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RESEARCH

Fractional Derivative Description of the Bloch Space

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Abstract

We establish new characterizations of the Bloch space $\mathcal{B}$ which include descriptions in terms of classical fractional derivatives. Being precise, for an analytic function $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ in the unit disc $\mathbb{D}$, we define the fractional derivative $D^{\mu}(f)(z) = \sum\limits_{n=0}^{\infty} \frac{\widehat{f}(n)}{\mu_{2n+1}} z^n$ induced by a radial weight μ, where $\mu_{2n+1} = \int_0^1 r^{2n+1}\mu(r)\,dr$ are the odd moments of μ. Then, we consider the space $\mathcal{B}^{\mu}$ of analytic functions f in $\mathbb{D}$ such that $\|f\|_{\mathcal{B}^{\mu}} = \sup_{z\in\mathbb{D}} \widehat{\mu}(z)|D^{\mu}(f)(z)| < \infty$, where $\widehat{\mu}(z) = \int_{|z|}^1 \mu(s)\,ds$. We prove that $\mathcal{B}^{\mu}$ is continously embedded in $\mathcal{B}$ for any radial weight μ, and $\mathcal{B} = \mathcal{B}^{\mu}$ if and only if $\mu \in \mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$. A radial weight $\mu \in \widehat{\mathcal{D}}$ if $\sup_{0\le r<1} \frac{\widehat{\mu}(r)}{\widehat{\mu}\left(\frac{1+r}{2}\right)} < \infty$ and a radial weight $\mu \in \check{\mathcal{D}}$ if there exist $K = K(\mu) > 1$ such that $\inf_{0\le r<1} \frac{\widehat{\mu}(r)}{\widehat{\mu}\left(1-\frac{1-r}{K}\right)} > 1$.

Keywords Fractional derivative · Bloch space · Radial weight · Doubling weight

Mathematics Subject Classification (2010) 26A33 · 30H30

1 Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For a nonnegative function $\omega \in L^1([0,1))$, the extension to $\mathbb{D}$, defined by $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, is called a radial weight. For $0 < p < \infty$ and such an ω, the Lebesgue space L^p_{ω} consists of complex-valued measurable functions f on $\mathbb{D}$ such that

$$\|f\|^p_{L^p_{\omega}} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\,dA(z) < \infty,$$

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where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. The corresponding weighted Bergman space is $A^p_\omega = L^p_\omega \cap \mathcal{H}(\mathbb{D})$. Throughout this paper we assume $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)\,ds > 0$ for all $z \in \mathbb{D}$, for otherwise $A^p_\omega = \mathcal{H}(\mathbb{D})$. As usual, for the standard weights $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$, $\alpha > -1$, we simply write L^p_α and A^p_α for the corresponding Lebesgue and Bergman spaces. In addition, we denote $L^p = L^p_0$ and $A^p = A^p_0$.

For a radial weight μ, the fractional derivative of $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathcal{H}(\mathbb{D})$ (induced by μ) is

$$D^\mu(f)(z) = \sum_{n=0}^{\infty} \frac{\widehat{f}(n)}{\mu_{2n+1}} z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

Here and from now on, μ_{2n+1} are the odd moments of μ, and in general we write $\mu_x = \int_0^1 r^x \mu(r)\,dr$ for a radial weight μ and $x \geq 0$. It is clear that $D^\mu(f)$ is a polynomial if f is a polynomial and it follows from the inequality

$$\mu_{2n+1} \geq \varepsilon^{n+\frac{1}{2}} \widehat{\mu}(\sqrt{\varepsilon}), \quad 0 < \varepsilon < 1, \quad n \in \mathbb{N},$$

that $D^\mu(f) \in \mathcal{H}(\mathbb{D})$ for each $f \in \mathcal{H}(\mathbb{D})$. If μ is the standard weight $\mu(z) = \beta(1-|z|^2)^{\beta-1}$, $\beta > 0$, $D^\mu(f)$ is nothing but

$$D^\beta(f)(z) = \frac{2}{\Gamma(\beta+1)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \widehat{f}(n) z^n, \quad z \in \mathbb{D}, \tag{1.2}$$

which basically coincides with the fractional derivative of order $\beta > 0$ introduced by Hardy and Littlewood in [3, p. 409]. The differences between Eq. 1.2 and [3, (3.13)] are in the multiplicative factor $\frac{2}{\Gamma(\beta+1)}$ and the inessential factor z^β. See [12, 13] for related definitions or reformulations of classical and generalized fractional derivative.

The fractional derivative D^μ was introduced in [11] by the second and third authors to study Littlewood-Paley formulas for Bergman spaces, and in particular they proved that

$$\|f\|^p_{A^p} \asymp \int_{\mathbb{D}} |D^\mu(f)(z)|^p \widehat{\mu}(z)^p\,dA(z), \quad f \in \mathcal{H}(\mathbb{D}), \quad \mu \in \mathcal{D}. \tag{1.3}$$

Let us recall that a radial weight $\mu \in \widehat{\mathcal{D}}$ if there exists $C = C(\mu) > 0$ such that $\widehat{\mu}(r) \leq C\widehat{\mu}(\frac{1+r}{2})$, $0 \leq r < 1$, and a radial weight $\mu \in \check{\mathcal{D}}$ if there exist $K = K(\mu) > 1$ and $C = C(\mu) > 1$ such that $\widehat{\mu}(r) \geq C\widehat{\mu}(1 - \frac{1-r}{K})$, $0 \leq r < 1$. We denote $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$.

In this paper, we are interested in obtaining a $p = \infty$ version of Eq. 1.3, therefore for each radial weight μ we consider the normed space

$$\mathcal{B}^\mu = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathcal{B}^\mu} = \sup_{z \in \mathbb{D}} \widehat{\mu}(z) |D^\mu(f)(z)| < \infty \right\}.$$

A first natural approach to this problem lead us to the following question: Which are the radial weights μ such that $\mathcal{B}^\mu = H^\infty$? As usual H^∞ denotes the space of bounded analytic functions in $\mathbb{D}$. As it could be expected, our first result provides a stark negative answer to this question.

Proposition 1 *Let μ be a radial weight. Then, $H^\infty \neq \mathcal{B}^\mu$.*

As for the proof Proposition 1 we show that the embedding $H^\infty \subset \mathcal{B}^\mu$ implies $\mu \in \widehat{\mathcal{D}}$, and then we draw on this property of the weight to construct a function $f \in \mathcal{B}^\mu \setminus H^\infty$.

In view of Proposition 1, the Littlewood-Paley formula

$$\|f\|_{A^p}^p \asymp \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np}\, dA(z), \quad n \in \mathbb{N},$$

and Eq. 1.3, it is also natural to study the relationship between $\mathcal{B}^\mu$ and the classical Bloch space of $f \in \mathcal{H}(\mathbb{D})$ such that $\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|)|f'(z)| < \infty$. We will also give thought to the analogous question for the little versions of both spaces; $\mathcal{B}_0$ the space of $f \in \mathcal{H}(\mathbb{D})$ such that $\lim\limits_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0$ and $\mathcal{B}_0^\mu = \left\{ f \in \mathcal{H}(\mathbb{D}) : \lim\limits_{|z|\to 1^-} \widehat{\mu}(z)|D^\mu(f)(z)| = 0 \right\}$.

Our first main result is the following.

Theorem 2 *Let μ be a radial weight. Then, $\mathcal{B}^\mu$ is continously embedded in $\mathcal{B}$, that is*

$$\|f\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{B}^\mu}, \quad f \in \mathcal{H}(\mathbb{D}). \tag{1.4}$$

Moreover, $\mathcal{B}_0^\mu$ is continously embedded in $\mathcal{B}_0$.

In order to prove the first part of Theorem 2 we use an appropriate integral representation of the fractional derivative D^μ and the well-known identification $\mathcal{B} \simeq (A^1)^\star$ via the A^2-pairing [14, Theorem 5.3]. Consequently, the proof boils down to showing that each $f \in \mathcal{B}^\mu$ induces a bounded linear functional $L_f(g) = \langle g, f\rangle_{A^2} = \lim_{r\to 1^-}\int_{\mathbb{D}} f(rz)\overline{g(z)}\, dA(z)$ on A^1 and $\|L_f\| \lesssim \|f\|_{\mathcal{B}^\mu}$. The proof of the second part of Theorem 2 is based on the fact that $f \in \mathcal{B}_0^\mu$ if and only if $\lim_{r\to 1^-}\|f - f_r\|_{\mathcal{B}^\mu} = 0$. Here and on the following $f_r(z) = f(rz)$, $0 \le r < 1$.

On the other hand, it is obvious that $\mathcal{B} = \mathcal{B}^\mu$ if $\mu = 1$ and it is known that $f \in \mathcal{B}$ if and only if $f^{(n)}(1-|z|)^n \in L^\infty$, $n \in \mathbb{N}$, [14, Theorem 4]. In addition, we recall that the Bloch space can be characterized in terms the multiplier transformation $f^{[\beta]}(z) = \sum_{n=0}^\infty (n+1)^\beta \widehat{f}(n) z^n$, which may also be regarded as fractional derivative of order $\beta > 0$ [1].

Our next main result shows that characterizations of the Bloch space in terms of classical fractional derivatives are examples of a general phenomenon rather than particular cases, and moreover it provides a neat characterization of the radial weights μ such that $\mathcal{B} = \mathcal{B}^\mu$.

Theorem 3 *Let μ be a radial weight. Then, the following conditions are equivalent:*

(i) *$\mu \in \mathcal{D}$;*

(ii) *$\mathcal{B}$ is continously embedded in $\mathcal{B}^\mu$, that is*

$$\|f\|_{\mathcal{B}^\mu} \lesssim \|f\|_{\mathcal{B}}, \quad f \in \mathcal{H}(\mathbb{D}); \tag{1.5}$$

(iii) *$\mathcal{B}^\mu = \mathcal{B}$ and*

$$\|f\|_{\mathcal{B}^\mu} \asymp \|f\|_{\mathcal{B}}, \quad f \in \mathcal{H}(\mathbb{D});$$

(iv) *$\mathcal{B}_0$ is continously embedded in $\mathcal{B}_0^\mu$, that is*

$$\|f\|_{\mathcal{B}^\mu} \lesssim \|f\|_{\mathcal{B}}, \quad f \in \mathcal{B}_0; \tag{1.6}$$

(v) *$\mathcal{B}_0^\mu = \mathcal{B}_0$ with equivalence of norms.*

The proof of Theorem 3 is lengthy and involved and requires some preparatory results. In fact, the proof of (i)⇒(ii) is based on asymptotic estimates of the integral means of order one of $D^\mu(B_\zeta)$, where $B_\zeta(z) = (1-\overline{\zeta}z)^{-2}$ is the Bergman reproducing kernel of A^2. It is also worth mentioning that we use smooth properties of universal Cesáro basis of polynomials introduced by Jevtić and Pavlović [4] to obtain the aforementioned asymptotic estimates .

Reciprocally, in order to prove (ii)⇒(i) we use that $\mathcal{D}=\widehat{\mathcal{D}}\cap\check{\mathcal{D}}=\widehat{\mathcal{D}}\cap\mathcal{M}$, where $\mu\in\mathcal{M}$ if there exist constants $C=C(\mu)>1$ and $K=K(\mu)>1$ such that $\mu_x\ge C\mu_{Kx}$ for all $x\ge 1$. Moreover, we prove that for every $f\in\mathcal{B}$, there exists $g\in L^\infty$ such that $P_\mu(g)=f$ and $\|g\|_{L^\infty}\lesssim\|f\|_{\mathcal{B}}$, where P_μ is the Bergman projection from L^2_μ to A^2_μ. Then, [10, Theorem 2] ensures that $\mu\in\mathcal{M}$. We get the condition $\mu\in\widehat{\mathcal{D}}$ by testing Eq. 1.5 on monomials. The rest of the proof follows from Theorem 2 and standard properties of the space $\mathcal{B}^\mu_0$.

The next few lines are dedicated to offer a brief insight to classes of radial weights $\widehat{\mathcal{D}},\check{\mathcal{D}},\mathcal{D}$ and $\mathcal{M}$. Doubling weights appear in a natural way in many questions on operator theory. For instance, the Bergman projection P_μ, induced by a radial weight μ, acts as a bounded operator from the space L^∞ of bounded complex-valued functions to the Bloch space $\mathcal{B}$ if and only if $\mu\in\widehat{\mathcal{D}}$, and $P_\mu:L^\infty\to\mathcal{B}$ is bounded and onto if and only if $\mu\in\mathcal{D}=\widehat{\mathcal{D}}\cap\check{\mathcal{D}}=\widehat{\mathcal{D}}\cap\mathcal{M}$. It is known that $\check{\mathcal{D}}\subset\mathcal{M}$, [10, Proof of Theorem 3] and $\check{\mathcal{D}}\subsetneq\mathcal{M}$ [10, Proof of Theorem 3 and Proposition 14]. Further, each standard radial weight obviously belongs to $\mathcal{D}$, while $\check{\mathcal{D}}\setminus\mathcal{D}$ contains exponential type weights such as

$$\mu(r)=\exp\left(-\frac{\alpha}{(1-r^l)^\beta}\right)\quad 0<\alpha,l,\beta<\infty.$$

The class of rapidly increasing weights, introduced in [8], lies entirely within $\widehat{\mathcal{D}}\setminus\mathcal{D}$, and a typical example of such a weight is

$$\mu(z)=\frac{1}{(1-|z|^2)\left(\log\frac{e}{1-|z|^2}\right)^\alpha},\quad 1<\alpha<\infty.$$

To this end we emphasize that the containment in $\widehat{\mathcal{D}}$ or $\check{\mathcal{D}}$ does not require differentiability, continuity or strict positivity. In fact, weights in these classes may vanish on a relatively large part of each outer annulus $\{z:r\le|z|<1\}$ of $\mathbb{D}$. For basic properties of the aforementioned classes, concrete nontrivial examples and more, see [6, 8, 10] and the relevant references therein.

The rest of the paper is organized as follows: Section 2 is devoted to proving the preliminary results needed in the proofs of the main results of the paper. Theorems 2 and 3 are proved in Section 3 and Section 4 contains a proof of Proposition 1.

Finally, we introduce the following notation that has already been used above in the introduction. The letter $C=C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a\lesssim b$ if there exists a constant $C=C(\cdot)>0$ such that $a\le Cb$, and $a\gtrsim b$ is understood in an analogous manner. In particular, if $a\lesssim b$ and $a\gtrsim b$, then we write $a\asymp b$ and say that a and b are comparable.

2 Preliminary Results

2.1 Background on Weights

Throughout the paper we will employ different descriptions of the classes of radial weights $\widehat{\mathcal{D}},\check{\mathcal{D}}$ and $\mathcal{M}$. The next result gathers several characterizations of $\widehat{\mathcal{D}}$ proved in [6, Lemma 2.1].

Lemma A *Let ω be a radial weight. Then, the following statements are equivalent:*

(i) $\omega\in\widehat{\mathcal{D}}$;

(ii) *There exist* $C = C(\omega) \geq 1$ *and* $\alpha_0 = \alpha_0(\omega) > 0$ *such that*

$$\widehat{\omega}(s) \leq C\left(\frac{1-s}{1-t}\right)^{\alpha}\widehat{\omega}(t), \quad 0 \leq s \leq t < 1,$$

for all $\alpha \geq \alpha_0$*;*

(iii)

$$\omega_x = \int_0^1 s^x \omega(s)ds \asymp \widehat{\omega}\left(1-\frac{1}{x}\right), \quad x \in [1, \infty);$$

(iv) *There exists* $C(\omega) > 0$ *such that* $\omega_n \leq C\omega_{2n}$*, for any* $n \in \mathbb{N}$.

We will use a couple of descriptions of the class $\check{\mathcal{D}}$ which are known for experts. We include a proof for the sake of completeness.

Lemma 4 *Let* ω *be a radial weight. Then, the following statements are equivalent:*

(i) $\omega \in \check{\mathcal{D}}$*;*

(ii) *There exist* $C = C(\omega) > 0$ *and* $\beta = \beta(\omega) > 0$ *such that*

$$\widehat{\omega}(s) \leq C\left(\frac{1-s}{1-t}\right)^{\beta}\widehat{\omega}(t), \quad 0 \leq t \leq s < 1;$$

(iii) *For each (or some)* $\gamma > 0$ *there exists* $C = C(\gamma, \omega) > 0$ *such that*

$$\int_0^r \frac{ds}{\widehat{\omega}(s)^{\gamma}(1-s)} \leq \frac{C}{\widehat{\omega}(r)^{\gamma}}, \quad 0 \leq r < 1.$$

Proof The equivalence (i)⇔(ii) was proved in [11, Lemma B]. Next, assume (ii) holds. Then,

$$\int_0^r \frac{ds}{\widehat{\omega}(s)^{\gamma}(1-s)} \leq \frac{C^{\gamma}(1-r)^{\gamma\beta}}{\widehat{\omega}(r)^{\gamma}}\int_0^r \frac{ds}{(1-s)^{1+\gamma\beta}} \lesssim \frac{1}{\widehat{\omega}(r)^{\gamma}}, \quad 0 \leq r < 1.$$

Now assume that (iii) holds. Then, if $0 \leq t \leq r < 1$

$$\frac{1}{\widehat{\omega}(t)^{\gamma}}\log\frac{1-t}{1-r} \leq \int_t^r \frac{ds}{\widehat{\omega}(s)^{\gamma}(1-s)} \leq \int_0^r \frac{ds}{\widehat{\omega}(s)^{\gamma}(1-s)} \leq \frac{C}{\widehat{\omega}(r)^{\gamma}}.$$

So if $r = 1 - \frac{1-t}{K}$ and $K > 1$, we get

$$\widehat{\omega}(t) \geq \left(\frac{\log K}{C}\right)^{1/\gamma}\widehat{\omega}\left(1-\frac{1-t}{K}\right), \quad 0 \leq t < 1.$$

Consequently, taking $K > e^C$, we get $\omega \in \check{\mathcal{D}}$. This finishes the proof. □

2.2 Estimates of the Integral Means of Fractional Derivative of Bergman Reproducing Kernel

For any radial weight ω, the norm convergence in A^2_{ω} implies the uniform convergence on compact subsets of $\mathbb{D}$, and hence each point evaluation L_z is a bounded linear functional on A^2_{ω}. Therefore there exist Bergman reproducing kernels $B^{\omega}_z \in A^2_{\omega}$ such that

$$L_z(f) = f(z) = \langle f, B^{\omega}_z\rangle_{A^2_{\omega}} = \int_{\mathbb{D}} f(\zeta)\overline{B^{\omega}_z(\zeta)}\omega(\zeta)\,dA(\zeta), \quad f \in A^2_{\omega}.$$

In this section we are going to obtain asymptotic estimates of the integral means of order one of $D^{\mu}(B_z^{\omega})$ for $\omega, \mu \in \widehat{\mathcal{D}}$. In order to get this result, we need establish some notation and previous results.

Let $W(z) = \sum_{k \in J} b_k z^k$ be a polynomial, where J denote a finite subset of $\mathbb{N}$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{D})$. The Hadamard product

$$(W * f)(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \quad z \in \mathbb{D},$$

is well defined. Furthermore, it is easy to observe that

$$(W * f)(e^{it}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{i(t-\theta)}) f(e^{i\theta})\, d\theta. \tag{2.1}$$

For a given C^{∞}-function $\Phi : \mathbb{R} \to \mathbb{C}$ with compact support, set

$$A_{\Phi,m} = \max_{x \in \mathbb{R}} |\Phi(x)| + m \max_{x \in \mathbb{R}} |\Phi^{(m)}(x)|, \quad m \in \mathbb{N} \cup \{0\},$$

and define the polynomials

$$W_n^{\Phi}(z) = \sum_{k \in \mathbb{Z}} \Phi\left(\frac{k}{n}\right) z^k, \quad n \in \mathbb{N}. \tag{2.2}$$

The Hardy space H^p consists of $f \in \mathcal{H}(\mathbb{D})$ for which $\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f) < \infty$, where $M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta\right)^{\frac{1}{p}}$, $0 < p < \infty$, and $M_{\infty}(r, f) = \max_{0 \le \theta \le 2\pi} |f(re^{i\theta})|$. The next result can be found in [5, pp. 111–113].

Theorem B *Let $\Phi : \mathbb{R} \to \mathbb{C}$ be a compactly supported C^{∞}-function. Then, for each $0 < p < \infty$ and $m \in \mathbb{N}$ with $mp > 1$, there exists a constant $C = C(p) > 0$ such that*

$$\|W_n^{\Phi} * f\|_{H^p} \le C A_{\Phi,m} \|f\|_{H^p}$$

for all $f \in H^p$ and $n \in \mathbb{N}$.

A particular case of the previous construction is useful for our purposes. By following [4, Section 2] (see also [11, Proposition 4]), let $\Psi : \mathbb{R} \to \mathbb{R}$ be a C^{∞}-function such that $\Psi \equiv 1$ on $(-\infty, 1]$, $\Psi \equiv 0$ on $[2, \infty)$ and Ψ is decreasing and positive on $(1, 2)$. Set $\psi(t) = \Psi\left(\frac{t}{2}\right) - \Psi(t)$ for all $t \in \mathbb{R}$. Let $V_0(z) = 1 + z$ and

$$V_n(z) = W_{2^{n-1}}^{\psi}(z) = \sum_{k=0}^{\infty} \psi\left(\frac{k}{2^{n-1}}\right) z^j = \sum_{k=2^{n-1}}^{2^{n+1}-1} \psi\left(\frac{k}{2^{n-1}}\right) z^j, \quad n \in \mathbb{N}. \tag{2.3}$$

these polynomials have the following properties (see [4, p. 175–177]):

$$\begin{aligned} &f(z) = \sum_{n=0}^{\infty} (V_n * f)(z), \quad f \in \mathcal{H}(\mathbb{D}), \\ &\|V_n * f\|_{H^p} \le C \|f\|_{H^p}, \quad f \in H^p, \quad 0 < p < \infty, \\ &\|V_n\|_{H^p} \asymp 2^{n(1-1/p)}, \quad 0 < p < \infty. \end{aligned} \tag{2.4}$$

Proposition 5 *Let $\omega,\ \mu \in \widehat{\mathcal{D}}$. Then,*

$$M_1(r, D^{\mu}(B_a^{\omega})) \asymp 1 + \int_0^{r|a|} \frac{1}{\widehat{\omega}(t)\widehat{\mu}(t)(1-t)}dt, \quad a \in \mathbb{D},\ r \in [0,1).$$

Proof Firstly, assume $\frac{1}{2} \le |a|, r < 1$. Bearing in mind Eq. 2.4,

$$M_1(r, D^{\mu}(B_a^{\omega})) \le \sum_{n=0}^{\infty} \|V_n * (D^{\mu}(B_a^{\omega}))_r\|_{H^1}, \tag{2.5}$$

On the one hand, by Lemma A and the formula $B_a^{\omega}(z) = \sum_{n=0}^{\infty} \frac{(\bar{a}z)^n}{2\omega_{2n+1}}$, it follows that

$$\begin{aligned}\|V_0 * (D^{\mu}(B_a^{\omega}))_r\|_{H^1} &= \left\| \frac{1}{2\omega_1\mu_1} + \frac{\bar{a}rz}{2\omega_3\mu_3} \right\|_{H^1} \le \frac{1}{2\omega_1\mu_1} + \frac{|a|r}{2\omega_3\mu_3} \\ &\lesssim \frac{1}{\widehat{\mu}(r|a|)\widehat{\omega}(r|a|)(1-r|a|)} \int_{\frac{4r|a|-1}{3}}^{r|a|} dt \\ &\asymp \int_{\frac{4r|a|-1}{3}}^{r|a|} \frac{1}{\widehat{\mu}(t)\widehat{\omega}(t)(1-t)}dt \le \int_0^{r|a|} \frac{1}{\widehat{\mu}(t)\widehat{\omega}(t)(1-t)}dt, \quad r, |a| \ge \frac{1}{2}.\end{aligned} \tag{2.6}$$

Analogously,

$$\|V_1 * (D^{\mu}(B_a^{\omega}))_r\|_{H^1} \lesssim \int_0^{r|a|} \frac{1}{\widehat{\mu}(t)\widehat{\omega}(t)(1-t)}dt, \quad r, |a| \ge \frac{1}{2}. \tag{2.7}$$

Now, for each $n \in \mathbb{N} \setminus \{1\}$ let us consider the functions

$$\varphi_{1,n}(x) = \frac{r^x}{\omega_{2x+1}} \chi_{[2^{n-1}, 2^{n+1}-1]}(x), \quad \frac{1}{2} \le r < 1,$$

$$\varphi_{2,n}(x) = \frac{|a|^x}{\mu_{2x+1}} \chi_{[2^{n-1}, 2^{n+1}-1]}(x), \quad \frac{1}{2} \le |a| < 1,$$

and choose C^{∞}-functions $\Phi_{1,n}$ and $\Phi_{2,n}$ with compact support contained in $[2^{n-2}, 2^{n+2}]$ such that $\Phi_{1,n} = \varphi_{1,n}$ and $\Phi_{2,n} = \varphi_{2,n}$ in $[2^{n-1}, 2^{n+1}-1]$. By following the proof of [11, (3.12)], there exist $C_1 = C_1(\omega) > 0$ and $C_2 = C_2(\mu) > 0$ such that

$$A_{\Phi_{1,n},2} \le C_1 \frac{r^{2^{n-1}}}{\omega_{2^n}} \quad \text{and} \quad A_{\Phi_{2,n},2} \le C_1 \frac{|a|^{2^{n-1}}}{\mu_{2^n}}, \quad \frac{1}{2} \le r, |a| < 1. \tag{2.8}$$

Then, if $a = |a|e^{i\theta}$

$$\begin{aligned} V_n * (D^{\mu}(B_a^{\omega}))_r(z) &= \frac{1}{2} \sum_{k=2^{n-1}}^{2^{n+1}-1} \psi\left(\frac{k}{2^{n-1}}\right) \frac{r^k}{\omega_{2k+1}} \frac{|a|^k}{\mu_{2k+1}} (ze^{-i\theta})^k \\ &= \frac{1}{2} \sum_{k=2^{n-1}}^{2^{n+1}-1} \psi\left(\frac{k}{2^{n-1}}\right) \Phi_{1,n}(k)\Phi_{2,n}(k)(ze^{-i\theta})^k \\ &= (W_1^{\Phi_{1,n}} * W_1^{\Phi_{2,n}} * V_n)(ze^{-i\theta}).\end{aligned}$$

Therefore, Theorem B, Eqs. 2.8 and 2.4 implies that there is $C = C(\mu, \omega) > 0$ such that for each $n \in \mathbb{N} \setminus \{1\}$

$$\|V_n * (D^\mu(B_a^\omega))_r\|_{H^1} \le C A_{\Phi_{1,n},2} A_{\Phi_{2,n},2} \|V_n\|_{H^1} \le C \frac{(r|a|)^{2^{n-1}}}{\omega_{2^n}\mu_{2^n}}, \quad \frac{1}{2} \le r, |a| < 1.$$

Then, arguing as in the proof of [11, Lemma 6] and using the hypotheses $\omega, \mu \in \widehat{\mathcal{D}}$ it follows that

$$\begin{aligned}\sum_{n=2}^{\infty} \|V_n * (D^\mu(B_a^\omega))_r\|_{H^1} &\lesssim \left(\sum_{n=2}^{\infty} \frac{(r|a|)^{2^{n-1}}}{\omega_{2^n}\mu_{2^n}}\right) \\ &\lesssim \int_0^{r|a|} \frac{1}{\widehat{\mu}(t)\widehat{\omega}(t)(1-t)}dt, \quad r, |a| \ge \frac{1}{2}.\end{aligned} \tag{2.9}$$

By joining Eqs. 2.5, 2.6, 2.7 and 2.9,

$$M_1(r, D^\mu(B_a^\omega)) \lesssim \int_0^{r|a|} \frac{1}{\widehat{\omega}(t)\widehat{\mu}(t)(1-t)}dt, \quad r, |a| \ge \frac{1}{2}. \tag{2.10}$$

Moreover, if $|a| \le \frac{1}{2}$ or $r \le \frac{1}{2}$,

$$M_1(r, D^\mu(B_a^\omega)) \le \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{2\mu_{2n+1}\omega_{2n+1}} \lesssim 1. \tag{2.11}$$

Thus Eqs. 2.10 and 2.11 yields

$$M_1(r, D^\mu(B_a^\omega)) \lesssim 1 + \int_0^{r|a|} \frac{1}{\widehat{\omega}(t)\widehat{\mu}(t)(1-t)}dt.$$

The reverse inequality follows from Hardy's inequality [2, Section 3.6] and the proof of [11, Lemma 6]. This finishes the proof. □

2.3 Previous Results on $\mathcal{B}^\mu$ and $\mathcal{B}_0^\mu$.

For a radial weight ω the orthogonal projection from L^2_ω to A^2_ω is given by

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B_z^\omega(\zeta)}\omega(\zeta)\, dA(\zeta), \quad z \in \mathbb{D}.$$

If $\omega = 1$, we simply write $P_\omega = P$ and $B_z^\omega(\zeta) = B_z(\zeta) = (1 - \overline{z}\zeta)^{-2}$, $z.\zeta \in \mathbb{D}$.

We begin with a useful representation of the fractional derivative of $P_\omega(f)$, $f \in L^p_\omega$, $1 < p < \infty$.

Proposition 6 *Let μ be a radial weight and $\omega \in \widehat{\mathcal{D}}$. Then,*

$$D^\mu(f)(z) = \int_{\mathbb{D}} f(\zeta) D^\mu(B_\zeta^\omega)(z)\omega(\zeta)dA(\zeta), \quad f \in A^1_\omega, \quad z \in \mathbb{D}, \tag{2.12}$$

and

$$D^\mu P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta) D^\mu(B_\zeta^\omega)(z)\omega(\zeta)dA(\zeta), \quad f \in L^p_\omega, \quad 1 < p < \infty, \quad z \in \mathbb{D}.$$

Proof Fix $z \in \mathbb{D}$ and observe that

$$\begin{aligned}\int_{\mathbb{D}} f(\zeta) D^{\mu}(B_{\zeta}^{\omega})(z)\omega(\zeta)dA(\zeta) &= \int_0^1 2r\omega(r)\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=0}^{\infty}\widehat{f}(n)r^n e^{in\theta}\right)\left(\sum_{k=0}^{\infty}\frac{z^k r^k e^{-ik\theta}}{2\mu_{2k+1}\omega_{2k+1}}\right)d\theta dr\\ &= \int_0^1 \omega(r)\sum_{n=0}^{\infty}\frac{\widehat{f}(n)z^n}{\mu_{2n+1}\omega_{2n+1}}r^{2n+1}dr\\ &= \sum_{n=0}^{\infty}\frac{\widehat{f}(n)}{\mu_{2n+1}}z^n = D^{\mu}(f)(z), \quad f \in A_{\omega}^1.\end{aligned}$$

Therefore if $f \in A_{\omega}^1$, Eq. 2.12 holds. Next, if $f \in L_{\omega}^p$ then $P_{\omega}(f) \in A_{\omega}^p$ by [10, Theorem 7]. Therefore, two applications of Eq. 2.12 and Fubini's Theorem yield

$$\begin{aligned}D^{\mu}P_{\omega}(f)(z) &= \int_{\mathbb{D}} P_{\omega}(f)(\zeta)D^{\mu}(B_{\zeta}^{\omega})(z)\omega(\zeta)dA(\zeta)\\ &= \lim_{\rho\to 1}\int_{\mathbb{D}} P_{\omega}(f)(\rho\zeta)D^{\mu}(B_{\zeta}^{\omega})(z)\omega(\zeta)dA(\zeta)\\ &= \lim_{\rho\to 1}\int_{\mathbb{D}}\left(\int_{\mathbb{D}} f(u)B_u^{\omega}(\rho\zeta)\omega(u)dA(u)\right)D^{\mu}(B_{\zeta}^{\omega})(z)\omega(\zeta)dA(\zeta)\\ &= \lim_{\rho\to 1}\int_{\mathbb{D}} f(u)\omega(u)\left(\int_{\mathbb{D}} B_u^{\omega}(\rho\zeta)D^{\mu}(B_{\zeta}^{\omega})(z)\omega(\zeta)dA(\zeta)\right)dA(u)\\ &= \lim_{\rho\to 1}\int_{\mathbb{D}} f(u)D^{\mu}(B_u^{\omega})(\rho z)\omega(u)dA(u),\end{aligned}$$

Finally, bearing in mind that $D^{\mu}(B_u^{\omega})(\rho z)$ converges uniformly in $u \in \mathbb{D}$ to $D^{\mu}(B_u^{\omega})(z)$ as $\rho \to 1^-$, it follows that

$$\lim_{\rho\to 1}\int_{\mathbb{D}} f(u)D^{\mu}(B_u^{\omega})(\rho z)\omega(u)dA(u) = \int_{\mathbb{D}} f(u)D^{\mu}(B_u^{\omega})(z)\omega(u)dA(u).$$

This finishes the proof. □

Now, we provide some useful descriptions of $\mathcal{B}_0^{\mu}$ which can be proved by standard techniques, so we omit its proof.

Proposition 7 *Let μ be a radial weight. Then,*

(i) *$\mathcal{B}_0^{\mu}$ is a closed subspace of $\mathcal{B}^{\mu}$.*
(ii) *Let $f \in \mathcal{H}(\mathbb{D})$, then $f \in \mathcal{B}_0^{\mu}$ if and only if*

$$\lim_{r\to 1^-}\|f - f_r\|_{\mathcal{B}^{\mu}} = 0,$$

where $f_r(z) = f(rz)$, $0 \le r < 1$.
(iii) *$\mathcal{B}_0^{\mu}$ is the closure in $\mathcal{B}^{\mu}$ of the set of polynomials.*

3 Main Results

Proof of Theorem 2 In order to prove the first part of our statement we are going to see that

$$\|f\|_{\mathcal{B}} \lesssim \|zV_{\mu}(f)\|_{L^{\infty}}, \quad f \in \mathcal{H}(\mathbb{D}),$$

where $V_\mu(f)(z) = \frac{\widehat{\mu}(z)}{|z|} D^\mu(f)(z)$, $z \in \mathbb{D}$. Precisely, we will prove that the functional

$$L_f(g) = \langle g, f \rangle_{A^2} = \lim_{r \to 1^-} \langle g, f_r \rangle_{L^2},$$

belongs to $(A^1)^*$ and $\|L_f\| \lesssim \|zV_\mu(f)\|_{L^\infty}$, this fact together with [14, Theorem 5.3] will finish the proof.

Let $h \in A^1$, then by Eq. 2.12

$$\begin{aligned} \big\langle h, V_\mu(f_r) \big\rangle_{L^2} &= \int_{\mathbb{D}} h(z) \frac{\widehat{\mu}(z)}{|z|} \int_{\mathbb{D}} \overline{f_r(\zeta)} D^\mu(B_z)(\zeta)\, dA(\zeta)\, dA(z) \\ &= \int_{\mathbb{D}} \overline{f_r(\zeta)} \left(\int_{\mathbb{D}} h(z) D^\mu(B_z)(\zeta) \frac{\widehat{\mu}(z)}{|z|} dA(z) \right) dA(\zeta) \\ &= \langle T(h), f_r \rangle_{L^2}, \quad 0 \le r < 1, \end{aligned} \tag{3.1}$$

where

$$T(h)(z) = \int_{\mathbb{D}} h(\zeta) D^\mu(B_\zeta)(z) \frac{\widehat{\mu}(\zeta)}{|\zeta|} dA(\zeta), \quad z \in \mathbb{D}.$$

Observe that

$$\begin{aligned} T(h)(z) &= \int_{\mathbb{D}} h(\zeta) D^\mu(B_\zeta)(z) \frac{\widehat{\mu}(\zeta)}{|\zeta|} dA(\zeta) \\ &= \int_0^1 2\widehat{\mu}(r) \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{n=0}^\infty \hat{h}(n) r^n e^{in\theta} \right) \left(\sum_{k=0}^\infty \frac{z^k r^k e^{-ik\theta}}{\mu_{2k+1}} (k+1) \right) d\theta\, dr \\ &= \int_0^1 2\widehat{\mu}(r) \sum_{n=0}^\infty \frac{n+1}{\mu_{2n+1}} \hat{h}(n) z^n r^{2n}\, dr \\ &= \sum_{n=0}^\infty \frac{2(n+1)\widehat{\mu}_{2n}}{\mu_{2n+1}} \hat{h}(n) z^n = \sum_{n=0}^\infty \frac{2(n+1)}{2n+1} \hat{h}(n) z^n, \quad z \in \mathbb{D}. \end{aligned}$$

So if $\tilde{g}(z) = \sum_{n=0}^\infty \frac{2n+1}{2(n+1)} \hat{g}(n) z^n \in A^1$, $g(z) = T(\tilde{g})(z)$ and by Eq. 3.1

$$\langle g, f_r \rangle_{L^2} = \big\langle \tilde{g}, V_\mu(f_r) \big\rangle_{L^2}, \quad 0 \le r < 1. \tag{3.2}$$

Let us prove

$$\|\tilde{g}\|_{A^1} \lesssim \|g\|_{A^1}, \quad g \in \mathcal{H}(\mathbb{D}).$$

Notice that $\tilde{g}(z) = \sum_{n=0}^\infty \frac{2n+1}{2(n+1)} \hat{g}(n) z^n = g(z) - \sum_{n=0}^\infty \frac{1}{2(n+1)} \hat{g}(n) z^n = g(z) - h(z)$ and $2zh(z) = \sum_{n=0}^\infty \frac{\hat{g}(n)}{n+1} z^{n+1}$ is the primitive of g with value 0 at the origin. So, by the Littlewood-Paley equivalence for A^1 [14, Theorem 4.28]

$$\|h\|_{A^1} = \int_{\mathbb{D}} |h(z)|\, dA(z) \asymp \int_{\mathbb{D}} |2zh(z)|\, dA(z) \asymp \int_{\mathbb{D}} |g(z)|(1-|z|)\, dA(z) \le \|g\|_{A^1},$$

and therefore $\|\tilde{g}\|_{A^1} \leq \|g\|_{A^1} + \|h\|_{A^1} \lesssim \|g\|_{A^1}$. This fact together with Eq. 3.2 yields

$$\begin{aligned} |\langle g, f_r\rangle_{L^2}| &= |\langle \tilde{g}, V_\mu(f_r)\rangle_{L^2}| \leq \int_{\mathbb{D}} |\tilde{g}(z)||V_\mu(f_r)(z)|dA(z) \\ &= \int_{\mathbb{D}} \frac{|\tilde{g}(z)|}{|z|}|zV_\mu(f_r)(z)|dA(z) \lesssim \|\tilde{g}\|_{A^1}\|zV_\mu(f_r)\|_{L^\infty} \lesssim \|g\|_{A^1}\|zV_\mu(f_r)\|_{L^\infty}. \end{aligned} \tag{3.3}$$

Now, let us see that for every $0 < r < 1$

$$\|zV_\mu(f_r)\|_{L^\infty} = \sup_{z\in\mathbb{D}} \widehat{\mu}(z)|D^\mu(f_r)(z)| \leq \|f\|_{\mathcal{B}^\mu}. \tag{3.4}$$

In order to prove this last inequality, notice that $\widehat{\mu}(z)D^\mu(f_r)(z)$ is a continuous function in $\overline{\mathbb{D}}$ for every $0 < r < 1$, so there exists $z_r \in \overline{\mathbb{D}}$ where the supremum is reached. In addition, $z_r \in \mathbb{D}$ because $\widehat{\mu} = 0$ on $\mathbb{T}$. So,

$$\begin{aligned} \sup_{z\in\mathbb{D}} \widehat{\mu}(z)|D^\mu(f_r)(z)| &= \widehat{\mu}(z_r)|D^\mu(f_r)(z_r)| \\ &\leq \widehat{\mu}(z_r) \sup_{|u|=|z_r|} |D^\mu(f)(u)| = \sup_{|u|=|z_r|} \widehat{\mu}(u)|D^\mu(f)(u)| \leq \|f\|_{\mathcal{B}^\mu}. \end{aligned}$$

Therefore, joining Eqs. 3.3 and 3.4, $L_f \in (A^1)^*$ and $\|L_f\| \lesssim \|f\|_{\mathcal{B}^\mu}$.

Now, let us prove that $\mathcal{B}_0^\mu \subset \mathcal{B}_0$. By Proposition 7 $\lim_{r\to1^-} \|f - f_r\|_{\mathcal{B}^\mu} = 0$ for each $f \in \mathcal{B}_0^\mu$, which together with Eq. 1.4 implies

$$\lim_{r\to1^-} \|f - f_r\|_{\mathcal{B}} \lesssim \lim_{r\to1^-} \|f - f_r\|_{\mathcal{B}^\mu} = 0, \quad \text{for each } f \in \mathcal{B}_0^\mu.$$

This finishes the proof. □

Proof of Theorem 3 (i)⇒(ii). Let $f \in \mathcal{B}$. Since $\mathcal{B} \subset A^1$, Eq. 2.12 yields

$$D^\mu(f)(z) = \int_{\mathbb{D}} f(\zeta)D^\mu(B_\zeta)(z)dA(\zeta).$$

Moreover, since the Bergman projection $P : L^\infty \to \mathcal{B}$ is bounded and onto there exists $h \in L^\infty$ such that $P(h) = f$ and $\|f\|_{\mathcal{B}} \asymp \|h\|_{L^\infty}$. Then,

$$\begin{aligned} D^\mu(f)(z) &= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} h(u)B_u(\zeta)dA(u)\right) D^\mu(B_\zeta)(z)dA(\zeta) \\ &= \int_{\mathbb{D}} h(u) \left(\int_{\mathbb{D}} D^\mu(B_\zeta)(z)B_u(\zeta)dA(\zeta)\right) dA(u) \\ &= \int_{\mathbb{D}} h(u)\overline{D^\mu(B_z)(u)}dA(u). \end{aligned}$$

Hence, by using Proposition 5 and Lemma 4(iii)

$$\begin{aligned} |D^\mu(f)(z)| &\leq \|h\|_{L^\infty} \int_{\mathbb{D}} |D^\mu(B_z)(u)|dA(u) \asymp \|f\|_{\mathcal{B}} \int_0^1 \left(1 + \int_0^{s|z|} \frac{1}{\widehat{\mu}(t)(1-t)^2}dt\right) ds \\ &\leq \|f\|_{\mathcal{B}} \left(1 + \int_0^1 \frac{|z|}{\widehat{\mu}(s|z|)(1-s|z|)}ds\right) \lesssim \frac{\|f\|_{\mathcal{B}}}{\widehat{\mu}(z)}, \quad z \in \mathbb{D}. \end{aligned}$$

Therefore, (ii) holds.

(ii)⇒(i). Firstly, let us prove that $\mu \in \widehat{\mathcal{D}}$. Testing Eq. 1.5 on monomials we obtain a constant $C = C(\mu) > 0$ such that

$$\widehat{\mu}(r)r^n \le C\mu_{2n+1}, \quad n \in \mathbb{N}\cup\{0\}, \quad 0 \le r < 1.$$

Therefore

$$\mu_{\frac{3}{2}n+1} = \left(\frac{3}{2}n+1\right)\int_0^1 r^{\frac{3}{2}n}\widehat{\mu}(r)dr \le C\left(\frac{3}{2}n+1\right)\mu_{2n+1}\int_0^1 r^{\frac{1}{2}n}dr \le 3C\mu_{2n+1}, \quad n \in \mathbb{N}\cup\{0\}.$$

So, if $x \ge 1$ and $n \in \mathbb{N}$ are such that $n \le x < n+1$,

$$\mu_{\frac{3}{2}x} \le C\mu_{\frac{3}{2}x+1} \le C\mu_{\frac{3}{2}n+1} \le C\mu_{2n+1} \le C\mu_{2x-2} \le C\mu_{2x}.$$

where $C > 0$ is a constant that changes in every inequality only depending on μ. Naming $y = \frac{3}{2}x$,

$$\mu_y \le C\mu_{\frac{4}{3}y} \le C^2\mu_{\frac{16}{9}y} \le C^3\mu_{\frac{64}{27}y} \le C^3\mu_{2y}, \quad y \ge \frac{3}{2}.$$

Consequently, $\mu \in \widehat{\mathcal{D}}$ by Lemma A (iv) .

Now, let us prove that $\mu \in \mathcal{M}$. By [10, Theorem 2] it is sufficient to prove that for every $f \in \mathcal{B}$, there exists $g \in L^\infty$ such that $P_\mu(g) = f$ and $\|g\|_{L^\infty} \lesssim \|f\|_{\mathcal{B}}$. Let $f(z) = \sum_{n=0}^{\infty}\hat{f}(n)z^n \in \mathcal{B}$ and $h(z) = \sum_{n=0}^{\infty}\hat{h}(n)z^n \in \mathcal{B}^\mu$. Then,

$$\begin{aligned}
P_\mu(\widehat{\mu}\cdot D_\mu(h))(z) &= \int_{\mathbb{D}} \widehat{\mu}(\zeta)D_\mu(h)(\zeta)\overline{B_z^\mu(\zeta)}\mu(\zeta)dA(\zeta)\\
&= \int_{\mathbb{D}}\left(\sum_{n=0}^{\infty}\frac{\hat{h}(n)\zeta^n}{\mu_{2n+1}}\right)\left(\sum_{k=0}^{\infty}\frac{\overline{\zeta}^k z^k}{2\mu_{2k+1}}\right)\mu(\zeta)\widehat{\mu}(\zeta)dA(\zeta)\\
&= \int_0^1 r\mu(r)\widehat{\mu}(r)\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=0}^{\infty}\frac{\hat{h}(n)r^n e^{in\theta}}{\mu_{2n+1}}\right)\left(\sum_{k=0}^{\infty}\frac{r^k e^{-ik\theta}z^k}{\mu_{2k+1}}\right)d\theta dr\\
&= \int_0^1 \mu(r)\widehat{\mu}(r)\sum_{n=0}^{\infty}\frac{\hat{h}(n)r^{2n+1}}{\mu_{2n+1}^2}z^n dr\\
&= \sum_{n=0}^{\infty}\frac{(\mu\widehat{\mu})_{2n+1}}{\mu_{2n+1}^2}\hat{h}(n)z^n, \quad z \in \mathbb{D}.
\end{aligned}$$

So, if we take $g(z) = \widehat{\mu}(z)D^\mu(h)(z)$ with h defined by $\hat{h}(n) = \frac{\mu_{2n+1}^2}{(\mu\widehat{\mu})_{2n+1}}\hat{f}(n)$, all that we need to prove is that $\lambda_n = \frac{(\mu_{2n+1})^2}{(\mu\widehat{\mu})_{2n+1}}$, $n \in \mathbb{N}\cup\{0\}$, is a coefficient multiplier of $\mathcal{B}$. Then, by hipothesis

$$\|g\|_{L^\infty} = \|h\|_{\mathcal{B}^\mu} \lesssim \|h\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{B}}$$

and $P_\mu(g) = f$. In order to prove that $\{\lambda_n\}_{n=0}^{\infty}$ is a coefficient multiplier of $\mathcal{B}$ it is enough to see that

$$\sup_{0<r<1}(1-r)M_1(r,\lambda^{[1]}) < \infty. \tag{3.5}$$

This follows from the equivalence $\|f\|_{\mathcal{B}} \asymp \sup_{z\in\mathbb{D}}(1-|z|)^{\beta}||f^{[\beta]}(z)|$ [1] for the multiplier transformation $f^{[\beta]}(z)=\sum_{n=0}^{\infty}(n+1)^{\beta}\widehat{f}(n)z^n,\ \beta>0$, and the inequalities

$$\begin{aligned}|(\lambda * f)^{[2]}(r^2e^{it})|(1-r)^2 &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\lambda^{[1]}(re^{i(t+\theta)})f^{[1]}(re^{-i\theta})\,d\theta\right|(1-r)^2\\ &\le M_\infty(r,f^{[1]})M_1(r,\lambda^{[1]})(1-r)^2 \lesssim \|f\|_{\mathcal{B}}M_1(r,\lambda^{[1]})(1-r),\end{aligned}$$

where $\lambda(z)=\sum\limits_{n=0}^{\infty}\lambda_n z^n$.

By Eq. 2.4,

$$M_1(r,\lambda^{[1]}) = \|(\lambda^{[1]})_r\|_{H^1} \le C(\mu)+\sum_{n=2}^{\infty}\|V_n*(\lambda^{[1]})_r\|_{H^1}, \quad 0<r<1. \tag{3.6}$$

For each $n\in\mathbb{N}\setminus\{1\}$ and $r\in\left[\frac{1}{2},1\right)$ consider

$$F_n(x)=\frac{\mu_{2x+1}^2}{(\mu\widehat{\mu})_{2x+1}}r^x\chi_{[2^{n-1},2^{n+1}]}(x), \quad x\in\mathbb{R},$$

and $G_n(x)=(x+1)F_n(x)$. On the other hand, by Lemma A(iii)

$$(\mu\widehat{\mu})_{2x+1}=\frac{1}{2}\widehat{\mu}\left(1-\frac{1}{2x+1}\right)^2 \asymp \mu_{2x+1}^2, \quad x\ge 0. \tag{3.7}$$

In addition, for each radial weight ν there exists a constant $C=C(\nu)>0$ such that

$$\int_0^1 s^x\left(\log\frac{1}{s}\right)^n\nu(s)ds \le C\nu_x, \quad n\in\{1,2\}, \quad x\ge 2.$$

So, a direct calculation implies that there exists a constant $C=C(\mu)>0$ such that

$$|G_n''(x)|\le C|G_n(x)|, \quad n\in\mathbb{N}\setminus\{1\}, \quad r\in\left[\frac{1}{2},1\right), \quad x\ge 2. \tag{3.8}$$

Therefore, Eqs. 3.8 and 3.7 yield

$$\begin{aligned}A_{F_n,2} &= \max_{x\in[2^{n-1},2^{n+1}]}|G_n(x)| + \max_{x\in[2^{n-1},2^{n+1}]}|G_n''(x)| \lesssim \max_{x\in[2^{n-1},2^{n+1}]}|G_n(x)|\\ &\lesssim \max_{x\in[2^{n-1},2^{n+1}]}(x+1)r^x \lesssim 2^n r^{2^{n-1}}, \quad n\in\mathbb{N}\setminus\{1\}.\end{aligned}$$

For each $n\in\mathbb{N}\setminus\{1\}$, choose a C^∞−function Φ_n with compact support contained in $[2^{n-2},2^{n+2}]$ such that $\Phi_n=G_n$ on $[2^{n-1},2^{n+1}]$ and

$$A_{\Phi_n,2}=\max_{x\in\mathbb{R}}|\Phi_n(x)|+\max_{x\in\mathbb{R}}|\Phi_n''(x)| \lesssim 2^n r^{2^{n-1}}, \quad n\in\mathbb{N}\setminus\{1\}. \tag{3.9}$$

Since

$$W_1^{\Phi_n}(z)=\sum_{k\in\mathbb{Z}}\Phi_n(k)z^k=\sum_{k\in\mathbb{Z}\cap[2^{n-2},2^{n+2}]}\Phi_n(k)z^k,$$

Eq. 2.3 yields

$$\begin{aligned}V_n*(\lambda^{[1]})_r(z) &= \sum_{k=2^{n-1}}^{2^{n+1}-1}\phi\left(\frac{k}{2^{n-1}}\right)(k+1)\frac{\mu_{2k+1}^2}{(\mu\widehat{\mu})_{2k+1}}r^kz^k = \sum_{k=2^{n-1}}^{2^{n+1}-1}\phi\left(\frac{k}{2^{n-1}}\right)\Phi_n(k)z^k\\ &= \left(W_1^{\Phi_n}*V_n\right)(z), \quad n\in\mathbb{N}\setminus\{1\}.\end{aligned}$$

This together with Theorem B, Eqs. 3.9 and 2.4, implies

$$\begin{aligned}\|V_n * (\lambda^{[1]})_r\|_{H^1} &= \|W_1^{\Phi_n} * V_n\|_{H^1} \lesssim A_{\Phi_n,2}\|V_n\|_{H^1} \lesssim 2^n r^{2^{n-1}}\|V_n\|_{H^1}\\ &\lesssim 2^n r^{2^{n-1}}, \quad r \in \left[\frac{1}{2}, 1\right), \quad n \in \mathbb{N}\setminus\{1\}.\end{aligned}$$

which combined with Eq. 3.6 gives

$$M_1(r, \lambda^{[1]}) \lesssim \sum_{n=2}^{\infty} 2^n r^{2^{n-1}} \lesssim \frac{1}{1-r}, \quad r \in \left[\frac{1}{2}, 1\right).$$

Therefore Eq. 3.5 holds, and thus $\mu \in \mathcal{M}$.

The equivalence between (ii) and (iii) follows from Theorem 2. Therefore, we have already proved (i)⇔(ii)⇔(iii).

Next, let us prove (i)⇒(iv). Take $f \in \mathcal{B}_0$, then $\lim_{r\to 1^-} \|f_r - f\|_{\mathcal{B}_0} = 0$, which together with (ii) implies that $f \in \mathcal{B}^\mu$ and $\lim_{r\to 1^-} \|f_r - f\|_{\mathcal{B}_0^\mu} = 0$. So, $f \in \mathcal{B}_0^\mu$ by Proposition 7.

Now let us prove (iv)⇒(ii). If (iv) holds, then

$$\|f_r\|_{\mathcal{B}^\mu} \lesssim \|f_r\|_{\mathcal{B}} \le \|f\|_{\mathcal{B}}, \quad f \in \mathcal{B}, \quad 0 \le r < 1.$$

Moreover, for any $z \in \mathbb{D}$

$$\widehat{\mu}(z)|D^\mu(f)(z)| = \lim_{r\to 1^-} \widehat{\mu}(r)|D^\mu(f)(rz)| \le \liminf_{r\to 1^-} \|f_r\|_{\mathcal{B}^\mu},$$

so

$$\|f\|_{\mathcal{B}^\mu} \le \liminf_{r\to 1^-} \|f_r\|_{\mathcal{B}^\mu} \lesssim \|f\|_{\mathcal{B}}, \quad f \in \mathcal{B}.$$

That is (ii) holds.

The equivalence between (iv) and (v) follows from Theorem 2. This finishes the proof. □

4 Relationship Between $\mathcal{B}^\mu$ and H^∞

For each $x \in \mathbb{R}$, $E(x)$ denotes the integer such that $E(x) \le x < E(x)+1$. The following technical lemma will be used in the proof of Proposition 1. It can be deduced from the proof of [7, Lemma 8] but we prove it for the convenience of the reader.

Lemma 8 *Let* $\omega \in \widehat{\mathcal{D}}$ *such that* $\widehat{\omega}(0) = 1$. *For every* $n \in \mathbb{N}\cup\{0\}$ *denote by* r_n *the smallest* $\infty \in [0,1)$ *such that* $\widehat{\omega}(\infty) = \frac{1}{2^n}$, *and* $M_n = E\left(\frac{1}{1-r_n}\right)$. *Then,*

$$1 + \sum_{n=0}^{\infty} 2^n r^{M_n} \asymp \frac{1}{\widehat{\omega}(r)}, \quad 0 \le r < 1.$$

Proof We may assume that $r_1 \le r < 1$. Then, there exists $N \in \mathbb{N}$ such that $r_N \le r < r_{N+1}$, and

$$\sum_{n=0}^{\infty} 2^n r^{M_n} \ge \sum_{n=0}^{\infty} 2^n r_N^{M_n} \ge 2^N r_N^{M_N} \asymp 2^N \asymp 2^{N+1} = \frac{1}{\widehat{\omega}(r_{N+1})} \ge \frac{1}{\widehat{\omega}(r)}.$$

In order to prove the reverse inequality. Firstly, we deal with $r = r_N$ and we split the series in two terms:

$$\sum_{n=0}^{N} 2^n r_N^{M_n} \le r_N^{M_N} \sum_{n=0}^{N} 2^n \asymp 2^{N+1} \asymp \frac{1}{\widehat{\omega}(r_N)}.$$

For the remaining term, by Lemma A (ii) there exists $\alpha > 0$ and $C > 0$ such that

$$\frac{1-r_n}{1-r_{n+j}} \geq C\left(\frac{\widehat{\omega}(r_n)}{\widehat{\omega}(r_{n+j})}\right)^{\frac{1}{\alpha}} = C2^{\frac{j}{\alpha}}, \quad n, j \in \mathbb{N} \cup \{0\}.$$

This, the inequality

$$\log\frac{1}{x} \geq 1-x, \quad 0 < x \leq 1,$$

and the definition of r_n give

$$\begin{aligned}\sum_{n=N+1}^{\infty} 2^n r_N^{M_n} &= 2^N \sum_{j=1}^{\infty} 2^j r_N^{M_{N+j}} = 2^N \sum_{j=1}^{\infty} 2^j e^{-M_{N+j}\log\frac{1}{r_N}} \\ &\leq 2^N \sum_{j=1}^{\infty} 2^j e^{-\frac{1-r_N}{1-r_{N+j}}} \leq 2^N \sum_{j=1}^{\infty} 2^j e^{-C2^{\frac{j}{\alpha}}} \asymp 2^N = \frac{1}{\widehat{\omega}(r_N)}.\end{aligned}$$

For $r_1 \leq r < 1$, take $N \in \mathbb{N}$ with $r_N \leq r < r_{N+1}$, and then

$$\sum_{n=0}^{\infty} 2^n r^{M_n} \leq \sum_{n=0}^{\infty} 2^n {r_{N+1}}^{M_n} \lesssim \frac{1}{\widehat{\omega}(r_{N+1})} = \frac{2}{\widehat{\omega}(r_N)} \leq \frac{2}{\widehat{\omega}(r)}.$$

This finishes the proof. □

Proof of Proposition 1 If $H^\infty \not\subset \mathcal{B}^\mu$, then its obvious that $H^\infty \neq \mathcal{B}^\mu$.
Next assume $H^\infty \subset \mathcal{B}^\mu$. Then, bearing in mind Theorem 2 we have that convergence in $\mathcal{B}^\mu$ implies uniform convergence on compact subsets of $\mathbb{D}$, and then standard arguments imply that H^∞ is continuously embedded in $\mathcal{B}^\mu$, that is there exists a constant $C > 0$ such that

$$\|f\|_{\mathcal{B}^\mu} \leq C\|f\|_{H^\infty}, \quad f \in H^\infty.$$

Then, by testing this inequality on monomials and arguing as in the proof of (ii)⇒(i) of Theorem 3 it follows that $\mu \in \widehat{\mathcal{D}}$. Next, without loss of generality assume that $\widehat{\mu}(0) = 1$ and consider the function

$$f(z) = \mu_1 + \sum_{n=0}^{\infty} \mu_{2M_n+1} 2^n z^{M_n} \in \mathcal{H}(\mathbb{D}),$$

where $\{M_n\}$ are associated to μ via the statement of Lemma 8. Let us prove $f \in \mathcal{B}^\mu \setminus H^\infty$. By Lemma 8,

$$\begin{aligned}\|f\|_{\mathcal{B}^\mu} &= \sup_{z\in\mathbb{D}} \widehat{\mu}(z)|D^\mu(f)(z)| = \sup_{z\in\mathbb{D}} \widehat{\mu}(z)\left|1 + \sum_{n=0}^{\infty} 2^n z^{M_n}\right| \\ &\lesssim \sup_{0\leq r<1} \widehat{\mu}(r)\left(1 + \sum_{n=0}^{\infty} 2^n r^{M_n}\right) < \infty,\end{aligned}$$

that is $f \in \mathcal{B}^\mu$. On the other hand, by Lemma 8 and Lemma A(ii)

$$\begin{aligned}\mu_1 + \sum_{n=0}^{\infty} \mu_{2M_n+1} 2^n &= \int_0^1 t\mu(t)\left(1 + \sum_{n=0}^{\infty} 2^n t^{2M_n}\right) dt \\ &\asymp \int_0^1 t\frac{\mu(t)}{\widehat{\mu}(t^2)}\, dt \asymp \int_0^1 t\frac{\mu(t)}{\widehat{\mu}(t)}\, dt = \infty,\end{aligned}$$

which implies that $f \notin H^\infty$ because it has positive Taylor coefficients. This finishes the proof. $\square$

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Declarations

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