

The hyperdeterminant vanishes on all but two Schur functors

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Abstract

We recall the notion of hyperdeterminant of a multidimensional matrix (tensor). We prove that if we restrict the hyperdeterminant to a skew-symmetric tensor $\bigwedge^p V \subseteq V^{\otimes p}$ with $p \geq 3$ then it vanishes. The hyperdeterminant also vanishes when we restrict it to the space $\Gamma_\lambda \otimes S_\lambda V \subseteq V^{\otimes p}$ where λ is a Young diagram with p boxes and $\lambda_2 \geq 2$ or $\lambda_3 \geq 1$.

Keywords: hyperdeterminant, degenerate multidimensional matrices, Schur functor

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1. Introduction

In [1] A. Cayley described for the first time the notion of hyperdeterminant of a multidimensional matrix (tensor) which is analogous to the determinant of a square matrix. In [5] L. Oeding computed the hyperdeterminant for symmetric
5 tensors, $Sym^p V \subseteq V^{\otimes p}$, describing all its irreducible factors. G. Ottaviani proposed in [6] to compute the irreducible factors of the hyperdeterminant of a skew-symmetric tensor $\bigwedge^p V \subseteq V^{\otimes p}$. In this paper we give a solution not only for skew-symmetric tensors but also for any Schur functor $\Gamma_\lambda \otimes S_\lambda V \subseteq V^{\otimes p}$ where λ is a Young diagram with $|\lambda| = p$ and $\lambda_2 \geq 2$ or $\lambda_3 \geq 1$.

10 **Main Theorem.** *When $A \in \Gamma_\lambda \otimes S_\lambda V \subseteq V^{\otimes p}$ and $p \geq 2$, $Det(A)$ can be*

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nonzero only for $\lambda = (p)$ (corresponding to the symmetric power $\text{Sym}^p V$) and $\lambda = (p-1, 1)$, where Γ_λ (resp. $S_\lambda V$) is the Σ_p -module (resp. $GL(V)$ -module) associated to λ .

The case $p = 2$ of *Main Theorem* corresponds to skew symmetric matrices in $\bigwedge^2 V$. In this case the determinant restricts to the square of the Pfaffian.

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2. Notations and preliminaries

Let V be a complex vector space of dimension n and let Σ_p be the symmetric group of permutations over p elements. We define

tensor product: $V^{\otimes p} = \{A : \underbrace{V^\vee \times \dots \times V^\vee}_p \rightarrow \mathbb{C} \text{ multilinear} \}$

skew-symmetric tensor product: $\bigwedge^p V =$

$\{A : \underbrace{V^\vee \times \dots \times V^\vee}_p \rightarrow \mathbb{C} \text{ multilinear s.t. } A(x_1, \dots, x_p) = \text{sign}(\sigma)A(x_{\sigma(1)}, \dots, x_{\sigma(p)}),$
 $\forall \sigma \in \Sigma_p\}$

symmetric tensor product: $\text{Sym}^p V =$

$\{A : \underbrace{V^\vee \times \dots \times V^\vee}_p \rightarrow \mathbb{C} \text{ multilinear s.t. } A(x_1, \dots, x_p) = A(x_{\sigma(1)}, \dots, x_{\sigma(p)}), \forall \sigma \in \Sigma_p\}$

We recall the definition of Schur functor, see [2]. A *Young diagram* denoted by $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0$ consists of a collection of boxes ordered in consecutive rows, where the i -th row has exactly λ_i boxes. The number of boxes of λ is denoted by $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_m$. We also denote by λ^* the conjugate Young diagram of λ .

Any filling of λ with numbers is called a *tableau*. Just to fix convention, for a given Young diagram, number the boxes consecutively (from left to right and top to bottom). Here we use all numbers from 1 to p in order to fill p boxes. More generally, a tableau can allow repetitions of numbers. Each filling describes a vector in $V^{\otimes p}$.

Let Σ_p be the symmetric group of permutations over p elements. Due to the filling, we can consider the elements of Σ_p as permuting the boxes. Let

$$P = P_\lambda = \{\sigma \in \Sigma_p : \sigma \text{ preserves each row}\}$$

$$Q = Q_\lambda = \{\sigma \in \Sigma_p : \sigma \text{ preserves each column}\}$$

They depend on λ but also on the filling of λ .

40 In the group algebra $\mathbb{C}\Sigma_p$, we introduce two elements corresponding to these subgroups

$$a_\lambda = \sum_{\sigma \in P} e_\sigma$$

$$b_\lambda = \sum_{\sigma \in Q} \text{sign}(\sigma) \cdot e_\sigma$$

As Σ_p acts on the p -th tensor power $V^{\otimes p}$ by permuting factors, then

$$\text{Im}(a_\lambda) = S^{\lambda_1} V \otimes \dots \otimes S^{\lambda_m} V \subseteq V^{\otimes p}$$

$$\text{Im}(b_\lambda) = \bigwedge^{\mu_1} V \otimes \dots \otimes \bigwedge^{\mu_l} V \subseteq V^{\otimes p}$$

where μ is the conjugate partition of λ .

Finally, we set $c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}\Sigma_p$ that is called the *Young symmetrizer* corresponding to λ . We denote the image of c_λ on $\mathbb{C}\Sigma_p$ by Γ_λ , it is a irreducible Σ_p -module.

$$\Gamma_\lambda = \text{Im}(c_\lambda|_{\mathbb{C}\Sigma_p})$$

We call the functor that associates $V \rightsquigarrow \Gamma_\lambda$ the *Schur functor*. We also denote by $S_\lambda V$ the image of c_λ acting on $V^{\otimes p}$ which is an irreducible $\text{GL}(V)$ -module and is non zero if and only if the number of rows is less or equal than n . With
45 these definitions we can state the Schur-Weyl duality (see [7]):

There is a $\Sigma_p \times \text{SL}(V)$ -decomposition $V^{\otimes p} = \bigoplus_\lambda \Gamma_\lambda \otimes S_\lambda V$ where the sum is extended on all Young diagrams with p boxes

With the purpose of giving the equations for $\Gamma_\lambda \otimes S_\lambda V$ where λ is a Young
50 diagram with p boxes we will state Theorem 2.1. To know how the equations

work we need some notations and definitions from [4]. We will denote by π the partitions of the set $\{1, 2, \dots, p\}$ and for any permutation $\sigma \in \Sigma_p$ we denote by $par(\sigma)$ the partition obtained by the cycle decomposition of σ . From a partition π we can get the corresponding Young diagram and we denote it by $shape(\pi) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, where every λ_i are the sizes of the blocks of π (arranged in non-increasing order).

We need to define a partial order in the set of Young diagrams to give the main definition of the theorem. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ be two Young diagrams (if it is necessary, the last entries of each diagram are filled with zero). Write $\lambda \leq \mu$ if and only if the following conditions hold:

- $\lambda_1 + \lambda_2 + \dots + \lambda_m = \mu_1 + \mu_2 + \dots + \mu_r$,
- $\lambda_1 \leq \mu_1$,
- $\lambda_1 + \lambda_2 \leq \mu_1 + \mu_2$,
- ...

According to [4], the set of minimal elements of the complement of the partially ordered set $\{\mu : \mu \leq \lambda\}$ is called *critical set* of the Young diagram λ . To state Theorem 2.1 we also define $Pos(\pi)$ as an element of the group algebra $\mathbb{C}\Sigma_p$ given by $\Sigma\sigma$ where the sum is extended to all permutations σ such that $par(\sigma)$ is a refinement of π .

Theorem 2.1. (Metropolis-Rota-Stein [4], Theorem at pag. 848) If $A \in \Gamma_\lambda \otimes S_\lambda V$, and π is a partition such that $shape(\pi)$ belongs to the critical set of λ , then

$$\sum_{\sigma \in Pos(\pi)} A(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = 0 \quad (1)$$

Each equation stated in the previous theorem is called a *positive equation* and we will denote it for brevity as $Pos(\pi) \cdot A = 0$.

We are interested in the product $\underbrace{V \otimes \dots \otimes V}_p$, where the group $GL(V) \times \dots \times GL(V)$ acts in a natural way. Once a basis is fixed in V , the tensors can

be represented as multidimensional matrices of format $\underbrace{(n) \times \dots \times (n)}_p$.

A feature of the Segre variety is that it contains a lot of linear subspaces. For any point $x = x_1 \otimes \dots \otimes x_p$, the linear space $x_1 \otimes \dots \otimes V_i \otimes \dots \otimes x_p$ passes through x for $i = 1, \dots, p$. Let $x \in X = \mathbb{P}^{k_1} \times \dots \times \mathbb{P}^{k_p} = \mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_p)$. Then the tangent space at x , $T_x X$ is the projectivization of $\oplus_i \langle x_1 \rangle \otimes \dots \otimes V_i \otimes \dots \otimes \langle x_p \rangle$ (see [6]).

Now consider $X \subseteq \mathbb{P}(V)$ a projective smooth irreducible variety. We say that a hyperplane H is *tangent* to X if H contains the tangent space to X at some point $x \in X$. This means $T_x X \subseteq H$. The set of tangent hyperplanes to X has a natural structure of projective irreducible variety, denoted by $X^\vee \subseteq \mathbb{P}(V^\vee)$ and called the *dual variety* of X .

Consider the product $X = \underbrace{\mathbb{P}^{n-1} \times \dots \times \mathbb{P}^{n-1}}_p$ of several projective spaces in the Segre embedding into the projective space $\mathbb{P}(V^{\otimes p})$. The *hyperdeterminant* of format $\underbrace{(n) \times \dots \times (n)}_p$ is a homogeneous polynomial function on $V^{\otimes p}$ which is a defining equation of the projectively dual variety X^\vee . We denote the hyperdeterminant by *Det*. For its main properties see [3] and [6].

From now on we will refer to a multidimensional matrix A as simply a matrix and we will write $Det(A)$ for its hyperdeterminant (when it exists). The following definition is the main point of the final argument.

According to [3], a matrix A is called *degenerate* if there exists $x^1 \otimes \dots \otimes x^p \in V \otimes \dots \otimes V$ nonzero such that $A(x^1, \dots, \underbrace{V}_i, \dots, x^p) = 0 \quad \forall i = 1, \dots, p$.

By definition the *kernel* is

$$K(A) = \{(x^1 \otimes \dots \otimes x^p) \in V \otimes \dots \otimes V \text{ such that } A(x^1, \dots, V, \dots, x^p) = 0, \forall i = 1, \dots, p\}$$

From these definitions we get, always from [3], that:

Proposition 2.2. *The following are equivalent:*

- $Det(A) = 0$
- A is degenerate

105 • $K(A) \neq \phi$

This gives a geometric reformulation of the definition of hyperdeterminant.

3. Proof of main theorem

Using the previous characterization we will get that the hyperdeterminant vanishes if we restrict it to skew-symmetric multidimensional matrices. We use
 110 the skewness properties:

if $A \in \bigwedge^p V \subseteq V^{\otimes p}$ then $A(x_1, \dots, x_p) = 0$ if $x_i = x_j$ for some $i, j = 1, \dots, p$.

Proposition 3.1. *If $p \geq 3$, every $A \in \bigwedge^p V$ is a degenerate matrix, indeed every $x \otimes \dots \otimes x \in K(A)$ for all $x \in V$.*

PROOF. $A(x, \dots, x, y, x, \dots, x) = 0$ for every $x, y \in V$ because two elements
 115 coincide and $p \geq 3$. Hence $x \otimes \dots \otimes x \in K(A)$ and A is degenerate.

In order to have the same result for the case $A \in \Gamma_\lambda \otimes S_\lambda V \subseteq V^{\otimes p}$ where λ is a Young diagram with p boxes and $\lambda_2 \geq 2$ or $\lambda_3 \geq 1$ we need two lemmas before giving the proof of the main theorem.

Lemma 3.2. *If $Pos(\pi) \cdot A = 0$ where π ranges over all partitions of $\{1, 2, \dots, p\}$
 120 such that $shape(\pi)$ is a Young diagram with at least two rows and p boxes, then A is degenerate.*

PROOF. Let $x, y \in V$. We consider the p unknowns $A(y, x, \dots, x)$, $A(x, y, x, \dots, x)$,
 \dots , $A(x, \dots, x, y)$. Since $shape(\pi) = \mu$ has at least two rows, we can consider $\mu_1 + 1$ of these unknowns and construct a homogeneous linear system.

125 In the first place, we consider the first $\mu_1 + 1$ unknowns.

By taking the partition constituted by $I = \{2, 3, \dots, \mu_1 + 1\}$ and its complement in $\{1, \dots, p\}$, and substituting $x_2 = y$ and $x_i = x$ for $i \in I$ with $i \neq 2$, and also $x_i = x$ for $i \notin I$ on the positive equations (1), we get an equation between these unknowns (that correspond with the first row of the following matrix M
 130). By taking the partition constituted by the subset $I = \{1, 2, \dots, \hat{k}, \dots, \mu_1 + 1\}$

with $k \neq 1$ and its complement in $\{1, \dots, p\}$, and substituting $x_1 = y$ and $x_i = x$ for $i \in I$ with $i \neq 1$, and also $x_i = x$ for $i \notin I$ on the positive equations (1), we get μ_1 equations between these unknowns (each of them corresponding to the k -th row of the following matrix M). In this way, we get $\mu_1 + 1$ equations and
135 a homogeneous linear system with the following matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 0 \end{pmatrix} \quad (2)$$

We can easily see that M is non degenerate. To do this we have to compute the determinant of $M - t \cdot I$. For $t = -1$ we get that $\det(M - t \cdot I) = 0$. As the rank is 1, the multiplicity of -1 is μ_1 , and since the trace is zero, we get that the eigenvalues are just $\underbrace{-1, \dots, -1}_{\mu_1}, (\mu_1)$. Therefore $\det(M) = (-1)^{\mu_1}(\mu_1)$ and
140 the $\mu_1 + 1$ unknowns are all zero.

We may repeat the same argument for any other subset of $\{\mu_1 + 1\}$ unknowns among the p unknowns. We conclude that $A(y, x, \dots, x) = A(x, y, x, \dots, x) = \dots = A(x, \dots, x, y) = 0$, so that $x \otimes \dots \otimes x \in K(A)$ and A is degenerate.

Lemma 3.3. *For every partition π of $\{1, 2, \dots, p\}$, with $\text{shape}(\pi) \neq (p), (p - 1, 1)$, the critical set of $\text{shape}(\pi)$ is composed by Young diagrams with at least
145 two rows.*

PROOF. By using the partial order explained before we have the following sequence within the set of the Young diagrams with p boxes:

$$\underbrace{(1, \dots, 1)}_p \leq \underbrace{(2, 1, \dots, 1)}_{p-2} \leq \underbrace{(2, 2, 1, \dots, 1)}_{p-4} \leq \dots \\ \dots \leq (p-2, 2) \leq (p-1, 1) \leq (p)$$

The critical set of each diagram is the next one to the right in this sequence. From the definition of partial order there is no manner in which we can have a

150 diagram with one row inside the sequence. In this context, the only possibilities are $(p - 1, 1)$ (the critical set is (p)) and (p) (the critical set is \emptyset).

Main Theorem. *When $A \in \Gamma_\lambda \otimes S_\lambda V \subseteq V^{\otimes p}$ and $p \geq 2$, $\text{Det}(A)$ can be nonzero only for $\lambda = (p)$ (corresponding to the Segre variety $\text{Sym}^p V$) and $\lambda = (p - 1, 1)$, where Γ_λ (resp. $S_\lambda V$) is the Σ_p -module (resp. $GL(V)$ -module) associated to λ .*

155 **PROOF.** The critical set of λ is always a Young diagram with at least two rows by Lemma 3.3. The proof is complete applying Lemma 3.2.

Remark 3.4. *One remark is in order. First, let us fix x and y of V . We have already shown that $A(y, x, \dots, x) = A(x, y, x, \dots, x) = \dots = A(x, \dots, x, y) = 0$ for any $A \in \Gamma_\lambda \otimes S_\lambda V$ with $\lambda \neq (p), (p - 1, 1)$. Then the same is true for any $A \in$
160 $\bigoplus_{\lambda \neq (p), (p-1,1)} \Gamma_\lambda \otimes S_\lambda V$. Therefore $x \otimes \dots \otimes x \in K(A)$ for any A in the previous direct sum, hence $\text{Det}(A) = 0$. This means that the hypersurface defined by $\text{Det}(A) = 0$ in $\mathbb{P}(V^{\otimes p})$ contains the projectivization of $\bigoplus_{\lambda \neq (p), (p-1,1)} \Gamma_\lambda \otimes S_\lambda V$, which is a linear space of higher dimension.*

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