all $0 < p < \infty$ and all $w \in A_\infty$ if, and only if, it holds for some fixed exponent $0 < p_0 < \infty$ and all $w \in A_\infty$. Therefore, fix $p_0 \in (1, \infty)$, $w \in A_\infty$ and $f \in L_c^\infty$, and assume without loss of generality that $\|M_{\alpha,\overline{\mathcal{A}}}f\|_{L^{p_0}(w)}$ and $\|T_\alpha f\|_{L^{p_0}(w)}$ are both finite. Since $w \in A_\infty$, then there exists $r > 1$ (that can be taken greater than p_0) such that $w \in A_r$. Observe that for all $0 < \delta < p_0/r < 1$, we have that $1 < r < p_0/\delta$ and thus, $w \in A_{p_0/\delta}$. Then

$$
\|M_\delta(T_\alpha f)\|_{L^{p_0}(w)} = \left\|M\big(|T_\alpha f|^\delta\big)\right\|_{L^{\frac{p_0}{\delta}}(w)}^{\frac{1}{\delta}} \leq C\,\|T_\alpha f\|_{L^{p_0}(w)} < \infty.
$$

The right hand side is finite by assumption. So by (4.3) and (4.2),

$$
\int_{\mathbb{R}^n} |T_\alpha f|^{p_0} w \leq C\int_{\mathbb{R}^n} (M_\delta(T_\alpha f))^{p_0} w \leq C\int_{\mathbb{R}^n} (M_\delta^\sharp(T_\alpha f))^{p_0} w \leq C\int_{\mathbb{R}^n} (M_{\alpha,\overline{\mathcal{A}}}f)^{p_0} w.
$$

□

Remark 4.2. *If* $K_\alpha \in S_\alpha \cap H_{\alpha,\infty}$ *we get* (4.1) *for any* $0 < p < \infty$ *and for all* $w \in A_\infty$ *with* $\overline{\mathcal{A}}(t) = t$*, which means* M_α *on the right hand side.*

Commutators. Commutators of the classical fractional integral I_α with a BMO function have been extensively studied (see, for example [7], [3], [35], [15]). Now we are going to study commutators of general fractional operators. As before, T_α will be a fractional operator given by convolution with a kernel K_α.
Recall that a locally integrable functions b is said to belong to BMO if

$$
\|b\|_{\mathrm{BMO}} = \sup_B \frac{1}{|B|}\int_B |b(x) - b_B|\,dx < \infty,
$$

where the sup runs over all balls (or cubes) $B \subset \mathbb{R}^n$ and b_B denotes the integral average of b over B.

Given T_α and $b \in \mathrm{BMO}$, define the k-th order commutator, $k \geq 0$, by

$$T^k_{\alpha,b} f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^k \, K_\alpha(x-y) \, f(y) \, dy.$$

Note that for $k = 0$, $T^k_{\alpha,b} = T_\alpha$. Also observe that $T^k_{\alpha,b} f = b T^{k-1}_{\alpha,b} f - T^{k-1}_{\alpha,b}(bf)$, $k \geq 1$.

Theorem 4.3. *Let T_α be a fractional operator with kernel K_α and suppose that T_α is bounded from $L^{q_0}(dx)$ to $L^{p_0}(dx)$, for some $1 < p_0, q_0 < \infty$. Let $b \in \mathrm{BMO}$ and $k \in \mathbb{N}$. Let $\mathcal{A}$ and $\mathcal{B}$ be Young functions such that $\overline{\mathcal{A}}^{-1}(t) \, \mathcal{B}^{-1}(t) \leq \mathcal{C}_k^{-1}(t)$ with $\mathcal{C}_k(t) = t(1 + \log^+ t)^k$. If $K_\alpha \in S_\alpha \cap H_{\alpha,\mathcal{B},k}$ then, for any $0 < p < \infty$ and any $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T^k_{\alpha,b} f(x)|^p \, w(x) \, dx \leq C \, \|b\|^{p\,k}_{\mathrm{BMO}} \int_{\mathbb{R}^n} M_{\alpha,\overline{\mathcal{A}}} f(x)^p \, w(x) \, dx, \qquad f \in L^\infty_c, \tag{4.4}$$

whenever the left-hand side is finite.

Proof. The proof of this Theorem follows the same steps as in Theorem 3.3, part (a) in [21]. So we only point out the differences. The corresponding inequality in Lemma 5.1 of [21] for the operator $T^k_{\alpha,b}$ is the following

$$M^\sharp_\delta\big(T^k_{\alpha,b} f\big)(x) \leq C \sum_{j=0}^{k-1} \|b\|^{k-j}_{\mathrm{BMO}} M_\varepsilon(T^j_{\alpha,b} f)(x) + C \, \|b\|^k_{\mathrm{BMO}} M_{\alpha,\overline{\mathcal{A}}} f(x), \tag{4.5}$$

for $0 < \delta < \varepsilon < 1$ and $k \geq 1$ (the case $k = 0$ is inequality (4.2)).
In order to estimate the term II in (5.4), p. 1413 of [21], for the operator T_α, proceed as in the proof of Theorem 4.1. In fact, using that $K_\alpha \in S_\alpha$, Hölder's inequality with $\overline{\mathcal{C}}_k$ and $\mathcal{C}_k$ and the generalized Hölder's inequality for $\overline{\mathcal{A}}$, $\mathcal{B}$, $\mathcal{C}_k$,

$$\begin{aligned}
II &\leq \frac{1}{|B|} \int_B \left(\int_{\widetilde{B}} |K_\alpha(y-z)(b(z) - b_{\widetilde{B}})^k f(z)| dz \right) dy \\
&\leq \frac{1}{|B|} \int_{\widetilde{B}} |b(z) - b_{\widetilde{B}}|^k |f(z)| \left(\int_{|y-z| \leq 3R} |K_\alpha(y-z)| dy \right) dz \\
&\leq C \frac{(3R)^\alpha}{|B|} \int_{\widetilde{B}} |b(z) - b_{\widetilde{B}}|^k |f(z)| dz \\
&\leq C |B|^{\frac{\alpha}{n}} \|(b - b_{\widetilde{B}})^k\|_{\overline{\mathcal{C}}_k, \widetilde{B}} \|f\|_{\overline{\mathcal{A}}, \widetilde{B}} \leq C \, \|b\|^k_{BMO} M_{\alpha,\overline{\mathcal{A}}} f(x).
\end{aligned}$$

For the rest of the proof, observe that, by extrapolation, it suffices to obtain the theorem for some fixed exponent $0 < p_0 < \infty$ and all $w \in A_\infty$. Therefore, choose p_0 such that the operator T_α is bounded from $L^{q_0}(dx)$ to $L^{p_0}(dx)$. Then, it is possible to obtain that $||T^j_{\alpha,b} f||_{L^{p_0}(w)} < \infty$ for all $0 \leq j \leq k-1$. The rest of the proof follows the proof of Theorem 3.1, part (a) of [21].

□

Remark 4.4. *As in* [21]*, changing in Theorem 4.3 the hypothesis on the kernel by the condition $K_\alpha \in S_\alpha \cap H_{\alpha,\infty,k}$, then* (4.4) *still holds with $\overline{\mathcal{A}}(t) = \mathcal{C}_k(t) = t(1 + \log^+ t)^k$.*

5. Applications

The fractional integral operator. Note that the kernel of the fractional integral I_α, $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$, belongs to $S_\alpha \cap H^*_{\alpha,\infty} \subset S_\alpha \cap H_{\alpha,\infty}$. Consequently, by Remark 4.2 estimate (4.1) holds for I_α with $M_{\alpha,\overline{\mathcal{A}}} = M_\alpha$. We thus recover the result in [25].
On the other hand, it is easy to show that $H^*_{\alpha,\infty} \subset H_{\alpha,\infty,k}$, for all $k \geq 0$. Consequently, by Remark 4.4, estimate (4.4) holds for the commutator of I_α with $\overline{\mathcal{A}}(t) = t(1+\log^+ t)^k$.

The fractional integral operator with rough kernel. Denote by S^{n-1} the unit sphere of $\mathbb{R}^n$. For $x \neq 0$, we write $x' = x/|x|$. Consider a function Ω defined on S^{n-1}. This function can be extended to $\mathbb{R}^n \setminus \{0\}$ as $\Omega(x) = \Omega(x')$ (notice the abuse in also calling the extension Ω). Thus Ω is a homogeneous function of degree 0.

Given a Young function $\mathcal{B}$ we define the $L^{\mathcal{B}}$-modulus of continuity of Ω as

$$\varpi_{\mathcal{B}}(t) = \sup_{|y|\leq t} \|\Omega(\cdot + y) - \Omega(\cdot)\|_{\mathcal{B},S^{n-1}}.$$

Set $K_\alpha(x) = \Omega(x)/|x|^{n-\alpha}$ and let $T_{\Omega,\alpha}$ be the corresponding operator with kernel K_α. We can then prove the following proposition for K_α.

Proposition 5.1. *Let $\Omega \in L^{\mathcal{B}}(S^{n-1})$ and $k \geq 0$. If*

$$\int_0^1 \left(1+\log\frac{1}{t}\right)^k \varpi_{\mathcal{B}}(t)\,\frac{dt}{t} < \infty, \tag{5.1}$$

then $K_\alpha \in S_\alpha \cap H_{\alpha,\mathcal{B},k}$.

Proof. First, notice that $\Omega \in L^{\mathcal{B}}(S^{n-1})$ implies that $K(x) = \frac{\Omega(x)}{|x|^n} \in S_{\mathcal{B}}$. In fact, since

$$\begin{aligned}\frac{1}{|B(0,s)|}\int_{|x|\sim s} \mathcal{B}\left(\frac{|K(x)|}{\lambda}\right)dx &= \frac{C}{s^n}\int_{S^{n-1}}\int_s^{2s} \mathcal{B}\left(\frac{|\Omega(x')|}{\lambda\rho^n}\right)\rho^{n-1}d\rho\, d\sigma(x')\\ &\leq C\int_{S^{n-1}} \mathcal{B}\left(\frac{|\Omega(x')|}{\lambda s^n}\right)d\sigma(x'),\end{aligned}$$

then $||K||_{\mathcal{B},|x|\sim s} \leq s^{-n}||\Omega||_{\mathcal{B},S^{n-1}}$.
On the other hand, it was proved in [21] that if $\Omega \in L^{\mathcal{B}}(S^{n-1})$ and if Ω satisfy (5.1), then $K \in H_{\mathcal{B},k}$. Now, from Remark 3.5, $K_\alpha \in H_{\alpha,\mathcal{B},k} \cap S_{\alpha,\mathcal{B}} \subset H_{\alpha,\mathcal{B},k} \cap S_\alpha$.

□

An immediate consequence of this proposition is that if $\Omega \in L^{\mathcal{A}}(S^{n-1})$ and (5.1) holds with $k=0$ and $\varpi_{\mathcal{A}}$ in place of $\varpi_{\mathcal{B}}$ then $T_{\Omega,\alpha}$ verifies (4.1). In the particular case that $\mathcal{A}(t) = t^r$, inequality (4.1) holds with $M_{\alpha,r'}$ in the right hand side.
On the other hand, if $k > 0$, let Ω be such that $T_{\Omega,\alpha}$ is bounded from $L^p(dx)$ to $L^q(dx)$ for some $1 < p,q < \infty$ (for example, any Ω in $L^{\frac{n}{n-\alpha}}(S^{n-1})$ works fine). Let $\mathcal{A}$, $\mathcal{B}$ be Young functions such that $\overline{\mathcal{A}}^{-1}(t)\,\mathcal{B}^{-1}(t) \leq \mathcal{C}_k^{-1}(t)$ with $\mathcal{C}_k(t) = t(1+\log^+ t)^k$. Then, if Ω verifies the hypothesis of the above proposition, the commutators of $T_{\Omega,\alpha}$ satisfy (4.4). In the particular case that $\mathcal{B}(t) = t^r$ inequality (4.4) holds with $M_{\alpha,L^{r'}(\log L)^{kr'}}$; if $\mathcal{B}(t) = t^r(1+\log^+ t)^{kr}$, (4.4) holds with $M_{\alpha,r'}$; if $\mathcal{B}(t) = t^r(1+\log^+ t)^k$, (4.4) holds with $M_{\alpha,L^{r'}(\log L)^k}$ (see table 2 in[21]).

Multipliers. Given a function m defined in $\mathbb{R}^n$ consider the multiplier operator T_m defined *a priori* for functions f in the Schwartz class by $\widehat{T_m f}(\xi) = m(\xi)\,\widehat{f}(\xi)$. Let $\beta = (\beta_1, \cdots, \beta_n)$ denote a multi-index of non-negative integers and $|\beta| = \beta_1 + \cdots + \beta_n$. As in [19], given $1 \le s < \infty$, $l \in \mathbb{N}$ and $0 < \alpha < n$, we say that $m \in M(s, l, \alpha)$ if there exists a constant B such that $|m(x)| \le B|x|^{-\alpha}$ and

$$\sup_{R>0} R^{|\beta|+\alpha}\, \|D^\beta m\|_{L^s, |\xi|\sim R} < +\infty, \qquad \text{for all } |\beta| \le l.$$

As a consequence of Theorem 4.3 we obtain the following result.

Corollary 5.2. *Assume that* $m \in M(s, l, \alpha)$*, where* $1 < s \le 2$*,* $l \in \mathbb{N}$ *and* $l > \frac{n}{s}$*. Then, for all* $k \ge 0$ *and any* $\epsilon > 0$ *we have that for all* $0 < p < \infty$ *and* $w \in A_\infty$*,*

$$\int_{\mathbb{R}^n} |T^k_{\alpha,b} f(x)|^p w(x)dx \le C \int_{\mathbb{R}^n} (M_{\alpha,\frac{n}{l}+\epsilon} f(x))^p w(x)dx, \qquad f \in L^\infty_c\,, \tag{5.2}$$

whenever the left-hand side is finite.

Proof. Decompose the operator T_m as in [20]. To do that, let $\phi \in C^\infty$ be a nonnegative function supported in $\{\xi : 1/2 < |\xi| < 2\}$ so that $\sum_{j\in\mathbb{Z}} \phi_j(\xi) = \sum_{j\in\mathbb{Z}} \phi(2^{-j}\,\xi) = 1$, $\xi \ne 0$. Write $m_j(\xi) = \phi_j(\xi)\, m(\xi)$ and so $m(\xi) = \sum_{j\in\mathbb{Z}} m_j(\xi)$ for $\xi \ne 0$. Set $K_{\alpha,j} = (m_j)\check{}$ and

$$m^N(\xi) = \sum_{|j|\le N} m_j(\xi), \qquad K^N_\alpha(x) = (m^N)\check{}(x) = \sum_{|j|\le N} K_{\alpha,j}(x).$$

Proceeding as in the final part of the proof of Lemma 1 of [20], only that working with K^N_α instead of K_N, $d = l$, $t = s$ and $p = 1$ we obtain that if $l > \frac{n}{s}$, then

$$\int_{|x|\sim R} |K^N_\alpha(x)|\, dx \le C R^\alpha,$$

where C does not depend on N. This implies that $K^N_\alpha \in S_\alpha$.
By the same Lemma 1 of [20] and the same replacements as above, we get that if $m \in M(s, l, \alpha)$ and $\frac{n}{s} < l < \frac{n}{s} + 1$ then

$$\|K^N_\alpha(\cdot - y) - K^N_\alpha(\cdot)\|_{L^{s'}, |x|\sim R} \le C\, R^{-n+\alpha} \left(\frac{|y|}{R}\right)^{l-\frac{n}{s}}, \qquad |y| < \frac{R}{2}, \tag{5.3}$$

where C does not depend on N. This implies that $K^N_\alpha \in H_{\alpha,s',k}$ for all $k \ge 0$ and this happens uniformly on N: for all $R > 0$ and $|y| < R$,

$$\sum_{j=1}^{\infty} (2^j\, R)^{n-\alpha}\, j^k \|K^N_\alpha(\cdot - y) - K^N_\alpha(\cdot)\|_{L^{s'}, |x|\sim 2^j\, R} \le C \sum_{j=1}^{\infty} j^k \left(\frac{|y|}{2^j\, R}\right)^{l-\frac{n}{s}}$$

$$\le C \sum_{j=1}^{\infty} j^k 2^{-j\,(l-\frac{n}{s})} \le C,$$

where C does not depend on N. Observe that by the same arguments than in Proposition 6.2 of [21], $K^N_\alpha \in H_{\alpha, L^r\,(\log L)^{k\,r}, k}$ uniformly in N, for all $1 < r < (n/l)'$.

To finish the proof of the corollary, take $N > 1$ and consider the operator T^N_m whose kernel is K^N_α. Since $K^N_\alpha \in S_\alpha \cap H_{\alpha,r,k}$, for all $1 < r < (\frac{n}{l})'$, then T^N_m verify (4.1) with

$M_{\alpha,\overline{\mathcal{A}}} = M_{\alpha,\frac{n}{l}+\varepsilon}$ for all $\varepsilon > 0$ with a constant independent of N.
Write now $r' = \frac{n}{l} + \epsilon$ and observe that $1 < r < (\frac{n}{l})'$. Set $\mathcal{B}(t) = t^r(1+\log^+ t)^{k\,r}$ and $\mathcal{A}(t) = t^r$, then $\overline{\mathcal{A}}^{-1}(t)\,\mathcal{B}^{-1}(t)\,\overline{\mathcal{C}}_k^{-1}(t) \leq C\,t$. Since $K_\alpha^N \in H_{\alpha,\mathcal{B},k}$ and T_m^N map $L^p(dx)$ to $L^q(dx)$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$ (see [19]), then Theorem 4.3 applies and therefore (4.4) holds with $M_{\alpha,\overline{\mathcal{A}}} = M_{\alpha,r'}$ with a constant independent of N. A standard approximation argument as in [20] leads to the desired estimate for T_m and $T_{m,b}^k$. □

6. TWO WEIGHTS INEQUALITIES

For operators such that their adjoints satisfy a Coifman type inequality it is possible to obtain two-weight norm inequalities, using a duality argument (see for example [27] and [21]).

Theorem 6.1. *Let $\mathcal{A}$ be a Young function and $1 < p < \infty$. Suppose that there exist Young functions $\mathcal{E}$ and $\mathcal{D}$ such that $\mathcal{E} \in B_{p'}$, $\mathcal{E}^{-1}(t)\,\mathcal{F}^{-1}(t) \leq \overline{\mathcal{A}}^{-1}(t)$ with $\mathcal{F}(t) = \mathcal{D}(t^p)$ and the function $\Phi(t) = t\mathcal{D}'(t) - \mathcal{D}(t)$ for $t > 1$ is also a Young function. If T is a linear operator such that its adjoint T^* satisfies that for all $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} |T^*f(x)|^{p'}\,w(x)\,dx \leq C \int_{\mathbb{R}^n} M_{\alpha,\overline{\mathcal{A}}}f(x)^{p'}\,w(x)\,dx, \qquad f \in L_c^\infty \tag{6.1}$$

then, for any weight u,

$$\begin{aligned}\int_{\mathbb{R}^n} |Tf(x)|^p\,u(x)\,dx &\leq C \int_{\mathbb{R}^n} |f(x)|^p\,M_{\alpha p,\mathcal{D}}u(x)\,dx,\\ &= C \int_{\mathbb{R}^n} |f(x)|^p\,M_\alpha(M_\Phi u)(x)\,dx, \qquad f \in L_c^\infty\,.\end{aligned} \tag{6.2}$$

Remark 6.2. For the applications below, and since all the operators considered here are of convolution type, proving (6.1) for T^* or T turns out be equivalent.

Proof of Theorem 6.1. Fix a weight u. The key point here is to prove that $(M_{\alpha p,\mathcal{D}}u)^\delta \in A_1$, for all $0 < \delta < 1$. Then, the rest of the proof will follow standard arguments. Therefore let us start observing that the conditions on the Young functions $\mathcal{D}$ and Φ, and Theorem 1.1 in [5] give that $M_{\alpha p,\mathcal{D}}u \approx M_\alpha(M_\Phi u)$. On the other hand, we can restrict ourselves to the set $\{x : M_{\alpha p,\mathcal{D}}u(x) < \infty\}$ or suppose that the functions f have support contained in this set. Moreover, we only have to consider $\alpha p < n$ (see, for example, the beginning of the proof of Theorem 1.2 in [3]). By duality, (6.2) turns out to be equivalent to

$$\int_{\mathbb{R}^n} |T^*f(x)|^{p'}\,M_{\alpha p,\mathcal{D}}u(x)^{1-p'}\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p'}u(x)^{1-p'}\,dx, \qquad f \in L_c^\infty.$$

Again observe that $M_{\alpha p,\mathcal{D}}u(x) = \infty$ would imply $M_{\alpha p,\mathcal{D}}u(x)^{1-p'} = 0$. Therefore, by Corollary 1.2 and Remark 1.3 in [5] we know that $(M_{\alpha p,\mathcal{D}}u)^\delta$ belongs to A_1 for all $0 < \delta < 1$. Thus, choosing $r > p'$ and $\delta = (p'-1)/(r-1)$, $M_{\alpha p,\mathcal{D}}u(x)^{1-p'} =$

$\{(M_{\alpha p,\mathcal{D}}u)^{\delta}\}^{1-r} \in A_r \subset A_\infty$, and so (6.1) can be applied. This and the generalized Hölder's inequality for $\overline{\mathcal{A}}$, $\mathcal{E}$ and $\mathcal{F}$, yields

$$\begin{aligned}
\int_{\mathbb{R}^n} |T^* f(x)|^{p'}\, M_{\alpha p,\mathcal{D}}u(x)^{1-p'}\, dx &\le C \int_{\mathbb{R}^n} M_{\alpha,\overline{\mathcal{A}}} f(x)^{p'}\, M_{\alpha p,\mathcal{D}}u(x)^{1-p'}\, dx \\
&\le C \int_{\mathbb{R}^n} M_{\mathcal{E}}(f\, u^{-\frac{1}{p}})(x)^{p'}\, M_{\alpha,\mathcal{F}}(u^{\frac{1}{p}})(x)^{p'}\, M_{\alpha p,\mathcal{D}}u(x)^{1-p'}\, dx \\
&= C \int_{\mathbb{R}^n} M_{\mathcal{E}}(f\, u^{-\frac{1}{p}})(x)^{p'}\, M_{\alpha p,\mathcal{D}}u(x)^{\frac{p'}{p}}\, M_{\alpha p,\mathcal{D}}u(x)^{1-p'}\, dx \\
&= C \int_{\mathbb{R}^n} M_{\mathcal{E}}(f\, u^{-\frac{1}{p}})(x)^{p'}\, dx \le C \int_{\mathbb{R}^n} |f(x)\, u(x)^{-\frac{1}{p}}|^{p'}\, dx \\
&= C \int_{\mathbb{R}^n} |f(x)|^{p'}\, u(x)^{1-p'}\, dx,
\end{aligned}$$

where we have used that $\mathcal{E} \in B_{p'}$ and so $M_{\mathcal{E}}$ is bounded on $L^{p'}(dx)$ (see [30]).

□

In order to apply Theorem 6.1, we shall consider operators T_α which are bounded from $L^p(dx)$ to $L^q(dx)$, where $1 < p, q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Notice that this implies that $1 < p < \frac{n}{\alpha}$ (as pointed out in the proof of Theorem 6.1, we only have to consider this case) and $q > \frac{n}{n-\alpha}$. Therefore, we want to obtain inequality (6.2) for $1 < p < \frac{n}{\alpha}$, and thus we need (6.1) to hold for p'. Let us choose p_0 so that $\frac{1}{p'} = \frac{1}{p_0} - \frac{\alpha}{n}$. We shall be considering that our operator satisfies (6.1) for all $w \in A_\infty$, and $f \in L^\infty_c$, whenever the left hand side is finite. Observe that if $w \in A_\infty \cap L^\infty$, then $||T_\alpha f||_{L^{p'}(w)} \le ||w||_\infty^{1/p'} ||T_\alpha f||_{L^{p'}(dx)} \le C ||w||_\infty^{1/p'} ||f||_{L^{p_0}(dx)} < \infty$, for all $f \in L^\infty_c$. Then, inequality (6.1) holds for $w \in A_\infty \cap L^\infty$. For $w \in A_\infty$ we define $w_N = \min\{w, N\}$, $N \in \mathbb{N}$, then $w_N \in A_\infty$ with a constant independent on N. Thus, (6.1) holds for w_N. Letting N tends to infinity, we get (6.1), for all $f \in L^\infty_c$, and therefore, we can use Theorem 6.1.

The classical fractional integral. In the case that $\overline{\mathcal{A}}(t) = t$, i.e., $K_\alpha \in H_{\alpha,\infty}$, we have that the hypotheses of Theorem 6.1 hold, if $p \ge 2$, for the functions $\mathcal{E}(t) = t^{p'}(1+\log^+ t)^{-1-\epsilon}$ and $\mathcal{D}(t) = t(1+\log^+ t)^{p-1+\varepsilon}$, because $p - 1 + \varepsilon > 1$, and so Remark 1.4 of [5] can be applied. If $1 < p < 2$, then the hypotheses of Theorem 6.1 hold for $\mathcal{E}(t) = t^{p'}(1+\log^+ t)^{-p'/p}$ and $\mathcal{D}(t) = t(1+\log^+ t)$, because again Remark 1.4 of [5] can be applied (with $p = 1$ and $\beta = 1$).

Since the kernel of the classical fractional integral $K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$ belongs to $H^*_{\alpha,\infty} \subset H_{\alpha,\infty}$, we obtain inequality (6.2) with $\mathcal{D}(t) = t(1+\log^+ t)^{[p]}$ for I_α. This was obtained by Pérez in [31]. In this case, $M_{\alpha p,\mathcal{D}}$ is equivalent to $M_{\alpha p}(M^{[p]})$. For the k-th order commutator of I_α, observe that $H^*_{\alpha,\infty} \subset H_{\alpha,\infty} \subset H_{\alpha,\infty,k}$, so Theorem 6.1 again holds with $\mathcal{D}(t) = t(1+\log^+ t)^{[(k+1)p]}$. This was first obtained in [3].

The fractional integral with rough kernel. Let us consider the fractional operator $T_{\Omega,\alpha} f = K_\alpha * f$, where $K_\alpha(x) = \frac{\Omega(x)}{|x|^{n-\alpha}}$ and Ω is as in the previous sections. If $\Omega \in L^{\mathcal{A}}(S^{n-1}) \cap L^{\frac{n}{n-\alpha}}(S^{n-1})$ and satisfies (5.1) with $k = 0$, then we have that the

Coifman type inequality holds with $M_{\alpha,\overline{\mathcal{A}}}$ on the right hand side. Then Theorem 6.1 can be applied to this operator. In the particular case that $\mathcal{A}(t) = t^r$, $r > p$, (6.2) holds with $\mathcal{D}(t) = t^{(r/p)'}(1 + \log^+ t)^{(r/p)'(p-1)+\epsilon}$ and $\epsilon > 0$ small enough. It suffices to apply Theorem 6.1 with $\mathcal{E}(t) = t^{p'}(1 + \log^+ t)^{-1-\varepsilon}$, and $\mathcal{F}(t) = t^{\frac{rp}{r-p}}(1 + \log^+ t)^{(r/p)'(p-1)+\epsilon}$, where $\varepsilon > 0$ is some small enough number that is related with $\epsilon > 0$.

For the commutator of this operator, assume that $\Omega \in L^{\mathcal{B}}(S^{n-1}) \cap L^{\frac{n}{n-\alpha}}(S^{n-1})$ satisfies (5.1), where $\mathcal{B}(t) = t^r$, $r > p$. Then, the Coifman type inequality holds with $\overline{\mathcal{A}}(t) = t^{r'}(1 + \log^+ t)^{kr'}$ and thus, inequality (6.2) holds with $\mathcal{D}(t) = t^{(r/p)'}(1 + \log^+ t)^{(r/p)'((k+1)p-1)+\epsilon}$ and $\epsilon > 0$ small enough (see table 1 in [21]).

The fractional integral associated to a multiplier. Let us now restrict our attention to fractional integrals associated to a multiplier and its commutators, i.e., suppose that we are under the same hypotheses as in Corollary 5.2. For these operators we have that the Coifman type inequality holds with $M_{\alpha,n/l+\varepsilon}$ on the right hand side, for both, T_m and $T^k_{m,b}$. Therefore we obtain the following.

Corollary 6.3. *If* $1 < p < r < (n/l)'$ *and* u *is a weight, then*

$$\int_{\mathbb{R}^n} |T_m f(x)|^p u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M_{\alpha p,\mathcal{D}} u(x)dx, \qquad f \in L^\infty_c. \tag{6.3}$$

and

$$\int_{\mathbb{R}^n} |T^k_{m,b} f(x)|^p u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M_{\alpha p,\mathcal{D}} u(x)dx, \qquad f \in L^\infty_c, \tag{6.4}$$

where $\mathcal{D}(t) = t^{(r/p)'}(1 + \log^+ t)^{(r/p)'(p-1)+\epsilon}$ *and* $\epsilon > 0$ *is small enough.*

The proof is the same as for $T_{\Omega,\alpha}$, in the case $\mathcal{A}(t) = t^r$, since the Coifman type inequality holds with $M_{\alpha,r'}$, and $r' = n/l + \varepsilon$. Since $\epsilon > 0$ is arbitrarily small, then $\mathcal{D}(t) = t^{(r/p)'}(1 + \log^+ t)^{(r/p)'(p-1)+\epsilon} \lesssim t^{(\tilde{r}/p)'}$, for all $1 < p < \tilde{r} < r < (n/l)'$. Therefore, we may write $\mathcal{D}(t) = t^{(r/p)'}$ in (6.3) and (6.4).

7. Further results

Following standard arguments (see [32], [4] and [22]), a result analogous to (6.2) can be obtained for the endpoint case $p = 1$. We just state the Theorems, leaving the proofs to the interested reader.

Theorem 7.1. *Let* $T_\alpha f = K_\alpha * f$ *be a fractional operator. Suppose that there exists* $\delta > 0$ *such that for any* $p \in (1, 1 + \delta)$*, there exists a Young function* $\mathcal{D}_p$ *satisfying*

$$\int_{\mathbb{R}^n} |T_\alpha f|^p u \leq C \int_{\mathbb{R}^n} |f|^p M_{\alpha p,\mathcal{D}_p} u, \tag{7.1}$$

for all weights u*. If* $K_\alpha \in H_{\alpha,\mathcal{A}}$*, then for any weight* u*,*

$$u\left(\{x \in \mathbb{R}^n : |T_\alpha f(x)| > \lambda\}\right) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| (Mu + M_{\alpha,\overline{\mathcal{A}}} u + M_{\alpha p,\mathcal{D}_p} u), \tag{7.2}$$

for all $\lambda > 0$.

Remark 7.2. If T_α is an operator given by convolution with $K_\alpha \in H_{\alpha,\infty}$, then it satisfies inequality (7.1) with $\mathcal{D}_p(t) = t(1+\log^+ t)^{[p]}$, therefore inequality (7.2) holds with the operators $Mu + M_\alpha u + M_{\alpha p,\mathcal{D}}$, where $\mathcal{D}(t) = t(1+\log^+ t)$. Observe that $M_\alpha u < Mu + M_{\alpha p,\mathcal{D}}u$, then for p small enough we obtain on the right hand side of (7.2) $Mu + M_{\alpha(1+\epsilon),\mathcal{D}}u$, for $\epsilon > 0$ small enough. Also, as in Example 1 in [5], we get that $M_{\alpha(1+\epsilon),\mathcal{D}}u \le M_{\alpha(1+\epsilon)}(Mu)$. In the case of I_α we obtain the same result as in [4].

Now observe that if we have a Coifman type inequality for $T^k_{\alpha,b}$, (4.4), using Theorem 6.1, we can obtain a two weights inequality. With this result we can prove:

Theorem 7.3. *Let* T_α *be a fractional operator with kernel* K_α*. Let* $0 \le k \in \mathbb{Z}$*,* $b \in \mathrm{BMO}$ *and* $T^k_{\alpha,b}$ *the k-th order commutator of* T_α*. Suppose that there exists* $\delta > 0$ *such that for all* $p \in (1, 1+\delta)$ *there exists a Young function* $\mathcal{D}_p$ *satisfying*

$$\int_{\mathbb{R}^n} |T^k_{\alpha,b} f(x)|^p u(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p M_{\alpha p,\mathcal{D}_p} u(x)\,dx, \tag{7.3}$$

for all weights u*. Let* $\mathcal{A}$*,* $\mathcal{B}$ *be Young functions such that* $\overline{\mathcal{A}}^{-1}(t)\,\mathcal{B}^{-1}(t) \le \mathcal{C}_k^{-1}(t)$ *with* $\mathcal{C}_k(t) = t(1+\log^+ t)^k$*. If* $K_\alpha \in H_{\alpha,\mathcal{B},k}$*, then*

$$u\left(\{x \in \mathbb{R}^n : |T^k_{\alpha,b} f(x)| > \lambda\}\right) \le C \int_{\mathbb{R}^n} \mathcal{C}_k\left(\frac{|f|}{\lambda}\right)(Mu + M_{\alpha,\overline{\mathcal{A}}}\,u + M_{\alpha p,\mathcal{D}_p} u) \tag{7.4}$$

for all weights u*,* $\lambda > 0$ *and* $p \in (1, 1+\delta)$*.*

The proofs of these theorems follow the same steps as in [22].

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