ON INTERPRETATION OF UNDERSTANDING IN MATHEMATICS: BEYOND FRAGILITY

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> Interpretation is a fragile instrument which is often made to bear too much ontological weight (Tahta, 1996, p. 131)

Three decades ago, the weakness of interpretation as an instrument to access the reality of students' understanding of mathematical knowledge was recognised in mathematics education (Pirie, 1988; Tahta, 1996). To overcome this unfavorable situation, the research considered it appropriate to focus efforts on the elaboration of interpretative models with which to observe visible aspects of understanding, such as its evolution during mathematical activity in the classroom. One of the most relevant contributions in this line was the Pirie–Kieren Dynamical Theory for the Growth of Mathematical Understanding (Pirie & Kieren, 1989, republished in this volume; Pirie & Kieren,1994a).

In the present study, we return to the problem of interpreting understanding in mathematics with the purpose of providing operational solutions to overcome its apparent fragility. We consider interpretation as an unavoidable way to approach students' mathematical understanding, focusing on its positive aspects, on what the student understands and how he/she understands it. We also justify the suitability of interpretation, not in the search for objectivity, but in the idea of being fair with what is interpreted and, above all, with respect to whom it is interpreted, recognising the key role of the teacher in this. To this end, we propose to show how the PK Theory can also be effective to interpret understanding for evaluation purposes in diagnosis and assessment contexts. We do so by reviewing the different dimensions that we recognise in its configuration, highlighting some of its main achievements and also suggesting possible ways to extend its interpretative proposal. As in other studies based on the PK Theory (Kieren, Pirie, & Calvert, 1999; Pirie & Martin, 2000; Thom & Pirie, 2006), we describe understanding interpretation in the course of classroom interaction episodes, with examples of written production and dialogue excerpts reflecting the shared activity performed by students from different educational levels on various mathematical topics.

Phenomenon-epistemological dimension

The PK Theory adopts a functional approach by interpreting understanding from the effective actions carried out by the learner in his or her attempt to solve tasks that require the use of mathematical knowledge (Pirie & Kieren, 1989 republished in this volume). Furthermore, this Theory recognises specificity in interpretation, through the detailed records provided by diagrammatic representation at levels of understanding linked to each concrete person and to each particular mathematical practice (Pirie & Kieren, 1994a). However, when faced with the challenge of assessing what a person understands and how he or she understands it, we believe that characterising the specific uses given to mathematical knowledge in each particular task can be useful to further clarify the processes of understanding that students demonstrate at each level of the pathway of growth proposed by the Pirie–Kieren Model. This applies especially to cases where the assessment results convey similar portraits of understanding in action. For this purpose, we suggest to record and interpret the mathematical activity, taking into account the phenomenon-epistemological analysis of the mathematical knowledge used to solve the task (Gallardo & González, 2006; Gallardo & Quintanilla, 2019). Below is an example of this interpretation that was applied jointly with the PK Model to assess the students' understanding.

We asked to the couple formed by Antonio and Eva (17–18 years old) to study the relative position of the r and s straight lines of equations $y = 2x + 1$ and $x + y + 2 = 0$, respectively. The task was posed to students as an initial assessment prior to the formal study of analytic geometry in space. We illustrate how their collective understanding was manifested through the use of different mathematical knowledge in the following set of excerpts.

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First episode of interpretation of understanding in mathematics

Excerpt 1.1

1	Antonio:	I can’t remember.
2	Eva:	No.
3	Teacher:	How can two lines be in the plane?
4	Antonio:	Lines can be secant, coincident or parallel, right? I’m lost here. On top of that, if the equations form a dependent system, the lines are coincident. If they are parallel, it’s an inconsistent system and if they are secants, it’s independent.

Excerpt 1.2

5	Teacher:	Aren’t you trying to solve it?
6	Antonio:	I can’t remember how to solve it, I remember this.
7	Teacher:	What do you have here?

$$\left.\begin{array}{l} y = 2x+1 \\ x+y+2=0 \end{array}\right\}$$

8	Antonio:	A system with two variables. [nervous laughter]
9	Eva:	Well, $y = 2x + 1$, and from here we clear x, right?
10	Antonio:	You have to make the system.
11	Eva:	Ok. We already have the two equations, right? And now, from here we have to clear one. And you substitute it in the other.

$$x+(2x+1)+2=0$$
$$3x+3=0$$
$$3x=-3$$
$$\boxed{x=-1}$$

12	Antonio:	–2 + 1 – 1, ... So, x is –1, equal to –1, ... The two are the same!

$$y = 2\cdot(-1)+1=-1$$

$$\boxed{\begin{array}{l} x=-1 \\ y=-1 \end{array}}$$

Excerpt 1.3

13	Antonio:	We won’t have to look for the matrix, right?
14	Eva:	Of course not! I don’t like making things more complicated than they have to be.
15	Antonio:	I think that when a system is inconsistent, ... what was it? A divided by A, equal to B divided by B, equal to C divided by C ......But I can't remember what the last one was.
16	Eva:	I can’t remember either.
17	Antonio:	Wait, it’s dependent when the first is equal to the second, different is inconsistent and when all three are different, independent, I think.
18	Eva:	Okay, and now what do we do with that?
19	Antonio:	So, it would be.... The three are not equal, so the system is independent. So, the lines are.... what’s the word?

$$\left.\begin{array}{l} 2x-y+1=0 \\ x+y+2=0 \end{array}\right\}$$

$$\frac{2}{1} \neq \frac{-1}{1} \neq \frac{1}{2}$$

20	Eva:	Secant?
21	Antonio:	Yes, secant. In other words, they intersect.
22	Eva:	So, the point can be this?
23	Antonio:	Yes, that's it! That's the point! This is to calculate the point at which they pass through. We've done the whole thing.

Initially (Excerpt 1.1), the students are not sure how to proceed, but they remember the theory regarding the relative positions of two straight lines in the plane and the geometric interpretation of a system of linear equations according to their set of solutions (lines 1–4). We interpreted the intentional consideration of both knowledge elements as evidence of Image Having (IH_1, IH_2). Thereafter (Excerpt 1.2), following the teacher's suggestion, the pair focused on solving the system formed by the two linear equations, momentarily forgetting the previous characterisation mentioned by Antonio. In fact, by pointing out the new arrangement of the equations, now written one below the other and related with a bracket, we interpret the teacher's actions are prompting the students to visually identify the system (lines 5–8) and also to apply the substitution method effectively, as one variable is already cleared (lines 9–12). For this reason, we considered here that the teacher's questions caused them to fold back to Image Making (IM_1). Such an occurrence was also observed by Pirie and Kieren (1994a) and Thom and Pirie (2006) when they interpreted similar teacher interventions during classroom episodes. After solving the system, the students failed to give a geometric interpretation of the obtained solution (lines 11–12), so they resumed the discussion in other ways (Excerpt 1.3). They went on to suggest using matrices associated with the system (lines 13–14), although ultimately, they explored a new characterisation proposed by Antonio, centring this time on the relationship between the equation coefficients (lines 15–17). Students were able to relate the use of this knowledge (matrix, coefficients, types of systems) to the concrete system of linear equations they were facing, possibly because they have used it in similar cases on previous occasions. We interpreted this new expressed knowledge, identified by Antonio and Eva as possible resolution options, as evidence of Image Having (IH_3, IH_4). From this point onwards, they focused on applying the characterisation to the particular case of the coefficients associated with the system proposed in the task (lines 18–21). We consider this new activity as an illustration of Image Making (IM_2), since it required performing different actions in order to be able to interpret the inequalities obtained, such as expressing equations in general form or calculating ratios between coefficients. Only at the end of the episode, the couple ends up linking their discussion, based on the given characterisations, to the system solution obtained previously (lines 22–23). This last connection illustrates a level of understanding that we interpret as Property Noticing (PN_1), as students observe that the point of intersection of two straight lines is precisely the solution of the system of linear equations. Table 1 summarises the main elements of mathematical knowledge, the relationships, and the essential strategies used by Antonio and Eva, which characterise their joint understanding pathway during the problem solving.

Mathematical knowledge	- Relative position of two straight lines in the plane (IH_1). - Types of systems of linear equations (inconsistent, independent, dependent) (IH_2, IH_4). - Solution to a linear system (IM_1). - Matrix associated to a linear system (IH_3).
Relations	- Relative position of straight lines and types of systems (IH_2). - Coefficients of linear equations and types of systems (IH_4). - Solution to a linear system and point of intersection of two straight lines (PN_1).
Strategies	- Solving a system of two linear equations in two variables (IM_1). - Comparing coefficients of linear equations (IM_2).

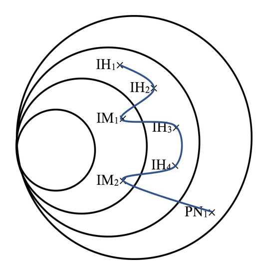


Table 1. Antonio and Eva's pathway of understanding through the uses of mathematical knowledge

Dialogical dimension

The PK Theory shows that the development of each student's mathematical understanding is closely related to the student's different interactions with the rest of their classmates and/or teacher in the specific context of a mathematical task (Pirie & Martin, 2000). On the one hand, individual understanding depends on the degree of interaction effectiveness and coordination. On the other, the understanding can potentially change the nature of the effective communication between learners (Kieren, Pirie, & Calvert, 1999). This externalisation of understanding, which unfolds through the recognised complementarity between acting and expressing (Pirie & Kieren, 1994a), creates a dialogic environment that is conducive to interpretation.

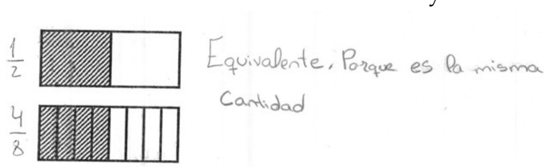
When applying the PK Model to assess students' understanding, the interpreter can be aware that certain ways of acting can facilitate or inhibit understanding and that dialogue excerpts, even brief ones, can be sufficient to notice changes in students' mathematical thinking (Pirie & Martin, 2000). From this dialogic viewpoint, we also contemplated the possibility that students are directly involved in the interpretation processes of their own mathematical understanding, sharing findings, discrepancies, possible obstacles, or inconsistencies identified during the mathematical evaluation activity with the interpreter. This interpretation gives students a key role. Indeed, they are provided with a common environment that facilitates discourse, critical discussion, and the necessary exchanges aimed, ultimately, at reaching the *consent with the other*, whose transformative role in the assessment of students' understanding we also claim (Gallardo & Quintanilla, 2019).

We implemented our proposal in an interpretation of the understanding of the fraction equivalence concept, applying the PK Model in parallel. The episode took place in the mathematics classroom of a group of first-year secondary education students (aged 12–13 years). The mathematical activity was broken down into three phases. In the first, each student built two equivalent fractions through paper folding. In the second, the students worked in pairs and were asked to analyse (describe and discuss) the suitability of the equivalent fractions elaborated by the rest of their classmates. In the third and last phase, the teacher sought to reach the consent with some students by discussing the analysis they had conducted. In the next set of excerpts, we show some excerpts of how this last phase unfolded, together with our interpretation of the understanding of the couple formed by Adriana and Carmen.

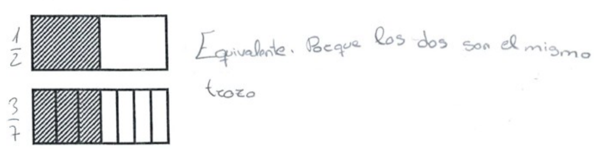
Second episode of interpretation of understanding in mathematics

Excerpt 2.1

1 Teacher: I've reviewed what you've done. Why is one half equivalent to four-eighths?



2 Carmen: When it's divided, it falls into equal parts.
3 Adriana: No! It's because this here and that there *(the coloured parts)* are equal.
4 Teacher: So why did you say that they are equivalent here?

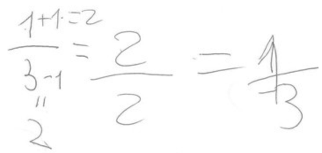


5 Adriana: Because the shaded parts are the same.
6 Carmen: No, they're not! They're not equivalent.
7 Adriana: Yes, they are.
8 Carmen: No, they're not, because these parts [the three coloured parts of three-sevenths] are not the same as these parts [the four uncoloured parts of the same fraction].
9 Adriana: Ah!
10 Teacher: The coloured area is equal, but Carmen's criterion...
11 Carmen: The parts are not equal.

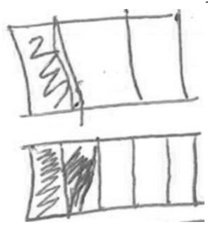
Excerpt 2.2

12 Teacher: If I now told you to give a fraction that is equivalent to one third. What would you do?
13 Carmen: Look, a third is the same as a half. There you go!
14 Adriana: Not at all! A half is bigger.

15 Carmen: Of course not! Look, let's take a part from here and put it here [in the fraction $\frac{1}{3}$, subtract 1 from the denominator and add it to the numerator]. Two-halves equal one-third.

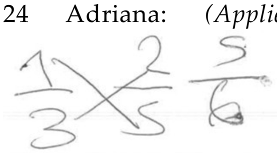


16 Adriana: No! It's not the same. That's weird. But how are you going to take 1 away from 3?
17 Carmen: You have to take the same parts. It's the same.
18 Adriana: Let's see, you colour a third... and two-fifths. Here you take two and that's it. Two-fifths.



19 Teacher: Is one-third equivalent to two-fifths because the coloured parts are equal?
20 Adriana: Yes.

Excerpt 2.3
21 Teacher: Last year, didn't you learn other methods to check whether two fractions are equivalent?
22 Adriana: Yes, the cross method! [On her own initiative, she applies equivalence method to two pairs of fractions analysed above: $\frac{1}{2}$ and $\frac{3}{6}$, and also, $\frac{2}{5}$ and $\frac{4}{10}$]. 2 times 3, 6. That is, it is equal. 2 times 10, 20, and 5 times 4, 20.]
23 Teacher: And does it work with one-third and two-fifths?
24 Adriana: *(Applies equivalence method)* No it doesn't.



25 Teacher: And how could you build a fraction equivalent to one third?
26 Adriana: Three ninths.

Initially (Excerpt 2.1), the students established two different partial criteria to identify and characterise equivalent fractions which were represented graphically. On the one hand, Adriana recognised that the coloured parts of both fractions had to be equal (lines 3, 5, 7). On the other, Carmen defended that the unit parts in each fraction had to be equal (lines 2, 6, 8). After a brief discussion mediated by the teacher, they seemed to accept both criteria as complementary conditions that needed be met in the graphic representation of the fraction (lines 9–11). On this occasion, in line with the findings of Kieren, Pirie, and Calvert (1999), the quest for the consent promoted by the teacher, who fostered an effective communication between both students, seemed to help them to connect their different ways of understanding fraction equivalence, and also to improve their understanding. In this part of the episode, we situated both students' mathematical activity at the understanding levels of Image Making and Image Having (IM_1 and IH_1).

Later however (Excerpt 2.2), during the construction of new equivalent fractions represented numerically, the students' respective pathways of understanding diverge and their interactions were not effective (lines 13–14, 16–17). On the one hand, Carmen does not transfer with understanding to the new situation the previous knowledge that proved useful in the physical and graphical representations of the fraction. On the contrary, she tried to adapt her graphic equal parts criterion to the new numerical context, and eventually generated an alternative algorithm of her own. In doing so, she applied an erroneous type of compensation between the denominator and the numerator, independently of any other knowledge element which she could have recognised and accepted as a fraction equivalence condition (lines 15, 17). Pirie (1988) had noticed this way of proceeding and described it as characteristic of certain students who fail to transfer their manipulations or visual actions into numerically valid calculation procedures. In the case of Carmen, this may have been the result of an evolution achieved without reaching an agreement, from Image Having (IH_1) to Property Noticing (PN_1). In this student, we do not perceive a favourable predisposition to consider the observations and accept the doubts of her partner. Carmen persists in developing her own calculation strategy that does not admit external revisions or criticisms. From our interpretative proposal, we argue that reaching consent with the other through the shared use of mathematical knowledge can foster the growth of understanding, preventing situations such as the one manifested by Carmen in this case.

Adriana, on the other hand, maintained her strategy of graphically representing fractions in units with equal-coloured parts. For her, the unit parts in each fraction also had to be equal, but her graphic representations were inaccurate and led her to an erroneous conclusion (lines 18–20). Her interventions here were interpreted as belonging to the Image Making (IM_2) layer of understanding. To summarise, at this stage, the students had not mutually consent on the suitability of the mathematical knowledge to be applied. Consequently, in the end, no positive transformation—which is expected when another person's understanding is interpreted—was perceived.

In the final dialogue excerpt (Excerpt 2.3), Carmen definitively disconnects from the task without reaching any consensus. The teacher moves on to focus on interpreting Adriana's understanding. We can see how the teacher helps the student to remember a property she already knew: two fractions are equivalent if cross multiplying them produces the same result (lines 21–22). This recovery of Primitive Knowing (PK), in a process that we interpreted as an example of collecting (Pirie & Martin, 2000), allows Adriana to solve the problem. Indeed, she replaced her initial graphic strategy with the new numerical procedure that she ends up applying intentionally and in a generalised way (lines 24–26). Thus, she moves from the level of Image Making (IM_2), with the construction of her own graphical representations, to that of *Property Noticing* (PN_2), with the use of the cross-multiplication method, not only to check, but also to construct new equivalent fractions. In her final intervention, Ariadne no longer needs her first concrete images and properties, but replaces them by a new numerical method whose use she extends to other cases. Precisely because of this, we interpret that her mathematical activity also provides evidence of *Formalising* (F_1) (Pirie & Kieren, 1994b). Moreover, in the last agreement reached between her and the teacher, concerning the existence and use of the equivalence method, we perceive the positive transformative effect on understanding that we claim with the search for consent with the other during the dialogical phase of interpretation. Figure 1 illustrates the students' different understanding pathways during the episode, based on the PK Model.

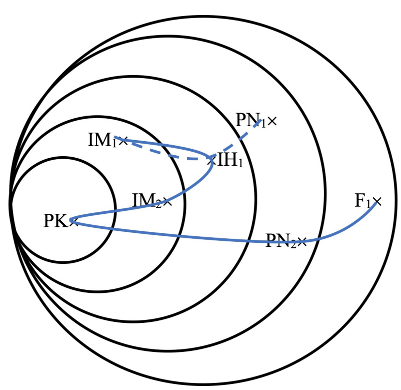


Figure 1. Carmen's (dashed line) and Adriana's (continuous line) pathways of understanding

Affective dimension

We are aware of the interest of the PK Theory in identifying possible reasons that justify the transition between different levels of understanding, so we propose to explore affective phenomena in interpretation and their relationship with the development of understanding during mathematical activity. Other studies based on this Theory have highlighted the affective dimension of understanding (Kieren, Pirie, & Calvert, 1999). In this line, we considered that identifying the relationship between affective domain and understanding would enable us to assess the growth of students' mathematical understanding more accurately. To this end, focusing on the key role of emotions in the development of understanding in mathematics, we proposed an approach that enables exploring the understanding of mathematical knowledge systematically, through different components of affect (Quintanilla & Gallardo, 2022). The actions deployed in the concrete situation, including the uses of mathematical knowledge, are directly related to a decision process in which emotions are key. We recognise the existence of mental processes that are strongly linked to the emotions underlying the decisions about the uses of mathematical knowledge and which explain the student's understanding. By integrating the different components of the affective domain and the understanding of mathematical knowledge, we sought to characterise the student's distinctive cognitive and affective features within a single interpretative process as he or she perform mathematical activity in the classroom.

In our third set of excerpts, we show an example in which we rely on the PK Model to simultaneously interpret both the affective traces that accompany actions and motivate them, and the traces of understanding displayed by two preservice elementary teachers when they solve in pair a task to estimate the length of a spiral. To address the affective dimension, we followed the methodological recommendation of Thom and Pirie (2006) and used video as well as audio recordings to identify the students' gestures, facial expressions, and body language that were associated with their emotional experiences.

Third episode of interpretation of understanding in mathematics

Excerpt 3.1

1	Rocío:	We could calculate the radius. [Serene tone of voice, looks at the folio. Tranquillity (emotional response).]
2	Ivan:	The hard part is that the radius varies because it's a spiral. They've never taught me this! [High tone of voice, drooping head, rigid body, shoulders forward. Distress (emotion) and tension (emotional response).]
3	Rocío:	[Reads the statement.] How long is it? Why would I need to know the radius? [Puts her hand to her head, lowers her tone of voice. Uncertainty (emotion).]
4	Ivan:	There's no point in knowing the radius, because what we want to know is the length. [Low voice, rigid body, drooping head resting on one hand, shoulders forward, staring at paper. Frustration (emotion) and blockage (emotional response).]
5	Rocío:	If we have to bring this to school or if we have to do it, we should take a string, lay it on top and then open it. [Watches carefully at partner's paper while sharing a possible strategy. Trust (emotion) and security (emotional response).]
6	Ivan:	Let's see if there's another way to do it. There has to be another way. But I don't see how! I think I saw something about this in technical drawing one day. [High tone of voice. Rigid body, shoulders forward, head down and leaning on one hand, looking intently at the paper. Distress (emotion) and tension (emotional response).]
7	Rocío:	Well, let's do it in a practical way. It's the best way. That would give us the solution! [High tone of voice. Satisfaction (emotion) and security (emotional response).]
8	Ivan:	I just don't know! First you have to know how to make a spiral. You have to use a compass and small and large set squares and you obtain a series of points and ... I can't remember how it was done! [High tone of voice. Frustration (emotion).]
9	Rocío:	Come on, let's use what we do know. First, we thought of the radius, but then we saw that we don't need it... And we looked for the most practical solution. [Erect body, looks directly at her partner, smiles. Trust (emotion) and tranquillity (emotional response).]
10	Ivan:	Wait. The spiral is inside a square. Let's see if the square has something to do with it. [High tone of voice.]

Excerpt 3.2

11	Ivan:	I was stuck. I didn't know what to do, there was no way I could solve it.

12 Teacher: Maybe you were out of strategies. And your feelings?
13 Ivan: Frustration, uncertainty [laughs softly]. I don't know, anger because I ... I can't! I want to, but I can't.
14 Teacher: Your partner suggested taking a string and shaping it into a spiral, stretching it out on a ruler and measuring it. In the end, you stuck to that strategy.
15 Ivan: I wasn't convinced.
16 Teacher: What would convince you?
17 Ivan: I don't know, a formula perhaps. I've always been taught formulas all my life, I see a formula and it convinces me. It's true that if you can measure a string, and you spread it as it is on a ruler, you get the result.
18 Teacher: Maybe it's incomplete?
19 Ivan: It's incomplete because I have been taught that there must be a formula in mathematics. I need the formula to prove it.

At the beginning of the episode (Excerpt 3.1), the students identify length as the magnitude involved in the task, they notice some characteristics of the spiral and its measurement, and propose a first strategy based on the radius, though they soon discard the strategy believing it is not being effective (lines 1–4). Thereafter, seeking greater operability, Rocío proposes an alternative solution: using a string as a physical object that allows transferring the length of the spiral to a standard measurement instrument which is easy to use (line 5). She justifies with conviction and confidence the relevance of this way of proceeding throughout the task (lines 5, 7, 9), in addition without having to carry it out in a physical way in real life. Rocío regarded her unconventional but efficient strategy as valid. This means that for her, solving mathematical problems is a flexible process where creativity also has its place. In her case, there was no discrepancy between the task reality and her belief system, her emotions contributing to the evolution of her understanding through the layers of Primitive Knowing (PK_1), Image Making (IM_1) and Image Having (IH_1). Moreover, we believe that her strategy demonstrates her understanding of length invariance versus curvature of a flexible object, a geometric property that we place at the Property Noticing (PN_1) level of understanding. In Figure 2, Rocío's pathway of understanding is represented by a continuous line.

Ivan's performance, however, differed from that of his partner. His interventions reflected emotional responses of distress and frustration from the very beginning (lines 2, 4). He felt distrust towards Rocío's strategy owing to his own belief that formulas had to be used to calculate the length. As a result, he insisted on looking for various alternatives based on other previous knowledge elements that allowed him to solve the task using more formal procedures consistent with these beliefs (lines 6, 8, 10). Based on our interpretative approach, his different emotions (distress, uncertainty, frustration) throughout the episode were triggered, on the one hand, by the discrepancies between the task characteristics and Rocío's convincing proposal, and on the other, Ivan's particular beliefs. These emotions, illustrated by the emotional response reiterations (tension, blockage), ultimately conditioned his decisions about his actions and uses of the mathematical knowledge put into play. In a complementary way, in the episode, we conducted a subsequent interpretation phase (Excerpt 3.2), in which we focused on seeking the consent with Ivan. The main aim was to confirm our findings on the emotions he experienced and his associated emotional responses (lines 11–19). These affective factors were the main reason for Ivan's back and forth circulation between the levels of Primitive Knowing (PK_1, PK_2, PK_3) and Image Making (IM_1, IM_2), and failure to move out to more external levels of understanding. Figure 2 illustrates the Ivan's pathway of understanding by a dashed line.

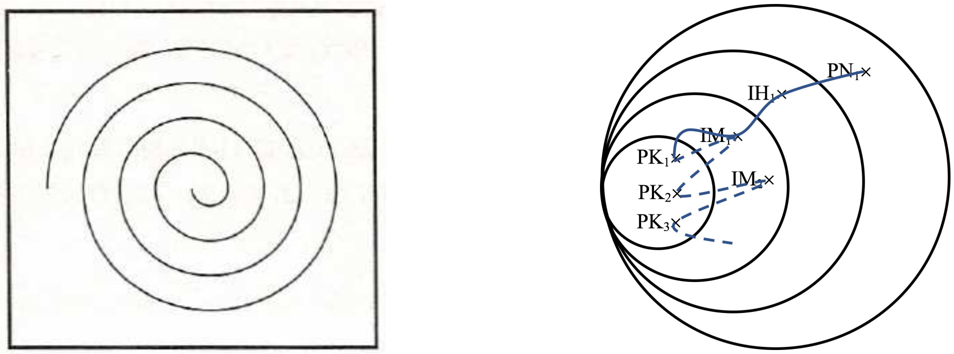


Figure 2. Task to estimate the length of a spiral and Rocío's (continuous) and Ivan's (dashed) pathways of understanding

Ethical dimension

The PK Theory has a clear ethical dimension which is mostly noticeable in the desire to increase students' mathematical understanding based on their own individual perspective. Indeed, the informal and/or unconventional mathematical practices and products of students who are sometimes categorised as "not good" at maths are valued (Pirie & Kieren, 1994a). The Theory shows remarkable sensitivity towards students themselves as knowledgeable persons, towards their practices and products, their integral development within the classroom through the shared use of mathematical knowledge and the key role of context in the development of their understanding (Kieren, Pirie, & Calvert, 1999). Along the journey described by the above dimensions, we introduced complementary interpretation strategies which also have positive ethical consequences for the assessment of students' understanding. Through the episodes described, we demonstrated classroom situations in which students attempt to solve a mathematical problem together, through searching the consent with the other, and where the interpreter fosters mutual agreement in order to transform the protagonists' understanding (Gallardo & Quintanilla, 2019).

We believe that the integration of all these aspects contributes to what we could call a *fair interpretation* of understanding in mathematics. In essence, the exercise consists of cultivating curiosity towards others and disinterested admiration for their actions and productions, guided by the desire for inclusion, reciprocity and equity in the interpretation. Such an interpretation values diversity of thought through mathematically argued dialogue and respect for such differences, thus contributing to positive contexts that foster social justice in the classroom.

Concluding remarks

Our specific aim was to advance an operative proposal that would allow to clarify, in practice, the complex processes of understanding in mathematics. To do so, we explored how the PK Theory, and their possible extensions, could be applied to the interpretation of several understanding episodes in the case of secondary school students and preservice elementary teachers engaged in different mathematical tasks. The proposed interpretation allowed us to delimit *what* students understood and *how* they understood, based on what they were able to do with the mathematical knowledge. The interpretation proposal also provided plausible explanations for their particular way of understanding, leading to suggestions on how to improve this understanding. In this way, we intended to show that interpretation can be a robust and justifiable instrument with which to approach students' mathematical understanding. At the same time, we privileged the ethical dimension of the interpretation and downplayed its ontological weight. Indeed, we shifted the main purpose from the quest for an assumed objective truth about understanding to ensuring justice in the classroom as students perform their mathematical tasks.

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