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Research Article

On One Generalization of the Multipoint Nonlocal Contact Problem for Elliptic Equation in Rectangular Area

Tinatin Davitashvili,[1] **Hamlet Meladze**,[2] **Francisco Criado-Aldeanueva**,[3] **and Jose Maria Sanchez**[4]

[1]*Iv.Javakhishvili Tbilisi State University, FENS, Tbilisi, Georgia*
[2]*Muskhelishvili Institute of Computational Mathematics, Georgian Technical University, Tbilisi, Georgia*
[3]*Department of Applied Physics II, Polytechnic School, Malaga University, Málaga, Spain*
[4]*Department of Didactic of Mathematics, Education Faculty, Malaga University, Málaga, Spain*

Correspondence should be addressed to Francisco Criado-Aldeanueva; fcaldeanueva@ctima.uma.es

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Nonlocal contact problem for two-dimensional linear elliptic equations is stated and investigated. The method of separation of variables is used to find the solution of a stated problem in the case of Poisson's equation. Then, the more general problem with nonlocal multipoint contact conditions for elliptic equation with variable coefficients is considered, and the iterative method to solve the problem numerically is constructed and investigated. The uniqueness and existence of the regular solution are proved. The iterative method allows reducing the solution of a nonlocal contact problem to the solution of a sequence of classical boundary value problems.

1. Introduction

It can be stated that the study of nonlocal boundary and initial-boundary problems and the development and analysis of numerical methods for their solution are important areas of applied mathematics. Nonlocal problems are naturally obtained in mathematical models of real processes and phenomena in biology, physics, engineering, ecology, etc. One can get acquainted with related questions in works [1–3] and the references cited there.

The publication of the articles [4–7] laid the foundation for further research in the area of nonlocal boundary problems and their numerical solution. In 1969, the work of A.Bitsadze and A.Samarskii [5] was published, which was related to the mathematical modeling of plasma processes. In this paper, a new type of nonlocal problem for elliptic equations was considered, hereinafter referred to as the Bitsadze-Samarsky problem. But the intensive research into nonlocal boundary value problems began in the 80s of the 20th century (see, for instance, [8–21] and references herein).

The work [12] is devoted to the formulation and investigation of a nonlocal contact problem for a parabolic-type linear differential equation with partial derivatives. In the first part of the work, the linear parabolic equation with constant coefficients is considered. To solve a nonlocal contact problem, the variable separation method (Fourier method) is used. Analytic solutions are built for this problem. Using the iterative method, the existence and uniqueness of the classical solution to the problem is proved. The effectiveness of the method is confirmed by numerical calculations. In paper [19], based on the variation approach, the definition of a classical solution is generalized for the Bitsadze-Samarskii nonlocal boundary value problem, posed in a rectangular area. In [21] the Bitsadze-Samarskii nonlocal boundary value problem for the two-dimensional Poisson equation is considered for a rectangular domain. The solution to this problem is defined as a solution to the local Dirichlet boundary value problem, by constructing a special method to find a function as the boundary value on the side of the rectangle, where the nonlocal condition was given. In paper [20], the two-dimensional Poisson equation with nonlocal integral boundary conditions in one of the directions, for a rectangular domain, is considered. For this problem, a difference scheme of increased order of approximation is constructed, its solvability is studied, and an iterative method for solving the corresponding system of difference equations is justified. In paper [8], a two-step difference scheme with the second order of accuracy for an approximate solution of the nonlocal boundary value problem for the elliptic differential equation in an arbitrary Banach space with the strongly positive operator is considered. In paper [11], the optimal control problem for the Helmholtz equation with nonlocal boundary conditions and quadratic functional is considered. The necessary and sufficient optimal conditions in a maximum principle form have been obtained. In [9], the Bitsadze-Samarskii boundary value problem is considered for a linear differential equation of first order for the bounded domain of the complex plane. The existence of a generalized equation is proved, and an a priori estimate is obtained. Then, the corresponding theorem on the existence and uniqueness of a generalized solution is proved. A boundary-value problem with a nonlocal integral condition is considered in [10] for a two-dimensional elliptic equation with mixed derivatives and constant coefficients. The existence and uniqueness of a weak solution is proved in a weighted Sobolev space. A difference scheme is constructed, and its convergence is proved. In paper [15], the one class of nonlocal in time problems for first-order evolution equations is considered. The solvability of the stated problem is investigated. All of them are, basically, related to the problems with nonlocal conditions considered only at the border of the area of the definition of the differential operator.

In the present paper, the multipoint nonlocal contact problem for linear elliptic equations is stated and investigated in two-dimensional domains. The method of separation of variables is used to find the solution to a stated problem in the case of Poisson's equation. Then, the more general problem with nonlocal contact conditions for an elliptic equation with variable coefficients is considered, and the iterative method to numerically solve the problem is constructed and investigated. The iterative method allows reducing the solution of a nonlocal contact problem to the solution of a sequence of classical boundary problems. The numerical experiment is conducted. The results fully agree with the theoretical conclusions and show the efficiency of the proposed iterative procedure.

2. Method of Separation of Variables for Poisson Equation

2.1. Formulation of the Problem. Let us consider a rectangular domain in a two-dimensional space R^2 with boundary Γ.

$$\{(x_1, x_2) | 0 \le x_1 \le 1,\ 0 \le x_2 \le 1\}. \tag{1}$$

Suppose $0 < \xi^0 < 1$ and define the segment $\{(x_1, x_2) | x_1 = \xi^0,\ 0 \le x_2 \le 1\}$ (see Figure 1).

We consider the following nonlocal contact problem to find the continuous function:

$$u(x_1, x_2) = \begin{cases} u^-(x_1, x_2), & 0 \le x_1 < \xi^0,\ 0 \le x_2 \le 1, \\ u(\xi^0, x_2), & x_1 = \xi^0,\ 0 \le x_2 \le 1, \\ u^+(x_1, x_2), & \xi^0 < x_1 \le 1,\ 0 \le x_2 \le 1, \end{cases} \tag{2}$$

which satisfies the following equations:

$$\begin{aligned} \Delta u^-(x_1, x_2) &= f^-(x_1, x_2), \quad 0 < x_1 < \xi^0,\ 0 < x_2 < 1, \\ \Delta u^+(x_1, x_2) &= f^+(x_1, x_2), \quad \xi^0 < x_1 < 1,\ 0 < x_2 < 1, \end{aligned} \tag{3}$$

the boundary conditions are as follows:

$$\begin{aligned} &\left.\begin{aligned} u^-(x_1, 0) &= 0, \quad 0 \le x_1 \le \xi^0, \\ u^+(x_1, 0) &= 0, \quad \xi^0 \le x_1 \le 1, \end{aligned}\right\}, \\ &\left.\begin{aligned} u^-(x_1, 1) &= 0, \quad 0 \le x_1 \le \xi^0, \\ u^+(x_1, 1) &= 0, \quad \xi^0 \le x_1 \le 1, \end{aligned}\right\}, \\ &\left.\begin{aligned} u^-(0, x_2) &= 0, \quad 0 \le x_2 \le 1, \\ u^+(1, x_2) &= 0, \quad 0 \le x_2 \le 1, \end{aligned}\right\}, \end{aligned} \tag{4}$$

and the nonlocal contact condition is as follows:

$$\begin{aligned} u^-(\xi^0, x_2) &= u^+(\xi^0, x_2) = u(\xi^0, x_2) = u_0(x_2) \\ &= \gamma^- u^-(\xi^-, x_2) + \gamma^+ u^+(\xi^+, x_2) + \phi_0(\xi^0, x_2), \\ &\quad 0 < x_2 < 1, \end{aligned} \tag{5}$$

where $f^-(x_1, x_2), f^+(x_1, x_2)$ are known as sufficiently smooth functions, and

$$0 < \xi^- < \xi^0, \xi^0 < \xi^+ < 1, \gamma^- > 0, \gamma^+ > 0, \gamma^- + \gamma^+ \le 1. \tag{6}$$

Note that in previously published articles, the nonlocal conditions were mainly formulated under the restriction of the following type: $\gamma^- + \gamma^+ < 1$. In this article, the results are achieved considering the following conditions $\gamma^- + \gamma^+ \le 1$.

2.2. Separation of Variables. Using the method of separation of variables, we can build the solution to nonlocal contact problems (2)–(6). Note that this technique can be extended for a more general case.

We will find the solution of nonlocal contact problems (2)–(6) in the following form:

$$u^-(x_1, x_2) = \sum_{k=1}^{\infty} a_k^-(x_1) \sin k\pi x_2, 0 \le x_1 \le \xi^0, 0 \le x_2 \le 1, \tag{7}$$

$$u^+(x_1, x_2) = \sum_{k=1}^{\infty} a_k^+(x_1) \sin k\pi x_2, \xi^0 \le x_1 \le 1, 0 \le x_2 \le 1. \tag{8}$$

It is evident that the functions (7) and (8) satisfy the following boundary conditions:

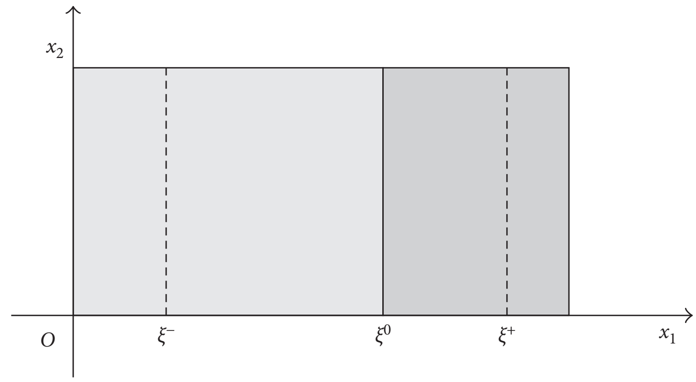

Figure 1: Domain for Poisson equation.

$$u^-(x_1,0)=u^-(x_1,1)=0,\quad 0\le x_1\le\xi^0,$$
$$u^+(x_1,0)=u^+(x_1,1)=0,\quad \xi^0\le x_1\le 1. \tag{9}$$

We must choose the coefficients $a_k^+(x_1)$ and $a_k^-(x_1)$, $k=1,2,3,\ldots$, so that the functions (7) and (8) satisfy the equation (3), and the rest boundary conditions are as follows:

$$u^-(0,x_2)=0,\ u^+(1,x_2)=0,\quad 0\le x_2\le 1, \tag{10}$$

and the nonlocal contact condition as well. Thus, $a_k^+(x_1)$ and $a_k^-(x_1)$, $k=1,2,3,\ldots$, should be solutions of the following problems:

$$\frac{d^2a_k^-(x_1)}{dx_1^2}-\lambda_k^2a_k^-(x_1)=f_k^-(x_1),\quad 0<x_1<\xi^0,\ a_k^-(0)=0,\ a_k^-(\xi^0)=\Phi_k, \tag{11}$$

$$\frac{d^2a_k^+(x_1)}{dx_1^2}-\lambda_k^2a_k^+(x_1)=f_k^+(x_1),\xi^0<x_1<1,\ a_k^+(\xi^0)=\Phi_k,\ a_k^+(1)=0, \tag{12}$$

where $\lambda_k=\pi^2k^2$, $f_k^-(x_1)$, $f_k^+(x_1)$ are the coefficients of Fourier series expansion of the functions $f^-(x_1,x_2)$ and $f^+(x_1,x_2)$.

$$f^-(x_1,x_2)=\sum_{k=1}^{\infty}f_k^-(x_1)\sin\lambda_kx_2,$$
$$f^+(x_1,x_2)=\sum_{k=1}^{\infty}f_k^+(x_1)\sin\lambda_kx_2, \tag{13}$$

and Φ_k, $k=1,2,\ldots$, are so far unknown constants.

$$u(\xi^0,x_2)=u_0(x_2)=\sum_{k=1}^{\infty}\Phi_k\sin\lambda_kx_2. \tag{14}$$

2.3. Existence and Uniqueness of the Solution. Using the method of constructing general solutions of differential equations with homogeneous boundary conditions and method of variation of constants or method of undetermined coefficients of Lagrange (see, for example, [22]), it is possible to build the general solutions of equations (11) and (12).

At first, we will consider the problem (11). The general solution of the problem (11) can be written in the following form:

$$a_k^-(x_1)=\tilde{c}_1^-(x_1)e^{\sqrt{\lambda_k}x_1}+\tilde{c}_2^-(x_1)e^{-\sqrt{\lambda_k}x_1}+\frac{1}{\sqrt{\lambda_k}}\int_0^{x_1}\sinh\left(\sqrt{\lambda_k}(x_1-s)\right)f_k^-(s)ds,\quad k=1,2,\ldots, \tag{15}$$

where we can define $\tilde{c}_1^-(x_1)$, $\tilde{c}_2^-(x_1)$, considering the boundary conditions $a_k^-(0)=0$, $a_k^-(\xi^0)=\Phi_k$.

Finally, the general solution of the problem (11) will get the following form:

$$a_k^-(x_1)=\frac{(\Phi_k-I^-)\sinh(\sqrt{\lambda_k}x_1)}{\sinh(\sqrt{\lambda_k}\xi^0)}+\frac{1}{\sqrt{\lambda_k}}\int_0^{x_1}\sinh\left(\sqrt{\lambda_k}(x_1-s)\right)f_k^-(s)ds,\quad k=1,2,\ldots, \tag{16}$$

where

$$I^-=\frac{1}{\sqrt{\lambda_k}}\int_0^{\xi^0}\sinh\left(\sqrt{\lambda_k}(\xi^0-s)\right)f_k^-(s)ds. \tag{17}$$

Consequently, the formal solution of the problem (3) and (4) on the left subarea is the following function:

$$u^{-}(x_1,x_2)=\sum_{k=1}^{\infty}\left[\frac{(\Phi_k-I^{-})\sinh(\sqrt{\lambda_k}x_1)}{\sinh(\sqrt{\lambda_k}\xi^0)}+\frac{1}{\sqrt{\lambda_k}}\int_0^{x_1}\sinh\left(\sqrt{\lambda_k}(x_1-s)\right)f_k^{-}(s)ds\right]\sin\lambda_k x_2, \quad (18)$$

where I^{-} is defined using (17).

Analogously, the general solution of the problem (12) can be written in the following form:

$$a_k^{+}(x_1)=\tilde{c}_1^{+}(x_1)e^{\sqrt{\lambda_k}x_1}+\tilde{c}_2^{+}(x_1)e^{-\sqrt{\lambda_k}x_1}+\frac{1}{\sqrt{\lambda_k}}\int_{\xi^0}^{x_1}\sinh\left(\sqrt{\lambda_k}(x_1-s)\right)f_k^{+}(s)ds, \quad (19)$$

$$k=1,2,\ldots.$$

Considering the boundary conditions $a_k^{+}(\xi^0)=\Phi_k$, $a_k^{+}(1)=0$, we can define $\tilde{c}_1^{+}$, $\tilde{c}_2^{+}$ uniquely.

Finally, we can describe the formal solution of the problem (3) and (4) on the right subarea in the following way:

$$u^{+}(x_1,x_2)=\sum_{k=1}^{\infty}\left[\frac{\left(\Phi_k\sinh\left(\sqrt{\lambda_k}(x_1-1)\right)-I^{+}\sinh\left(\sqrt{\lambda_k}(x_1-\xi^0)\right)\right)}{\sinh\left(\sqrt{\lambda_k}(\xi^0-1)\right)}+\frac{1}{\sqrt{\lambda_k}}\int_{\xi^0}^{x_1}\sinh\left(\sqrt{\lambda_k}(x_1-s)\right)f_k^{+}(s)ds\right]\sin\lambda_k x_2, \quad (20)$$

where

$$I^{+}=\frac{1}{\sqrt{\lambda_k}}\int_{\xi^0}^{1}\sinh\left(\sqrt{\lambda_k}(1-s)\right)f_k^{+}(s)ds. \quad (21)$$

We can define the coefficients Φ_k, $k=1,2,\ldots$, using the equality (14) and nonlocal contact condition (5).

$$\begin{aligned}\Phi_k&=\gamma^{-}\frac{\Phi_k\sinh\left(\sqrt{\lambda_k}\xi^{-}\right)}{\sinh\left(\sqrt{\lambda_k}\xi^0\right)}+\gamma^{+}\frac{\Phi_k\sinh\left(\sqrt{\lambda_k}(\xi^{+}-1)\right)}{\sinh\left(\sqrt{\lambda_k}(\xi^0-1)\right)}-\left(F_k^{-}+F_k^{+}\right)\\&+\frac{\gamma^{-}}{\sqrt{\lambda_k}}\int_0^{\xi^{-}}\sinh\left(\sqrt{\lambda_k}(\xi^{-}-s)\right)f_k^{-}(s)ds\\&+\frac{\gamma^{+}}{\sqrt{\lambda_k}}\int_{\xi^0}^{\xi^{+}}\sinh\left(\sqrt{\lambda_k}(\xi^{+}-s)\right)f_k^{+}(s)ds+\phi_{0k},\end{aligned} \quad (22)$$

where

$$\begin{aligned}\phi_0\left(\xi^0,x_2\right)&==\sum_{k=1}^{\infty}\phi_{0k}\sin\lambda_k x_2,\\F_k^{-}&=\gamma^{-}\frac{\sinh(\sqrt{\lambda_k}\xi^{-})}{\sqrt{\lambda_k}\sinh\left(\sqrt{\lambda_k}\xi^0\right)}\int_0^{\xi^0}\sinh\left(\sqrt{\lambda_k}(\xi^0-s)\right)f_k^{-}(s)ds,\\F_k^{+}&=\gamma^{+}\frac{\sinh\left(\sqrt{\lambda_k}(\xi^{+}-\xi^0)\right)}{\sqrt{\lambda_k}\sinh\left(\sqrt{\lambda_k}(\xi^0-1)\right)}\int_{\xi^0}^{1}\sinh\left(\sqrt{\lambda_k}(1-s)\right)f_k^{+}(s)ds.\end{aligned} \quad (23)$$

Then, we will get

$$\Phi_k \left\{ 1 - \left[\gamma^- \frac{\sinh(\sqrt{\lambda_k}\,\xi^-)}{\sinh(\sqrt{\lambda_k}\,\xi^0)} + \gamma^+ \frac{\sinh(\sqrt{\lambda_k}\,(\xi^+ - 1))}{\sinh(\sqrt{\lambda_k}\,(\xi^0 - 1))} \right] \right\} = -F_k + \phi_{0k}, \tag{24}$$

where

$$F_k = F_k^- + F_k^+ - \left[\frac{\gamma^-}{\sqrt{\lambda_k}} \int_0^{\xi^-} \sinh\left(\sqrt{\lambda_k}\,(\xi^- - s)\right) f_k^-(s)ds + \frac{\gamma^+}{\sqrt{\lambda_k}} \int_{\xi^0}^{\xi^+} \sinh\left(\sqrt{\lambda_k}\,(\xi^+ - s)\right) f_k^+(s)ds \right]. \tag{25}$$

As

$$\frac{\sinh(\sqrt{\lambda_k}\,\xi^-)}{\sinh(\sqrt{\lambda_k}\,\xi^0)} < 1 \text{ and } \frac{\sinh(\sqrt{\lambda_k}\,(\xi^+ - 1))}{\sinh(\sqrt{\lambda_k}\,(\xi^0 - 1))} < 1, \tag{26}$$

then we will have

$$1 - \left[\gamma^- \frac{\sinh(\sqrt{\lambda_k}\,\xi^-)}{\sinh(\sqrt{\lambda_k}\,\xi^0)} + \gamma^+ \frac{\sinh(\sqrt{\lambda_k}\,(\xi^+ - 1))}{\sinh(\sqrt{\lambda_k}\,(\xi^0 - 1))} \right] > 1 - (\gamma^- + \gamma^+) \geq 0. \tag{27}$$

Consequently, from the equality (24), we get

$$\Phi_k = \left\{ 1 - \left[\gamma^- \frac{\sinh(\sqrt{\lambda_k}\,\xi^-)}{\sinh(\sqrt{\lambda_k}\,\xi^0)} + \gamma^+ \frac{\sinh(\sqrt{\lambda_k}\,(\xi^+ - 1))}{\sinh(\sqrt{\lambda_k}\,(\xi^0 - 1))} \right] \right\}^{-1} (-F_k + \phi_{0k}), \quad k = 1, 2, \ldots, \tag{28}$$

where F_k is defined from (25). Finally, the formal solution of the problem (3)–(5) is the following function:

$$u(x_1, x_2) = \begin{cases} u^-(x_1, x_2), & 0 \leq x_1 < \xi^0,\ 0 \leq x_2 \leq 1, \\ u(\xi^0, x_2), & x_1 = \xi^0,\ 0 \leq x_2 \leq 1, \\ u^+(x_1, x_2), & \xi^0 < x_1 \leq 1,\ 0 \leq x_2 \leq 1, \end{cases} \tag{29}$$

where

$$\begin{aligned} u^-(x_1, x_2) &= \sum_{k=1}^{\infty} \left[\frac{(\Phi_k - I^-)\sinh(\sqrt{\lambda_k}\,x_1)}{\sinh(\sqrt{\lambda_k}\,\xi^0)} + \frac{1}{\sqrt{\lambda_k}} \int_0^{x_1} \sinh\left(\sqrt{\lambda_k}\,(x_1 - s)\right) f_k^-(s)ds \right] \sin \lambda_k x_2, \\ u(\xi^0, x_2) &= \sum_{k=1}^{\infty} \Phi_k \sin \lambda_k x_2, \\ u^+(x_1, x_2) &= \sum_{k=1}^{\infty} \left[\frac{\left(\Phi_k \sinh\left(\sqrt{\lambda_k}\,(x_1 - 1)\right) - I^+ \sinh\left(\sqrt{\lambda_k}\,(x_1 - \xi^0)\right)\right)}{\sinh\left(\sqrt{\lambda_k}\,(\xi^0 - 1)\right)} + \frac{1}{\sqrt{\lambda_k}} \int_{\xi^0}^{x_1} \sinh\left(\sqrt{\lambda_k}\,(x_1 - s)\right) f_k^+(s)ds \right] \sin \lambda_k x_2, \end{aligned} \tag{30}$$

Φ_k is defined from (28), and I^-, I^+ is defined from (17) and (21), respectively.

Thus, the following theorem is true.

Theorem 1. *If $f^-(x_1, x_2)$, $f^+(x_1, x_2)$, and $\phi_0(\xi^0, x_2)$ are sufficiently smooth functions, then the problem (3)–(5) has a unique regular solution.*

Note that the applied technique can be successfully used for more general problems, but in this case, the use of the spectral theory of linear operators will be needed.

3. Nonlocal Contact Problem for Equation with Variable Coefficients

3.1. Designations. Now, let us consider the problem with nonlocal contact conditions for the elliptic equation with variable coefficients.

Suppose D is a rectangular domain in two-dimensional space R^2 (see Figure 2) $D = \{(x_1, x_2)|0 < x_1 < a,\ 0 < x_2 < b\}$ with a piecewise boundary $\gamma = \cup_{i=1}^{4}\gamma_i$, where

$$\begin{aligned} \gamma_1 &= \{(x_1, x_2)|0 \le x_1 \le a,\ x_2 = 0\}, \\ \gamma_2 &= \{(x_1, x_2)|0 \le x_1 \le a,\ x_2 = b\}, \\ \gamma_3 &= \{(x_1, x_2)|x_1 = 0, 0 \le x_2 \le b\}, \\ \gamma_4 &= \{(x_1, x_2)|\ x_1 = a,\ 0 \le x_2 \le b\}. \end{aligned} \tag{31}$$

Suppose $0 < \xi^0 < a$ and define the following equations:

$$\begin{aligned} \Gamma^0 &= \{(x_1, x_2)|x_1 = \xi^0, 0 \le x_2 \le b\}, \\ \gamma_1^- &== \{(x_1, x_2)|0 \le x_1 \le \xi^0, x_2 = 0\}, \\ \gamma_1^+ &= \{(x_1, x_2)|\xi^0 \le x_1 \le a, x_2 = 0\}, \\ \gamma_2^+ &== \{(x_1, x_2)|\xi^0 \le x_1 \le a, x_2 = b\}, \\ \gamma_2^- &== \{(x_1, x_2)|0 \le x_1 \le \xi^0, x_2 = b\}. \end{aligned} \tag{32}$$

It is obvious that $\gamma_1^- \cup \gamma_1^+ = \gamma_1$, $\gamma_2^- \cup \gamma_2^+ = \gamma_2$, and Γ^0 divides the domain D into two parts (subdomains), D^- and D^+, where

$$\begin{aligned} &D^- = \{(x_1, x_2)|0 < x_1 < \xi^0, 0 < x_2 < b\}, \qquad D^+ = \{(x_1, x_2)\ |\ \xi^0 < x_1 < a, 0 < x_2 < b\}, \\ &\Gamma_i^- = \{(x_1, x_2)|x_1 = \xi_i^-,\ 0 < \xi_i^- < \xi^0,\ 0 \le x_2 \le b\}, \\ &\quad i = \overline{(1, n)},\ 0 < \xi_n^- < \cdots < \xi_1^- < \xi^0, \\ &\Gamma_j^+ = \{(x_1, x_2)|x_1 = \xi_j^+, \xi^0 < \xi_j^+ < a, 0 \le x_2 \le b\}, \\ &\quad j = \overline{(1, m)},\ \xi^0 < \xi_1^+ < \xi_2^+ < \cdots < \xi_m^+ < b, \end{aligned} \tag{33}$$

where Γ^0, Γ_i^-, $i = \overline{(1, n)}$ and Γ_j^+, $j = \overline{(1, m)}$ intersect γ_1 and γ_2, respectively, in the following points:

$$\begin{aligned} &A^0(\xi^0, 0),\ B^0(\xi^0, b),\ A_i^-(\xi_i^-, 0), \\ &B_i^-(\xi_i^-, b),\ A_i^+(\xi_i^+, 0), \text{ and } B_i^+(\xi_i^+, b). \end{aligned} \tag{34}$$

3.2. Statement of the Problem. Let us consider the following problem to find in the domain $\overline{D} = D \cup \gamma$ (where γ is defined in 3.1) a continuous function $u(x_1, x_2)$:

$$\begin{aligned} &u(x_1, x_2) = \begin{cases} u^-(x_1, x_2), & \text{if } (x_1, x_2) \in D^-, \\ u^0(x_1, x_2), & \text{if } (x_1, x_2) \in \Gamma^0, \\ u^+(x_1, x_2), & \text{if } (x_1, x_2) \in D^+, \end{cases} \\ &u^-(x_1, x_2) \in C^2(D^-),\ u^+(x_1, x_2) \in C^2(D^+), \\ &\qquad u^0(x_1, x_2) \in C(\Gamma^0), \end{aligned} \tag{35}$$

which satisfies the equations

$$L^- u^-(x_1, x_2) \equiv \sum_{\alpha,\beta=1}^{2} \frac{\partial}{\partial x_\alpha}\left(K_{\alpha\beta}^-(x_1, x_2)\frac{\partial u^-}{\partial x_\beta}\right) - k^-(x_1, x_2)u^- = -f^-(x_1, x_2),\ (x_1, x_2) \in D^-, \tag{36}$$

$$L^+ u^+(x_1, x_2) \equiv \sum_{\alpha,\beta=1}^{2} \frac{\partial}{\partial x_\alpha}\left(K_{\alpha\beta}^+(x_1, x_2)\frac{\partial u^+}{\partial x_\beta}\right) - k^+(x_1, x_2)u^+ = -f^+(x_1, x_2),\ (x_1, x_2) \in D^+. \tag{37}$$

The function $u(x_1, x_2)$ also satisfies the boundary conditions

$$\begin{aligned} u^-(x_1, x_2) &= \phi^-(x_1, x_2),\ (x_1, x_2) \in \gamma_1^- \cup \gamma_2^- \cup \gamma_3, \\ u^+(x_1, x_2) &= \phi^+(x_1, x_2),\ (x_1, x_2) \in \gamma_1^+ \cup \gamma_2^+ \cup \gamma_4, \end{aligned} \tag{38}$$

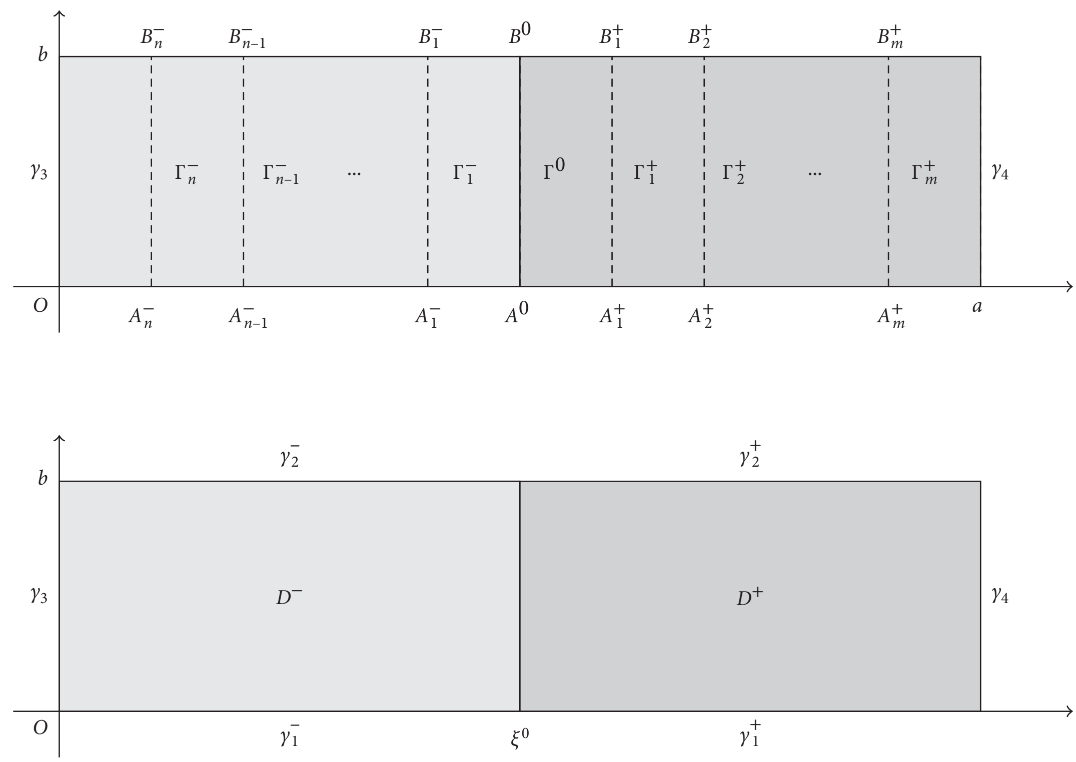

FIGURE 2: Domain for contact problem for equation with variable coefficients $\overline{D} = D \cup \gamma$.

the nonlocal contact conditions

$$u^-\left(\Gamma^0\right) = u^+\left(\Gamma^0\right) = u\left(\Gamma^0\right) = \sum_{i=1}^{n} \beta_i^- u^-\left(\Gamma_i^-\right) + \sum_{j=1}^{m} \beta_j^+ u^+\left(\Gamma_j^+\right) + \phi^0\left(\Gamma^0\right), \tag{39}$$

and the coordination conditions

$$\begin{aligned} u\left(A^0\right) &= \sum_{i=1}^{n} \beta_i^- u^-\left(A_i^-\right) + \sum_{j=1}^{m} \beta_j^+ u^+\left(A_j^+\right) + \phi^0\left(A^0\right), \\ u\left(B^0\right) &= \sum_{i=1}^{n} \beta_i^- u^-\left(B_i^-\right) + \sum_{j=1}^{m} \beta_j^+ u^+\left(B_j^+\right) + \phi^0\left(B^0\right), \end{aligned} \tag{40}$$

where $\beta_i^- = \text{const} > 0$, $\beta_j^+ = \text{const} > 0$, $0 < \sum_{i=1}^{n} \beta_i^- + \sum_{j=1}^{m} \beta_j^+ \le 1$. $K_{\alpha\beta}^{\pm}(\cdot)$, $k^{\mp}(\cdot)$ $f^{\pm}(\cdot)$, $\phi^{\pm}(\cdot)$ and $\phi^0(\cdot)$ are known functions, which satisfy all the conditions, that provide the existence of the unique solutions of the Dirichlet problem in D^- and D^+ [23].

Suppose that the equations (36) and (37) are uniformly elliptic. Then, their coefficients satisfy the following conditions [23]:

$$\begin{aligned} &4K_{11}^-(x_1,x_2) \cdot K_{22}^-(x_1,x_2) > \left(K_{12}^-(x_1,x_2) + K_{21}^-(x_1,x_2)\right)^2, \\ &4K_{11}^+(x_1,x_2) \cdot K_{22}^+(x_1,x_2) > \left(K_{12}^+(x_1,x_2) + K_{21}^+(x_1,x_2)\right)^2. \end{aligned} \tag{41}$$

$K_{\alpha\beta}^-(x_1,x_2)$ and $u^-(x_1,x_2)$ can be considered as coefficient of heat conductivity and temperature of the first body (D^-), and $K_{\alpha\beta}^+(x_1,x_2)$ and $u^+(x_1,x_2)$ can be considered as coefficient of heat conductivity and temperature of the second body (D^+). Thus, the stated problem can be considered as a mathematical model of the stationary distribution of heat in two contacting isotropic bodies.

We will call the problem (35)–(37) and (40) nonlocal contact one since it is a generalization of a classical contact problem.

3.3. Uniqueness of a Solution of Problems (35)–(37) and (40).

The following theorem is true.

Theorem 2. *If the regular solution of problems (35)–(37) and (40) exists and condition* $0 < \sum_{i=1}^{n} \beta_i^- + \sum_{j=1}^{m} \beta_j^+ \le 1$ *is fulfilled, then then the solution is unique.*

Proof. Suppose that problem (35)–(37) and (40) has two solutions: $v(x_1,x_2)$ and $w(x_1,x_2)$. Then, for the function $z(x_1,x_2) = v(x_1,x_2) - w(x_1,x_2)$, we will have the following problem:

$$\begin{aligned} L^- z^-(x_1,x_2) &= 0, \text{ if } (x_1,x_2) \in D^-, \\ L^+ z^+(x_1,x_2) &= 0, \text{ if } (x_1,x_2) \in D^+, \\ z^-(x_1,x_2) &= 0 \text{ if } (x_1,x_2) \in \gamma_1^- \cup \gamma_2^- \cup \gamma_3, \\ z^+(x_1,x_2) &= 0 \text{ if } (x_1,x_2) \in \gamma_1^+ \cup \gamma_2^+ \cup \gamma_4, \end{aligned} \tag{42}$$

$$z\left(\Gamma^0\right) = z^-\left(\Gamma^0\right) = z^+\left(\Gamma^0\right) = \sum_{i=1}^{n} \beta_i^- z^-\left(\Gamma_i^-\right) + \sum_{j=1}^{m} \beta_j^+ z^+\left(\Gamma_j^+\right). \tag{43}$$

From the equality (43), it follows that

$$\max\left|z\left(\Gamma^0\right)\right| \le \max\sum_{i=1}^{n}\beta_i^-\left|z^-\left(\Gamma_i^-\right)\right| + \max\sum_{j=1}^{m}\beta_j^+\left|z^+\left(\Gamma_j^+\right)\right|$$
$$\le \max_{0\le i\le n}\left|z^-\left(\Gamma_i^-\right)\right|\sum_{i=1}^{n}\beta_i^- + \max_{0\le j\le m}\left|z^+\left(\Gamma_j^+\right)\right|\sum_{j=1}^{m}\beta_j^+. \tag{44}$$

Taking into account the condition $0<\sum_{i=1}^{n}\beta_i^- + \sum_{j=1}^{m}\beta_j^+ \le 1$, we obtain

$$\max\left|z\left(\Gamma_0\right)\right| \le \max_{0\le i\le n}\left|z^-\left(\Gamma_i^-\right)\right| \text{ or } \max\left|z\left(\Gamma_0\right)\right| \le \max_{0\le j\le m}\left|z^+\left(\Gamma_j^+\right)\right|. \tag{45}$$

This means that the function z does not attain a maximum on Γ_0, but attains its maximum on D^- or D^+, which contradicts the maximum principle. So, $z\equiv$ const, and taking into account condition (42), we obtain $z\equiv 0$; that is, the solution of the problem (35)–(37) and (40) is unique. □

3.4. Iterative Method for Problems (35)–(37) and (40). Let us consider the following iteration process for the problem (35)–(37) and (40):

$$L^-\left[u^-\left(x_1,x_2\right)\right]^{(k)} = -f^-\left(x_1,x_2\right), \text{if } \left(x_1,x_2\right)\in D^-, \tag{46}$$

$$L^+\left[u^+\left(x_1,x_2\right)\right]^{(k)} = -f^+\left(x_1,x_2\right), \text{if } \left(x_1,x_2\right)\in D^+, \tag{47}$$

$$\left[u^-\left(x_1,x_2\right)\right]^{(k)} = \phi^-\left(x_1,x_2\right), \text{if } \left(x_1,x_2\right)\in \gamma_1^-\cup\gamma_2^-\cup\gamma_3, \tag{48}$$

$$\left[u^+\left(x_1,x_2\right)\right]^{(k)} = \phi^+\left(x_1,x_2\right), \text{ if } \left(x_1,x_2\right)\in \gamma_1^+\cup\gamma_2^+\cup\gamma_4, \tag{49}$$

$$u^{(k)}\left(\Gamma^0\right) = \left[u^-\left(\Gamma^0\right)\right]^{(k)} = \left[u^+\left(\Gamma^0\right)\right]^{(k)}$$
$$= \sum_{i=1}^{n}\beta_i^-\left[u^-\left(\Gamma_i^-\right)\right]^{(k-1)} + \sum_{j=1}^{m}\beta_j^+\left[u^+\left(\Gamma_j^+\right)\right]^{(k-1)} + \phi^0\left(\Gamma^0\right), \tag{50}$$

where $k=1,2,\ldots$. Initially, we can take, for example, $[u^-(\Gamma_i^-)]^{(0)}=0$, $[u^+(\Gamma_j^+)]^{(0)}=0$, $i=\overline{(1,n)}$, $j=\overline{(1,m)}$.

Given the initial approximations in nonlocal contact condition (50) of the iterative process $\overline{(42)–(50)}$, $[u^-(\Gamma_i^-)]^{(k-1)}$ and $[u^+(\Gamma_j^+)]^{(k-1)}$, $i=\overline{(1,n)}$, $j=\overline{(1,m)}$, we can calculate the values of u on Γ^0 and, thus, get two classical boundary problems. After solving these problems, we can define the consequent values of u on Γ^0 from (50) for the next iteration.

Theorem 3. *If the solution of problems (35)–(37) and (40) exists, then the iterative process (46)–(50) converges to this solution at the rate of an infinitely decreasing geometric progression.*

Proof. Denote by $z^{(k)}(x_1,x_2)=u^{(k))}(x_1,x_2)-u(x_1,x_2)$, where u is a solution of the problem (35)–(37) and (40) and $u^{(k)}$ is a solution of the problem (46)–(50).

Then, we obtain the following problems:

$$L^-\left[z^-\left(x_1,x_2\right)\right]^{(k)} = 0 \text{ if } \left(x_1,x_2\right)\in D^-,$$
$$L^+\left[z^+\left(x_1,x_2\right)\right]^{(k)} = 0 \text{ if } \left(x_1,x_2\right)\in D^+,$$
$$\left[z^-\left(x_1,x_2\right)\right]^{(k)} = 0 \text{ if } \left(x_1,x_2\right)\in D^+,$$
$$\left[z^+\left(x_1,x_2\right)\right]^{(k)} = 0 \text{ if } \left(x_1,x_2\right)\in \gamma_1^+\cup\gamma_2^+\cup\gamma_4,$$
$$\left[z\left(\Gamma^0\right)\right]^{(k)} = \left[z^-\left(\Gamma^0\right)\right]^{(k)}$$
$$= \sum_{i=1}^{n}\beta_i^-\left[z^-\left(\Gamma_i^-\right)\right]^{(k-1)} + \sum_{j=1}^{m}\beta_j^+\left[z^+\left(\Gamma_j^+\right)\right]^{(k-1)}, \tag{51}$$

where $k=1,2,\ldots$, and $[z^-(\Gamma_i^-)]^{(0)}=0$, $[z^+(\Gamma_j^+)]^{(0)}=0$, $i=\overline{(1,n)}$, $j=\overline{(1,m)}$.From the equality [20], we have

$$\max_{\Gamma^0}\left|\left[z\left(\Gamma^0\right)\right]^{(k)}\right| \le \max_{1\le i\le n}\left|\left[z^-\left(\Gamma_i^-\right)\right]^{(k-1)]}\right|$$
$$\cdot\sum_{i=1}^{n}\beta_i^- + \max_{1\le j\le m}\left|\left[z^+\left(\Gamma_j^+\right)\right]^{(k-1)}\right|\sum_{j=1}^{m}\beta_j^+. \tag{52}$$

If we use Schwarz's lemma [24], we will get inequalities.

$$\max_{1\le i\le n}\left|\left[z^-\left(\Gamma_i^-\right)\right]^{(k-1)}\right| \le q^-\max_{\Gamma^0}\left|\left[z\left(\Gamma^0\right)\right]^{(k-1)}\right|, \tag{53}$$

$$\max_{1\le j\le m}\left|\left[z^+\left(\Gamma_j^+\right)\right]^{(k-1)}\right| \le q^+\max_{\Gamma^0}\left|\left[z\left(\Gamma^0\right)\right]^{(k-1)}\right|, \tag{54}$$

where $q^+=$ const, $0<q^+<1$, $q^-=$ const, $0<q^-<1$. Note that these constants depend only on the geometric properties of domains D^- and D^+.

If we use inequalities (53) and (54), then we have

$$\max_{\Gamma^0}\left|\left[z\left(\Gamma^0\right)\right]^{(k)}\right| \le q^+\sum_{j=1}^{m}\beta_j^+\cdot\max_{\Gamma^0}\left|\left[z\left(\Gamma^0\right)\right]^{(k-1)}\right|$$
$$+ q^-\sum_{i=1}^{n}\beta_i^-\cdot\max_{\Gamma^0}\left|\left[z\left(\Gamma^0\right)\right]^{(k-1)}\right|, \tag{55}$$

or

$$\max_{\Gamma^0}\left|\left[z\left(\Gamma^0\right)\right]^{(k)}\right| \le Q\max_{\Gamma^0}\left|\left[z\left(\Gamma^0\right)\right]^{(k-1)}\right|, \tag{56}$$

where $Q=q^+\sum_{j=1}^{m}\beta_j^+ + q^-\sum_{i=1}^{n}\beta_i^-$.

Taking into account the conditions $\beta_i^-=$ *const* >0, $\beta_j^+=$ *const* >0, $0<\sum_{j=1}^{m}\beta_j^+ + \sum_{i=1}^{n}\beta_i^- \le 1$, we obtain $0<Q<1$. This implies that

$$\lim_{k\longrightarrow\infty}\left[z\left(\Gamma^0\right)\right]^{(k)} = 0. \tag{57}$$

If the solution of the problem (35)–(37) and (40) exists, then by the maximum principle we obtain

$$\max_{D^-}\left|\left[u^-\left(x_1,x_2\right)\right]^{(k)} - u^-\left(x_1,x_2\right)\right| = O\left(Q^k\right),$$
$$\max_{D^+}\left|\left[u^+\left(x_1,x_2\right)\right]^{(k)} - u^+\left(x_1,x_2\right)\right| = O\left(Q^k\right), \tag{58}$$

and, accordingly,

$$\max_{D}\left|\left[u(x_1,x_2)\right]^{(k)} - u(x_1,x_2)\right| = O\left(Q^k\right). \qquad (59)$$

Thus, the iterative process (46)–(50) converges to this solution of the problem (35)–(37) and (40)at the rate of an infinitely decreasing geometric progression with ratio Q. □

Remark 1. By using the described iterative algorithm (46)–(50), the solution of a nonclassical contact problem (35)–(37) and (40) is reduced to the solution of a sequence of classical boundary problems, which can be solved by any well-studied method.

3.5. Existence of a Solution of Problems (35)–(37) and (40). Let us now prove the existence of a regular solution of the problems (35)–(37) and (40) in case of $f^-(x_1,x_2) \equiv 0$ and $f^+(x_1,x_2) \equiv 0$. We introduce the notation $\varepsilon^{(k)}(x_1,x_2) = u^{(k)}(x_1,x_2) - u^{(k-1)}(x_1,x_2)$. Then, for the function $\varepsilon^{(k)}$, we obtain the following problem:

$$\begin{aligned} L^-\left[\varepsilon^-(x_1,x_2)\right]^{(k)} &= 0, \quad \text{if } (x_1,x_2) \in D^-,\\ L^+\left[\varepsilon^+(x_1,x_2)\right]^{(k)} &= 0, \quad \text{if } (x_1,x_2) \in D^+,\\ \left[\varepsilon^-(x_1,x_2)\right]^{(k)} &= 0, \quad \text{if } (x_1,x_2) \in \gamma_1^- \cup \gamma_2^- \cup \gamma_3,\\ \left[\varepsilon^+(x_1,x_2)\right]^{(k)} &= 0, \quad \text{if } (x_1,x_2) \in \gamma_1^+ \cup \gamma_2^+ \cup \gamma_4,\\ \left[\varepsilon\left(\Gamma^0\right)\right]^{(k)} = \left[\varepsilon^-\left(\Gamma^0\right)\right]^{(k)} = \left[\varepsilon^+\left(\Gamma^0\right)\right]^{(k)} &= \sum_{i=1}^{n}\beta_i^-\left[\varepsilon^-\left(\Gamma_i^-\right)\right]^{(k-1)} + \sum_{j=1}^{m}\beta_j^+\left[\varepsilon^+\left(\Gamma_j^+\right)\right]^{(k-1)}, \end{aligned} \qquad (60)$$

where $k = \overline{1,2,\ldots,}$ and $[\varepsilon^-(\Gamma_i^-)]^{(0)} = 0$, $[\varepsilon^+(\Gamma_j^+)]^{(0)} = 0$, $i = \overline{(1,n)}$, $j = \overline{(1,m)}$.

Then, analogously to (56), we obtain the estimation

$$\max_{\Gamma^0}\left|\left[\varepsilon\left(\Gamma^0\right)\right]^{(k)}\right| \leq Q\max_{\Gamma^0}\left|\left[\varepsilon\left(\Gamma^0\right)\right]^{(k-1)}\right|, \quad 0<Q<1, \qquad (61)$$

or

$$\max_{\Gamma^0}\left|u^{(k)}\left(\Gamma^0\right) - u^{(k-1)}\left(\Gamma^0\right)\right| \leq Q\max_{\Gamma^0}\left|u^{(k-1)}\left(\Gamma^0\right) - u^{(k-2)}\left(\Gamma^0\right)\right|, \quad 0<Q<1. \qquad (62)$$

This means that the sequence $\{u^{(k)}(x_1,x_2)\}$ converges uniformly on Γ^0. Then, the functions $[u^+(x_1,x_2)]^{(k)}$ and $[u^-(x_1,x_2)]^{(k)}$ converge to the functions $u^+(x_1,x_2)$ and $u^-(x_1,x_2)$, respectively, on the domains D^- and D^+ on the base of Harnack's first theorem [25, 26].

From this, we conclude that the limit function is the regular solution of the problem (35)–(37) and (40).

$$\lim_{k\longrightarrow\infty} u^{(k)}(x_1,x_2) = u(x_1,x_2). \qquad (63)$$

4. Numerical Example

Let us consider the area $D = \{(x_1,x_2) | 0 < x_1 < 1,\ 0 < x_2 < 1\}$ (see Figure 3).

We consider the following test problem for the numerical solution to find in $\overline{D}$ a continuous function (35) $u(x_1,x_2)$, which satisfies the following equations:

$$\begin{aligned} \frac{\partial}{\partial x_1}\left[\left(1+x_1^2\right)\frac{\partial u^-}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\left(1+x_2^2\right)\frac{\partial u^-}{\partial x_2}\right] &== f^-(x_1,x_2), \quad \text{if } (x_1,x_2) \in D^-,\\ \frac{\partial}{\partial x_1}\left[\left(1+2x_1^2\right)\frac{\partial u^+}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\left(1+2x_2^2\right)\frac{\partial u^+}{\partial x_2}\right] &= f^+(x_1,x_2), \quad \text{if } (x_1,x_2) \in D^+, \end{aligned} \qquad (64)$$

where

$$\begin{aligned} f^-(x_1,x_2) &= -\frac{1}{4}x_1x_2\left(-16+\pi^2\left(1+x_2^2\right)\right)\cos\frac{\pi x_2}{2} - \pi x_1\left(1+2x_2^2\right)\sin\frac{\pi x_2}{2},\\ f^+(x_1,x_2) &= -4x_1x_2\cos\frac{\pi x_2}{2} + \frac{1}{4}(x_1-1)\left[x_2\left(-16+\pi^2\left(1+2x_2^2\right)\right)\cos\frac{\pi x_2}{2} + 4\pi\left(1+4x_2^2\right)\sin\frac{\pi x_2}{2}\right]. \end{aligned} \qquad (65)$$

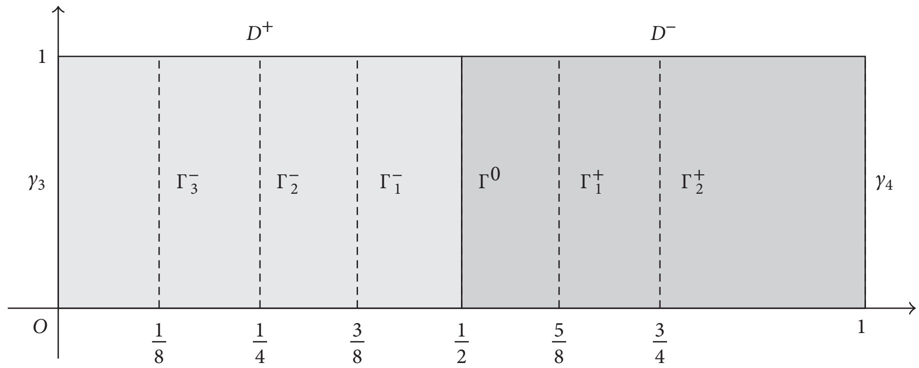

FIGURE 3: Domain for a numerical example.

The function $u(x_1, x_2)$ also satisfies the boundary conditions

$$\begin{aligned} u^-(x_1,x_2) &= 0, \quad \text{if } (x_1,x_2) \in \gamma_1^- \cup \gamma_2^- \cup \gamma_3, \\ u^+(x_1,x_2) &= 0, \quad \text{if } (x_1,x_2) \in \gamma_1^+ \cup \gamma_2^+ \cup \gamma_4, \end{aligned} \tag{66}$$

the nonlocal contact conditions

$$\begin{aligned} u^-\left(\frac{1}{2},x_2\right) = u^+\left(\frac{1}{2},x_2\right) = u(\Gamma^0) &= \frac{1}{8}u^+\left(\frac{5}{8},x_2\right) + \frac{1}{8}u^+\left(\frac{3}{4},x_2\right) \\ &+ \frac{1}{8}u^-\left(\frac{1}{8},x_2\right) + \frac{1}{8}u^-\left(\frac{1}{4},x_2\right) \\ &+ \frac{1}{8}u^-\left(\frac{3}{8},x_2\right) + \frac{21}{64}x_2\cos\frac{\pi x_2}{2}, \end{aligned} \tag{67}$$

and the coordination conditions are fulfilled.

The exact solution of this problem is

$$u(x_1,x_2) = \begin{cases} x_1x_2\cos\dfrac{\pi x_2}{2}, & \text{if } (x_1,x_2) \in D^-, \\ \dfrac{1}{2}x_2\cos\dfrac{\pi x_2}{2}, & \text{if } (x_1,x_2) \in \Gamma^0, \\ (1-x_1)x_2\cos\dfrac{\pi x_2}{2}, & \text{if } (x_1,x_2) \in D^+. \end{cases} \tag{68}$$

Let us consider the following iterative process:

$$\begin{aligned} &\frac{\partial}{\partial x_1}\left[\left(1+x_1^2\right)\frac{\partial (u^-)^{(k)}}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\left(1+x_2^2\right)\frac{\partial (u^-)^{(k)}}{\partial x_2}\right] = f^-(x_1,x_2), \text{ if } (x_1,x_2) \in D^-, \\ &\frac{\partial}{\partial x_1}\left[\left(1+2x_1^2\right)\frac{\partial (u^+)^{(k)}}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\left(1+2x_2^2\right)\frac{\partial (u^+)^{(k)}}{\partial x_2}\right] = f^+(x_1,x_2), \text{ if } (x_1,x_2) \in D^+, \\ &[u^-(x_1,x_2)]^{(k)} = 0, \text{ if } (x_1,x_2) \in \gamma_1^- \cup \gamma_2^- \cup \gamma_3, \\ &[u^+(x_1,x_2)]^{(k)} = 0, \text{ if } (x_1,x_2) \in \gamma_1^+ \cup \gamma_2^+ \cup \gamma_4, \end{aligned} \tag{69}$$

$$\begin{aligned} \left[u(\Gamma^0)\right]^{(k)} &= \left[u^-\left(\frac{1}{2},x_2\right)\right]^{(k)} \\ &= \frac{1}{8}\left[u^-\left(\frac{1}{8},x_2\right)\right]^{(k-1)} + \frac{1}{8}\left[u^-\left(\frac{1}{4},x_2\right)\right]^{(k-1)} + \frac{1}{8}\left[u^-\left(\frac{3}{8},x_2\right)\right]^{(k-1)} \\ &\quad + \frac{1}{8}\left[u^+\left(\frac{5}{8},x_2\right)\right]^{(k-1)} + \frac{1}{8}\left[u^+\left(\frac{3}{4},x_2\right)\right]^{(k-1)} + \frac{21}{64}x_2\cos\frac{\pi x_2}{2}, \end{aligned} \tag{70}$$

where $k = 1, 2, \ldots$, and the initial value $[u(\Gamma^0)]^{(1)}$ is equal to 0.

The numerical calculations were carried out using the program Wolfram Mathematica (see Figure 4). In computations, the following Wolfram functions were used with the corresponding arguments: NDSolveValue (to assign the computed value to each component of the vector function—the solution of the system of two equations with partial derivatives), Dirichlet condition (to specify boundary values within the function NDSolveValue), Piecewise (to

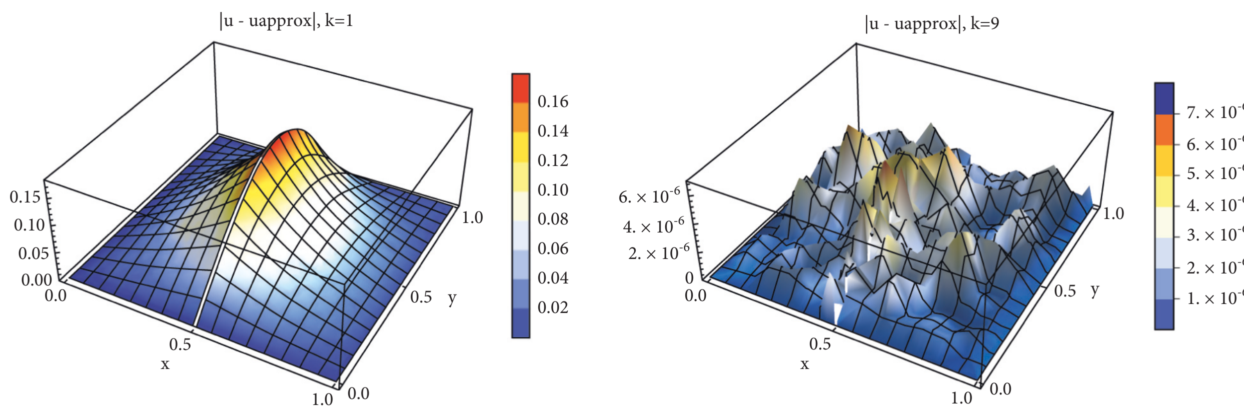

FIGURE 4: The magnitude of the difference between exact and approximate solutions for $k = 1$ and 9. For $k = 1$, range is $0.02 - 0.16$; for $k = 9$, range is $1.0 \times 10^{-6} - 6.0 \times 10^{-6}$.

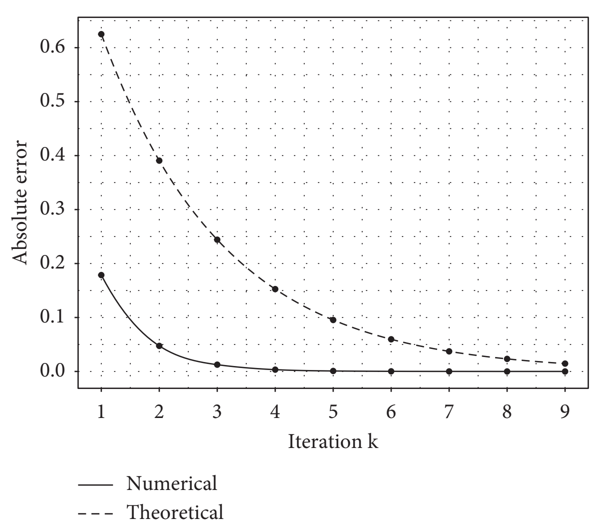

FIGURE 5: Absolute error (numerical) and Q^k versus iteration k.

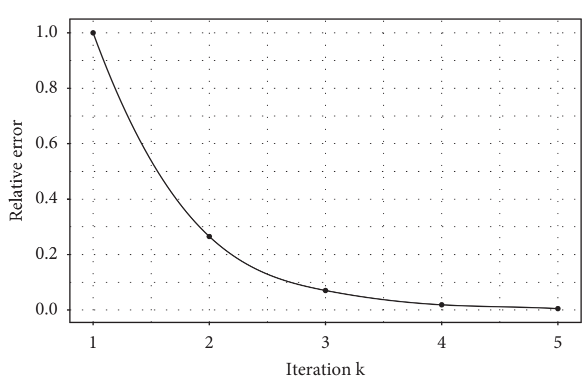

FIGURE 6: Relative error versus iteration k $k = 1 \ldots 5$.

represent the solution on the whole area for further visualization), NMaxValue (to calculate uniform norm), Do (to organize the outer loop for iterations), along with other supplementary functions to calculate norms and get respective graphs.

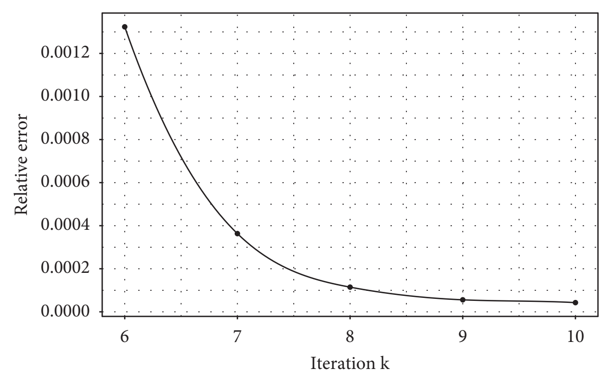

FIGURE 7: Relative error versus iteration k $k = 6 \ldots 10$.

TABLE 1: Relative error.

Iteration k	Relative error
1	0.999954
2	0.265173
3	0.0703862
4	0.018678
5	0.00495362
6	0.00132358
7	0.000363221
8	0.000114644
9	0.0000558231
10	0.0000428056
11	0.0000389617

From Theorem 3, we have $Q = q^+ \sum_{j=1}^{M} \beta_j^+ + q^- \sum_{i=1}^{N} \beta_i^-$, where $0 < q^+, q^- < 1$. The absolute error decreases approximately as $O(Q^k)$. Considering the nonlocal contact condition of the numerical example, $Q < 5/8$.

Figure 5 compares the absolute error (in C-norm) with the theoretical value $(5/8)^k$.

Figures 6 and 7 show the behavior of relative error $\|u_{exact} - u_{appr}\|_C \|u_{exact}\|_C$ (uniform C-norm is taken on the open area D) versus iteration k, for $k = 1, \ldots, 5$ and $k = 6, \ldots, 10$, respectively (see also results in Table 1).

The results of numerical calculations fully agree with the theoretical conclusions and show the efficiency of the proposed iterative procedure.

5. Conclusion

The theory of contact problems is widely used in many fields of mechanics (including construction mechanics) and mechanical engineering. In these problems, various contact conditions are considered along the contact line (see, for example, [27, 28]).

In the present article, a new type of nonclassical boundary-value problem with nonlocal contact conditions along the contact line is considered for an elliptic equation with variable coefficients and mixed derivatives. Thus, using the results of the present article, one can expand the class of contact problems.

The main results of the proposed article can be formulated as follows.

The existence and uniqueness of the solution of a nonlocal contact problem for the elliptic equation with variable coefficients and mixed derivatives is proved. For this aim, the convergent iterative method (46)–(50) is constructed, which also is used to find the numerical solution. The method converges to the solution of the problem (35)–(37) and (40) at the rate of infinitely decreasing geometric progression. By using this iterative algorithm, the solution of a nonclassical contact problem is reduced to the solution of a sequence of classical boundary problems, which can be solved by applying well-studied methods. The results of numerical calculations agree with theoretical results.

The analytical solution in a form of a series is received for the same problem, but with constant coefficients to avoid huge formulae. Moreover, the applied technique can be successfully used for more general problems, but, in this case, the use of the spectral theory of linear operators will be needed.

In contrast to boundary nonlocal problems, convergence is achieved under the more general conditions: $0 < \gamma^- + \gamma^+ \leq 1$ (in the case of the Fourier method) and $0 < \sum_{j=1}^{M} \beta_j^+ + \sum_{j=1}^{N} \beta_i^- \leq 1$ (in the case of the general equation with variable coefficients).

The technique used in the present article can also be applied to problems with parabolic-type equations.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed to all sections. All authors read and approved the final manuscript.

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