# MAXIMAL ABELIAN DIAGONALIZABLE GROUPS & FINE GRADINGS ON SIMPLE LIE ALGEBRAS

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*Algebraic Groups and Lie Algebras*

# INTRODUCTION TO FINE GRADINGS

# *What is a grading?*

$\mathbb{F}$ algebraically closed field of characteristic zero.

Let $G$ be an abelian group and $\mathcal{A}$ an $\mathbb{F}$-algebra.

A $G$-grading $\Gamma$ is a decomposition into vector subspaces $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ such that $\forall g, h \in G$,

$$\mathcal{A}_g\mathcal{A}_h \subset \mathcal{A}_{g+h}.$$

$G$=grading group, $\mathcal{A}_g$ homogeneous component of degree $g$,

$\mathrm{Supp}(\Gamma) = \{g \in G \mid \mathcal{A}_g \neq 0\}$ support of $\Gamma$.

Example.

$\mathcal{A} = \mathbb{F}[x]$ the polynomial algebra is $\mathbb{Z}$-graded with

$$\mathcal{A}_n = \begin{cases} \mathbb{F}x^n & \forall n \geq 0 \\ 0 & \text{remaining } n \end{cases}$$

# Example: on matrix algebras with few pieces

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$.

If $n = p + q$, the following block partition ($pp$, $pq$, $qp$ y $qq$ by rows) provides a $\mathbb{Z}_2$-grading:

$$\mathcal{A} = \underbrace{\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)}_{\mathcal{A}_{\bar{0}}} \oplus \underbrace{\left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right)}_{\mathcal{A}_{\bar{1}}}$$

$$\mathcal{A}_{\bar{0}}\mathcal{A}_{\bar{0}} \subset \mathcal{A}_{\bar{0}}, \quad \mathcal{A}_{\bar{0}}\mathcal{A}_{\bar{1}} \subset \mathcal{A}_{\bar{1}}, \quad \mathcal{A}_{\bar{1}}\mathcal{A}_{\bar{1}} \subset \mathcal{A}_{\bar{0}}$$

# *Example:* on matrix algebras with many pieces

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$.

Let us denote by $e_{ij}$ the matrix with 1 in the position $(i, j)$ and 0 in the others. These elements multiply as: $e_{ij}e_{kl} = \delta_{jk}e_{il}$.

$$\mathcal{A} = \langle e_{11}, e_{22}, ..., e_{nn}\rangle \oplus (\oplus_{i\neq j}\langle e_{ij}\rangle)$$

is a $\mathbb{Z}^{n-1}$-grading with the choice of degrees:

$$\deg e_{ii} = \bar{0}$$
$$\deg e_{ij} = (0, \ldots, 0, \underbrace{-1}_{i}, 0, \ldots, 0, \underbrace{1}_{j}, 0, \ldots, 0) \in \mathbb{Z}^n$$

# *Example:* *on a Lie algebra with many pieces*

$\mathcal{L} = \mathfrak{sl}(n, \mathbb{F}) = \{x \in \mathrm{Mat}_n(\mathbb{F}) \mid \mathrm{tr}(x) = 0\}$ simple Lie algebra of type $A_{n-1}$.

$$\mathcal{L} = \langle e_{11} - e_{22}, \ldots, e_{11} - e_{nn} \rangle \oplus \left( \oplus_{i \neq j} \langle e_{ij} \rangle \right)$$

is a $\mathbb{Z}^{n-1}$-grading induced by the above, since $\mathcal{L} \leq \mathcal{A}^- = (\mathcal{A}, [\ ,\ ])$ and $\mathcal{L}_g = \mathcal{A}_g \cap \mathcal{L}\ \forall g$.

# Example: on a Lie algebra with few pieces

The decomposition into symmetric and skew-symmetric matrices gives a $\mathbb{Z}_2$-grading:

$\mathcal{L} = \mathfrak{sl}_n(\mathbb{F}) = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$:

$$\mathcal{L}_{\bar{0}} = \{x \in \mathfrak{sl}_n(\mathbb{F}) \mid x^t = -x\} \cong \mathfrak{so}(n, \mathbb{F})$$
$$\mathcal{L}_{\bar{1}} = \{x \in \mathfrak{sl}_n(\mathbb{F}) \mid x^t = x\}$$

# Some algebraical motivation for studying gradings

- In the case of complex semisimple Lie algebras, the root decomposition is a grading over the group $\mathbb{Z}^{\mathrm{rank}}$ (which has played a key role in representation theory and so on).
- A superalgebra is an algebra graded over $\mathbb{Z}_2$.
- Gradings by finite abelian groups are involved in the definition of Lie color algebras.
- Symmetric spaces in Geometry correspond to $\mathbb{Z}_2$-gradings on Lie algebras (and lately to $\mathbb{Z}_m$-gradings).
- $\mathbb{Z}$-graded Lie algebra with pieces $1, 0, -1$:
  Tits-Kantor-Koecher construction applied to a Jordan algebra.
- $\mathbb{Z}$-graded Lie algebras with more pieces:
  related to structurable algebras.

# Basic definitions: fine gradings

Def. Let $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \oplus_{h\in H}\mathcal{A}'_h$ be two gradings. Then $\Gamma$ is said to be a refinement of $\Gamma'$ (or $\Gamma'$ a coarsening of $\Gamma$) if each $\mathcal{A}_g \subset$ some $\mathcal{A}'_h$.

Example. The above $\mathbb{Z}^{n-1}$-grading on $\mathrm{Mat}_n(\mathbb{F})$ was a refinement of the $\mathbb{Z}_2$-grading (chosen any $p$ and $q = n - p$).

Example. The above $\mathbb{Z}^{n-1}$-grading on $\mathfrak{sl}(n, \mathbb{F})$ was not a refinement of the $\mathbb{Z}_2$-grading (with even part the skewsymmetric matrices).

Def. A grading is fine if it has no proper refinements.

# Fundamental example:
## the root decomposition

If $\mathfrak{h}$ is a Cartan subalgebra (i.e., maximal semisimple subalgebra) of a finite dimensional semisimple Lie $\mathbb{F}$-algebra $L$,

$$L = \bigoplus_{\alpha \in \mathfrak{h}^*} L_\alpha,$$
$$L_\alpha = \{x \in L \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$$

$\Phi = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid L_\alpha \neq 0\}$ is called root system, $L_\alpha$ are called root spaces.

There is always $\Delta = \{\alpha_1, \dots, \alpha_l\} \subset \Phi$ a basis of $\Phi$, characterized by satisfying $\Phi \subset \sum_{i=1}^{l} \mathbb{Z}_{\geq 0}\alpha_i \cup \sum_{i=1}^{l} \mathbb{Z}_{\leq 0}\alpha_i$. The number $l$ is the rank of $L$.

This is a grading over $\sum_{i=1}^{l} \mathbb{Z}\alpha_i \cong \mathbb{Z}^l$ with support $\Phi \cup \{0\}$.

Example: the $\mathbb{Z}^{n-1}$-grading on $\mathfrak{sl}_n(\mathbb{F})$ of the above slide is just the root decomposition relative to $\mathfrak{h}$ = traceless diagonal matrices

# Motivation:
## for fine gradings in Mathematics

- They are analogues to the root decomposition: they could provide alternative approaches to the structure theory and representation theory of even the most common Lie algebras.
- They provide alternatives to the Chevalley basis of semisimple Lie algebras. The basis adapted to the grading has uncommon porperties: Every homogeneous element is either semisimple or nilpotent.
- It is a way to study the (group of) symmetries of the algebra.
- Each fine grading is related with a particular model or presentation of the algebra.

# Motivation:

## all the gradings can be obtained from the fine ones

- Obviously every grading is a coarsening of a fine grading.
- But it is easy to compute the coarsenings of a fixed grading:

  If $\Gamma : \mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ is a $G$-grading,
  and $\pi : G \to H$ is a group epimorphism,
  $\Longrightarrow \Gamma^\pi : \mathcal{L} = \bigoplus_{h\in H}\mathcal{M}_h$
  is an $H$-grading for $\mathcal{M}_h = \sum_{\pi(g)=h}\mathcal{L}_g$.
- And if $G$ is the *universal grading group*, every coarsening is obtained in this way.

# Some directions:
## and applications of fine gradings

- ♡ The problem of finding isomorphism classes of solvable Lie algebras of a given dimension;
- ♡ The study of deformations (non-equivalent transformations of the structure constants) of Lie algebras during which a chosen grading is preserved (particularly useful in applications).
- ♡ Construction of contractions of Lie algebras.
- ♡ They provide maximal sets of quantum observables with additive quantum numbers.

# EXAMPLE: Pauli gradings on matrix algebras

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$, $\xi$ primitive $n$th root of unit.

$$X_n = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & \xi & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \xi^{n-2} & 0 \\ 0 & \dots & \dots & 0 & \xi^{n-1} \end{pmatrix}, \quad Y_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

$$\boxed{X_n^n = 1 = Y_n^n, \qquad Y_nX_n = \xi X_nY_n}$$

$\mathcal{A}$ is $\mathbb{Z}_n^2$-graded: $\mathcal{A} = \oplus_{(\bar{i},\bar{j})\in\mathbb{Z}_n^2}\mathcal{A}_{(\bar{i},\bar{j})}$, $\qquad \mathcal{A}_{(\bar{i},\bar{j})} = \mathbb{F}X_n^iY_n^j$

(Pauli's grading)

$\Rightarrow \mathcal{A}$ is a graded division algebra (every nonzero hom. element is invertible).

# EXAMPLE: *fine gradings on* $\mathfrak{sl}(2,\mathbb{C})$

The root decomposition is a $\mathbb{Z}$-grading on $\mathfrak{sl}(2,\mathbb{C})$:

$$L_0 = \langle h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$$

$$L_2 = \langle e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle$$

$$L_{-2} = \langle f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle$$

$[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$

Also a $\mathbb{Z}_2^2$-grading given by the Pauli's matrices:

$$L_{(\bar{1},\bar{1})} = \langle X_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$$

$$L_{(\bar{0},\bar{1})} = \langle Y_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$$

$$L_{(\bar{1},\bar{0})} = \langle X_2 Y_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle$$

$$[e_i, e_{i+1}] = e_{i+2} \quad \forall i$$

# EXAMPLE: on Octonions

They are obtained by applying 3 times the Cayley-Dickson process:

$$\begin{aligned} \mathbb{K} &= \mathbb{F} \oplus \mathbb{F}i & i^2 &= -1 \\ \mathbb{H} &= \mathbb{K} \oplus \mathbb{K}j & j^2 &= -1 \\ \mathbb{O} &= \underbrace{\mathbb{H}}_{\bar{0}} \oplus \underbrace{\mathbb{H}l}_{\bar{1}} & l^2 &= -1 \end{aligned}$$

obtaining then a $\mathbb{Z}_2^3$-grading

$$\begin{aligned} \mathbb{O}_{(\bar{0},\bar{0},\bar{0})} &= \mathbb{F}1 & \mathbb{O}_{(\bar{0},\bar{0},\bar{1})} &= \mathbb{F}l \\ \mathbb{O}_{(\bar{1},\bar{0},\bar{0})} &= \mathbb{F}i & \mathbb{O}_{(\bar{1},\bar{0},\bar{1})} &= \mathbb{F}il \\ \mathbb{O}_{(\bar{0},\bar{1},\bar{0})} &= \mathbb{F}j & \mathbb{O}_{(\bar{0},\bar{1},\bar{1})} &= \mathbb{F}jl \\ \mathbb{O}_{(\bar{1},\bar{1},\bar{0})} &= \mathbb{F}ij & \mathbb{O}_{(\bar{1},\bar{1},\bar{1})} &= \mathbb{F}(ij)l \end{aligned}$$

# GRADINGS & MAD-GROUPS

# Finite order automorphisms

$(\omega \in \mathbb{F},\ \omega^3 = 1)$

♡ If $\mathcal{L} = \mathcal{L}_{\bar 0} \oplus \mathcal{L}_{\bar 1} \oplus \mathcal{L}_{\bar 2}$ is a $\mathbb{Z}_3$-grading,

$$\Longrightarrow f: \begin{array}{ccc} \mathcal{L} & \to & \mathcal{L} \\ x \in \mathcal{L}_{\bar i} & \mapsto & \omega^i x \end{array} \quad \text{order 3 automorphism}$$

♡ And if $f: \mathcal{L} \to \mathcal{L}$ order 3 automorphism, then the eigenspace decomposition $\mathcal{L} = \mathcal{L}_{\bar 0} \oplus \mathcal{L}_{\bar 1} \oplus \mathcal{L}_{\bar 2}$ is a $\mathbb{Z}_3$-grading.

♡ Every $\mathbb{Z}_3$-grading (inner) on $\mathcal{L}$ is as follows: there are $s_0, s_1, \dots, s_l \geq 0$ such that $3 = s_0 + \sum n_i s_i$, for $\Delta = \{\alpha_1, \dots, \alpha_l\}$ a basis of $\Phi$ (relative to some $\mathfrak{h}$) with $\sum n_i \alpha_i$ the maximal root and

$$\mathcal{L}_{\bar n} = \oplus_{\alpha \in \Phi \cup \{0\}} \{ L_\alpha \mid \alpha = \sum \alpha_i p_i, \sum s_i p_i \equiv n \pmod 3 \}$$

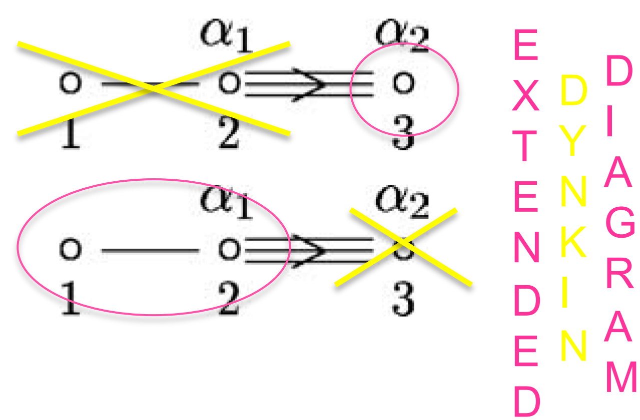
Semisimple part of $\mathcal{L}_{\bar 0}$ has Dynkin diagram the one obtained when removing from the extended Dynkin diagram the nodes whose labels $s_i \neq 0$.

# Finite order automorphisms: Examples on $\mathfrak{g}_2$

There are two order 3 automorphisms of $\mathfrak{g}_2$ (up to conjugacy):

Label $(1, 1, 0)$, fixed subalgebra: $\mathfrak{a}_1 + Z$

Label $(0, 0, 1)$, fixed subalgebra: $\mathfrak{a}_2$



# $\mathbb{Z}$-gradings

$$\begin{array}{rcl}
\mathbb{Z}\text{-grading on } \mathcal{L} & \Longleftrightarrow & \mathbb{F}^\times \hookrightarrow \mathrm{Aut}(\mathcal{L}) \\
\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n & \longrightarrow & z \mapsto \varphi_z : \mathcal{L} \to \mathcal{L} \\
 & & \text{if } x \in \mathcal{L}_n,\ \varphi_z(x) := z^n x \\
\mathcal{L}_n = \{x \in \mathcal{L} : \varphi(z)(x) = z^n x \,\forall z \in \mathbb{F}^\times\} & \longleftarrow & \varphi
\end{array}$$

# The correspondence

$G$ finitely generated abelian group.

| | | |
|---|---|---|
| $G$-grading on $A$ | $\Longleftrightarrow$ | Monomorphism of algebraic groups $\hat{G} = \mathrm{Hom}(G, \mathbb{F}) \to \mathrm{Aut}(A)$ |
| $A = \oplus_{g \in G} A_g$ | $\longrightarrow$ | $\chi \mapsto \varphi_\chi : A \to A$ <br> si $a \in A_g$, $\varphi_\chi(a) := \chi(g)a$ |

where if $G = \mathbb{Z}^n \times T$, for $T$ the torsion subgroup, then $\hat{G} = (\mathbb{F}^\times)^n \times T$ is a quasitorus.

Thus gradings on $A$ are in one-to-one correspondence with quasitori of $\mathrm{Aut}(A)$.

# The correspondence for *fine* gradings

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where if $G = \mathbb{Z}^n \times T$, for $T$ the torsion subgroup, then $\hat{G} = (\mathbb{F}^\times)^n \times T$ is a quasitorus.
Thus gradings on $A$ are in one-to-one correspondence with quasitori of $\mathrm{Aut}(A)$.

| | | |
|---|---|---|
| Fine grading on $A$ | $\Longleftrightarrow$ | Maximal quasitorus of $\mathrm{Aut}(A)$. <br> $=$ Maximal Abelian Diagonalizable subgroup <br> MAD-groups of $\mathrm{Aut}(A)$. |
| (up to equivalence) | $\longleftrightarrow$ | (up to conjugacy) |

# EXAMPLE: the root decomposition

The Cartan grading, or $\mathbb{Z}^l$-root decomposition relative to $\mathfrak{h}$, is produced by the maximal torus:

$$\begin{array}{rcl} \mathcal{T} = \{f \in \mathrm{Aut}(\mathcal{L}) : f|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}\} & \cong & (\mathbb{F}^\times)^l \\ f & \mapsto & (x_1, \ldots, x_l) \end{array}$$

where $x_i \in \mathbb{F}^\times$ are such that $f|_{\mathcal{L}_{\alpha_i}} = x_i \,\mathrm{id}$
(fixed $\{\alpha_1, \ldots, \alpha_l\}$ a basis of the root system relative to $\mathfrak{h}$).

The scalars determine $f$ since $f|_{\mathcal{L}_{\alpha=\sum s_i\alpha_i}} = x_1^{s_1} \ldots x_l^{s_l} \,\mathrm{id}_{\mathcal{L}_\alpha}$.

# EXAMPLE: fine gradings on $\mathfrak{sl}(2)$ in terms of MAD-groups

$$\mathrm{Aut}(\mathfrak{a}_1) \cong \mathrm{PSL}(2) = \frac{\mathrm{SL}(2)}{\langle -I \rangle}$$

★ The $\mathbb{Z}$-grading (the root decomposition) is produced by the torus:

$$\left\langle \left[ \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \right] : \alpha \in \mathbb{F}^\times \right\rangle$$

★ The $\mathbb{Z}_2^2$-grading is produced by

$$\langle [\mathbf{i}X_2], [\mathbf{i}Y_2] \rangle \ (X_2 \text{ and } Y_2 \text{ anticommute}, \mathbf{i} \in \mathbb{F}, \mathbf{i}^2 = -1)$$

# EXAMPLE: gradings on matrices

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$ $G$-graded.

There are $k, l_1, \ldots, l_r \in \mathbb{N}$ with $kl_1 \ldots l_r = n$ such that $\mathcal{A}$ is isomorphic (as graded algebra) to:

$$\underbrace{\mathrm{Mat}_k(\mathbb{F})}_{\substack{\text{elementary} \\ \text{coars. of } \mathbb{Z}^{k-1}\text{-grad}}} \otimes \underbrace{\mathrm{Mat}_{l_1}(\mathbb{F})}_{\substack{\text{Pauli} \\ \mathbb{Z}_{l_1}^2\text{-grading}}} \otimes \cdots \otimes \underbrace{\mathrm{Mat}_{l_r}(\mathbb{F})}_{\substack{\text{Pauli} \\ \mathbb{Z}_{l_r}^2\text{-grading}}}$$

# Gradings on matrices
## in terms of MAD-groups

$\mathcal{P}_k = \{\alpha X_k^i Y_k^j \mid i,j = 0,\ldots,k-1; \alpha \in \mathbb{F}^\times, \alpha^k = 1\} \leq \mathrm{GL}(k,\mathbb{F})$
$\mathcal{D}_k = \{\mathrm{diag}(d_1,\ldots,d_k) \mid \ d_i \in \mathbb{F}^\times\} \leq \mathrm{GL}(k,\mathbb{F})$

Any MAD-group of $\mathrm{Aut}(\mathrm{Mat}_n(\mathbb{F})) = \mathrm{Ad}(\mathrm{GL}(n,\mathbb{F})) \cong \frac{\mathrm{GL}(n,\mathbb{F})}{\mathbb{F}^\times I}$ is conjugated to

$$\mathrm{Ad}(\mathcal{P}_{l_1} \otimes \cdots \otimes \mathcal{P}_{l_r} \otimes \mathcal{D}_k)$$

for some $l_1,\ldots,l_r,k \in \mathbb{N}$ such that $l_1 \ldots l_r k = n$.

# *Tricks*

If $Q$ is a finite maximal quasitorus of a semisimple, connected and simply connected group $G$,

$$\boxed{\theta \in Q \Rightarrow \text{Fix}(\theta) \text{ is a semisimple subalgebra}}$$

In other words: the elements in $Q$ come from

removing ONLY ONE NODE of the Dynkin diagram.

# STATE OF THE ART

# *A bit of history* *of Lie gradings*

- 1989, Patera and Zassenhaus initiated a systematic study of gradings on Lie algebras;
- 1998, Havlícek, Patera and Pelantová, computation of MAD-groups (so, fine gradings on) the classical simple complex Lie algebras ($\neq \mathfrak{d}_4$);
- 2010, Bahturin and Kotchetov, classification (up to isomorphism) of gradings on classical simple Lie algebras ($\mathbb{F} = \overline{\mathbb{F}}$, char $\mathbb{F} \neq 2$);
- 2010, Elduque, fine gradings (up to equivalence) on classical simple Lie algebras ($\mathbb{F} = \overline{\mathbb{F}}$, char $\mathbb{F} \neq 2$);
- 2006, 2009, 2015, Martín, D., Viruel, $\mathfrak{g}_2$, ($\mathfrak{d}_4$), $\mathfrak{f}_4$ and $\mathfrak{e}_6$ ($\mathbb{F} = \overline{\mathbb{F}}$, char $\mathbb{F} = 0$);
- 2015 Yu? (with Elduque's results), announced $\mathfrak{e}_7$ and $\mathfrak{e}_8$;
- 2013, Kotchetov and Elduque, THE BOOK (including prime characteristic and more).


# Gradings on classical Lie algebras of types B, C and D

If $\mathcal{L} \in \begin{cases} \mathfrak{so}(n, \mathbb{F}) & n \geq 5, n \neq 8 \\ \mathfrak{sp}(n, \mathbb{F}) & n \geq 6, n \text{ even} \end{cases}$

$\Rightarrow$ there is a matrix algebra $\mathcal{A}$ and an involution $\varphi\colon \mathcal{A} \to \mathcal{A}$ s.t.

$$\mathcal{L} = \mathrm{Skew}(\mathcal{A}, \varphi).$$

Every grading on $\mathcal{L}$ is restriction of one on $\mathcal{A}$:

There is $\Gamma : \mathcal{A} = \oplus_{g \in G} \mathcal{A}_g$ such that $\mathcal{L}_g = \mathcal{A}_g \cap \mathcal{L}$.

# Gradings on classical Lie algebras of type A

If $\mathcal{L} = \mathfrak{sl}(n, \mathbb{F})$, then $\mathcal{L} \leq \mathcal{A}^-$ for $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$.
Observe that $\varphi(x) = -x^t$ satisfies $\varphi \in \mathrm{Aut}\,\mathcal{L}$ but $\varphi \notin \mathrm{Aut}\,\mathcal{A}$.

Type I- If $\mathcal{A} = \oplus_{g \in G} \mathcal{A}_g$ is a grading, then

$$\mathcal{L}_g = \mathcal{A}_g \cap \mathcal{L} \text{ gives a grading on } \mathcal{L}$$

Type II- If $\mathcal{A} = \oplus_{g \in G} \mathcal{A}_g$ is a grading and $\varphi$ a graded ($\varphi(A_g) \subset A_g$) antihomomorphism, then, given $h \in G$ of order 2,

$$\mathcal{L}_g = \mathrm{Skew}(\mathcal{A}_g, \varphi) \oplus (\mathrm{Sym}(\mathcal{A}_{g+h}, \varphi) \cap \mathcal{L})$$

gives a grading on $\mathcal{L}$

Every grading on $\mathcal{L}$ is of type either I or II
(of type I if $n = 2$)

# MAD-groups (type A) with inner automorphisms (type I)

$\mathcal{P}_k = \{\alpha X_k^i Y_k^j \mid i, j = 0, \ldots, k-1; \alpha \in \mathbb{C}^\times, |\alpha| = 1\}$
$\mathcal{D}_k = \{\mathrm{diag}(d_1, \ldots, d_k) \mid \alpha_i \in \mathbb{C}^\times\}$

Any MAD-group of $\mathrm{Aut}(\mathfrak{sl}(n, \mathbb{C}))$ containing only inner automorphisms is conjugated to

$$\mathrm{Ad}(\mathcal{P}_{k_1} \otimes \cdots \otimes \mathcal{P}_{k_s} \otimes \mathcal{D}_m)$$

for some $k_1 \ldots k_s m = n$.

# A MAD-group (type A) with outer automorphisms (type II)

$$\mathcal{T}_{2p,s} = \{\beta\, \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_s, \alpha_1, 1/\alpha_1, \ldots, \alpha_p, 1/\alpha_p) \mid \beta \in \{1, \mathbf{i}\}, \varepsilon_i = \pm 1, \alpha_j \in \mathbb{C}^\times\}$$

$\longrightarrow$ $\mathrm{Ad}(\mathcal{T}_{2p,s}) \rtimes \langle \varphi_C \rangle$ MAD-group of $\mathrm{Aut}(\mathfrak{sl}(2p+s, \mathbb{C}))$

$$\varphi_C(x) = -C^{-1} x^t C, \quad C = \mathrm{diag}(I_s, \underbrace{X_2, \ldots, X_2}_{p})$$

# "Outer" MAD-groups (type A)

Let $s, p, m$ be such that $(2p+s)m = n$.
Take $(\bar{P}, \bar{Q}) = \{(P_k, Q_k)\}_{k=1}^{m}$ $m$-pairs of matrices with $\nu_i^j, \rho_i^j = \pm 1, \pm \mathbf{i}$:

$$\begin{aligned} P_k &= \operatorname{diag}(\nu_1^k, \dots, \nu_s^k, \nu_{s+1}^k, \nu_{s+1}^k, \dots, \nu_{s+p}^k, \nu_{s+p}^k) \\ Q_k &= \operatorname{diag}(\rho_1^k, \dots, \rho_s^k, \rho_{s+1}^k, \rho_{s+1}^k, \dots, \rho_{s+p}^k, \rho_{s+p}^k) \end{aligned}$$

Let us denote

$$\begin{gathered} S_{m,2p,s}^{k}(\bar{P}, \bar{Q}) = \langle \{ Q_k \otimes I_{2^{k-1}} \otimes X_2 \otimes I_{2^{m-k}}, P_k \otimes I_{2^{k-1}} \otimes Y_2 \otimes I_{2^{m-k}} \} \rangle \\ \mathcal{T}_{2p,s}^{(m)}(\bar{P}, \bar{Q}) = \prod_{k=1}^{m} S_{m,2p,s}^{k}(\bar{P}, \bar{Q}) \cdot \mathcal{T}_{2p,s} \otimes I_{2^m}. \end{gathered}$$

Any MAD-group of $\mathrm{Aut}(\mathfrak{sl}(n, \mathbb{C}))$ with outer automorphisms is conjugated to $\mathrm{Ad}(\mathcal{T}_{2p,s}^{(m)}(\bar{P}, \bar{Q})) \rtimes \langle \varphi_C \rangle$ for a suitable $C$.

# EXAMPLE: fine gradings on $\mathfrak{sl}(3,\mathbb{C})$

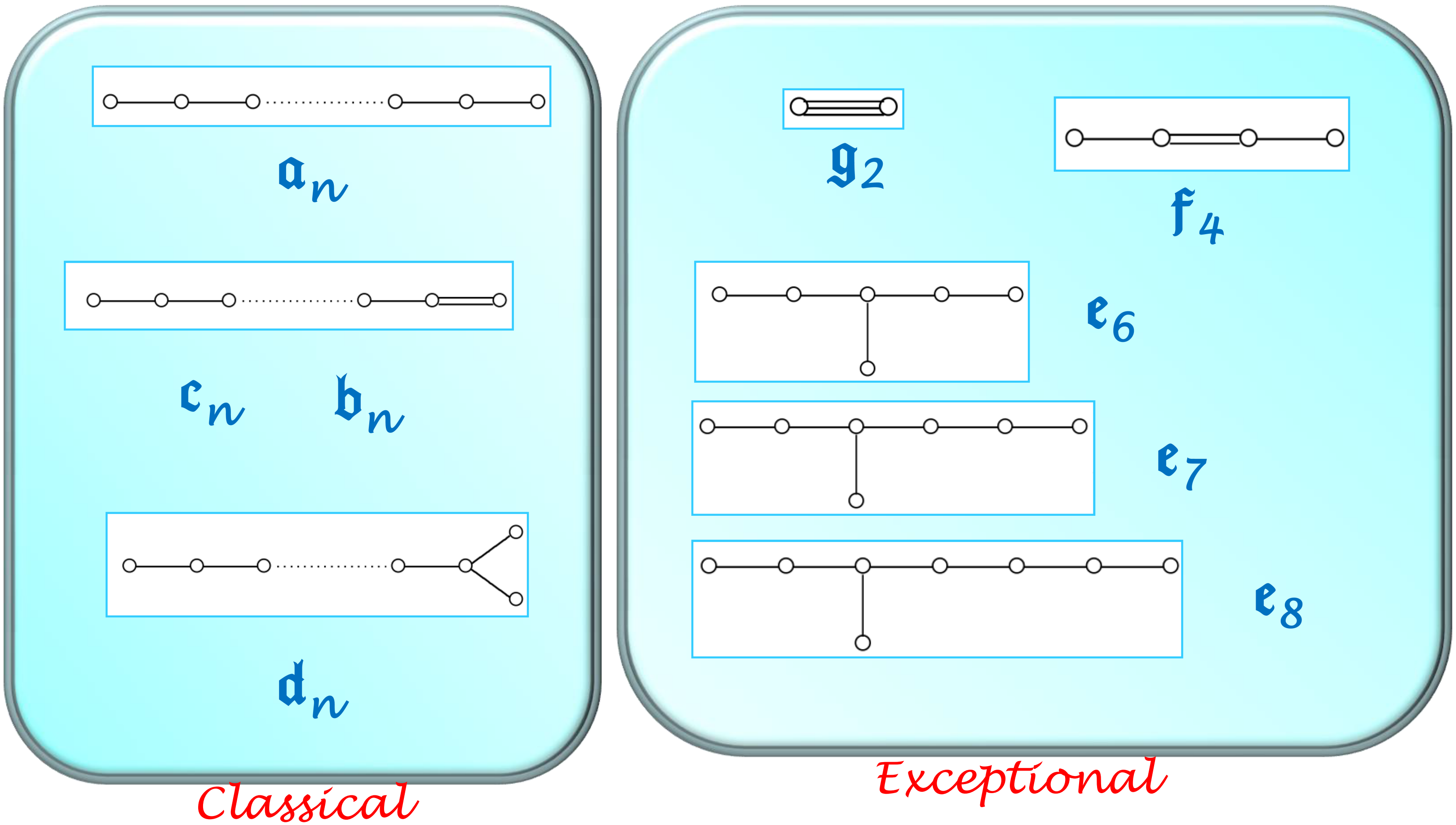
$$\mathrm{Aut}(\mathfrak{sl}(3)) \cong \mathrm{PGL}(3) \rtimes \mathbb{Z}_2$$
$$\varphi(x) = -x^t, \quad \varphi[C]\varphi = [(C^t)^{-1}]$$

| Inner<br>$\mathbb{Z}^2$ and $\mathbb{Z}_3^2$-gradings | Outer<br>$\mathbb{Z} \times \mathbb{Z}_2$ and $\mathbb{Z}_2^3$-gradings |
|---|---|
| $\left\{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & (\alpha\beta)^{-1} \end{bmatrix} \right\}$ | $\left\{ \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} : \alpha^2 + \beta^2 = 1 \right\} \times \langle \varphi \rangle$ |
| $\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle$ | $\left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \right\} \times \langle \varphi \rangle$ |

# MAD-GROUPS OF $G_2$

# About the existence of $\mathfrak{g}_2$

Killing (1887-90) and Cartan (1894, thesis) classified simple Lie algebras, in particular they found the root systems:



# *Two gradings on* $\mathfrak{g}_2$

The first model by Cartan (1914): $\mathfrak{g}_2 = \mathrm{Der}(\mathbb{O})$

There are two fine gradings on $\mathfrak{g}_2$:

♡ The $\mathbb{Z}^2$-Cartan grading (the root decomposition).

♡ The $\mathbb{Z}_2^3$-grading induced from $\mathbb{O}$.

| | |
|---|---|
| $\mathbb{O}_{(0,0,0)} = \mathbb{R}1$ | $\mathbb{O}_{(0,0,1)} = \mathbb{R}l$ |
| $\mathbb{O}_{(1,0,0)} = \mathbb{R}i$ | $\mathbb{O}_{(1,0,1)} = \mathbb{R}il$ |
| $\mathbb{O}_{(0,1,0)} = \mathbb{R}j$ | $\mathbb{O}_{(0,1,1)} = \mathbb{R}jl$ |
| $\mathbb{O}_{(1,1,0)} = \mathbb{R}ij$ | $\mathbb{O}_{(1,1,1)} = \mathbb{R}(ij)l$ |

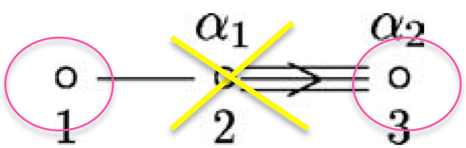
# *They are all (?)*

Not only gradings on $\mathbb{O}$ induce gradings on $\mathfrak{g}_2$, but conversely, because these two groups are isomorphic:

$$\begin{array}{ccc} \mathrm{Aut}(\mathbb{O}) & \stackrel{\cong}{\longleftrightarrow} & \mathrm{Aut}(\mathrm{Der}(\mathbb{O})) \\ f & \mapsto & \mathrm{Ad}(f)\colon \mathfrak{g}_2 \to \mathfrak{g}_2;\ d \mapsto fdf^{-1} \end{array}$$

Hence the correspondence between fine gradings ($\mathbb{O} = \mathfrak{der}(\mathbb{O})$) is 1-1.

# Another way of working on $\mathfrak{g}_2$




Any order two automorphism $\sigma \in \mathrm{Aut}(\mathfrak{g}_2)$ is:
There is $V$ ($\dim V = 2$) such that

$$\mathfrak{g}_2 \cong \underbrace{\mathfrak{sl}(V) \oplus \mathfrak{sl}(V)}_{L_{\bar{0}}} \oplus \underbrace{V \otimes S^3(V)}_{L_{\bar{1}}}$$

with $\sigma|_{L_{\bar{0}}} = \mathrm{id}$ and $\sigma|_{L_{\bar{1}}} = -\mathrm{id}$.

So $\mathrm{SL}(2) \times \mathrm{SL}(2) \to \mathrm{Aut}(\mathfrak{g}_2)$ is a homomorphism with image the centralizer of $\sigma$ and kernel $\{(\alpha I, \beta I) \mid \alpha\beta^3 = 1, \alpha^2 = 1 = \beta^2\} \Rightarrow$

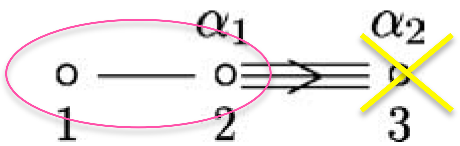
$$\mathrm{Cent}_{\mathrm{Aut}(\mathfrak{g}_2)}(\sigma) \cong \frac{\mathrm{SL}(2) \times \mathrm{SL}(2)}{\langle(-I, -I)\rangle} \qquad \sigma = [(-I, I)]$$

# MAD-groups of $\mathrm{Aut}(\mathfrak{g}_2)$

Any MAD-group with an order two element lives in $\mathrm{SL}(2)^2/\langle(-I,-I)\rangle$, and hence it is:

- $T = \left\{ \left[ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \right] : \alpha, \beta \in \mathbb{F}^\times \right\}$
- $\langle \{ [(\mathbf{i}X_2, \mathbf{i}X_2)], [(\mathbf{i}Y_2, \mathbf{i}Y_2)], [(-I, I)] \} \rangle \cong \mathbb{Z}_2^3$

# *Not more*




Another MAD-group would contain an order 3 element $\theta$ producing $\mathfrak{g}_2 \cong \mathfrak{sl}(U) \oplus U \oplus U^*$ (dim $U = 3$) and then it would live in

$$\mathsf{Cent}_{\mathsf{Aut}(\mathfrak{g}_2)}(\theta) \cong \mathrm{SL}(3)$$

# MAD-GROUPS OF $F_4$

# Models of $\mathfrak{f}_4$

- Borel (1950): $\mathfrak{f}_4 = \mathfrak{isom}(\mathbb{OP}^2)$
- Chevalley-Schafer (1950): $\mathfrak{f}_4 = \mathrm{Der}(\mathbb{A})$

where the Albert algebra is the 27-dim Jordan algebra

$$\mathbb{A} = \{x = (x_{ij}) \in \mathrm{Mat}_{3\times 3}(\mathbb{O}) : x_{ij} = \overline{x_{ji}}\}$$

with the product $x \cdot y = \frac{1}{2}(xy + yx)$. Distinguished elements for any $a \in \mathbb{O}$:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\iota_1(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & \bar{a} & 0 \end{pmatrix} \quad \iota_2(a) = \begin{pmatrix} 0 & 0 & \bar{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \quad \iota_3(a) = \begin{pmatrix} 0 & a & 0 \\ \bar{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

# Gradings on $\mathfrak{f}_4$ from the Albert algebra

Again every grading on $\mathbb{A}$ induce a grading on $\mathfrak{f}_4$ and conversely, because these two groups are isomorphic:

$$\begin{array}{ccc} \mathrm{Aut}(\mathbb{A}) & \xleftrightarrow{\cong} & \mathrm{Aut}(\mathrm{Der}(\mathbb{A})) \\ f & \mapsto & \mathrm{Ad}(f) : \mathfrak{f}_4 \to \mathfrak{f}_4 ; d \mapsto fdf^{-1} \end{array}$$

so that fine gradings are in one-to-one correspondence.

# $G_2 \subset F_4$

In particular, every grading on $\mathbb{O}$ induces a grading on $\mathbb{A}$: we have the group monomorphism:

$$\begin{array}{rcl} G_2 \cong \mathrm{Aut}(\mathbb{O}) & \hookrightarrow & \mathrm{Aut}(\mathbb{A}) \cong F_4 \\ f & \mapsto & \hat{f} : \mathbb{A} \to \mathbb{A} \\ & & E_i \mapsto E_i;\ \iota_i(a) \mapsto \iota_i(f(a)) \end{array}$$

For instance $\mathbb{Z}_2^3 \subset_{\mathrm{MAD}} G_2 \subset F_4$, although this abelian group is not maximal in $F_4$.

# Two fine gradings on $\mathfrak{f}_4$

The $\mathbb{Z}_2^3$-grading coming from $\mathbb{O}$ can be refined with:

* The $\mathbb{Z}_2^2$-grading on $\mathbb{A}$:

$$\begin{array}{ll} \mathbb{A}_{(\bar 0,\bar 0)} = \langle E_1, E_2, E_3 \rangle & \mathbb{A}_{(\bar 0,\bar 1)} = \iota_1(\mathbb{O}) \\ \mathbb{A}_{(\bar 1,\bar 0)} = \iota_2(\mathbb{O}) & \mathbb{A}_{(\bar 1,\bar 1)} = \iota_3(\mathbb{O}) \end{array}$$

* The $\mathbb{Z}$-grading on $\mathbb{A}$ produced by the derivation $2[R_{E_1}, R_{\iota_1(1)}] \in \mathfrak{Der}(\mathbb{A})$ (which has integer eigenvalues), for $R_x(y) = y \cdot x$ the multiplication operator.

$\longrightarrow$ $\mathbb{Z}_2^5, \mathbb{Z}_2^3 \times \mathbb{Z}$-fine gradings on $\mathbb{A}$ and on $\mathfrak{f}_4$

# The corresponding MAD-groups??

$F_4 \cong \mathrm{Isom}(\mathbb{O}\mathbb{P}^2) \rightsquigarrow \mathbb{O}\mathbb{P}^2 \cong F_4/\mathrm{Spin}(9) \rightsquigarrow$

$$\mathfrak{f}_4 \cong \underbrace{\mathfrak{so}(9)}_{L_{\bar{0}}} \oplus \underbrace{\text{spin module}}_{L_{\bar{1}}}$$

Any MAD-group with order two elements contains an automorphism conjugated to $\sigma$ producing such $\mathbb{Z}_2$-grading and hence it lives in

$$\mathrm{Cent}_{F_4}(\sigma) \cong \mathrm{Spin}(9).$$

# The corresponding MAD-groups: looking inside Spin(9)

Take $\{e_0, \ldots, e_8\}$ an orthonormal basis of $(\mathbb{F}^9, q)$. Recall that

$$\mathrm{Spin}(9) = \{\pm u_1 \ldots u_{2r} \mid u_j \in \mathbb{F}^9, q(u_j) = 1\}.$$

Hence the MAD-groups of $F_4$ with order two elements are:

* Maximal torus $\cong (\mathbb{F}^\times)^4$;
* $\langle e_1 e_2 e_3 e_4, e_1 e_2 e_5 e_6, e_0 e_1 e_3 e_5 \rangle \times \{\frac{(\varepsilon+\varepsilon^{-1})1+\mathbf{i}(\varepsilon-\varepsilon^{-1})e_7 e_8}{2}\} \cong \mathbb{F}^\times \times \mathbb{Z}_2^3$;
* $\langle -1, e_1 e_2 e_3 e_4, e_1 e_2 e_5 e_6, e_1 e_2 e_7 e_8, e_1 e_3 e_5 e_7 \rangle \cong \mathbb{Z}_2^5$.

# E6
# AND
# ITS RELATIVES E7 AND E8

# CONJECTURAL LIST: (Draper, Elduque, Kotchetov, 2013)

| $\mathfrak{e}_6$ | | $\mathfrak{e}_7$ | | $\mathfrak{e}_8$ | |
|---|---|---|---|---|---|
| Univ. group | Model | Univ. group | Model | Univ. group | Model |
| $\mathbb{Z}^6$ | Roots | $\mathbb{Z}^7$ | Roots | $\mathbb{Z}^8$ | Roots |
| $\mathbb{Z}^4 \times \mathbb{Z}_2$ | $T(\mathbb{K}, H_3(\mathbb{O}))$ | $\mathbb{Z}^4 \times \mathbb{Z}_2^2$ | $T(\mathbb{Q}, H_3(\mathbb{O}))$ | $\mathbb{Z}^4 \times \mathbb{Z}_2^3$ | $T(\mathbb{O}, H_3(\mathbb{O}))$ |
| $\mathbb{Z}^2 \times \mathbb{Z}_3^2$ | $T(\mathbb{O}, H_3(\mathbb{K}))$ | —— | | $\mathbb{Z}^2 \times \mathbb{Z}_3^3$ | $T(\mathbb{O}, H_3(\mathbb{O}))$ |
| $\mathbb{Z}^2 \times \mathbb{Z}_2^3$ | $T(\mathbb{O}, H_3(\mathbb{K}))$ | $\mathbb{Z}^3 \times \mathbb{Z}_2^3$ | $T(\mathbb{O}, H_3(\mathbb{Q}))$ | —— | |
| $\mathbb{Z}^2 \times \mathbb{Z}_2^3$ | $\text{Kan}(\text{C-D}(H_4(\mathbb{F})))$ | $\mathbb{Z}^2 \times \mathbb{Z}_2^4$ | $\text{Kan}(\text{C-D}(H_4(\mathbb{K})))$ | $\mathbb{Z}^2 \times \mathbb{Z}_2^5$ | $\text{Kan}(\text{C-D}(H_4(\mathbb{Q}))))$ |
| —— | | $\mathbb{Z} \times \mathbb{Z}_3^3$ | $T(\mathbb{Q}, H_3(\mathbb{O}))$ | —— | |
| $\mathbb{Z} \times \mathbb{Z}_2^5$ | $\text{Kan}(\text{C-D}(H_4(\mathbb{F})))$ | $\mathbb{Z} \times \mathbb{Z}_2^6$ | $\text{Kan}(\text{C-D}(H_4(\mathbb{K})))$ | $\mathbb{Z} \times \mathbb{Z}_2^7$ | $\text{Kan}(\text{C-D}(H_4(\mathbb{Q})))$ |
| $\mathbb{Z} \times \mathbb{Z}_2^4$ | $T(\mathbb{K}, H_3(\mathbb{O}))$ | $\mathbb{Z} \times \mathbb{Z}_2^5$ | $T(\mathbb{Q}, H_3(\mathbb{O}))$ | $\mathbb{Z} \times \mathbb{Z}_2^6$ | $T(\mathbb{O}, H_3(\mathbb{O}))$ |
| —— | | $\mathbb{Z} \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$ | $\text{Kan}(\text{C-D}(H_4(\mathbb{K})))$ | $\mathbb{Z} \times \mathbb{Z}_4^3$ | $\text{Kan}(\text{C-D}(H_4(\mathbb{Q})))$ |
| $\mathbb{Z}_3^4$ | $g(pK, Ok)$ | —— | | $\mathbb{Z}_3^5$ | $g(Ok, Ok)$ |
| $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$ | $T(\mathbb{O}, H_3(\mathbb{K}))$ | —— | | —— | |
| $\mathbb{Z}_2 \times \mathbb{Z}_3^3$ | $T(\mathbb{K}, H_3(\mathbb{O}))$ | $\mathbb{Z}_2^2 \times \mathbb{Z}_3^3$ | $T(\mathbb{Q}, H_3(\mathbb{O}))$ | $\mathbb{Z}_2^3 \times \mathbb{Z}_3^3$ | $T(\mathbb{O}, H_3(\mathbb{O}))$ |
| $\mathbb{Z}_2^7$ | $\mathfrak{stu}(\text{C-D}(H_4(\mathbb{F})))$ | $\mathbb{Z}_2^8$ | $\mathfrak{stu}(\text{C-D}(H_4(\mathbb{K})))$ | $\mathbb{Z}_2^9$ | $\mathfrak{stu}(\text{C-D}(H_4(\mathbb{Q})))$ |
| $\mathbb{Z}_2^6$ | $T(\mathbb{K}, H_3(\mathbb{O}))$ | $\mathbb{Z}_2^7$ | $T(\mathbb{Q}, H_3(\mathbb{O})))$ | $\mathbb{Z}_2^8$ | $T(\mathbb{O}, H_3(\mathbb{O})))$ |
| $\mathbb{Z}_4^3$ | $\text{Der}(\text{C-D}(H_4(\mathbb{Q})))$ | $\mathbb{Z}_4^3 \times \mathbb{Z}_2$ | $\mathfrak{str}_0(\text{C-D}(H_4(\mathbb{Q})))$ | $\mathbb{Z}_4^3 \times \mathbb{Z}_2^2$ | $\mathfrak{stu}_3(\text{C-D}(H_4(\mathbb{Q})))$ |
| $\mathbb{Z}_4 \times \mathbb{Z}_2^4$ | $\text{Der}(\text{C-D}(H_4(\mathbb{Q})))$ | $\mathbb{Z}_4 \times \mathbb{Z}_2^5$ | $\mathfrak{str}_0(\text{C-D}(H_4(\mathbb{Q})))$ | $\mathbb{Z}_4 \times \mathbb{Z}_2^6$ | $\mathfrak{stu}_3(\text{C-D}(H_4(\mathbb{Q})))$ |
| —— | | $\mathbb{Z}_4^2 \times \mathbb{Z}_2^3$ | $\mathfrak{stu}_3(\text{C-D}(H_4(\mathbb{K})))$ | —— | |
| —— | | —— | | $\mathbb{Z}_5^3$ | Jordan grading |

# Chevalley-Schafer (1950)

If $R_x : \mathbb{A} \to \mathbb{A}; y \mapsto yx$ denotes the operator multiplication, $[R_x, R_y] \in \mathrm{Der}(\mathbb{A})$, so that $\mathrm{Der}(\mathbb{A}) \oplus R_{\mathbb{A}_0} \leq \mathfrak{gl}(\mathbb{A})$. It is simple of type $E_6$!

$$\mathfrak{e}_6 \cong \underbrace{\mathrm{Der}(\mathbb{A})}_{L_{\bar{0}} \cong \mathfrak{f}_4} \oplus \underbrace{\mathbb{A}_0}_{L_{\bar{1}}}$$

Every $G$-grading on $\mathbb{A}$ can be extended to a $G \times \mathbb{Z}_2$-grading on $\mathfrak{e}_6$, getting:

fine gradings over $\mathbb{Z}^4 \times \mathbb{Z}_2$, $\mathbb{Z}_2^6$, $\mathbb{Z}_2^4 \times \mathbb{Z}$, $\mathbb{Z}_3^3 \times \mathbb{Z}_2$

# In terms of MAD-groups

Let $\tau \in \mathrm{Aut}(\mathfrak{e}_6)$ the automorphism producing the above $\mathbb{Z}_2$-grading (outer, since it fixes a rank 4 subalgebra).

$$\mathrm{Cent}_{\mathrm{Aut}(\mathfrak{e}_6)}(\tau) = F_4 \times \langle \tau \rangle$$

Hence, if $Q$ is a MAD-group of $F_4 \Rightarrow Q \times \langle \tau \rangle$ is a MAD-group of $\mathrm{Aut}(\mathfrak{e}_6)$.

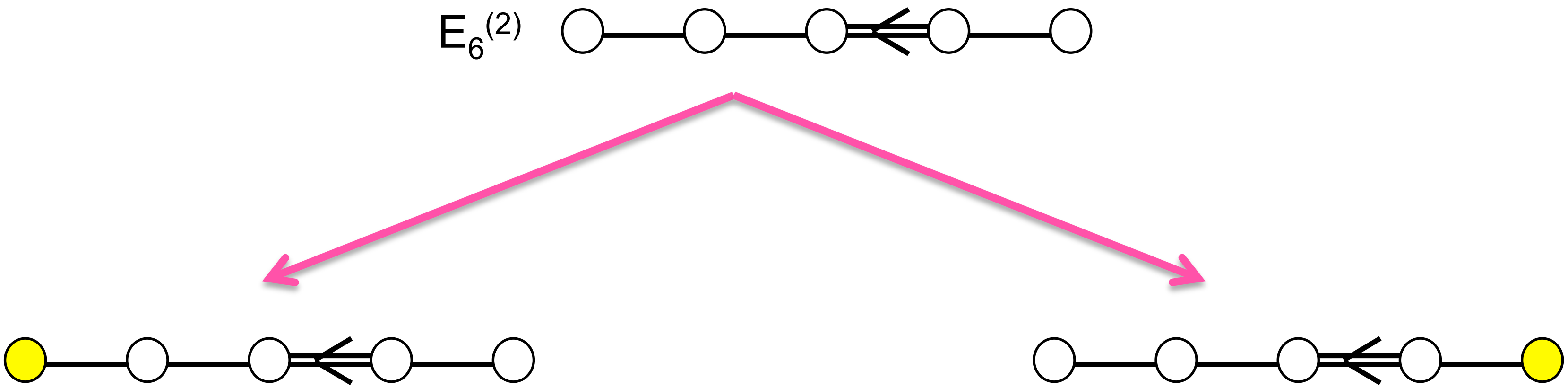
# About the group $Aut(\mathfrak{e}_6)$

Let $E_6$ be the connected and simply connected algebraic group of type $E_6$.

$$\begin{array}{ccc} E_6 \rtimes \langle \tau \rangle & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathfrak{e}_6) = \mathrm{Inn}(\mathfrak{e}_6) \rtimes \langle \tau \rangle \\ \downarrow & & \downarrow \\ E_6 & \xrightarrow[\mathrm{Ad}]{} & \mathrm{Inn}(\mathfrak{e}_6) \cong \frac{E_6}{ker\ \mathrm{Ad} \cong \mathbb{Z}_3}. \end{array}$$

# *With outer involutions*

$E_6^{(2)}$



$\tau$

$\mathrm{Fix}(\tau) \cong \mathfrak{f}_4$

$\mathrm{Cent}_{\mathrm{Aut}(\mathfrak{e}_6)}(\tau) = F_4 \times \langle \tau \rangle$

$\tau' = \tau t \quad (t \in \mathrm{Inn}(\mathfrak{e}_6))$

$\mathrm{Fix}(\tau') \cong \mathfrak{c}_4$

$\mathrm{Cent}_{\mathrm{Aut}(\mathfrak{e}_6)}(\tau') = \frac{\mathrm{Sp}(8)}{\langle -I \rangle} \times \langle \tau' \rangle$

# MAD-groups with outer automorphism τ'

If $Q \leq_{\mathrm{MAD}} \frac{\mathrm{Sp}(8)}{\langle -I \rangle} = \mathrm{Aut}(\mathfrak{c}_4) \Rightarrow Q \times \langle \tau' \rangle \leq_{\mathrm{MAD}} \mathrm{Aut}(\mathfrak{e}_6)$.

$\mathcal{T}^{(0)}_{8,0}$

$(\mathbb{F}^\times)^2 \times \mathbb{Z}_2^2$ $\quad \mathcal{T}^{(1)}_{4,0}(I_4, I_4) = \langle \mathrm{diag}\{\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}\} \otimes I_2, I_4 \otimes X_2, I_4 \otimes Y_2 \rangle$

$\mathcal{T}^{(1)}_{2,2}(I_4, I_4)$

$\mathcal{T}^{(1)}_{0,4}(I_4, I_4)$

$$\mathcal{T}^{(2)}_{0,2}\left(\left(\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{pmatrix}, I_2\right), (I_2, I_2)\right) = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{pmatrix} \otimes I_2 \otimes Y_2, I_2 \otimes X_2 \otimes I_2, I_2 \otimes Y_2 \otimes I_2, I_4 \otimes X_2 \right\rangle$$

$\mathbb{Z}_4 \times \mathbb{Z}_2^3$

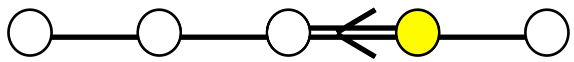
$$\mathcal{T}^{(2)}_{2,0}((I_2, I_2), (I_2, I_2)) = \langle \mathrm{diag}\{\alpha, \tfrac{1}{\alpha}\} \otimes I_4, I_4 \otimes X_2, I_4 \otimes Y_2, I_2 \otimes X_2 \otimes I_2, I_2 \otimes Y_2 \otimes I_2 \rangle$$

$\mathbb{F}^\times \times \mathbb{Z}_2^4$

$$\mathcal{T}^{(3)}_{0,1}((1,1),(1,1),(1,1)) = \langle I_4 \otimes X_2, I_4 \otimes Y_2, I_2 \otimes X_2 \otimes I_2, I_2 \otimes Y_2 \otimes I_2, X_2 \otimes I_4, Y_2 \otimes I_4 \rangle$$

$\mathbb{Z}_2^6$

# For outer MAD-groups:

The outer order four automorphism $\theta$ of $\mathfrak{e}_6$ corresponding to the diagram: 

produces the following $\mathbb{Z}_4$-grading: There are vector spaces $W, V$ ($\dim W = 4$, $\dim V = 2$) such that

$$\mathfrak{e}_6 \cong \underbrace{\mathfrak{sl}(W) \oplus \mathfrak{sl}(V)}_{L_{\bar{0}}} \oplus \underbrace{S^2(W) \otimes V}_{L_{\bar{1}}} \oplus L_{\bar{2}} \oplus L_{\bar{3}}$$

So $\mathrm{SL}(4) \times \mathrm{SL}(2) \to \mathrm{Aut}(\mathfrak{e}_6)$ is a homomorphism with image the centralizer of $\theta$ and kernel $\{(\alpha I, \beta I) \mid \alpha^2\beta = 1, \alpha^4 = 1 = \beta^2\} \Rightarrow$

$$\mathrm{Cent}_{\mathrm{Aut}(\mathfrak{e}_6)}(\theta) \cong \frac{\mathrm{SL}(4) \times \mathrm{SL}(2)}{\langle(\mathbf{i}I_4, -I_2)\rangle} \times \langle\theta\rangle \qquad \theta^2 = [(\mathbf{i}I_4, I_2)]$$

# Outer MAD-groups: without outer involutions

Any MAD-group with such order four element lives in $\frac{\mathrm{SL}(4)\times\mathrm{SL}(2)}{\langle(\mathbf{i}I_4,-I_2)\rangle}\times\langle\theta\rangle$, and hence it is conjugate to ($\zeta^2=\mathbf{i}$):

$$\langle\,[(\zeta X_4,\mathbf{i}X_2)],[(\zeta Y_4,\mathbf{i}\,Y_2)]\,,\theta\,\rangle\cong\mathbb{Z}_4^3$$

# Again the gradings viewpoint:

How one can see the related $\mathbb{Z}_4^3$-grading on $\mathfrak{e}_6$?

★ By using the model (Pauli's gradings on $\mathfrak{sl}(4)$ and $\mathfrak{sl}(2)$):

$$\mathfrak{e}_6 \cong \mathfrak{sl}(W) \oplus \mathfrak{sl}(V) \oplus S^2(W) \otimes V \oplus L_{\bar{2}} \oplus L_{\bar{3}}$$

★ As $\mathfrak{e}_6 \cong \mathfrak{der}(CD(H_4(\mathbb{H}))$, there must be a $\mathbb{Z}_4^3$-grading on the structurable algebra $A = CD(H_4(\mathbb{H}))$.

★ From this last approach,
as $\mathfrak{e}_7 \cong \mathfrak{str}_0(CD(H_4(\mathbb{H})) \Rightarrow \mathfrak{e}_7$ has a $\mathbb{Z}_4^3 \times \mathbb{Z}_2$-grading;
as $\mathfrak{e}_8 \cong \mathfrak{stu}_3(CD(H_4(\mathbb{H})) \Rightarrow \mathfrak{e}_8$ has a $\mathbb{Z}_4^3 \times \mathbb{Z}_2^2$-grading.

(The Structure algebra/the Steinberg construction are naturally $\mathbb{Z}_2/\mathbb{Z}_2^2$-graded).

# *Inner MAD-groups:*
## *with order 2 elements*

$$\mathbb{Z}_2^3 \subset_{\mathrm{MAD}} G_2 \subset F_4 \subset \mathrm{Inn}(\mathfrak{e}_6) \cong \frac{E_6}{\mathbb{Z}_3}$$

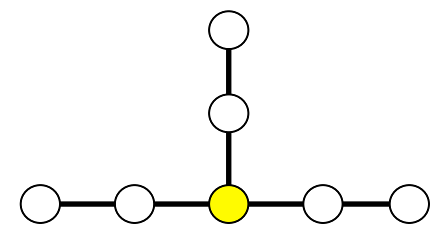
$$\longrightarrow \mathrm{Cent}_{\mathrm{Inn}(\mathfrak{e}_6)}(\mathbb{Z}_2^3) \cong \mathbb{Z}_2^3 \times \mathrm{PSL}(3)$$

In $\mathrm{PSL}(3)$ there are two MAD-groups:

$$\begin{cases} \langle [X_3], [Y_3] \rangle \cong \mathbb{Z}_3^2 \\ \left\langle \left[ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1/(\alpha\beta) \end{pmatrix} \right] \right\rangle \cong (\mathbb{F}^\times)^2 \end{cases}$$

$\longrightarrow$ MAD-groups of $\mathrm{Aut}(\mathfrak{e}_6)$: $\boxed{\mathbb{Z}_2^3 \times \mathbb{Z}_3^2 \text{ and } \mathbb{Z}_2^3 \times (\mathbb{F}^\times)^2.}$

# Inner MAD-groups:
## with order 3 elements



Adams model: $(\dim V = 3)$

$$\mathfrak{e}_6 \cong \underbrace{\mathfrak{sl}(V) \oplus \mathfrak{sl}(V) \oplus \mathfrak{sl}(V)}_{L_{\bar{0}}} \oplus \underbrace{V \otimes V \otimes V}_{L_{\bar{1}}} \oplus \underbrace{V^* \otimes V^* \otimes V^*}_{L_{\bar{2}}}$$

$\theta|_{L_{\bar{1}}} : v_1 \otimes v_2 \otimes v_3 \mapsto v_3 \otimes v_1 \otimes v_2$ $\rightarrow$ $\mathrm{Cent}(\theta, \mathbb{Z}_3) = \mathbb{Z}_3^2 \times \mathrm{PSL}(3)$
$\rightarrow$ Two MADs: $\mathbb{Z}_3^4$ and $\mathbb{Z}_3^2 \times (\mathbb{F}^\times)^2$.

(More non-toral $\mathbb{Z}_3^2$ lead to repeated MAD-groups).

$\tau|_{L_{\bar{1}}} : v_1 \otimes v_2 \otimes v_3 \mapsto v_2 \otimes v_1 \otimes v_3$; $\mathrm{Cent}(\tau) = \langle \tau \rangle \times F_4$
Here it is the required $\mathbb{Z}_3^3$-grading on $\mathfrak{f}_4$.

# Not more MAD-groups of $\mathrm{Aut}(\mathfrak{e}_6)$:
## Yu's approach

The difficult point is not trying to imagine fine gradings on maximal groups, but being sure that the list is complete (??).
The idea would be to find an inductive argument, but obviously:

$$Q \leq_{\mathrm{MAD}} H \leq G \not\Rightarrow Q \leq_{\mathrm{MAD}} G.$$

A recent approach (Yu, preprint arXiv):

★ To work with compact Lie groups.

★ To find closed abelian subgroups $F \leq G$ satisfying the condition $(*)$:

$$\dim F = \dim \mathrm{Cent}_G(F).$$

Of course if $Q$ is a MAD-group of $G \Rightarrow Q$ satisfies $(*)$.
The good point is that the condition $(*)$ has good functorial properties.

Translating the results $\rightarrow$ to the complex case
$\xrightarrow{\text{Elduque}}$ to $\mathbb{F} = \overline{\mathbb{F}}$, $\mathrm{char}\,\mathbb{F} = 0$ $\rightarrow$ the temptative list is now complete.

# The end