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Dynamical boundary value problem for a viscoelastic half-space with cut

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Abstract. The dynamical boundary value problem for viscoelastic half-space with cut in the form of a strip is considered. The problem is reduced to the singular integral equation of first kind. Using the method of orthogonal polynomials, the integral equation is reduced to an infinite system of linear algebraic equations. The quasi-completely regularity of the obtained system is proved and the reduction method for approximate solution is developed.

Mathematics Subject Classification. 74B05, 74K20, 74K15.

1. Statement of the problem

It is investigated the dynamical contact problem for a viscoelastic half-space ($-\infty < x$, $y > 0$, $z < \infty$) with cut in the form of strip ($0 \le y \le b$, $-\infty < z < \infty$) lying in the plane $x = 0$. The border of the cut is under the action of uniformly distributed shearing harmonic (acting along the OZ axis) load of intensity $\tau_{xz} = \tau_0 e^{-ikt}$, k is oscillation frequency, t is time parameter. In the linear theory of viscoelasticity for Kelvin-Voigt materials only displacement component $\omega = \omega(x, y, t)$ and tangential stresses components $\tau_{yz} = G\frac{\partial\omega}{\partial y} + G_0\frac{\partial\dot{\omega}}{\partial y}$, $\tau_{xz} = G\frac{\partial\omega}{\partial x} + G_0\frac{\partial\dot{\omega}}{\partial x}$ are other than zero (so called anti-plane deformation), where G and G_0 are the elastic and viscoelastic shear modulus, respectively. The dot means a derivative with respect to the variable t. $\dot{\omega} \equiv \frac{\partial\omega}{\partial t}$ (Fig. 1).

The problem is equivalent to the boundary value problem

$$G\Delta\omega + G_0\Delta\dot{\omega} = \rho\ddot{\omega}, \qquad |x| < \infty, \quad y > 0, \qquad \frac{\partial\omega(x,0,t)}{\partial y} + \frac{\partial\dot{\omega}(x,0,t)}{\partial y} = 0$$

(this equation is satisfied everywhere in cutting half-space). ρ is the material density of the half-space [2,3,5,6,9,16].

The displacements at the cut boundaries have discontinuities and the cut boundaries are loaded with shear harmonic stresses, that's why we have the following conditions

$$\begin{aligned} &< \omega(0,y,t) > \equiv \omega(-0,y,t) - \omega(+0,y,t) = \varphi(y,t), \qquad 0 < y \le 1, \quad \varphi(y,t) = 0, \quad y > 1 \\ &G\omega'(-0,y,t) + G_0\dot{\omega}'(-0,y,t) = G\omega'(+0,y,t) + G_0\dot{\omega}'(+0,y,t) = \tau_0 e^{-ikt}, \qquad 0 \le y \le 1. \end{aligned} \tag{1}$$

2. Reduction of the problem to the integral equation

Considering stationary oscillations of the half-space, we assume that

$$\omega(x,y,t) = \omega_0(x,y)e^{-itk}, \qquad \varphi(y,t) = \varphi_0(y)e^{-ikt}$$

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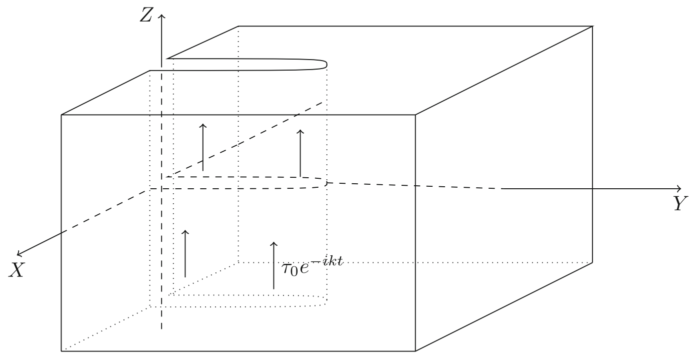

FIG. 1. Sketch of the posed problem

Thus one obtains the following boundary value problem

$$\begin{aligned} &(G - ikG_0)\Delta\omega_0 = -\rho k^2\omega_0, \qquad |x| < \infty, \qquad y > 0, \qquad \frac{\partial\omega_0(x,0)}{\partial y} = 0 \\ &< \omega_0(0,y) >= \varphi_0(y), \qquad 0 < y < 1, \quad \varphi_0(y) \equiv 0, \quad y \geq 1 \end{aligned} \tag{2}$$

Multiplying equations (2) by $e^{i\alpha x}$ and integrating by parts separately on the intervals $(-\infty, 0)$ and $(0, \infty)$, for Fourier transform we obtain the one-dimensional boundary value problem [11–14]

$$\omega''_\alpha(y) - (\alpha^2 - k_0^2)\omega_\alpha(y) = f(y), \qquad 0 < y < \infty, \quad \omega'_\alpha(0) = 0 \tag{3}$$

where

$$k_0^2 = \frac{\rho k^2}{\widetilde{G}}, \qquad f(y) = i\alpha\varphi_0(y), \widetilde{G} = G - ikG_0.$$

The decreasing at infinity fundamental function of equation (3) is defined by the methods of integral transformations and contour integration. Since the Green's function $G_\alpha(y, \eta)$ of the boundary value problem (3) must satisfy the equation $G_\alpha(0, \eta) = 0$, it can be constructed in the form of a simple combination of the above-mentioned fundamental function, that is,

$$G_\alpha(y, \eta) = \Phi(y, \eta) + \Phi(y, -\eta).$$

Thus

$$\omega_\alpha(y) = \int_0^1 [\Phi(y,\eta) + \Phi(y,-\eta)]\, f(\eta)\, \mathrm{d}\eta = \int_{-1}^1 \Phi(y,\eta) f(\eta)\, \mathrm{d}\eta.$$

We have taken into account the fact that the right-hand side of equation (3) is equal to zero for $y > 1$, and its continuation is verified in even form by negative values of the argument.

Consequently, a solution of the boundary value problem (3) can be represented in the form

$$\omega_\alpha(y) = \int_{-1}^1 \frac{\alpha e^{-\sqrt{\alpha^2-k_0^2}|y-\eta|}}{2i\sqrt{\alpha^2 - k_0^2}} \varphi_0(\eta)\, \mathrm{d}\eta = \frac{\alpha}{2i(\alpha^2 - k_0^2)} \int_{-1}^1 \frac{\operatorname{sgn}(y-\eta)}{e^{\sqrt{\alpha^2-k_0^2}|y-\eta|}} \varphi'_0(\eta)\, \mathrm{d}\eta$$

Here it is taken into account that $\varphi_0(\pm 1) = 0$ and it is assumed that $\gamma(\alpha) = \sqrt{\alpha^2 - k_0^2} \to |\alpha|$, as $|\alpha| \to \infty$, and when k_0 is real number $\sqrt{\alpha^2 - k_0^2} = -i\sqrt{k_0^2 - \alpha^2}$, that is, the real axis of the complex plane $z = \alpha + i\sigma$ goes around the branch points $-k_0$ from above and k_0 from below.

Using inverse transformation the functions $\omega_0(x,y)$ and $\frac{\partial \omega_0(x,y)}{\partial x}$ are represented so

$$\begin{aligned}\omega_0(x,y) &= -\int_{-1}^{1} \varphi_0'(\eta)\,\mathrm{d}\eta \frac{1}{2\pi}\int_0^\infty \frac{\alpha e^{-\sqrt{\alpha^2-k_0^2}|y-\eta|}\mathrm{sgn}(y-\eta)}{\alpha^2-k_0^2}\sin\alpha x\,\mathrm{d}\alpha\\ &= -\frac{1}{2\pi}\int_{-1}^{1}\varphi_0'(\eta)\,\mathrm{d}\eta\left\{\int_0^\infty\left(\frac{\alpha e^{-\sqrt{\alpha^2-k_0^2}|y-\eta|}}{(\alpha^2-k_0^2)}-\frac{e^{-\alpha|y-\eta|}}{\alpha}\right)\mathrm{sgn}(y-\eta)\sin\alpha x\,\mathrm{d}\alpha\right\}\\ &\quad -\frac{1}{2\pi}\int_{-1}^{1}\varphi_0'(\eta)\,\mathrm{d}\eta\int_0^\infty \frac{e^{-\alpha|y-\eta|}\mathrm{sgn}(y-\eta)}{\alpha}\sin\alpha x\,\mathrm{d}\alpha \end{aligned} \tag{4}$$

$$\begin{aligned}\frac{\partial\omega_0(x,y)}{\partial x} &= -\frac{1}{2\pi}\int_{-1}^{1}\varphi_0'(\eta)\,\mathrm{d}\eta\left\{\int_0^\infty\left(\frac{\alpha^2 e^{-\sqrt{\alpha^2-k_0^2}|y-\eta|}}{(\alpha^2-k_0^2)}-e^{-\alpha|y-\eta|}\right)\mathrm{sgn}(y-\eta)\cos\alpha x\,\mathrm{d}\alpha\right\}\\ &\quad -\frac{1}{2\pi}\int_{-1}^{1}\varphi_0'(\eta)\,\mathrm{d}\eta\int_0^\infty \frac{e^{-\alpha|y-\eta|}\cos\alpha x}{\mathrm{sgn}(y-\eta)}\,\mathrm{d}\alpha. \end{aligned} \tag{5}$$

Since the integrand of the interior integral in formula (4) can have at infinity the behavior α^{-1}, its Fourier transformation (in a sense of the theory of generalized functions) is represented as a sum of its principal and regular parts.

Calculating the last integral in formula (4), we obtain

$$\omega_0(x,y) = -\frac{1}{2\pi}\int_{-1}^{1} R(x,|y-\eta|)\varphi_0'(\eta)\,\mathrm{d}\eta - \frac{1}{2\pi}\int_{-1}^{1}\arctan\frac{x}{y-\eta}\varphi_0'(\eta)\,\mathrm{d}\eta,$$

where

$$R(x,|y-\eta|) = \int_0^\infty\left(\frac{\alpha e^{-\sqrt{\alpha^2-k_0^2}|y-\eta|}}{(\alpha^2-k_0^2)}-\frac{e^{-\alpha|y-\eta|}}{\alpha}\right)\mathrm{sgn}(y-\eta)\sin\alpha x\,\mathrm{d}\alpha$$

and from (5) we have:

$$\frac{\partial\omega_0(0,y)}{\partial x} = \frac{1}{2\pi}\int_{-1}^{1}R_0(|y-\eta|)\varphi_0'(\eta)\,\mathrm{d}\eta - \frac{1}{2\pi}\int_{-1}^{1}\frac{\varphi_0'(\eta)\,\mathrm{d}\eta}{y-\eta} \tag{6}$$

where

$$R_0(|y-\eta|) = \int_0^\infty\left(\frac{\alpha^2 e^{-\sqrt{\alpha^2-k_0^2}|y-\eta|}}{(\alpha^2-k_0^2)} - e^{-\alpha|y-\eta|}\right)\mathrm{sgn}(y-\eta)\,\mathrm{d}\alpha.$$

Taking into account the condition (1) and formula (6), we obtain the following singular integral equation of the first kind [10]

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{\varphi_0'(\eta)\,\mathrm{d}\eta}{\eta-y} + \int_{-1}^{1}R_0(|y-\eta|)\varphi_0'(\eta)\,\mathrm{d}\eta = \frac{\tau_0}{\widetilde{G}} \tag{7}$$

3. Reduction of integral Equation (7) to the infinite system of linear algebraic equations

A solution of the integral equation (7) will be sought in the form

$$\varphi_0'(\eta) = \frac{1}{\sqrt{1-\eta^2}} \sum_{m=1}^{\infty} a_m T_m(\eta) \tag{8}$$

where $T_m(\eta)$ is the first kind Chebyshev orthogonal polynomial, $\{a_n\}_{n\geq 1}$ is unknown sequences.

Using the following known spectral relation

$$\frac{1}{\pi}\int_{-1}^{1} \frac{T_m(\eta)\,\mathrm{d}\eta}{(\eta - y)\sqrt{1-\eta^2}} = U_{m-1}(y), \qquad -1 < y < 1$$

from integral equation (7) we have [15]

$$\sum_{m=1}^{\infty} a_m U_{m-1}(y) + 2\sum_{m=0}^{\infty} a_m \int_{-1}^{1} \frac{R_0(|y-\eta|)T_m(\eta)\,\mathrm{d}\eta}{\sqrt{1-\eta^2}} = \frac{2\tau_0}{\widetilde{G}}$$

where $U_{m-1}(y)$ are the second kind Chebyshev orthogonal polynomials.

Multiplying both parts of the above equality by $(1-y^2)^{1/2}U_{n-1}(y)$ integrating in the interval $(-1, 1)$ and on the virtue of orthogonality of the Chebyshev polynomials, one obtains the infinite system of linear algebraic equations

$$a_n + \sum_{m=1}^{\infty} R_{nm} a_m = f_n, \qquad n = 1, 2, 3, \ldots \tag{9}$$

where

$$R_{nm} = \frac{4}{\pi}\int_{-1}^{1} (1-y^2)^{1/2} U_{n-1}(y) \left(\int_{-1}^{1} R_0(|y-\eta|) \frac{T_m(\eta)\,\mathrm{d}\eta}{\sqrt{1-\eta^2}} \right) \mathrm{d}y,$$

$$f_n = \frac{4\tau_0}{\pi\widetilde{G}}\int_{-1}^{1} (1-y^2)^{1/2} U_{n-1}(y)\,\mathrm{d}y = \begin{cases} \frac{2\tau_0}{\widetilde{G}}, & n = 1 \\ 0, & n \neq 1 \end{cases}. \tag{10}$$

Let us prove that the system (9) is quasi-complete regularity in the class of bounded sequences. Really, the matrix elements R_{nm} can be written as

$$R_{nm} = \frac{1}{m} h_{nm},$$

where

$$h_{nm} = \frac{4}{\pi}\int_{-1}^{1}\int_{-1}^{1} \frac{\partial R_0(|y-\eta|)}{\partial \eta} \sqrt{1-\eta^2}\sqrt{1-y^2}\, U_{m-1}(\eta) U_{n-1}(y)\,\mathrm{d}y\,\mathrm{d}\eta$$

Then we have:

$$S_n = \sum_{m=1}^{\infty} |R_{nm}| = \sum_{m=1}^{\infty} \frac{1}{m}|h_{nm}|, \qquad n = 1, 2, \ldots$$

Note that the coefficients $\{h_{nm}\}_{n,m=1}^{\infty}$ are the Fourier coefficients of the square summable function $\partial R_0(|y-\eta|)/\partial\eta$ in the square $-1 \leq y,\ \eta \leq 1$ with respect to the complete orthogonal system of the functions $\{U_{n-1}(\eta)U_{m-1}(y)\}_{n,m=1}^{\infty}$. Based on Bessel's inequality we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|h_{nm}|^2<\infty.$$

Then, according to the well-known theorem of analysis, the following series is also convergent

$$\sum_{n=1}^{\infty}C_n, \qquad C_n=\sum_{m=1}^{\infty}|h_{nm}|^2,$$

respectively, at least $C_n=O(1/n^{1+\delta})$, $n\to\infty$ ($\delta>0$ is small positive number).

Then based on Cauchy-Bunyakowsky inequality we have

$$S_n\le\sqrt{\sum_{m=1}^{\infty}\frac{1}{m^2}}\sqrt{\sum_{n=1}^{\infty}|h_{nm}|^2}=\frac{\pi}{\sqrt{6}}\sqrt{C_n}$$

and sum S_n satisfies the following estimation:

$$S_n=O\left(n^{-\frac{1+\delta}{2}}\right), \qquad n\to\infty. \tag{11}$$

Thus, by virtue of (10), (11) the quasi-complete regularity of system (8) in the class of bounded sequences is proved.

On the other hand, based on the Rodrigues formula for Jakob's orthogonal polynomials and using partial integration we have the following representation

$$\begin{aligned}R_{nm}&=\frac{4}{\pi}\int_{-1}^{1}(1-y^2)^{1/2}U_{n-1}(y)\left(\int_{-1}^{1}R_0(|y-\eta|)\frac{T_m(\eta)\,\mathrm{d}\eta}{\sqrt{1-\eta^2}}\right)\mathrm{d}y\\&=\frac{\delta_{nm}}{(n-1)(n-2)m(m-1)}\int_{-1}^{1}(1-y^2)^{5/2}P_{n-3}^{(5/2,5/2)}(y)\\&\quad\left(\int_{-1}^{1}\frac{\partial^4R_0(|y-\eta|)}{\partial y^2\partial\eta^2}(1-\eta^2)^{3/2}P_{m-2}^{(3/2,2/2)}(\eta)\,\mathrm{d}\eta\right)\mathrm{d}y,\\&\quad n\ge4,\quad m\ge3\end{aligned} \tag{12}$$

where

$$\delta_{nm}=\frac{\Gamma(n+1)\Gamma(m+1)}{8\Gamma(n+0.5)\Gamma(m+0.5)},$$

By using Stirling formula [1] for $\Gamma(z)$ function we obtain

$$\delta_{nm}=\begin{cases}n^{1/2}, & n\to\infty\\ m^{1/2}, & m\to\infty\end{cases} \tag{13}$$

Based on the relations (12), (13) and Darboux asymptotic formula for R_{nm} we obtain the following estimates at least

$$R_{nm}=\begin{cases}O(n^{-2}), & n\to\infty\\ O(m^{-2}), & m\to\infty\end{cases}$$

Accordingly, the following conditions hold

$$\sum_{m=3,n=4}^{\infty}|R_{nm}|^2<\infty, \qquad \sum_{n=1}^{\infty}|f_n|^2<\infty. \tag{14}$$

The conditions (14) prove that the infinite systems (9) is quasi-completely regular in space l_2, that is, their solutions satisfy the condition $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ [7,8].

The results of [7] are applicable to an infinite system (9). On the basis of this fact, the system

$$a_n^N + \sum_{m=1}^{N} R_{nm} a_m^N = \tau_0 \frac{g_n}{\delta_n}, \qquad n = 1, 2, \ldots, N, \tag{15}$$

is solvable for sufficiently large N and convergence of approximate solutions $\{a_n^N\}_{n=1,\ldots,N}$ to exact solution $\{a_n\}_{n\geq 1}$ is valid in the sense of the norm of the space l_2.

The convergence rate is determined by the inequality

$$\|a - \varphi_0^{-1} \bar{a}^N\|_{l_2} \leq C_1 \left[\sum_{n=N+1}^{\infty} \sum_{m=1}^{\infty} |R_{nm}|^2\right]^{1/2} + C_2 \left(\frac{\sum_{n=N+1}^{\infty} |f_n|^2}{\sum_{n=1}^{\infty} |f_n|^2}\right)^{1/2},$$

where $a = \{a_n\}_{n\geq 1} = (a_1, a_2, \ldots, a_n, \ldots)$ is the solution of the system (8), $\bar{a}^N = (a_1^N, a_2^N, \ldots, a_N^N)$ is the solution of the system (15), $\varphi_0^{-1} \bar{a}^N = (a_1^N, a_2^N, \ldots, a_N^N, 0, 0, \ldots)$.

Considering the expression for R_{nm} and f_n we have

$$C_1 \left[\sum_{n=N+1}^{\infty} \sum_{m=1}^{\infty} |R_{nm}|^2\right]^{1/2} \leq C_1^* \left[\sum_{n=1}^{\infty} \frac{1}{(n+N)^4}\right]^{1/2} = C_1^* [\zeta(4, N)]^{1/2}, \qquad C_2 \left(\frac{\sum_{n=N+1}^{\infty} |f_n|^2}{\sum_{n=1}^{\infty} |f_n|^2}\right) = 0,$$

where $\zeta(s, N)$ is known generalized Zeta function.

Using the following asymptotic formula of the Zeta function [4]

$$\zeta(2m, N) \equiv \sum_{n=1}^{\infty} \frac{1}{(n+N)^{2m}} = \frac{N^{-2m+1}}{2m-1} + \frac{1}{2} N^{-2m} + \sum_{k=1}^{\infty} B_{2k} \frac{\Gamma(N+2k+1)}{(2k)! N^{2k+2m+1}} + O(N^{-2m-2N-1}),$$

for convergence rate we obtain

$$\|a - \varphi_0^{-1} \bar{a}^N\|_{l_2} \leq C N^{-3/2}$$

Thus, the solutions of the system (9) can be constructed by the reduction method with any accuracy [7,8]. The following theorem holds:

Theorem 3.1. *The infinite system of linear algebraic equations* (9) *is quasi-completely regular in the space* l_2.

Consequently, the integral equation (7) *has the unique solution in the form* (8).

The intensity factor of stress $\tau_{yz}(0, y, t)$ *at the end of cut is calculated using the formula*

$$K_I + i K_{II} = \lim_{y \to 1+} \sqrt{1-y}\, \tau_{yz}(0, y, t) = \frac{e^{-ikt}}{\sqrt{2}} (G - ikG_0) \sum_{n=1}^{\infty} a_n$$

4. Discussion and numerical results

Let us consider the following parameters for the problem:

$$G = 80 \cdot 10^9\,\text{Pa}, \quad G_0 = 65 \cdot 10^9\,\text{Pa}, \quad \tau_0 = 1\,\text{Pa}, \quad \rho = 2700\,\text{kg/m}^3, \quad k = 3\,\text{Hz}, \tag{16}$$

$$G = 65 \cdot 10^9\,\text{Pa}, \quad G_0 = 50 \cdot 10^9\,\text{Pa}, \quad \tau_0 = 1\,\text{Pa}, \quad \rho = 2700\,\text{kg/m}^3, \quad k = 3\,\text{Hz}, \tag{17}$$

$$G = 55 \cdot 10^9\,\text{Pa}, \quad G_0 = 40 \cdot 10^9\,\text{Pa}, \quad \tau_0 = 1\,\text{Pa}, \quad \rho = 2700\,\text{kg/m}^3, \quad k = 3\,\text{Hz}, \tag{18}$$

The corresponding reduced system consisting of 20 and 25 equations was solved by MATLAB to obtain the numerical results of present problem. Increasing the number of equations leads to a change in the eleven order after the decimal point in the solution, which confirms the rapid convergence of the process..

For complex factors of the intensity of the stress $\tau_{yz}(0, y, t)$, we have

$$K_I + iK_{II} = 0.14142135 - 0.34471461i$$
$$K_I + iK_{II}= 0.12031532 - 0.31280561i$$
$$K_I + iK_{II}= 0.11230531 - 0.30331562i$$

Absolute value of the complex coefficients of the stress $\tau_{yz}(0, y, t)$, respectively, to (16), (17, (18))

$$K = 0.37259652$$
$$K= 0.33514642$$
$$K= 0.32343909$$

Calculations also showed that the decrease of the values G and G_0 leads to a decrease of the intensity coefficient, what was to be expected naturally.

5. Conclusion

The stated boundary value problem is reduced to a one-dimensional boundary value problem for an ordinary differential equation of the second order using contour integration and integral transformation. This method allows us to represent the solution of the problem by an unknown jump of the displacement of the boundary points of the crack as a sum of the main and regular parts in an explicit form in quadratures, consequently, the problem is reduced to a singular integral equation of the first kind. The resulting integral equation is investigated by the method of orthogonal polynomials in order to obtain an approximate solution of an equivalent infinite system of linear algebraic equations. The advantage of this approach is that along with proving the existence of a solution to the problem, we obtain its structure, asymptotics and the rate of convergence of the approximate solution to the exact one.

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Declarations
Conflict of interest The authors declare no conflict of interest.

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References

[1] Abramowitz, M., Stegun, I.A.: Handbook of mathematical functions with formulas, graphs, and mathematical tables. Dover Publications, Inc., New York, 1992. Reprint of the 1972 edition
[2] Altenbach, H.: Creep analysis of thin-walled structures. ZAMM-J. Appl. Math. Mech./Zeitschrift für Angewandte Mathematik und Mechanik: Applied Mathematics and Mechanics **82**(8), 507–533 (2002)
[3] Altenbach, H., Gorash, Y., Naumenko, K.: Steady-state creep of a pressurized thick cylinder in both the linear and the power law ranges. Acta Mech. **195**, 263–274 (2008)
[4] Beitman, G., Erdein, A.: Higher Transcendent Functions, vol. 1. Nauka, Moscow (1973). (**in Russian**)
[5] Bland, D.R.: The theory of linear viscoelasticity, volume. International Series of Monographs on Pure and Applied Mathematics, vol. 10. Pergamon Press, New York-London-Oxford-Paris (1960)
[6] Fabrizio, M., Morro, A.: Mathematical Problems in Linear Viscoelasticity. Number 12 in Studies in Applied and Numerical Mathematics. Society for Industrial and Applied Mathematics (1992)
[7] Kantorovich, L.V., Akilov, G.P.: Functional analysis. Izdat. "Nauka", Moscow, second edition (1977)
[8] Kantorovič, L.V., Krylov, V.I.: Approximate Methods of Higher Analysis, corrected Gosudarstv. Izdat. Fiz.-Mat. Lit, Moscow-Leningrad (1962)
[9] Kupradze, V.D., Burchuladze, T.V.: The dynamical problems of the theory of elasticity and thermoelasticity. J. Math. Sci. **7**, 415–500 (1977)
[10] Muskhelishvili, N.I.: Singular integral equations. Wolters-Noordhoff Publishing, Groningen, 1972. Boundary problems of functions theory and their applications to mathematical physics, Reprinted
[11] Shavlakadze, N.: Dynamical contact problem for an elastic half-space with a rigid and elastic inclusion. Proc. A. Razmadze Math. Inst **159**, 87–94 (2012)
[12] Shavlakadze, N.: The effective solution of two-dimensional integro-differential equations and their applications in the theory of viscoelasticity. ZAMM Z. Angew. Math. Mech. **95**(12), 1548–1557 (2015)
[13] Shavlakadze, N., Odishelidze, N., Criado-Aldeanueva, F.: The investigation of singular integro-differential equations relating to adhesive contact problems of the theory of viscoelasticity. Z. Angew. Math. Phys. **72**(2), 15 (2021)
[14] Snedon, I.: Fourier Transformation. Inostr. Literat, Moscow (1955). (**in Russian**)
[15] Szego, G.: Orthogonal Polynomials. Fizmatgiz, Moscow (1962). (**in Russian**)
[16] Von Christensen, R.M.: Theory of Viscoelasticity. Academic Press, New York (1971)

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