A $Geo^{[X]}/G^{[X]}/1$ retrial queueing system with removal work and total renewal discipline

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Abstract

In this paper we consider a discrete-time retrial queueing system with batch arrivals of geometric type and general batch services. The arriving group of customers can decide to go directly to the server expelling out of the system the batch of customers that is currently being served, if any, or to join the orbit. After a successful retrial all the customers in the orbit get service simultaneously. An extensive analysis of the model is carried out, and using a generating functions approach some performance measures of the model, such as the first distribution's moments of the number of customers in the orbit and in the system, are obtained. The generating functions of the sojourn time of a customer in the orbit and in the system are also given. Finally, in the section of conclusions and research results the main contributions of the paper are commented.

Keywords: Discrete-time system, retrials, expulsions, total renewal discipline, sojourn time.

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1. Introduction

There is a great potential for using the discrete-time queues in the performance analyses of computer and communication networks. The discrete-time queueing system has been found to be more appropriate in modeling computer

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and telecommunication systems than their continuous counterpart because the basic unit time in the discrete case is a binary code. Indeed, much of the usefulness of discrete-time queues derives from the fact that they can be used in the performance analysis of Digital Network and related computer communication technologies wherein the continuous-time models do not adapt [1], [2].

Queueing systems with repeated attempts are characterized by the fact that a customer finding the server busy upon arrival must leave the service area and repeat its request for service after some random time. Between trials, the blocked customers joins a group of unsatisfied customers called *orbit*. Retrial queues have been widely used to model many practical problems in telephone switching systems, telecommunication networks and computers competing to gain service from a central processing unit. For a detailed review of the main results and the literature on this topic the reader is referred to [3], [4], [5], [6], [7] and [8].

In many real telecommunication systems, it is frequently observed that the server processes the packets in groups. In such batch-service systems, jobs that arrive one at a time must wait in the queue until a sufficient number of jobs gets accumulated. A variety of batch-service queues with infinite waiting space has been studied by many researchers e.g. [9], [10], [11], [12] and [13]. This service discipline is closely related to other disciplines described in the queueing literature like G-networks, clearing systems, catastrophes, etc, see for example [14], [15], [16], [17], [18].

An interesting feature that is considered in this model is the total renewal discipline, that is, jobs or customers are served in groups (batch queues) but leaving the orbit empty at the moment of their batch service. Batch arrivals have been used to describe large deliveries and batch services to model a hospital out-patient department holding a clinic once a week, a transport link with capacity. Nowadays, home automation has greatly increased in popularity over the past several years, which refers to the automatic and electronic control of household features, activities, and appliances. Various control systems are utilized in this residential extension of building automation. The actions in this type of systems can be individual or in batches with total renewal discipline, for example, turning on the lights of a certain area at the same time when the alarm is turned on, combined with other actions. The automation of features in one's home helps to promote security, comfort, energy efficiency, and convenience.

Another feature that is usually found when a message is being processed in computers, in communications switching queues, etc, is that sometimes the information incoming to the server is more actual than the one on service. In that case, the message is moved to another place if the contained information can be used later on, or if the information is not any more valuable it is deleted, in both cases the server is upgraded. The mechanism of moving messages by the arrival of one of them is called synchronized or triggered motion. There are several mechanisms on how and where the messages are moved, for a survey on them refer to [14] and [19], [20, 21]. This mechanism of service interruptions was first studied in [22] that is considered an $M/M/1$ pre-emptive two-priority queueing model with exponentially distributed service interruption. An extensive study on such models can be consulted, for example, in [23, 24], [25], [26] and for a detailed review on queues with service interruptions the reader is referred to [27].

The remainder of this paper is structured as follows. The assumptions of the queueing system under study are given in the next section. In section 3 the Markov chain associated to our model is studied. The queue and system size distribution are obtained together with several performance measures of the system. In section 4 the busy period (BP) is analysed and in section 5 the sojourn times distribution of a customer in the queue and in the system with its respective means are given. Finally, a section of conclusions summarizes the main results for the system.

2. The mathematical model

In this paper a discrete-time queueing system in which the time axis is segmented into a sequence of equal intervals, called slots, is considered. It is assumed that all queueing activities (arrivals, departures and retrials) occur at the slot boundaries, and therefore, they may occur at the same time. So we suppose that the departures occur at the moment immediately before the slot boundaries, but external arrivals and retrials, in this order, occur at the moment after the slot boundaries.

Batches of customers arrive according to a geometrical arrival process with probability a, that is, a is the probability that an arrival occurs in a slot. The number of individual external customers arriving in each batch is k, $k \geq 1$, with probability c_k, and generating function (GF) $C(z) = \sum_{k=1}^{\infty} c_k z^k$, $0 < z \leq 1$. If an arriving batch of customers finds the server free, it begins immediately and jointly its service, otherwise, with probability θ it expels out of the system the group of customers that is currently being served, and starts immediately its service, or, with complementary probability $\bar{\theta}$ it joins the orbit in order to try its luck some time later.

The service times are independent and distributed with arbitrary distribution $\{s_i\}_{i=1}^{\infty}$, and generating function (GF) $S(x) = \sum_{i=1}^{\infty} s_i x^i$, $0 < x \leq 1$. Hence, s_i is the probability that a service lasts i slots. Let $S_k = \sum_{i=k}^{\infty} s_i$ denote the probability that the service lasts not less than k slots.

The retrials are jointly made by all the customers of the orbit. The retrial time (the time between two successive attempts) follows a geometrical law with probability $1 - r$, where r is the probability that the group of customers in the orbit does not make a retrial in a slot.

Once a service is finished, if no arrival occurs, and a successful retrial has taken place, all the customers of the orbit get service jointly and simultaneously.

3. The Markov chain associated to the system

At time k^+, the instant immediately after slot k, the state of the system can be described by the process $\{X_k\ ,\ k \in \mathbb{N}\}$ with $X_k = (C_k, \xi_k, N_k^{(1)}, N_k^{(2)})$ where C_k denotes the state of the server 0 or 1 according to whether the server is free or busy, and $N_k^{(i)}$, $i = 1, 2$, is the number of customers in the server and in the orbit respectively. If $C_k = 1$, then ξ_k corresponds to the remaining service time of the group being served.

It can be shown that $\{X_k\,,\ k \in \mathbb{N}\}$ is the Markov chain of the queueing system under consideration, whose states space is

$$\{(0,n),\ n \geq 0; (i,m,n) : i,m \geq 1,\ n \geq 0\}.$$

Our first task is to find the stationary distribution:

$$\begin{aligned}
\pi_{0,n} &= \lim_{k\to\infty} P[C_k = 0,\ N_k^{(2)} = n],\ n \geq 0, \\
\pi_{i,m,n} &= \lim_{k\to\infty} P[C_k = 1,\ \xi_k = i,\ N_k^{(1)} = m,\ N_k^{(2)} = n],\ \ i,m \geq 1,\ n \geq 0,
\end{aligned}$$

of the Markov chain $\{X_k\,,\ k \in \mathbb{N}\}$.

The Kolmogorov equations for the stationary distribution are

$$\begin{aligned}
\pi_{0,0} &= \bar{a}\pi_{0,0} + \bar{a}\sum_{m=1}^{\infty}\pi_{1,m,0} \Leftrightarrow a\pi_{0,0} = \bar{a}\sum_{m=1}^{\infty}\pi_{1,m,0}, && (1)\\
\pi_{0,n} &= \bar{a}r\pi_{0,n} + \bar{a}r\sum_{m=1}^{\infty}\pi_{1,m,n},\ \ n \geq 1, && (2)\\
\pi_{i,m,0} &= ac_m s_i \pi_{0,0} + \bar{a}(1-r)s_i\pi_{0,m} + ac_m s_i\sum_{l=1}^{\infty}\pi_{1,l,0} + \\
&+ \bar{a}(1-r)s_i\sum_{l=1}^{\infty}\pi_{1,l,m} + \bar{a}\pi_{i+1,m,0} + \\
&+ a\theta c_m s_i\sum_{j=2}^{\infty}\sum_{l=1}^{\infty}\pi_{j,l,0},\ \ i,m \geq 1, && (3)\\
\pi_{i,m,n} &= a\bar{\theta}\sum_{k=0}^{n-1}\pi_{i+1,m,k}c_{n-k} + \bar{a}\pi_{i+1,m,n} + ac_m s_i\pi_{0,n} + \\
&+ ac_m s_i\sum_{l=1}^{\infty}\pi_{1,l,n} + a\theta c_m s_i\sum_{j=2}^{\infty}\sum_{l=1}^{\infty}\pi_{j,l,n},\ i,m,n \geq 1, && (4)
\end{aligned}$$

where $\bar{a} = 1 - a$ and the normalization condition is

$$\sum_{n=0}^{\infty}\pi_{0,n} + \sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\pi_{i,m,n} = 1.$$

In order to solve (1)-(4) the following generating functions corresponding to the sequences $\pi_{0,n}$, $\pi_{i,m,n}$ and $\pi_{1,m,n}$ are introduced

$$\begin{aligned}
\varphi_0(z) &= \sum_{n=1}^{\infty} \pi_{0,n} z^n, \\
P_0(x,z_1) &= \sum_{i=1}^{\infty}\sum_{m=1}^{\infty} \pi_{i,m,0}\, x^i\, z_1^m, \\
P(x,z_1,z_2) &= \sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \pi_{i,m,n}\, x^i\, z_1^m\, z_2^n, \\
\Pi_1(z) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \pi_{1,m,n}\, z^n.
\end{aligned}$$

Multiplying eq. (2) by z^n and summing over n, gives

$$\varphi_0(z) = \frac{\bar{a}r}{1-\bar{a}r}\Pi_1(z). \tag{5}$$

Multiplying eq (3) by $x^i\, z_1^m$, and taking into account eqs. (1) and (5) we get

$$\begin{aligned}
\frac{x-\bar{a}}{x}P_0(x,z_1) &= \frac{a}{\bar{a}}(1-a\theta)c(z_1)S(x)\pi_{0,0} + \alpha(r)S(x)\Pi_1(z_1) + \\
&+ a\theta c(z_1)S(x)P_0(1,1) - \bar{a}\sum_{m=1}^{\infty}\pi_{1,m,0}z_1^m, \qquad (6)
\end{aligned}$$

where $\alpha(r) = \frac{\bar{a}(1-r)}{1-\bar{a}r}$.

Choosing $x=1$, $z_1=1$ in eq. (6) yields

$$a\bar{\theta}P_0(1,1) = \frac{a^2}{\bar{a}}\bar{\theta}\pi_{0,0} + \alpha(r)\Pi_1(1), \tag{7}$$

and substituting (7) into (6) leads to

$$\begin{aligned}
\bar{\theta}\frac{x-\bar{a}}{x}P_0(x,z_1) &= \frac{a}{\bar{a}}\bar{\theta}c(z_1)S(x)\pi_{0,0} + \alpha(r)S(x)[\bar{\theta}\Pi_1(z_1) + \theta c(z_1)\Pi_1(1)] - \\
&- \bar{a}\bar{\theta}\sum_{m=1}^{\infty}\pi_{1,m,0}z_1^m. \qquad (8)
\end{aligned}$$

Setting $x=\bar{a}$ in eq. (8), we get

$$\begin{aligned}
\bar{a}\bar{\theta}\sum_{m=1}^{\infty}\pi_{1,m,0}z_1^m &= \frac{a}{\bar{a}}\bar{\theta}c(z_1)S(\bar{a})\pi_{0,0} + \alpha(r)S(\bar{a})[\bar{\theta}\Pi_1(z_1) + \\
&+ \theta c(z_1)\Pi_1(1)], \qquad (9)
\end{aligned}$$

by substituting the above equation into eq. (6) it is, finally, obtained

$$\bar{\theta}P_0(x,z_1)=\frac{S(x)-S(\bar{a})}{x-\bar{a}}xk(z_1), \tag{10}$$

where $k(z_1)=\frac{a}{\bar{a}}\bar{\theta}c(z_1)\pi_{0,0}+\alpha(r)[\bar{\theta}\Pi_1(z_1)+\theta c(z_1)\Pi_1(1)]$.

From eq. (9), for $z_1=1$, we have

$$\Pi_1(1)=\frac{\bar{a}-S(\bar{a})}{\bar{a}\alpha(r)S(\bar{a})}a\bar{\theta}\pi_{0,0}, \tag{11}$$

and setting $x=1$, $z_1=1$ in eq. (10) gives

$$P_0(1,1)=\frac{1-S(\bar{a})}{S(\bar{a})}\pi_{0,0}. \tag{12}$$

By multiplying eq. (4) by $x^i z_1^m z_2^n$ and summing over i,m,n, we obtain

$$\begin{aligned}\frac{x-(\bar{a}+a\bar{\theta}c(z_2))}{x}P(x,z_1,z_2) &= \frac{a\bar{\theta}c(z_2)}{x}P_0(x,z_1)+\\ &+ a\theta c(z_1)S(x)P(1,1,z_2)+\\ &+ ac(z_1)S(x)\left[\bar{\theta}+\frac{\bar{a}r}{1-\bar{a}r}\right]\Pi_1(z_2)-\\ &- \Big[(\bar{a}+a\bar{\theta}c(z_2))\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\pi_{1,m,n}z_1^m z_2^n+\\ &+ a\bar{\theta}c(z_2)\sum_{m=1}^{\infty}\pi_{1,m,0}z_1^m\Big].\end{aligned} \tag{13}$$

Choosing $x=1, z_1=1$, in the above equation, and using (12), yields

$$P(1,1,z_2)=\Pi_1(z_2)+\frac{c(z_2)\Pi_1(1)-\Pi_1(z_2)}{a\bar{\theta}[1-c(z_2)]}\alpha(r), \tag{14}$$

and substituting (14) into (13) gives

$$\begin{aligned}&\bar{\theta}[1-c(z_2)]\frac{x-[\bar{a}+a\bar{\theta}c(z_2)]}{x}P(x,z_1,z_2)=\\ =\;&\frac{a\bar{\theta}^2c(z_2)[1-c(z_2)]}{x}P_0(x,z_1)+\\ +\;&c(z_1)S(x)\Big[a\bar{\theta}[1-c(z_2)]\frac{1}{1-\bar{a}r}\Pi_1(z_2)-\alpha(r)\theta[\Pi_1(z_2-c(z_2)\Pi_1(1))]\Big]-\\ -\;&\bar{\theta}[1-c(z_2)]\Big[[\bar{a}+a\bar{\theta}c(z_2)]\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Pi_{1,m,n}z_1^m z_2^n+\\ +\;&a\bar{\theta}c(z_2)\sum_{m=1}^{\infty}\Pi_{1,m,0}z_1^m\Big].\end{aligned} \tag{15}$$

Setting $x = \bar{a} + a\bar{\theta}c(z_2)$ in (15) leads to

$$\begin{aligned}
&\bar{\theta}[1-c(z_2)]\Big[(\bar{a}+a\bar{\theta}c(z_2))\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\pi_{1,m,n}z_1^m z_2^n + a\bar{\theta}c(z_2)\sum_{m=1}^{\infty}\pi_{1,m,0}z_1^m\Big] = \\
= \quad & \frac{a\bar{\theta}^2 c(z_2)[1-c(z_2)]}{\bar{a}+a\bar{\theta}c(z_2)}P_0(\bar{a}+a\bar{\theta}c(z_2), z_1) + \\
+ \quad & c(z_1)S[\bar{a}+a\bar{\theta}c(z_2)]\Big[a\bar{\theta}[1-c(z_2)]\frac{1}{1-\bar{a}r}\Pi_1(z_2) - \\
- \quad & \alpha(r)\theta[\Pi_1(z_2) - c(z_2)\Pi_1(1)]\Big], \qquad (16)
\end{aligned}$$

and, finally, substituting (16) into (15), one has

$$\begin{aligned}
\bar{\theta}[1-c(z_2)]P(x,z_1,z_2) = \; & \frac{a\bar{\theta}^2 c(z_2)[1-c(z_2)]}{\bar{a}+a\bar{\theta}c(z_2)} \times \\
\times \quad & \frac{(\bar{a}+a\bar{\theta}c(z_2))P_0(x,z_1) - xP_0(\bar{a}+a\bar{\theta}c(z_2), z_1)}{x-[\bar{a}+a\bar{\theta}c(z_2)]} + \\
+ \quad & \frac{S(x)-S[\bar{a}+a\bar{\theta}c(z_2)]}{x-[\bar{a}+a\bar{\theta}c(z_2)]}xc(z_1)\Big[a\bar{\theta}[1-c(z_2)]\frac{1}{1-\bar{a}r}\Pi_1(z_2) - \\
- \quad & \alpha(r)\theta[\Pi_1(z_2) - c(z_2)\Pi_1(1)]\Big], \qquad (17)
\end{aligned}$$

the last formula can be written as

$$\begin{aligned}
P(x,z_1,z_2) \quad = \quad & \frac{a\bar{\theta}c(z_2)[S(x)-S(\bar{a})] - (x-a)[S[\bar{a}+a\bar{\theta}c(z_2)] - S(\bar{a})]}{(x-\bar{a})[x-[\bar{a}+a\bar{\theta}c(z_2)]]}\frac{x}{\bar{\theta}}k(z_1) + \\
+ \quad & \frac{S(x)-S[\bar{a}+a\bar{\theta}c(z_2)]}{x-[\bar{a}+a\bar{\theta}c(z_2)]}xc(z_1) \times \\
\times \quad & \Big[\frac{a}{1-\bar{a}r}\Pi_1(z_2) - \frac{\Pi_1(z_2)-c(z_2)\Pi_1(1)}{\bar{\theta}[1-c(z_2)]}\alpha(r)\theta\Big]. \qquad (18)
\end{aligned}$$

From eq. (16), with $z_1 = 1$, we obtain

$$\begin{aligned}
\Pi_1(z_2) \quad = \quad & \frac{1}{D_\Pi(z_2)}\Big[[1-c(z_2)]\Big[\bar{a}[S[\bar{a}+a\bar{\theta}c(z_2)] - S(\bar{a})] - a\bar{\theta}c(z_2)S(\bar{a})\Big] + \\
+ \quad & \theta c(z_2)[\bar{a}-S(\bar{a})]S[\bar{a}+a\bar{\theta}c(z_2)]\Big][1-\bar{a}r]a\bar{\theta}\pi_{0,0},
\end{aligned}$$

where $D_\Pi(z_2) = \Big[\bar{\theta}[1-c(z_2)]\big[(1-\bar{a}r)(\bar{a}+a\bar{\theta}c(z_2)) - aS[\bar{a}+a\bar{\theta}c(z_2)]\big] + \bar{a}\theta(1-r)S[\bar{a}+a\bar{\theta}c(z_2)]\Big]\bar{a}S(\bar{a})$.

In order to find the unknown constant $\pi_{0,0}$ the normalization condition $\pi_{0,0} + \varphi_0(1) + P_0(1,1) + P(1,1,1) = 1$ will be used.

The value of $P(1,1,1)$ can be found from formula (14), that for $z_2 = 1$, takes the form

$$P(1,1,1) = \Pi_1(1) + \frac{\Pi'(1) - c'(1)\Pi_1(1)}{a\bar{\theta}c'(1)}\alpha(r),$$

where $\Pi'(1)$ is given by

$$\begin{aligned}
\Pi'(1) &= \frac{1}{\bar{a}^3\theta S(\bar{a})S(\bar{a}+a\bar{\theta})(1-r)^2}\Big[\bar{a}\big[a\bar{\theta}S(\bar{a}) - \bar{a}[S(\bar{a}+a\bar{\theta}) - S(\bar{a})](1-r)\big] + \\
&+ (\bar{a} - S(\bar{a}))S(\bar{a}+a\bar{\theta})[\bar{a}\theta(1-r) - a\bar{\theta}] + \bar{\theta}(\bar{a} - S(\bar{a}))(1-\bar{a}r)(\bar{a}+a\bar{\theta})\Big] \times \\
&\times a\bar{\theta}(1-\bar{a}r)c'(1)\pi_{0,0}.
\end{aligned}$$

Now, after some algebra, the expression of $\pi_{0,0}$ is obtained:

$$\pi_{0,0} = \frac{\bar{a}^2 S(\bar{a}+a\bar{\theta})S(\bar{a})(1-r)}{D_\pi},$$

where $D_\pi = \bar{a}^2\theta S(\bar{a}+a\bar{\theta})(1-r) + \bar{\theta}(\bar{a} - S(\bar{a}))(\bar{a}+a\bar{\theta})(1-\bar{a}r) + \bar{a}\big[a\bar{\theta}S(\bar{a}) + \bar{a}[S(\bar{a}+a\bar{\theta}) - S(\bar{a})]\big]$.

The above results can be summarized in the following theorem:

Theorem 1. *The generating functions of the stationary distribution of the chain of the system are given by*

$$\begin{aligned}
\varphi_0(z) &= \frac{\bar{a}r}{1-\bar{a}r}\Pi_1(z), \\
\bar{\theta}P_0(x,z_1) &= \frac{S(x) - S(\bar{a})}{x - \bar{a}}xk(z_1), \\
P(x,z_1,z_2) &= \frac{a\bar{\theta}c(z_2)[S(x) - S(\bar{a})] - (x-a)[S[\bar{a}+a\bar{\theta}c(z_2)] - S(\bar{a})]}{(x-\bar{a})[x - [\bar{a}+a\bar{\theta}c(z_2)]]}\frac{x}{\bar{\theta}}k(z_1) + \\
&+ \frac{S(x) - S[\bar{a}+a\bar{\theta}c(z_2)]}{x - [\bar{a}+a\bar{\theta}c(z_2)]}xc(z_1) \times \\
&\times \Big[\frac{a}{1-\bar{a}r}\Pi_1(z_2) - \frac{\Pi_1(z_2) - c(z_2)\Pi_1(1)}{\bar{\theta}[1-c(z_2)]}\alpha(r)\theta\Big],
\end{aligned}$$

where

$$\begin{aligned}
\Pi_1(z) &= \frac{1}{D_\Pi(z)}\Big[[1-c(z)]\Big[\bar{a}[S[\bar{a}+a\bar{\theta}c(z)] - S(\bar{a})] - a\bar{\theta}c(z)S(\bar{a})\Big] + \\
&+ \theta c(z)[\bar{a} - S(\bar{a})]S[\bar{a}+a\bar{\theta}c(z)]\Big][1-\bar{a}r]a\bar{\theta}\pi_{0,0}, \\
k(z_1) &= \frac{a}{\bar{a}}\bar{\theta}c(z_1)\pi_{0,0} + \alpha(r)[\bar{\theta}\Pi_1(z_1) + \theta c(z_1)\Pi_1(1)],
\end{aligned}$$

$$\pi_{0,0} = \frac{\bar{a}^2 S(\bar{a}+a\bar{\theta})S(\bar{a})(1-r)}{D_\pi},$$

and $D_\Pi = \bar{a}^2\theta S(\bar{a}+a\bar{\theta})(1-r)+\bar{\theta}(\bar{a}-S(\bar{a}))(\bar{a}+a\bar{\theta})(1-\bar{a}r)+\bar{a}\big[a\bar{\theta}S(\bar{a})+\bar{a}[S(\bar{a}+a\bar{\theta})-S(\bar{a})]\big]$, $\alpha(r)=\frac{\bar{a}(1-r)}{1-\bar{a}r}$.

Corollary 1. 1. *The probability generating function of the number of customers in the server is given by*

$$\begin{aligned}\psi_1(z) &= \pi_{0,0}+\varphi_0(1)+P_0(1,z)+P(1,z,1)=\pi_{0,0}+\frac{1-S(\bar{a}+a\bar{\theta})}{a\theta\bar{\theta}}\times \\ &\times \Big[k(z)+c(z)\frac{a\bar{\theta}-(1-\bar{a}r)c'(1)\Pi_1(1)+(1-\bar{a}r)\Pi_1'(1)}{(1-\bar{a}r)c'(1)}\alpha(r)\Big].\end{aligned}$$

2. *The probability generating function of the number of customers in the orbit is given by*

$$\begin{aligned}\psi_2(z) &= \pi_{0,0}+P_0(1,1)+\varphi_0(z)+P(1,1,z)=\pi_{0,0}+\frac{1}{1-\bar{a}z}\Pi_1(z)+ \\ &+ \frac{c(z)\Pi_1(1)-\Pi_1(z)}{a\bar{\theta}[1-c(z)]}\alpha(r).\end{aligned}$$

3. *The probability generating function of the number of customers in the system is given by*

$$\Phi(z)=\psi_1(z)\cdot\psi_2(z).$$

Corollary 2. 1. *The mean number of customers in the server is given by*

$$E[N^{(1)}]=\psi_1'(1)=\frac{a}{\bar{a}}\bar{\theta}c'(1)\pi_{0,0}+\frac{(1-\bar{a}r)(1+\bar{\theta})\Pi_1'(1)+a\theta c'(1)\Pi_1(1)}{1-\bar{a}r}\alpha(r),$$

2. *The mean number of customers in the orbit is given by*

$$E[N^{(2)}]=\psi_2'(1)=\Pi_1'(1)+\frac{\Pi''(1)c'(1)-\Pi_1'(1)c''(1)}{2a\bar{\theta}c'(1)^2}\alpha(r).$$

where

$$\Pi''(1)=\frac{1}{\bar{a}\theta(1-r)S(\bar{a}+a\bar{\theta})}\Big[\frac{N''}{\bar{a}\theta(1-r)S(\bar{a}+a\bar{\theta})}-\Pi_1(1)D''-2\Pi_1'(1)D'\Big],$$

where

$$
\begin{aligned}
N'' &= c''(1)\Big[\theta(\bar{a}-S(\bar{a}))[S(\bar{a}+a\bar{\theta})+a\bar{\theta}S'(\bar{a}+a\bar{\theta})]+\\
&+ a\bar{\theta}S(\bar{a})-\bar{a}[S(\bar{a}+a\bar{\theta})-S(\bar{a})]\Big]+\\
&+ a\bar{\theta}c'(1)^2\Big[\theta(\bar{a}-S(\bar{a}))[2S'(\bar{a}+a\bar{\theta})+a\bar{\theta}S''(\bar{a}+a\bar{\theta})]+\\
&+ 2[S(\bar{a})-\bar{a}S'(\bar{a}+a\bar{\theta})]\Big],\\
D' &= \bar{\theta}c'(1)\Big[aS(\bar{a}+a\bar{\theta})-(1-\bar{a}r)(\bar{a}+a\bar{\theta})+a\bar{a}\theta(1-r)S'(\bar{a}+a\bar{\theta})\Big],\\
D'' &= \bar{\theta}c''(1)\Big[a\bar{a}\theta(1-r)S'(\bar{a}+a\bar{\theta})-(1-\bar{a}r)(\bar{a}+a\bar{\theta})+aS(\bar{a}+a\bar{\theta})\Big]+\\
&+ a\bar{\theta}c'(1)^2\Big[a\bar{a}\theta\bar{\theta}(1-r)S''(\bar{a}+a\bar{\theta})-2\theta[1-\bar{a}r-aS'(\bar{a}+a\bar{\theta})]\Big].
\end{aligned}
$$

3. *The mean number of customers in the system is given by*

$$E[N]=E[N^{(1)}]+E[N^{(2)}].$$

4. Busy period

A busy period (BP) is defined as the period starting with the arrival of a customer that finds the system empty and ends at the first service completion epoch at which the system becomes empty again.

This section considers the busy period of an auxiliary system in which the arriving customers go directly to the server, which will be useful to study the customers delay in the original system. Specifically, we will suppose that the probability of an arrival is $a\theta$, and as in the original model the arriving customer or group of customers expels out of the system the customers that are currently being served, if any. Let's denote by $h_k,\ \ k\geq 0$, the probability that the busy period of our auxiliary system lasts exactly k slots. Then we have:

$$
\begin{aligned}
h_0 &= 0,\\
h_k &= (1-a\theta)^k s_k+\sum_{i=1}^{k}(1-a\theta)^{i-1}s_i a\theta h_{k-i}+\\
&+ \sum_{i=1}^{k}(1-a\theta)^{i-1}S_{i+1}a\theta h_{k-i},\quad k\geq 1.
\end{aligned}
$$

The above formulae can be explained in the following way:

1. In the first $k-1$ slots no new customer arrives (with probability $(1-a\theta)^{k-1}$), and in the slot k the customer, or group of customers, that opened the BP finishes his service (with probability s_k) and no new customers arrive (with probability $1-a\theta$).
2. In the first $i-1$ slots no new customer arrives (with probability $(1-a\theta)^{i-1}$), and in the slot i: the customer, or group of customers, that opened the BP finishes his service (with probability s_i), a new customer, or group of customers, arrives (with probability $a\theta$) and the BP opened by this new customer, or group of customers, lasts $k-i$ slots (with probability h_{k-i}).
3. With probability S_{i+1} the service of the customer, or group of customers, that opened the BP lasts not less of $i+1$ slots, in the first $i-1$ slots no new customer arrives (with probability $(1-a\theta)^{i-1}$), and in the slot i a new customer, or group of customers, arrives expelling out of the system the customer, or group of customers, that are currently being served, (with probability $a\theta$) opening a new BP of length $k-i$ slots (with probability h_{k-i})

A recursive procedure of the above formula can lead to obtain numerically the distribution $\{h_k,\, k \geq 0\}$ but in order to find the moments of the distribution we will use the GF $h(x) = \sum_{k=0}^{\infty} h_k x^k$, that is given by

$$\begin{aligned} h(x) &= S[(1-a\theta)x] + \frac{a\theta}{1-a\theta} S[(1-a\theta)x]h(x) + \\ &+ \frac{a\theta}{1-a\theta} \frac{(1-a\theta)x - S[(1-a\theta)x]}{1-(1-a\theta)x} h(x), \end{aligned}$$

that is

$$h(x) = \frac{[1-(1-a\theta)x]S[(1-a\theta)x]}{1-x+a\theta x S[(1-a\theta)x]}.$$

The mean length of a busy period is given by

$$\bar{h} = h'(1) = \frac{1-S(1-a\theta)}{a\theta S(1-a\theta)}.$$

In order to find the generating function of the sojourn time that a customer spends in the orbit, the GF $h(x;i)$ of the distribution of the busy period that starts with a customer in the server to which remains i slots to finish its service will be needed. This GF has the following expression

$$h(x;i) \quad = \quad \frac{[(1-a\theta)x]^i}{1-a\theta}[1-a\theta+a\theta h(x)] + a\theta x \frac{1-[(1-a\theta)x]^{i-1}}{1-(1-a\theta)x} h(x),\ i \geq 1.$$

Let us explain the above formula:

If after the first $i-1$ slots no customer arrives to the system and in the slot i, either a new customer does not arrive and then the BP ends with probability $1-a\theta$, or another customer arrives and then with probability $a\theta$ a new BP is opened with GF $h(x)$. This accounts for the first term of the right hand side of the formula.

Now, with respect to the second term, if after $k-1$ slots, $k = 1,\ldots,i-1$, a new customer does not arrive (with probability $(1-a\theta)^{k-1}$) and in the slot k a new customer arrives (with probability $a\theta$), a BP is opened with GF $h(x)$. Summing over k from 1 to $i-1$, the given formula of $h(x;i)$ is obtained.

5. Sojourn times

5.1. Sojourn time of a customer in the server

In this section the distribution of the time that a customer spends in the server will be obtained. With this aim let b_k be the probability that the sojourn time of a customer in the server lasts exactly k slots. The distribution $\{b_k,\ k \geq 0\}$ is given by

$$\begin{aligned} b_0 &= 0, \\ b_k &= (1-a\theta)^{k-1}s_k + a\theta(1-a\theta)^{k-1}S_{k+1},\ \ k \geq 1. \end{aligned}$$

The corresponding GF $b(x) = \displaystyle\sum_{k=0}^{\infty} b_k x^k$ is given by

$$b(x) = \frac{a\theta x + (1-x)S[(1-a\theta)x]}{1-(1-a\theta)x},$$

and the mean time that a customer spends in the server is

$$\bar{b} = b'(1) = \frac{1-S(1-a\theta)}{a\theta}.$$

5.2. Sojourn time of a customer in the orbit

The GF of the stationary distribution of the waiting time of a customer in the orbit is given by

$$
\begin{aligned}
W(x) &= \pi_{0,0}+\varphi_0(1)+\theta[P_0(1,1)+P(1,1,1)]+\\
&+ (1-\theta)\sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\pi_{i,m,n}\omega(x).
\end{aligned}
$$

The above formula has the following explanation:

An arriving customer spends 0 slots in the orbit with probability $\pi_{0,0}+\varphi_0(1)+\theta[P_0(1,1)+P(1,1,1)]$ and with probability $(1-\theta)\pi_{i,m,n}$, $i,m\geq 1$, $n\geq 0$, it finds m customers in the server with a remaining service time of i slots, and n customers in the orbit. Then this customer will wait in the orbit till the beginning of its service for a period of time with GF $h(x;i)\omega(x)$, where $\omega(x)$ is the GF of the elapsed time in the orbit since the ending of the BP $h(x;i)$.

The GF $W(x)$ can be expressed in the form:

$$
\begin{aligned}
W(x) &= \pi_{0,0}+\varphi_0(1)+\theta[P_0(1,1)+P(1,1,1)]+\\
&+ (1-\theta)\Big[\frac{a\theta h(x)}{1-(1-a\theta)x}[P_0(1,1)+P(1,1,1)]+\\
&+ \frac{1-x(1-a\theta+a\theta h(x))}{1-(1-a\theta)x}\big[P_0[(1-a\theta)x,1]+P[(1-a\theta)x,1,1]\big]\Big]\omega(x).
\end{aligned}
$$

In order to find the GF $\omega(x)$, we will denote by ω_k, $k\geq 0$, the probability that the sojourn time of a customer in the orbit since the end of a BP lasts exactly k slots, then we have

$$
\begin{aligned}
\omega_0 &= 1-r,\\
\omega_k &= (\bar{a}\,r)^k(1-r)+(1-\delta_{1,k})a\,r\sum_{l=1}^{k-1}(\bar{a}\,r)^{l-1}\sum_{i=1}^{k-l}h_i\omega_{k-i-l},\ k\geq 1,
\end{aligned}
\tag{19}
$$

where $\delta_{a,b}$ is Kronecker's delta.

The GF $\omega(x)$ of the distribution $\{\omega_k,\ k\geq 0\}$ is given by

$$\omega(x)=\sum_{k=0}^{\infty}\omega_k x^k=\frac{1-r}{1-rx[\bar{a}+a\,h(x)]},$$

with mean

$$\bar{\omega} = \omega'(1) = \frac{r(1 + a\,\bar{h})}{1 - r}.$$

Let us explain the formula (19):

Consider the slot in which a BP ends, say the slot 0, the customers in the orbit will spend there 0 slots if in the slot 0 no customer arrives to the server (which is ensured by the fact that in the slot 0 a BP ended) and a retrial occurs (with probability $1 - r$).

The customers in the orbit will wait exactly k, $k \geq 1$, slots since the ending of a BP, if: in the slot 0 no customer arrives to the server (which is insured by the fact that the BP has ended in the slot 0) and no retrial occurs (with probability r), and in the following $k - 1$ slots no new customers arrive to the server and no retrials occur (with probability $(\bar{a}\,r)^{k-1}$) and in the slot k no new customer arrives to the server and a retrial occurs (with probability $\bar{a}\,(1 - r)$), this accounts for the first term of the formula.

Now, let us explain the second term of the formula (19): before the slot l, $1 \leq l \leq k - 1$, no customer arrives to the system and no retrial occurs (with probability $\bar{a}^{\,l-1}\,r^{l}$), in the slot l a new customer, or group of customers, arrives to the system opening a BP with length of i slots, and once this BP is finished the customers in the orbit will wait there till the beginning of their service a period of time of $k - l - i$ slots (all with probability $a\,h_i\,\omega_{k-l-i}$).

5.3. Sojourn time of a customer in the system

The GF $V(x)$ of the stationary distribution of the sojourn time of a customer in the system is given by

$$V(x) = W(x)\,b(x),$$

and the corresponding mean time is given by

$$\bar{V} = V'(1) = \bar{W} + \bar{b}.$$

6. Conclusions and research results

In this paper a discrete-time retrial queueing system has been studied. Customers arrive in batches according to a geometrical law and the service times are general. If an arriving group of customers finds the server free, it begins jointly and simultaneously its service, otherwise the group can opt with a certain probability to go directly to the server expelling out of the system the customers that are in the server or to join the orbit. The retrials from the orbit are made jointly by all the customers in the orbit.

A thorough analysis of the model has been carried out, obtaining generating functions for the distributions of the number of customers in the server and in the orbit, but the main research contribution of the paper is a complete study of the sojourn time of a customer in the orbit not only for its relevance concerning to this system but also because it opens a new research approach to its treatment in more general discrete-time retrial systems.

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