THE HILBERT TRANSFORM DOES NOT MAP $L^1(Mw)$ TO $L^{1,\infty}(w)$

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Abstract. We disprove the following a priori estimate for the Hilbert transform H and the Hardy Littlewood maximal operator M:

$$\sup_{t>0} tw\{x \in \mathbb{R} : |Hf(x)| > t\} \leq C \int |f(x)| Mw(x)\, dx \quad .$$

This is a sequel to paper [5] by the first author, which shows the existence of a Haar multiplier operator for which the inequality holds.

1. Introduction and statement of main result

In [2], C. Fefferman and E. Stein observed the following a priori estimate for the Hardy Littlewood maximal operator M:

$$\sup_{t>0} t\, w\{x \in \mathbb{R} : |Mf(x)| > t\} \leq C \int |f(x)| Mw(x)\, dx \quad .$$

Here the weight w is a non-negative, locally integrable function, and $w(E)$ denotes the integral of the weight over the set E. We give a negative answer to the question whether such an inequality holds when the Hardy Littlewood maximal operator on the left hand side is replaced by the Hilbert transform. For a discussion of the history of this question we refer to [5].

Theorem 1.1. *For each constant $C > 0$ there is a weight function w on the real line and an integrable compactly supported function f and a $t > 0$ such that*

$$t\, w\{x \in \mathbb{R} : |Hf(x)| > t\} \geq C \int |f(x)| Mw(x)\, dx \quad .$$

Similarly as in [5], we prove Theorem 1.1 as a consequence of the following:

Proposition 1.2. *For each constant $C > 0$ there is an everywhere positive weight function w on the real line and an integrable compactly supported function f and a $t > 0$ such that*

$$t^2\, w\{x \in \mathbb{R} : |Hf(x)| > t\} \geq C \int |f(x)|^2 \left(\frac{Mw(x)}{w(x)}\right)^2 w(x)\, dx \quad . \tag{1}$$

The reduction to Proposition 1.2 is taken from [1], we sketch the argument at the end of this paper. Following [5] further, we reduce Proposition 1.2 to the dual proposition:

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Proposition 1.3. *For each constant C there is a nontrivial weight w on the real line such that*

$$\|H(w 1_{[0,1)})\|_{L^2(w/(Mw)^2)} \geq C \|1_{[0,1)}\|_{L^2(w)} \quad .$$

Our construction of the weight w is a somewhat simpler variant of the construction in [5]. It was discovered during a stimulating summer school on "Weighted estimates for singular integrals" at Lake Arrowhead, Oct 3-8. 2010.

2. Proof of Theorem 1.2

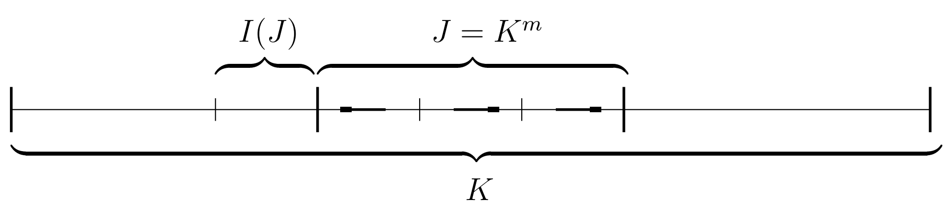
Recall that a triadic interval I is of the form $[3^j n, 3^j(n+1))$ with integers j, n. Denote by I^m the triadic interval of one third the length of I which contains the center of I.

Fix an integer k which will be chosen large enough depending on the constant C in Proposition 1.3. Define $\mathbf{K}_0$ to be $\{[0,1)\}$ and recursively for $i \geq 1$:

$$\mathbf{J}_i := \{K^m : K \in \mathbf{K}_{i-1}\} \quad ,$$

$$\mathbf{K}_i := \{K : K \text{ triadic}, \ |K| = 3^{-ik}, \ K \subset \bigcup_{J \in \mathbf{J}_i} J\} \quad .$$

Proceeding recursively from the larger to the smaller intervals, we choose for each $J \in \mathbf{J} := \bigcup_{i \geq 1} \mathbf{J}_i$ a sign $\epsilon(J) \in \{-1, 1\}$. More precisely, $\epsilon(J)$ depends on the values $\epsilon(J')$ with $|J'| > |J|$. The exact choice will be specified below. Define for each $J \in \mathbf{J}$ the interval $I(J)$ to be the triadic interval of length $3^{1-k}|J|$ whose right endpoint equals the left endpoint of J if $\epsilon(J) = 1$, and whose left endpoint equals the right endpoint of J if $\epsilon(J) = -1$. Note that $I(J)$ has the same length as the intervals in $\mathbf{K}_i$.



Next we define a sequence of absolutely continuous measures on $[0,1]$. We continue to use the same symbol for a measure and its Lebesgue density. Let w_0 be the uniform measure on $[0,1)^m \cup I([0,1))^m$ with total mass 1. Recursively we define the measure w_i by the following properties: It coincides with w_{i-1} on the complement of $\bigcup_{K \in \mathbf{K}_i} K$. For $K \in \mathbf{K}_i$ we have $w_i(K) = w_{i-1}(K)$ and the restriction of w_i to K is supported and uniformly distributed on $K^m \cup I(K^m)$.

Let w be the weak limit of the sequence w_i and note that w is supported on $\bigcup_{J \in \mathbf{J}} I(J)$. For $K \in \mathbf{K}_i$, $J \in \mathbf{J}_i$, $x \in I(J)$, and any triadic interval K' with $|K'| \geq |K|$ we have

$$w(x) = \frac{w(I(J))}{|I(J)|} = \frac{w(K)}{|K|} \geq \frac{w(K')}{|K'|} \quad . \tag{2}$$

We claim that for $J \in \mathbf{J}$ and $x \in I(J)^m$ we have

$$Mw(x) \leq 7w(x) \quad . \tag{3}$$

To see this, let I be a (not necessarily triadic) interval containing x. If I is contained in $I(J)$, then by the first identity of (2) the average of w over I equals $w(x)$. If I is not contained in $I(J)$, then $|I| \geq |I(J)|/3$. Let $\mathbf{K}'$ be the collection of triadic intervals of length $|I(J)|$ which intersect I and note that

$$\sum_{K' \in \mathbf{K}'} |K'| \leq |I| + 2|I(J)| \leq 7|I|$$

because at most two intervals in $\mathbf{K}'$ are not entirely covered by I. With (2) we conclude that the average of w over I is no more than $7w(x)$, which completes the proof of (3).

Lemma 2.1. *For $K \in \mathbf{K}_i$, $J = K^m$, $x \in I(J)^m$, and $k > 3000$ we have*

$$|Hw(x)| \geq (k/3)w(x) \quad .$$

This Lemma proves Proposition 1.3, because with (3) and since w is constant on every $I(J)$ we have

$$49\|Hw\|^2_{L^2(w/(Mw)^2)} \geq (k^2/9) \sum_{J \in \mathbf{J}} \int_{I(J)^m} w(y)\, dy \geq (k^2/27)\|1_{[0,1)}\|^2_{L^2(w)} \quad .$$

Proof of Lemma 2.1: We split the principal value integral for $Hw(x)$ into six summands:

$$p.v. \int_{I(J)} \frac{w(y)}{y-x}\, dy \tag{4}$$

$$+ \int_J \frac{w(y)}{y-x}\, dy \tag{5}$$

$$+ \int_{K^c} \Big(\frac{w(y)}{y-x} - \frac{w(y)}{y-c(J)}\Big)\, dy \tag{6}$$

$$+ \int_{(\bigcup_{\mathbf{K}_i} K')^c} \frac{w(y)}{y-c(J)}\, dy \tag{7}$$

$$+ \sum_{K' \in \mathbf{K}_i \setminus \{K\}} \int_{K'} \frac{w(y)}{y-c(J)} - \frac{w(y)}{c(K')-c(J)}\, dy \tag{8}$$

$$+ \sum_{K' \in \mathbf{K}_i \setminus \{K\}} \int_{K'} \frac{w(y)}{c(K')-c(J)}\, dy \quad . \tag{9}$$

The terms (7) and (9) remain unchanged if we replace w by w_i and hence depend only on the choices of $\epsilon(J')$ with $|J'| > |J|$. The integrand of (5) is positive or negative depending on $\epsilon(J)$. Specify the choice of $\epsilon(J)$ so that the sign of (5) equals the sign of (7)+(9). If the latter is zero, we may arbitrarily set $\epsilon(J) = 1$. We estimate

$$|(5)| \geq \sum_{K' \in \mathbf{K}_{i+1}, K' \subset J} \int_{K'} \frac{w(y)}{|y-x|}\, dy$$

$$\geq \sum_{K' \in \mathbf{K}_{i+1}, K' \subset J} \frac{w(K')}{\sup_{y \in K'} |y-x|}$$

$$\geq \sum_{n=1}^{3^k} \frac{1}{n+1} \frac{w(I(J))}{|I(J)|} \geq (k/2)w(x) \quad .$$

The remaining terms are small error terms, we estimate with $\delta = |I(J)^m|$:

$$|(4)| = |\int_{I(J)\setminus[x-\delta,x+\delta]} \frac{w(y)}{y-x}\, dy| \leq 3w(x) \quad ,$$

$$|(6)| \leq 4 \sum_{|K'|=|K|,K'\neq K} \int_{K'} \frac{|x-c(J)|}{|y-c(J)|^2} w(y)\, dy$$

$$\leq 8 \sum_{|K'|=|K|,K'\neq K} \frac{|x-c(J)|}{|c(K')-c(J)|^2} w(K')$$

$$\leq 16 \sum_{n=1}^{\infty} \frac{1}{(n-3/4)^2} \frac{w(I(J))}{|I(J)|} \leq 200w(x) \quad ,$$

$$|(8)| \leq 4 \sum_{K'\in\mathbf{K}_i} \int_{K'} \frac{|y-c(K')|}{|c(K')-c(J)|^2} w(y)\, dy \quad ,$$

and the last expression is dominated by the same final bound as (6). Putting all estimates together, we have

$$|(4)+(5)+(6)+(7)+(8)+(9)|$$

$$\geq |(5)+(7)+(9)| - |(4)| - |(6)| - |(8)|$$

$$\geq |(5)| - |(4)| - |(6)| - |(8)|$$

$$\geq (k/2 - 403)w(x) \quad .$$

This completes the proof of Lemma 2.1 and thus Theorem 1.3.

3. Remarks

3.1. **More general kernels.** The construction can be generalized to apply to more general kernels, including those with even symmetry, such as for example $\mathrm{Re}(|x|^{-1+\alpha i})$ with $\alpha \neq 0$. Choose J to be the union of 3^{k-1} not necessarily adjacent but appropriately chosen intervals of length $3^{-k}|K|$ contained in K, and $I(J)$ an appropriate further interval of this length well inside K, so that the kernel of the Calderon Zygmund operator for $x \in I(J)^m$ has sufficient positive or negative bias on J.

3.2. **Weights in Theorem 1.1.** We specify weights satisfying Theorem (1.1). Fix a constant C as in Proposition (1.3) and consider k and the weight w constructed above. We slightly change w to make it positive by adding ce^{-x^2} for sufficiently small c so as to not change the conclusion of Proposition (1.3). We may normalize the measure to be probability measure and call the remaining measure w again. The conclusion of Proposition 1.3 can be written:

$$(\int (Hw(x))^2 \frac{w(x)}{(Mw(x))^2}\, dx)^{1/2} \geq C \quad . \tag{10}$$

Multiplying both sides of (10) by the left hand side of (10), setting $f = (Hw)w/(Mw)^2$ and using essential self-duality of H we obtain

$$|\int w(x)Hf(x)\,dx| \geq C(\int f(x)^2 \frac{(Mw(x))^2}{(w(x))^2} w(x)\,dx)^{1/2} \quad . \tag{11}$$

Letting f^* be the non-increasing rearrangement of Hf on $[0,1]$, we may estimate the left hand side of (11)

$$\int_0^1 f^*(y)\,dy \leq 2 \sup_{y\in[0,1]} y^{1/2} f^*(y) = 2\sup_{t>0} w(\{x : |Hf(x)| \geq t\})^{1/2} t \quad .$$

Hence Proposition 1.2 holds for the constant $C/2$ with the weight w and some existentially chosen t. Now let E be the set on the left hand side of Proposition 1.2 for the given w, f, and appropriate t, then we have

$$M(w1_E)(x) = \sup_{x\in I} \frac{\int_I w}{\int_I 1} \frac{\int_I 1_E w}{\int_I w} \leq Mw(x) M_w 1_E(x) \quad ,$$

where M_w denotes the Hardy Littlewood maximal function with respect to the weight w. With Hölder's inequality we obtain

$$\int |f(x)| M(w1_E)(x)\,dx \leq \left(\int |f(x)|^2 \frac{Mw(x)^2}{w(x)}\,dx \right)^{1/2} \|M_w 1_E\|_{L^2(w)} \quad .$$

With the Hardy Littlewood maximal theorem with respect to the weight w we can estimate $\|M_w 1_E\|_{L^2(w)}$ by $w(E)^{1/2}$. This shows that Theorem 1.1 holds for the weight $w1_E$.

3.3. **A1 weights.** It remains open to date whether the a priori inequality

$$t\, w\{x \in \mathbb{R} : |Hf(x)| > t\} \leq C\|w\|_{A_1} \int |f(x)| w(x)\,dx \tag{12}$$

holds, where the A_1 constant is defined as $\|w\|_{A_1} := \|Mw/w\|_\infty$. Our construction in this paper does not seem to address this question. The recent preprint [4] has announced that the analogue of (12) for Haar multipliers is false. In [3], a version of (12) has been proved with an additional logarithmic factor in the A_1 constant of the weight.

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