New conditions for testing necessarily/possibly efficiency of non-degenerate basic solutions based on the tolerance approach

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Abstract

In this paper, a specific type of multiobjective linear programming problem with interval objective function coefficients is studied. Usually, in such problems, it is not possible to obtain an optimal solution which optimizes simultaneously all objective functions in the interval multiobjective linear programming (IMOLP) problem, requiring the selection of a compromise solution. In conventional multiobjective programming problems these compromise solutions are called efficient solutions. However, the efficiency cannot be defined in a unique way in IMOLP problems. Necessary efficiency and possible efficiency have been considered as two natural extensions of efficiency to IMOLP problems. In this case, necessarily efficient solutions may not exist and the set of possibly efficient solutions usually has an infinite number of elements. Furthermore, it has been concluded that the problem of checking necessary efficiency is co-NP-complete even for the case of only one objective function. In this paper, we explore new conditions for testing necessarily/possibly efficiency of a basic non-degenerate solution in IMOLP problems. We show properties of the necessarily efficient solutions in connection with possibly and necessarily optimal solutions to the related single objective problems. Moreover, we utilize the tolerance approach and sensitivity analysis for testing the necessary efficiency.

Finally, based on the new conditions, a procedure to obtain some necessarily efficient and strictly possibly efficient solutions to multiobjective problems with interval objective functions is suggested.

Keywords: multiple criteria analysis, multiobjective interval linear programming, necessarily efficient solutions, possibly efficient solutions, tolerance approach, sensitivity analysis

1. Introduction

In multiobjective programming (MOP) problems, an optimal solution which optimizes all objective functions simultaneously does not usually exist, giving rise to the need of finding compromise solutions, i.e., feasible solutions which are not worse than any other feasible solutions. In interval multiobjective programming (IMOP) problems, the selection of compromise solutions becomes more complex because the efficiency cannot be defined in a unique way. Inuiguchi and Sakawa (1995) consider two different approaches to deal with interval objective functions: the satisficing approach and the optimizing approach. These two approaches are applied also to IMOLP problems. In the satisficing approach, similar to the robust optimization approach (Ben-Tal, et al., 2009), each interval objective function is transformed into one or several objective functions, i.e., optimizing the lower bound, the upper bound and the central value of the intervals, etc., in order to obtain a compromise solution (for further details see also Oliveira and Antunes (2007)). Although the compromise solution obtained in this way is efficient for some possible realizations of the uncertain coefficients, it might not be the most adequate one to the IMOLP problem. In fact, if the gradients of the objective functions chosen are highly correlated, the scope of the search might be reduced and ultimately the gradient cone of each objective function becomes a ray (Antunes and Clímaco, 2000).

On the other hand, the optimizing approach extends the concept of efficiency, in other words, Pareto optimality or noninferiority, used in traditional multiobjective linear programming (MOLP) problems to the interval objective function case. In this context, there are two kinds of optimal solutions to the IMOLP problem. Bitran (1980) studied the solutions which are efficient for any objective function coefficient vectors within the given range of conceivable variations. Inuiguchi and Kume (1989) named this type of solutions "necessarily efficient solutions" and they also studied other type of efficient solutions called "possibly efficient solutions" which are efficient for at least one of the objective function coefficient vectors within the given range of conceivable variations. On the other hand, the basic properties and theoretical foundations for necessary and possible efficiency were discussed in Bitran (1980), Inuiguchi and Kume (1989), Inuiguchi and Sakawa (1996) and Oliveira and Antunes (2007). In this framework, Hladík (2012) showed that the problem of checking necessary efficiency is co-NP-complete even for the case of only one objective. More recently, Rivaz and Yaghoobi (2015) introduced two new solution concepts: a solution is "possibly weak efficient" if it is weakly efficient for at least one of the given objective function coefficient vectors and "necessarily weak efficient" if it is weakly efficient for any given objective function coefficient vectors within their admissible range of variation. Additionally, Inuiguchi (2013) and Inuiguchi et al. (2016) also investigated the relation and properties of the necessarily efficient solutions with possibly and necessarily optimal solutions to the related single objective problems with uncertain coefficients of the objective function.

The "necessarily efficient" solutions are the most reasonable one, because they are efficient for all conceivable realizations of objective function coefficients within the given range. However, in most situations they may not exist; on the other hand, the "possibly efficient" solutions are minimally reasonable and are considered to be the optimistic ones (Inuiguchi and Kume, 1989), (Ida, 1999). The approaches presented in Steuer (1981), Ida (1999), Ida (2005), Wang and Wang (2001) and Oliveira, Antunes and Barrico (2014) allow for the enumeration of all possibly efficient solutions; however, the required computational burden might be considerable if the number of interval coefficients on the objective function is high. Another issue is that when the decision-maker (DM) is faced with a large set of solutions, in many cases with just slight differences among the objective function values, the decision problem becomes even more complex (Antunes, Clímaco, 2000).

The same difficulties can also be found in single objective optimization problems. In this case, the solutions are called possibly and necessarily optimal solutions, respectively. In order to overcome the main problems raised by these approaches, Inuiguchi and Sakawa (1995) have proposed a methodology to solve linear programming (LP) problems with interval coefficients on the objective function which considers a “minimax regret” objective function and uses relaxed LP problems. Inuiguchi and Sakawa (1997) and Mausser and Laguna (1999) suggested a similar approach, but considering “maximin achievement rate” and “minimax relative regret” objective functions, respectively. More recently, Rivaz and Yaghoobi (2013), Rivaz and Yaghoobi (2018) and Rivaz et al. (2016) introduced the “minimax regret” concept to MOLP problems with interval objective coefficients. Rivaz et al. (2016) showed that a kind of minimax regret solution is possibly efficient and coincides with the necessarily efficient solution when it exists. They succeeded to construct an algorithm for finding a necessarily efficient solution by solving a kind of “minimax regret” problem. However, it requires a big computational effort to obtain a solution to the kind of minimax regret problem, and moreover if no necessarily efficient solutions are available, no information is provided regarding the robustness of the obtained solution, i.e. the allowable variation ranges of the objective function coefficients to which the solution remains efficient/optimal.

Hladík (2008) proposes a procedure for computing a tolerance (an interval) for each objective function coefficient and applies the method for checking the necessary efficiency of a given solution of a MOLP problem. In this context, this paper goes beyond this purpose since it is aimed at exploring new conditions for testing both necessarily and possibly efficiency by combining the results obtained by Inuiguichi (2013) and Inuiguchi et al. (2016) with the tolerance region obtained using Filippi's method and sensitivity analysis. Subsequently, based on these new conditions, an approach to find some robust solutions to multiobjective problems with interval objective functions is proposed without a very big computational burden which allows identifying some necessarily efficient solutions and strictly possibly efficient solutions.

This paper is organized as follows: in Section 2 an overview of the tolerance approach is provided. In Sections 3 and 4 a brief description and analysis of the theoretical underpinning assumptions underlying LP and MOLP problems with interval coefficients in the objective functions is given, with the proposal of new conditions based on the sensitivity analysis and tolerance approach for testing the possibly/necessary optimality in interval linear programming (ILP) problems as well as possibly/necessary efficiency in IMOLP problems, respectively. Many of the obtained conditions are illustrated by numerical examples. In Section 5 a new approach for obtaining some of necessarily efficient solutions to IMOLP problems is proposed. Finally, in Section 6, some conclusions are drawn, and future work developments are indicated.

2. The tolerance approach

Sensitivity analysis in LP problems allows obtaining an interval for each coefficient or term under which the same basis remains optimal, assuming that all other coefficients and terms remain constant (Gal, 1995). However, this one-at-a-time restriction constitutes a major limitation in practice (Wendell, 2004). Only in some situations it is possible to use the intervals obtained from traditional sensitivity analysis in considering simultaneous variations (e.g., for variations in the objective function coefficients of non-basic variables). Nevertheless, when one or more of the corresponding variables are basic, the region over which the same basis remains optimal is a convex polyhedron (Wendell, 2004). In this context, the tolerance approach proposed in Wendell (1982; 1984; 1985; 1987) allows assessing the impact on the optimal basis of simultaneous and independent changes in selected parameters of an LP problem, overcoming the inherent limitations of sensitivity analysis. In particular, if the simultaneous and independent changes of the objective function coefficients are taken into account, it gives a maximum-tolerance percentage and a corresponding tolerance hyperbox within which selected parameters may vary from their estimated values while still retaining the same optimal basic feasible solution (Wendell, 2004). Subsequent work on the tolerance approach has included proposals to expand the tolerance hyperbox to a rectangular parallelepiped (see (Wondolowski, 1991) and (Wendell, 1992)). Wondolowski (1991) suggested the Individual Tolerance Range (ITR) method where it is possible to obtain different values both for the increase and decrease of each element of the estimated objective function coefficient vector, respectively. More recently, Filippi (2005) and Hladík (2011) proposed algorithmic procedures to further enlarge the ITR method, by computing those individual tolerances iteratively. Additionally, a critical review of the tolerance approach, including its extensions and applications has been provided in Wendell and Chen (2010). Finally, in Borgonovo et al. (2018) a merged approach is proposed bringing together the insights from Wendell's tolerance and Wagner's global sensitivity approaches.

In the context of MOLP, tolerance analysis has been mainly devoted to address the sensitivity of the objective functions' weights, producing a maximum tolerance percentage by which the weights may be perturbed simultaneously and independently, without changing the optimal solution. In this line of work, Hladík (2008) computes a tolerance such that all the coefficients can deviate within their tolerances, while maintaining the same efficient solution, also extending symmetric tolerances to unsymmetrical ones (called tolerance matrices). Then, the proposed procedures for computing the additive and multiplicative tolerances are used as a sufficient condition in testing the necessary efficiency in interval MOLP.

2.1. Underpinning assumptions and basic notation

A polyhedron $\boldsymbol{\mathcal{P}}^0$ is the solution set of a system of linear inequalities $\boldsymbol{\mathcal{P}}^0 = \{\mathbf{x} \in IR^d\colon\ A^0\mathbf{x} \le \mathbf{b}^0\}$, where A^0 is a real r×d matrix and $\mathbf{b}^0$ is a real r-vector. If $\mathbf{b}^0 = 0$ then $\boldsymbol{\mathcal{P}}^0$ is called polyhedral cone. A hyperbox is a polyhedron of the form $\{\mathbf{x} \in IR^d\colon\ \mathbf{l} \le \mathbf{x} \le \mathbf{u}\}$, where $\mathbf{l}$ and $\mathbf{u}$ are d-vectors defined over the extended reals $IR \cup \{\pm\infty\}$. $\boldsymbol{\mathcal{B}}(\mathbf{l}, \mathbf{u})$ denotes the hyperbox identified by vectors $\mathbf{l}$ and $\mathbf{u}$. Given a vector $\mathbf{v} \in IR^d$, diag($\mathbf{v}$) denotes the square matrix having the entries of $\mathbf{v}$ in its main diagonal. It is assumed that the maximum and minimum over the empty set are $-\infty$ and $+\infty$, respectively.

From a geometrical point of view, finding a maximum objective coefficient tolerance for an optimal solution to preserve the optimality is equivalent to building the largest hyperbox contained in a given polyhedral cone and centred in a given point of the cone (Filippi, 2005). If the given point is mapped onto the origin by a translation, then the required hypercube may be characterized as follows (Filippi, 2005):

Lemma 1: Let $\boldsymbol{\mathcal{P}}^0 = \{\mathbf{x} \in IR^d\colon\ A^0\mathbf{x} \le \mathbf{b}^0\}$ be any polyhedron where $A^0 \neq \mathbf{0}$ has r rows and $\mathbf{b}^0 \ge \mathbf{0}$. For all k = 1, …, r

$$\sigma_k = \begin{cases} \dfrac{b_k^0}{\sum_{j=1}^{d} \left|a_{kj}^0\right|}, \text{ if} \sum_{j=1}^{d} \left|a_{kj}^0\right| > 0 \\ \quad 0,\ \text{otherwise} \end{cases}, \tag{1}$$

and let $\sigma(\boldsymbol{\mathcal{P}}^0) = \min\limits_{k}\left\{\sigma_k\colon \sum_{j=1}^{d}\left|a_{kj}^0\right| > 0\right\}$. Then, $\boldsymbol{\mathcal{B}}\left(-\sigma(\boldsymbol{\mathcal{P}}^0)\mathbf{e}, +\sigma(\boldsymbol{\mathcal{P}}^0)\mathbf{e}\right)$ is the largest hyperbox centred in the origin and contained in $\boldsymbol{\mathcal{P}}^0$, where $\mathbf{e} \in IR$ denotes a vector of ones.

Consider the following LP problem:

$$\max \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}},$$
$$\mathrm{s.t.}\ \bar{\mathrm{A}}\bar{\mathbf{x}} = \mathbf{b},$$
$$\bar{\mathbf{x}} \geq \mathbf{0}, \tag{2}$$

wher e $\bar{\mathrm{A}}$ is an m×h matrix, **b** and **c** are m- and h-dimensional vectors and $\bar{\mathbf{x}}$ is an h-dimensional vector of decision variables.

Let the following LP problem with perturbed objective function coefficients be given as:

$$\max \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}} = (\hat{\mathbf{c}} + \mathrm{diag}(\mathbf{g})\gamma)^{\mathrm{T}}\bar{\mathbf{x}},$$
$$\mathrm{s.t.}\ \bar{\mathrm{A}}\bar{\mathbf{x}} = \mathbf{b},$$
$$\bar{\mathbf{x}} \geq \mathbf{0}, \tag{3}$$

where $\hat{\mathbf{c}}$ is an h-dimensional vector and $\mathrm{diag}(\mathbf{g})$ is a diagonal h×h matrix whose diagonal elements are $\mathrm{g_{jj}} = \mathrm{c}'_{\mathrm{j}}$, $\boldsymbol{\gamma}$ is an h-dimensional vector showing perturbation parameters of the objective function coefficient. When $\mathrm{c}'_{\mathrm{j}} = 1$ there is an additive perturbation in $\hat{\mathrm{c}}_j$ and when $\mathrm{c}'_{\mathrm{j}} = \hat{\mathrm{c}}_j$ there is a multiplicative perturbation in $\hat{\mathrm{c}}_j$, i.e. a percentage variation of γ_{j}×100% regarding the estimated value $\hat{\mathrm{c}}_j$ ($\hat{\mathrm{c}}_j$ is the j$^{\mathrm{th}}$ element of estimated objective function coefficient vector $\hat{\mathbf{c}}$). When $\hat{\mathrm{c}}_j = 0$ the percentage variation has no meaning and may arbitrarily be fixed to zero.

Then it is possible to obtain the following theorem about a maximum objective coefficient tolerance for an optimal solution to preserve the optimality from Lemma 1, which corresponds to Wendell's results.

Theorem 1. Consider an optimal basic solution **x** to problem (2). The maximum tolerance of the objective function coefficients is for all k ∊ S (Wendell, 1984):

$$\tau^* = \min_{k \notin S} \tau_k, \text{ where } \tau_k = \begin{cases} \dfrac{\hat{\mathrm{c}}_{\mathrm{B}}\,\mathrm{B}^{-1}\,\bar{\mathbf{A}}_{.\mathrm{k}} - \hat{\mathrm{c}}_{\mathrm{k}}}{|\mathrm{c}'_{\mathrm{k}}| + \sum_{\mathrm{i}\in\mathrm{J}} |\mathrm{c}'_{\mathrm{B_i}}\,\mathrm{y_{ik}}|}, & \text{if } |\mathrm{c}'_{\mathrm{k}}| + \sum_{\mathrm{i}\in\mathrm{J}} |\mathrm{c}'_{\mathrm{B_i}}\,\mathrm{y_{ik}}| > 0, \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

and S is the index set of (h–m) non-basic variables; J is the index set of m basic variables; $\hat{\mathbf{c}}_{\mathrm{B}}$ is the basic variable objective function m-dimensional coefficient vector; c'_{B_i} is the i$^{\mathrm{th}}$ basic variable perturbed coefficient; B^{-1} is the inverse of the basis (m×m) matrix B; $y_{ik} = \mathbf{B}^{-1}_{\mathrm{i.}}\,\bar{\mathbf{A}}_{.\mathrm{k}}$, $\mathbf{B}^{-1}_{\mathrm{i.}}$ is the i$^{\mathrm{th}}$ row of the inverse of the basis, $\bar{\mathbf{A}}_{.\mathrm{k}}$ is the k$^{\mathrm{th}}$ column of the $\bar{\mathrm{A}}$ matrix.

Consider $A^0 = (\begin{bmatrix} -(\mathrm{B}^{-1}\mathrm{N}) \\ \mathrm{I}_{\mathrm{h-m}} \end{bmatrix}^{\mathrm{T}} \mathrm{diag}(\mathbf{c}'))$ and $\mathbf{b}^0 = \begin{bmatrix} \mathrm{B}^{-1}\mathrm{N} \\ -\mathrm{I}_{\mathrm{h-m}} \end{bmatrix}^{\mathrm{T}} \hat{\mathbf{c}}$, where $\hat{\mathbf{c}} = \begin{bmatrix} \hat{\mathbf{c}}_{\mathrm{B}} \\ \hat{\mathbf{c}}_{\mathrm{N}} \end{bmatrix}$ and $\hat{\mathbf{c}}_{\mathrm{N}}$ is the non-basic variable objective function coefficient's vector, N is the m×(h-m) matrix of non-basic variables obtained from the m×h matrix $\bar{\mathrm{A}}$, $\mathrm{I}_{\mathrm{h-m}}$ is an (h-m)×(h-m) identity matrix and $\begin{bmatrix} \mathrm{B}^{-1}\mathrm{N} \\ -\mathrm{I}_{\mathrm{h-m}} \end{bmatrix}$ is an h×(h-m) matrix.

Let $\mathrm{q_{jk}}$ be the j$^{\mathrm{th}}$ element of the k$^{\mathrm{th}}$ column of $\begin{bmatrix} \mathrm{B}^{-1}\mathrm{N} \\ -\mathrm{I}_{\mathrm{h-m}} \end{bmatrix}$. Then, according to the notation used in Lemma 1, (4) can be equivalently given as:

$$\tau_k = \begin{cases} \frac{\sum_{j=1}^{h} q_{jk}\hat{c}_j}{\sum_{j=1}^{h} |q_{jk}| c'_j}, \text{if } \sum_{j=1}^{h} |q_{jk}| c'_j > 0 \\ 0, \text{otherwise} \end{cases}, \tag{5}$$

or

$$\tau_k = \begin{cases} \frac{b_k^0}{\sum_{j=1}^{h} |a_{kj}^0|}, \text{if } \sum_{j=1}^{h} |a_{kj}^0| > 0 \\ 0, \text{otherwise} \end{cases}, \tag{6}$$

where $b_k^0 = \sum_{j=1}^{h} q_{jk}\hat{c}_j$ corresponds to the element k of vector $\mathbf{b}^0$ and $\sum_{j=1}^{h} |a_{kj}^0| = \sum_{j=1}^{h} |q_{jk}| c'_j$, where a_{kj}^0 is the element of row k and column j of matrix A^0.

In what follows, we investigate a maximum objective coefficient tolerance for an optimal solution **x** to problem (2) to preserve the optimality.
Let $H_k^{\leq} \equiv \{\gamma: \hat{c}_k + \gamma\, \mathbf{g}_{.k} - \hat{\mathbf{c}}_B B^{-1} \bar{\mathbf{A}}_{.k} - \gamma_{\boldsymbol{B}} \text{diag}(\mathbf{g}_B) B^{-1} \bar{\mathbf{A}}_{.k} \leq 0\}$, for each $k \in S$, where $\mathbf{g}_{.k}$ is the k^{th} column of diag($\mathbf{g}$), diag($\mathbf{g}_B$) is the diagonal matrix whose main diagonal is the perturbed objective function coefficients of the basic variables and $\hat{c}_k$ is the k^{th} objective function coefficient ($k \in S$).
If the optimal basis B is dual degenerate, then $\tau^* = 0$ and since degeneracy often occurs in real problems this poses a drawback to the tolerance approach. This problem can be overcome by allowing ITR for each objective function coefficient (Wondolowski, 1991). However, as indicated in Figure 1 the hyperbox contained in a given polyhedral cone and centred in a given point of the cone obtained with the Wondolowski's approach can sometimes be further extended.

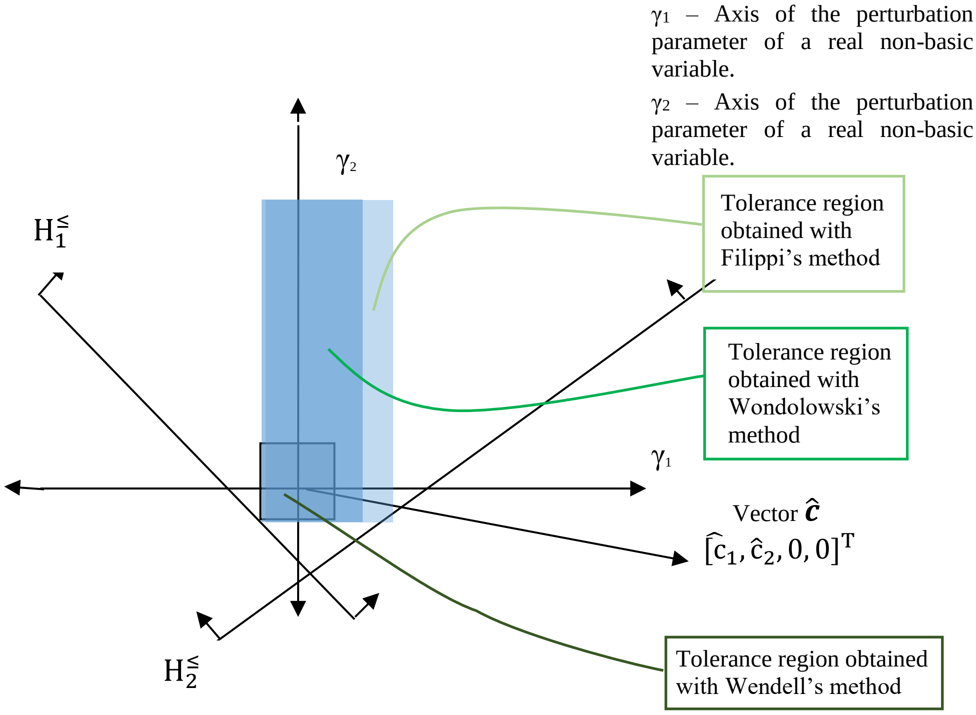

Figure 1. Illustrative example of the geometric interpretation of the tolerance approach for the objective function coefficients with an enlarged hyperbox.

In order to get larger tolerances an iterative procedure is needed. In this context, Filippi (2005) suggested an algorithm which allows building up a maximal inner hyperbox of a polyhedron containing the origin by an iterative application of Lemma 1.

Given a polyhedron $\boldsymbol{\mathcal{P}} \subseteq IR^d$ and a hyperbox $\boldsymbol{\mathcal{B}}(\mathbf{l}, \mathbf{u}) \subseteq IR^d$, we say that $\boldsymbol{\mathcal{B}}(\mathbf{l}, \mathbf{u})$ is a maximal inner hyperbox of $\boldsymbol{\mathcal{P}}$ if $\boldsymbol{\mathcal{B}}(\mathbf{l}, \mathbf{u}) \subset \boldsymbol{\mathcal{B}}(\mathbf{l}', \mathbf{u}')$ implies $\boldsymbol{\mathcal{B}}(\mathbf{l}', \mathbf{u}') \not\subset \boldsymbol{\mathcal{P}}$. Therefore, a maximal inner hyperbox is maximal with respect to inclusion (Filippi, 2005).

Filippi's algorithm provided below, expands the relaxation results given by Wendell (1992) and Wondolowski (1991), and increases the hyperbox by an iterative procedure which fixes the values of $\boldsymbol{\gamma}$ that reach a hyperplane of $\mathrm{H}_k^{\leq}$ and then lets the others further expand until they reach a hyperplane (or, even, until they expand to infinity).

Consider polyhedron $\boldsymbol{\mathcal{P}}^0 = \{\mathbf{x} \in IR^d \colon A^0\mathbf{x} \leq \mathbf{b}^0\}$, where $A^0 = (\begin{bmatrix} -(\mathrm{B}^{-1}\mathrm{N}) \\ \mathrm{I}_{\mathrm{h-m}} \end{bmatrix}^{\mathrm{T}} \mathrm{diag}(\mathbf{c}'))$, $\mathbf{b}^0 = \begin{bmatrix} \mathrm{B}^{-1}\mathrm{N} \\ -\mathrm{I}_{\mathrm{n-m}} \end{bmatrix}^{\mathrm{T}} \hat{\mathbf{c}}$, $\mathrm{d} = \mathrm{h} - \mathrm{m}$, $\boldsymbol{\mathcal{L}} = (\boldsymbol{\mathcal{P}}^t, \mathrm{l}^{\mathrm{t}}, \mathrm{u}^{\mathrm{t}}, \mathrm{L}^{\mathrm{t}}, \mathrm{U}^{\mathrm{t}})$, S is the index set of $(\mathrm{h} - \mathrm{m})$ non-basic variables and J is the index set of m basic variables.

1) Set $\mathrm{t} = 0$, such that $l_j^0 = -\infty$ and $u_j^0 = +\infty$ for all $\mathrm{j} \in \mathrm{J}$;

2) $\mathrm{L}^0 = \mathrm{U}^0 = \{\mathrm{j} \in \mathrm{J} \colon a_{\mathrm{kj}}^0 = 0 \text{ for all } k \in \mathrm{S}\}$.

3) Compute for each inequality of $\boldsymbol{\mathcal{P}}^t$

$$\sigma_k = \begin{cases} \frac{b_k^t}{\sum_{\mathrm{j}\in\mathrm{J}} |a_{\mathrm{kj}}^t|}, \text{ if} \sum_{\mathrm{j}\in\mathrm{J}} |a_{\mathrm{kj}}^t| > 0 \\ 0, \text{otherwise} \end{cases}$$

Let $\sigma(\boldsymbol{\mathcal{P}}^t) = \min_{\mathrm{k}} \{\sigma_k \colon \sum_{\mathrm{j}\in\mathrm{J}} |\mathrm{a}_{\mathrm{kj}}^{\mathrm{t}}| > 0\}$;

4) Define set $\mathrm{K}^{\mathrm{t}} = \{\mathrm{k} \in \mathrm{S} \colon \sigma_k = \sigma(\boldsymbol{\mathcal{P}}^t)\}$;

5) For all $\mathrm{k} \in \mathrm{S}$, we compute $\mathrm{L}_{\mathrm{k}}^{\mathrm{t}}$ and $\mathrm{U}_{\mathrm{k}}^{\mathrm{t}}$ by

$\mathrm{L}_{\mathrm{k}}^{\mathrm{t}} = \{\mathrm{j} \in \mathrm{J} \colon \mathrm{a}_{\mathrm{kj}}^{\mathrm{t}} < 0 \text{ and } \mathrm{a}_{\mathrm{kj}}^{\mathrm{t}} < 0 \text{ for some } \mathrm{h} \in \mathrm{K}^{\mathrm{t}}\}$,

$\mathrm{U}_{\mathrm{k}}^{\mathrm{t}} = \{\mathrm{j} \in \mathrm{J} \colon \mathrm{a}_{\mathrm{kj}}^{\mathrm{t}} > 0 \text{ and } \mathrm{a}_{\mathrm{kj}}^{\mathrm{t}} > 0 \text{ for some } \mathrm{h} \in \mathrm{K}^{\mathrm{t}}\}$.

6) Obtain $\mathrm{L}_{\mathrm{k}}^{\mathrm{t}} \cup \mathrm{U}_{\mathrm{k}}^{\mathrm{t}}$ for all $\mathrm{k} \in \mathrm{S}$.

7) Apply the following transformations:

$$a_{\mathrm{kj}}^{t+1} = \begin{cases} 0, \text{ if } j \in \mathrm{L}_{\mathrm{k}}^{\mathrm{t}} \cup \mathrm{U}_{\mathrm{k}}^{\mathrm{t}} \\ a_{\mathrm{kj}}^t, \text{ otherwise} \end{cases}, \mathrm{j} \in \mathrm{J}, \mathrm{k} \in \mathrm{S}$$

$$b_{\mathrm{k}}^{\mathrm{t+1}} = b_{\mathrm{k}}^{\mathrm{t+1}} - \sigma(\boldsymbol{\mathcal{P}}^t)\left(\sum_{\mathrm{j}\in\mathrm{L}_{\mathrm{k}}^{\mathrm{t}}} |a_{\mathrm{kj}}^t| + \sum_{\mathrm{j}\in\mathrm{U}_{\mathrm{k}}^{\mathrm{t}}} |a_{\mathrm{kj}}^t|\right), \mathrm{k} \in \mathrm{S},$$

$$l_{\mathrm{j}}^{\mathrm{t+1}} = \begin{cases} -\sigma(\boldsymbol{\mathcal{P}}^t), \text{if } \mathrm{j} \in \cup_{\mathrm{k}\in\mathrm{S}} \mathrm{L}_{\mathrm{k}}^{\mathrm{t}} \\ l_{\mathrm{j}}^{\mathrm{t}}, \text{otherwise.} \end{cases}, \mathrm{j} \in \mathrm{J}$$

$$u_{\mathrm{j}}^{\mathrm{t+1}} = \begin{cases} +\sigma(\boldsymbol{\mathcal{P}}^t), \text{if } \mathrm{j} \in \cup_{\mathrm{k}\in\mathrm{S}} \mathrm{U}_{\mathrm{k}}^{\mathrm{t}} \\ u_{\mathrm{j}}^{\mathrm{t}}, \text{otherwise.} \end{cases}, \mathrm{j} \in \mathrm{J}$$

$\mathrm{L}^{\mathrm{t+1}} = \mathrm{L}^{\mathrm{t}} \cup (\cup_{\mathrm{k}\in\mathrm{S}} \mathrm{L}_{\mathrm{k}}^{\mathrm{t}})$,

$\mathrm{U}^{\mathrm{t+1}} = \mathrm{U}^{\mathrm{t}} \cup (\cup_{\mathrm{k}\in\mathrm{S}} \mathrm{U}_{\mathrm{k}}^{\mathrm{t}})$.

8) Set $t = t + 1$;

9) If $\mathrm{A}^{\mathrm{t+1}} = 0$, then proceed to next step, else go to step 3.

10) Let $\tau_{\mathrm{j}}^{\mathrm{L}} = -l_{\mathrm{j}}^{\mathrm{t+1}}$ and $\tau_{\mathrm{j}}^{\mathrm{U}} = u_{\mathrm{j}}^{\mathrm{t+1}}$. Then, for each objective function coefficient, we obtain the range of c_j as:

$\hat{\mathrm{c}}_{\mathrm{j}} - \tau_{\mathrm{j}}^{\mathrm{L}} \mathrm{c}_{\mathrm{j}}' \leq \mathrm{c}_{\mathrm{j}} \leq \hat{\mathrm{c}}_{\mathrm{j}} + \tau_{\mathrm{j}}^{\mathrm{U}} \mathrm{c}_{\mathrm{j}}'$.

11) End.

In the procedure above, matrix A^t is composed of the (k, j)-component $a_{\mathrm{kj}}^{\mathrm{t}}, \mathrm{j} \in \mathrm{J}, \mathrm{k} \in \mathrm{S}$.

Theorem 2. We have $\tau_{\mathrm{j}}^{\mathrm{L}} \leq \tau_j^-$ and $\tau_{\mathrm{j}}^{\mathrm{U}} \geq \tau_j^+$ for all $\mathrm{j} \in \mathrm{J}$ such that $\mathrm{c}_{\mathrm{j}}' > 0$ (Theorem 4 in Filippi (2005)), where τ_j^- and τ_j^+ are the ITR of the objective function coefficients given as (Wondolowski, 1991):

$$\tau_j^- = \begin{cases} \min\limits_{k\in S}\{\tau_k : y_{jk} > 0\}, \text{ if } c'_j \neq 0 \text{ and } \exists k \in S \text{ such that } y_{jk} > 0, \\ -\infty, \text{ if } y_{jk} \leq 0, \text{for all } k \in S, \; j \in J, \\ 0, \text{ if } c'_j = 0 \text{ with } y_{jk} > 0 \text{ for some } k \in S, \\ -\infty, \text{ if } j \in S. \end{cases} \tag{7}$$

$$\tau_j^+ = \begin{cases} \min\limits_{k\in S}\{\tau_k : y_{jk} < 0\}, \text{ if } c'_j \neq 0 \text{ and } \exists k \in S \text{ such that } y_{jk} < 0, \\ +\infty, \text{ if } y_{jk} \geq 0, \text{for all } k \in S, \; j \in J, \\ 0, \text{ if } c'_j = 0 \text{ with } y_{jk} < 0 \text{ for some } k \in S, \\ \tau_j, \text{ if } j \in S. \end{cases} \tag{8}$$

As described in Filippi (2005), when an optimal solution **x** to problem (2) is degenerate, we should consider several optimal bases and we look for individual tolerances such that at least one of the optimal bases remains optimal, possibly enlarging the tolerances. In this context, it is worth mentioning that degenerate problems can also be handled (to some extent) by considering other invariances rather than considering the classical optimal basis invariance (Hladík, 2011).

3. LP problems with interval coefficients on the objective function

It is widely acknowledged that in interval linear programming (ILP) problems different formulations have different properties (Garajová et al., 2018). In our case, we deal with the following type of ILP problems:

$$\max z(\mathbf{x}) = \boldsymbol{c}^{\mathrm{T}}\mathbf{x},$$
$$\text{s.t. } \mathbf{x} \in \mathrm{X} = \{\mathbf{x} \in IR^n : \mathrm{A}\,\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}, \tag{9}$$

where $\boldsymbol{c}$ is an n-dimensional vector in a given range Θ and Θ is a closed and convex hyperbox defined by

$$\Theta = \{\boldsymbol{c} = (c_1, \dots, c_n)^{\mathrm{T}} \in IR^n : c_j \in [c_j^L, c_j^U], \mathrm{j} = 1, \dots, \mathrm{n}\}. \tag{10}$$

Namely, the j-th element c_j of $\boldsymbol{c}$ is not known exactly but it takes a value in the given interval $[c_j^L, c_j^U]$, i.e., $c_j \in [c_j^L, c_j^U], \mathrm{j} = 1, \dots, \mathrm{n}$. A is an m × n matrix, **b** is an m-dimensional vector, **x** is an n-dimensional vector and the superscripts L and U represent lower and upper bounds of the coefficients, respectively.

Let $\mathrm{X}(\boldsymbol{c}) = \left\{\boldsymbol{x} : \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \max\limits_{y\in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}, \boldsymbol{x} \in \mathrm{X}\right\}$ be the set of optimal solutions to ILP (9). From $\mathrm{X}(\boldsymbol{c})$ two types of optimal solutions regarding problem (9) can be obtained: “possibly optimal solutions” and “necessarily optimal solutions” (Inuiguchi and Sakawa, 1994).

Definition 1. A possibly optimal solution to ILP problem (9) is a solution which is optimal for at least one $\boldsymbol{c} \in \Theta$. The set of all possibly optimal solutions to Problem (9) is denoted by $\mathrm{P}_0(\Theta)$.

Definition 2. A necessarily optimal solution to ILP problem (9) is a solution which is optimal for any $\boldsymbol{c} \in \Theta$. The set of all necessarily optimal solutions to Problem (9) is denoted by $\mathrm{N}_0(\Theta)$.

The solution sets of Definitions 1 and 2 can be defined, respectively, as:

$$\mathrm{P}_0(\Theta) = \cup_{\boldsymbol{c}\in\Theta} X(\boldsymbol{c}), \tag{11}$$
$$\mathrm{N}_0(\Theta) = \cap_{\boldsymbol{c}\in\Theta} X(\boldsymbol{c}). \tag{12}$$

An element of $\mathrm{P}_0(\Theta)$ is a feasible optimal solution for at least one $\boldsymbol{c} \in \Theta$. Since Θ represents the region where $\boldsymbol{c}$ can assume values (possible region of $\boldsymbol{c}$), each element of $\mathrm{P}_0(\Theta)$ is possibly optimal. Therefore, each element of $\mathrm{P}_0(\Theta)$ is called "possibly optimal solution".

On the other hand, an element of $\mathrm{N}_0(\Theta)$ is a feasible optimal solution for any $\boldsymbol{c} \in \Theta$. Hence, each element of $\mathrm{N}_0(\Theta)$ is a "necessarily optimal solution". A necessarily optimal solution is also possibly optimal, i.e., $\mathrm{N}_0(\Theta) \subseteq \mathrm{P}_0(\Theta)$.

In an uncertain environment, necessarily optimal solutions are the most robust and conservative ones. Contrarily, a possibly optimal solution is an optimistic one. Nevertheless, necessarily optimal solutions may not always exist.

Let us now consider Ω with 2^n extreme points (where n is the number of interval coefficients and, without loss of generality, all the coefficients are given as intervals) defined in the following manner (Steuer, 1981):

$$\Omega = \left\{\mathbf{f}^{\mathrm{q}} \in IR^n \colon \mathbf{f}^1, \mathbf{f}^2, \ldots, \mathbf{f}^{2^n}\right\} = \left\{\mathbf{f} \in IR^n \colon f_j \in \left\{c_j^L, c_j^U\right\} \text{ for } \mathrm{j} = 1, \ldots, \mathrm{n}\right\} \tag{13}$$

Since Θ is closed, any $\boldsymbol{c} \in \Theta$ may be written as a convex combination of $\mathbf{f}^{\mathrm{q}}$. That is, for any $\boldsymbol{c} \in \Theta$ there is a $\mathbf{w} \in IR^{2^{\mathrm{n}}}, \mathbf{w} \geq \mathbf{0}, \sum_{\mathrm{q}=1}^{2^{\mathrm{n}}} \mathrm{w}_{\mathrm{q}} = 1$ such that:

$$\boldsymbol{c} = \sum_{\mathrm{q}=1}^{2^{\mathrm{n}}} \mathrm{w}_{\mathrm{q}}\, \mathbf{f}^{\mathrm{q}}. \tag{14}$$

On the other hand, any linear convex combination of $\mathbf{f}^{\mathrm{q}}$ is also an element of Θ.

Consider the $(2^{\mathrm{n}} \times \mathrm{n})$ F matrix having as rows the $\left(\mathbf{f}^{\mathrm{q}}\right)^{\mathrm{T}}$ vectors.

It is known that a solution $\mathbf{x}^* \in \mathrm{X}$ is weakly efficient regarding F if and only if there is a vector $\mathbf{w} \in IR^{2^{\mathrm{n}}}, \mathbf{w} \geq \mathbf{0}, \sum_{\mathrm{q}=1}^{2^{\mathrm{n}}} \mathrm{w}_{\mathrm{q}} = 1$ such that $\mathbf{x}^*$ optimizes the following LP problem with a surrogate scalar function (Steuer, 1981):

$$\begin{aligned} &\max \mathrm{z}(\mathbf{x}) = \mathbf{w}^{\mathrm{T}} \mathrm{F}\, \mathbf{x}, \\ &\mathrm{s.t.}\ \mathbf{x} \in \mathrm{X}, \\ &\mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{15}$$

Hence, we may conclude that problem (9) becomes equivalent in some sense to the following vector optimization problem:

$$\begin{aligned} &\max \mathrm{z}(\mathbf{x}) = \mathrm{F}\, \mathbf{x}, \\ &\mathrm{s.t.}\ \mathbf{x} \in \mathrm{X}, \\ &\mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{16}$$

Definition 3. A feasible solution to problem (16) is efficient if and only if there is no other feasible solution which allows improving the value of an objective function without worsening the value of at least another objective function, i. e., $\mathbf{x} \in \mathrm{X}$ is efficient if and only if there is no other $\mathbf{x}' \in \mathrm{X}$, such that $z_q(\mathbf{x}') \geq z_q(\mathbf{x})$ for all q and $z_q(\mathbf{x}') > z_q(\mathbf{x})$ for at least one q $(\mathrm{q} = 1, \ldots, 2^{\mathrm{n}})$.

Definition 4. A feasible solution to problem (16) is weakly efficient if and only if there is no other feasible solution which improves strictly all the objective function values, i.

e., $\mathbf{x} \in \mathrm{X}$ is weakly efficient if and only if there is no other $\mathbf{x}' \in \mathrm{X}$, such that $z_q(\mathbf{x}') > z_q(\mathbf{x})$ for all q $(\mathrm{q} = 1, \dots, 2^{\mathrm{n}})$.

Therefore, any extreme point that is possibly optimal to problem (9) is also weakly efficient to problem (16) and *vice-versa*.

Consider $\mathrm{P}(\mathbf{x})$ as the set of vectors $\boldsymbol{c}$ for which $\mathbf{x}$ is an optimal solution, i.e.

$$\mathrm{P}(\mathbf{x}) = \{\boldsymbol{c}\colon \boldsymbol{c}^{\mathrm{T}}\mathbf{x} = \max_{y \in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}\} \tag{17}$$

For any feasible solution $\mathbf{x}$

$$\mathbf{x} \in \mathrm{P_o}(\Theta) \Leftrightarrow P_{\Theta}(\mathrm{P}(\mathbf{x})) = 1 \Leftrightarrow \Theta \cap \mathrm{P}(\mathbf{x}) \neq \varnothing, \tag{18}$$

$$\mathbf{x} \in \mathrm{N_o}(\Theta) \Leftrightarrow N_{\Theta}(\mathrm{P}(\mathbf{x})) = 1 \Leftrightarrow \Theta \subseteq \mathrm{P}(\mathbf{x}), \tag{19}$$

where P and N are possibility and necessity measures, defined as follows:

$$P_A(\mathrm{B}) = \begin{cases} 1, \mathrm{if} \mathrm{A} \cap \mathrm{B} \neq \varnothing \\ 0, \mathrm{otherwise.} \end{cases} \text{ and } N_A(\mathrm{B}) = \begin{cases} 1, \mathrm{if}\ \mathrm{A} \subseteq \mathrm{B} \\ 0, \mathrm{otherwise.} \end{cases}$$

Let the midpoint of an interval be given by $\mathrm{c_j^C} = \mathrm{m}([\mathrm{c_j^L}, \mathrm{c_j^U}]) = \frac{1}{2}(\mathrm{c_j^L} + \mathrm{c_j^L})$ and consider the following surrogate LP problem to ILP (9):

$$\max z(\mathbf{x}) = (\mathbf{c}^{\mathrm{C}})^{\mathrm{T}}\mathbf{x},$$

$$\mathrm{s.t.}\, \mathrm{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \tag{20}$$

where $(\mathbf{c}^{\mathrm{C}}) = (\mathrm{c_1^C}, \mathrm{c_2^C}, \dots, \mathrm{c_n^C})^{\mathrm{T}}$ is the objective function vector with the midpoint of the interval objective coefficient values.

Let $\vartheta\,(\mathbf{c}'\colon \mathbf{c}^{\mathrm{C}}) = \{\mathbf{c} = (\mathrm{c_1}, \mathrm{c_2}, \dots, \mathrm{c_n})^{\mathrm{T}}\colon \mathrm{c_j^C} - \tau_j^{\mathrm{L}}\mathrm{c_j'} \leq \mathrm{c_j} \leq \mathrm{c_j^C} + \tau_j^{\mathrm{U}}\mathrm{c_j'}, \mathrm{j} = 1, 2, \dots, \mathrm{n}\}$ be the result of Filippi's method of the tolerance approach with respect to objective function coefficient vector of $\mathbf{x}$ based on an optimal basic solution $\bar{\mathbf{x}}^*$ to the following problem equivalent to problem (20):

$$\max z(\mathbf{x}) = (\mathbf{c}^{\mathrm{C}})^{\mathrm{T}}\mathbf{x},$$

$$\mathrm{s.t.}\, \bar{\mathrm{A}}\bar{\mathbf{x}} = \mathbf{b}, \bar{\mathbf{x}} = (\mathbf{x}^{\mathrm{T}}, \mathbf{s})^{\mathrm{T}} \geq \mathbf{0}, \tag{21}$$

where $\bar{\mathrm{A}} = (\mathrm{A}\ \mathrm{I_m})$ and $\mathbf{s} \in IR^{\mathrm{m}}$ is a slack variable vector.

Theorem 3. Let $\bar{\mathbf{x}}^*$ be an optimal basic solution to problem (21) which is not degenerate. Let $\mathbf{x}^*$ be the optimal solution to problem (20) corresponding to $\bar{\mathbf{x}}^*$. If $\Theta \subseteq \vartheta\,(\mathbf{c}'\colon \mathbf{c}^{\mathrm{C}})$ then solution $\mathbf{x}^* \in \mathrm{N_o}(\Theta)$.

Proof: Assume $\mathbf{x}^* \notin \mathrm{N_o}(\Theta)$, then there is a feasible solution $\mathbf{x}'$ such that $\sum_{\mathrm{j}=1}^{\mathrm{n}} \bar{\mathrm{c}}_{\mathrm{j}}\mathrm{x}_{\mathrm{j}}' > \sum_{\mathrm{j}=1}^{\mathrm{n}} \bar{\mathrm{c}}_{\mathrm{j}}\mathrm{x}_{\mathrm{j}}^*$ for some $\bar{\mathrm{c}}_{\mathrm{j}}$ such that $\mathrm{c_j^L} \leq \bar{\mathrm{c}}_{\mathrm{j}} \leq \mathrm{c_j^U}$, j=1,2,…,n. It is verified that $\mathrm{c_j^C} - \tau_j^{\mathrm{L}}\mathrm{c_j'} \leq \bar{\mathrm{c}}_{\mathrm{j}} \leq \mathrm{c_j^C} + \tau_j^{\mathrm{U}}\mathrm{c_j'}$ because $\Theta \subseteq \vartheta\,(\mathbf{c}'\colon \mathbf{c}^{\mathrm{C}})$ implies that $\mathrm{c_j^L} \geq \mathrm{c_j^C} - \tau_j^{\mathrm{L}}\mathrm{c_j'}$ and $\mathrm{c_j^U} \leq \mathrm{c_j^C} + \tau_j^{\mathrm{U}}\mathrm{c_j'}$. This fact implies that $\mathbf{x}^*$ is optimal when the objective function coefficients $\mathrm{c_j^C}$ are changed to $\bar{\mathrm{c}}_{\mathrm{j}}$, j=1,2,..,n. Namely, we have $\sum_{\mathrm{j}=1}^{\mathrm{n}} \bar{\mathrm{c}}_{\mathrm{j}}\mathrm{x}_{\mathrm{j}}^* \geq \sum_{\mathrm{j}=1}^{\mathrm{n}} \bar{\mathrm{c}}_{\mathrm{j}}\mathrm{x}_{\mathrm{j}}'$. This contradicts with $\sum_{\mathrm{j}=1}^{\mathrm{n}} \bar{\mathrm{c}}_{\mathrm{j}}\mathrm{x}_{\mathrm{j}}' > \sum_{\mathrm{j}=1}^{\mathrm{n}} \bar{\mathrm{c}}_{\mathrm{j}}\mathrm{x}_{\mathrm{j}}^*$.

Theorem 4. Let $\mathbf{x}^*$ be an optimal solution to problem (20). Then this solution is possibly optimal, i.e., $\mathbf{x}^* \in \mathrm{P_o}(\Theta)$.

Proof: It is obvious because $\mathbf{x}^*$ is an optimal solution to Problem (20) and $\mathbf{c}^{\mathrm{C}} \in \Theta$.

Corollary 1. If $\Theta \not\subset \vartheta\,(\mathbf{c}': \mathbf{c}^{\mathrm{C}})$ and $\Theta \not\subset \mathrm{P}(\mathbf{x}^*)$ are satisfied with an optimal basic solution $\mathbf{x}^*$ to problem (20), then this solution is only possibly optimal, i.e. $\mathbf{x}^* \notin \mathrm{N}_{\mathrm{o}}(\Theta)$ but $\mathbf{x}^* \in \mathrm{P}_{\mathrm{o}}(\Theta)$.

Proof: We have $\mathbf{x}^* \notin \mathrm{N}_{\mathrm{o}}(\Theta)$ from (19) and $\mathbf{x}^* \in \mathrm{P}_{\mathrm{o}}(\Theta)$ from Theorem 4.

Corollary 2. If $\Theta \not\subset \vartheta\,(\mathbf{c}': \mathbf{c}^{\mathrm{C}})$ and $\Theta \subseteq \mathrm{P}(\mathbf{x}^*)$ are satisfied with an optimal basic solution $\mathbf{x}^*$ to problem (20), then we have $\mathbf{x}^* \in \mathrm{N}_{\mathrm{o}}(\Theta)$ and $\mathbf{x}^* \in \mathrm{P}_{\mathrm{o}}(\Theta)$.

Proof: We have $\mathbf{x}^* \in \mathrm{N}_{\mathrm{o}}(\Theta)$ from (19) and $\mathbf{x}^* \in \mathrm{P}_{\mathrm{o}}(\Theta)$ from Theorem 4.

Let c_j^{Lsens} be the lower bound for individual changes of c_j^{C}, where the optimal basis of an optimal basic solution $\bar{\mathbf{x}}^*$ to problem (21) remains optimal, and c_j^{Usens} be the upper bound for individual changes of c_j^{C} where the optimal basis remains optimal by means of traditional sensitivity analysis (usually this is provided by common LP software), where $\bar{\mathbf{x}}^*$ to problem (21) corresponds to an optimal solution $\mathbf{x}^*$ to problem (20).

Let $\overline{[c_j^{\mathrm{Lsens}}, c_j^{\mathrm{Usens}}]}$ be the cylindrical extension of $[c_j^{\mathrm{Lsens}}, c_j^{\mathrm{Usens}}]$, i.e.,

$$\overline{[\mathbf{c}^{\mathrm{Lsens}}, \mathbf{c}^{\mathrm{Usens}}]} = \{\boldsymbol{c} = (c_1, c_2, \dots, c_n)^{\mathrm{T}} : \ c_j^{\mathrm{Lsens}} \leq c_j \leq c_j^{\mathrm{Usens}}\}.$$

Theorem 5. Let $\bar{\mathbf{x}}^*$ be an optimal basic solution to problem (21) which is not degenerate. Let $\mathbf{x}^*$ be the optimal solution to problem (20) corresponding to $\bar{\mathbf{x}}^*$. If $\Theta \subseteq \mathrm{P}(\mathbf{x}^*)$ then $\Theta \subseteq \overline{[c_j^{\mathrm{Lsens}}, c_j^{\mathrm{Usens}}]}$ for all j (j = 1…, n).

Proof. From $\Theta \subseteq \mathrm{P}(\mathbf{x}^*)$ and $\mathbf{c}^{\mathrm{C}} \in \Theta$, we have $c_j^{\mathrm{L}} \geq c_j^{\mathrm{Lsens}}$ and $c_j^{\mathrm{U}} \leq c_j^{\mathrm{Usens}}$, j =1, 2, …, n. This implies $\Theta \subseteq \overline{[c_j^{\mathrm{Lsens}}, c_j^{\mathrm{Usens}}]}$, j = 1, 2, …, n.

Corollary 3 (Theorem 5). Let $\bar{\mathbf{x}}^*$ be an optimal basic solution to problem (21) which is not degenerate. Let $\mathbf{x}^*$ be the optimal solution to problem (20) corresponding to $\bar{\mathbf{x}}^*$. If $\Theta \not\subset \overline{[c_j^{\mathrm{Lsens}}, c_j^{\mathrm{Usens}}]}$ for some j then $\Theta \not\subset \mathrm{P}(\mathbf{x}^*)$ and thus $\mathbf{x}^*$ is not a necessarily optimal solution, i.e., $\mathbf{x}^* \notin \mathrm{N}_{\mathrm{o}}(\Theta)$.

Example 1. Consider the following LP problem with interval objective functions (Inuiguchi and Sakawa, 1994):

$\max\, [2, 3]x_1 + [1.5, 2.5]x_2,$
$\mathrm{s.t.}\, 3x_1 + 4x_2 \leq 42,$
$3x_1 + x_2 \leq 24,$
$x_1 \geq 0, 0 \leq x_2 \leq 9.$

In this problem, $\Theta = \{(c_1, c_2)^{\mathrm{T}}: 2 \leq c_1 \leq 3, 1.5 \leq c_2 \leq 2.5\}$.
Solving Problem (20) when $c_1^{\mathrm{C}} = 2.5, c_2^{\mathrm{C}} = 2$, we obtain $(6, 6)^{\mathrm{T}}$ as an optimal solution.
The set of coefficients which allows obtaining solution $(6, 6)^{\mathrm{T}}$ is:
$\mathrm{P}((6, 6)^{\mathrm{T}}) = \{\, (c_1, c_2)^{\mathrm{T}}: c_1 - 3\, c_2)^{\mathrm{T}} \leq 0,\ 4c_1 - 3\, c_2)^{\mathrm{T}} \geq 0\}.$

From Figure 2, it can be concluded that $\Theta \subseteq \mathrm{P}((6,6)^{\mathrm{T}})$ and that $(6,6)^{\mathrm{T}}$ is an optimal solution for any $(c_1, c_2)^{\mathrm{T}} \in \Theta$ and, therefore, it is a necessarily optimal solution. Moreover, this result corroborates Theorem 3 since, according to the tolerance region obtained using Filippi's method, solution $(6,6)^{\mathrm{T}}$ is optimal within $\vartheta\,(\mathbf{c}'\!:\mathbf{c}^{\mathrm{C}}) = \{(c_1, c_2)^{\mathrm{T}}\!: 1.875 \le \mathrm{c}_1 \le 3.529, 1.176 \le \mathrm{c}_2 \le 2.5\}$ and $\Theta \subseteq \vartheta\,(\mathbf{c}'\!:\mathbf{c}^{\mathrm{C}})$.

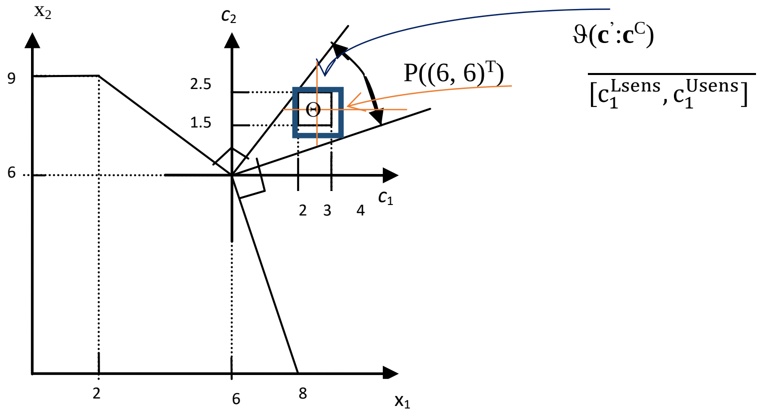

Figure 2. An illustrative example of a necessarily optimal solution which corroborates Theorem 3.

Consider that the objective function of the previous problem is changed to:
$\max\,[1,3]\mathrm{x}_1 + [1,3]\mathrm{x}_2$.
In this new problem, $\Theta = \{\boldsymbol{c} = (c_1, c_2)^{\mathrm{T}}\!: 1 \le \mathrm{c}_1 \le 3, 1 \le \mathrm{c}_2 \le 3\}$.
Solving Problem (20) when $c_1^{\mathrm{C}} = 2, c_2^{\mathrm{C}} = 2$, we obtain $(6,6)^{\mathrm{T}}$ as an optimal solution.

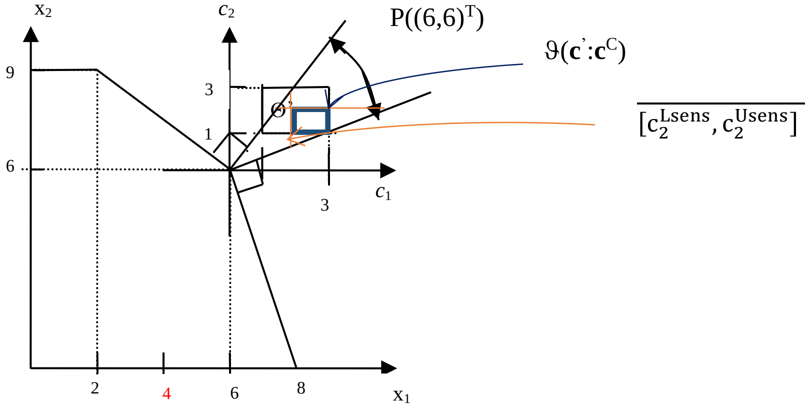

Figure 3. An illustrative example of a possibly optimal solution which corroborates Corollary 3.

From Figure 3, it can thus be seen that $\Theta \cap \mathrm{P}((6,6)^{\mathrm{T}})$ and this solution is now possibly optimal. Additionally, this result validates Corollary 1 since, according to the tolerance region obtained using Filippi's method, solution $(6,6)^{\mathrm{T}}$ is optimal within $\vartheta\,(\mathbf{c}'\!:\mathbf{c}^{\mathrm{C}}) = \{(c_1, c_2)^{\mathrm{T}}\!: 1.71 \le \mathrm{c}_1 \le 3 \text{ and } 1 \le \mathrm{c}_2 \le 2.29\}$ and $\Theta \not\subset \vartheta\,(\mathbf{c}'\!:\mathbf{c}^{\mathrm{C}})$ and $\Theta \not\subset \mathrm{P}((6,6)^{\mathrm{T}})$. Finally, Corollary 3 is also illustrated since $\Theta \not\subset \overline{[\mathrm{c}_2^{\mathrm{Lsens}}, \mathrm{c}_2^{\mathrm{Usens}}]}$, where

$\overline{[\mathbf{c}^{\mathrm{Lsens}}, \mathbf{c}^{\mathrm{Usens}}\,]} = \{(c_1, c_2)^{\mathrm{T}}\!: |\ 1.5 \le c_1 \le 6;\ \ 0.7 \le c_2 \le 2.7\}$, for j =1, 2, $\Theta \not\subset \mathrm{P}(\mathbf{x})$ and thus $(6, 6)^{\mathrm{T}}$ is not a necessarily optimal solution.

Consider that the objective function of the previous problem is changed to:

$\max\,[2.5, 3.5]\mathrm{x}_1 + [0.5, 0.8]\mathrm{x}_2$.

In this new problem, $\Theta = \{\boldsymbol{c} = (c_1, c_2)^{\mathrm{T}}\!: 2.5 \le c_1 \le 3.5, 0.5 \le c_2 \le 0.8\}$.

Solving Problem (20) when $c_1^{\mathrm{C}} = 3, c_2^{\mathrm{C}} = 0.65$, we obtain $(8, 0)^{\mathrm{T}}$ as an optimal solution. From Figure 4 it can be concluded that $\Theta \subseteq \mathrm{P}((8, 0)^{\mathrm{T}})$ and that $(8, 0)^{\mathrm{T}}$ is an optimal solution for any $(c_1, c_2)^{\mathrm{T}} \in \Theta$ and, therefore, it is a necessarily optimal solution. According to the tolerance region obtained by Filippi's method, solution $(8, 0)^{\mathrm{T}}$ is optimal within $\vartheta\,(\mathbf{c}'\!: \mathbf{c}^{\mathrm{C}}) = \{(c_1, c_2)^{\mathrm{T}}\!: 2.364 \le c_1 \le +\infty$ and $-\infty \le c_2 \le 0.7879\}$ and $\Theta \not\subset \vartheta\,(\mathbf{c}'\!: \mathbf{c}^{\mathrm{C}})$, but because $\Theta \subseteq \mathrm{P}((8, 0)^{\mathrm{T}})$ holds this solution is necessarily optimal (Corollary 2).

The basic properties and theoretical grounds for necessarily and possibly optimality are discussed in Inuiguchi and Sakawa (1994) and further developed in and Inuiguchi et al. (2016).

Although LP problems usually have unique optimal solutions, MOLP problems often allow obtaining multiple efficient solutions. Therefore, Inuiguchi (2013) suggested that the existence of necessarily efficient solutions in an IMOLP problem might occur more frequently than that of necessarily optimal solutions in an ILP problem.

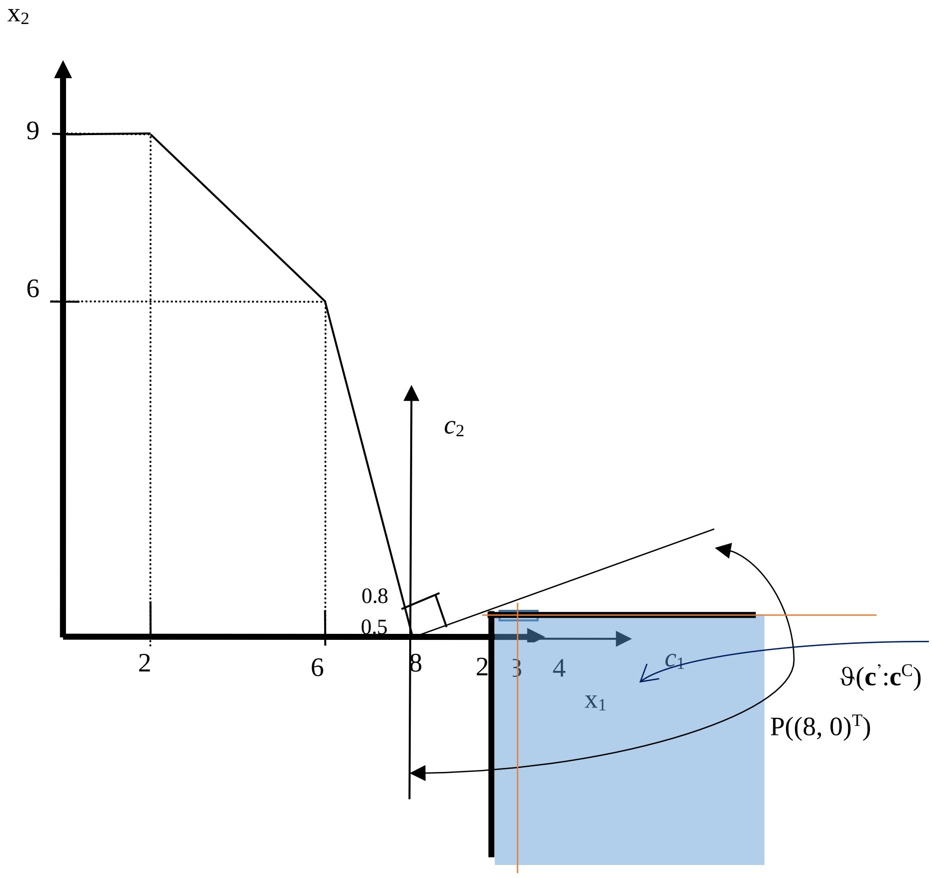

Figure 4. A necessarily optimal solution with $\Theta \not\subset \vartheta\ (\mathbf{c}'\!: \mathbf{c}^{\mathrm{C}})$

4. MOLP problems with interval coefficients in the objective functions

Consider the following interval MOLP (IMOLP) problem:
$\max Z(\mathbf{x}) = C\mathbf{x}$,
$\mathrm{s.t.}\, \mathbf{x} \in \mathrm{X} = \{\mathbf{x} \in IR^{\mathrm{n}}: \mathrm{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, (22)
where C is an m × n matrix in a given range Φ and Φ is a set of p × n matrices, whose generic components are $\mathrm{c}_{\mathrm{kj}} \in [\mathrm{c}_{\mathrm{kj}}^{\mathrm{L}}, \mathrm{c}_{\mathrm{kj}}^{\mathrm{U}}]$ for k =1, …, p, j = 1, …, n. Namely, the (k,j)-th component c_{kj} is not known exactly but it takes a value in the given interval $[\mathrm{c}_{\mathrm{kj}}^{\mathrm{L}}, \mathrm{c}_{\mathrm{kj}}^{\mathrm{U}}]$ ($\mathrm{k} \in \{1, 2, \dots, \mathrm{p}\}, \mathrm{j} \in \{1, 2, \dots, \mathrm{n}\}$). A is an m × n matrix, **b** is an m-dimensional vector, **x** is an n-dimensional vector. Let Φ^{k} be a set of n-dimensional vectors $\mathbf{c}_{\mathrm{k}} = (\mathrm{c}_{\mathrm{k1}}, \mathrm{c}_{\mathrm{k2}}, \dots, \mathrm{c}_{\mathrm{kn}})^{\mathrm{T}}$, whose transpose $\mathbf{c}_{\mathrm{k}}^{\mathrm{T}} = (\mathrm{c}_{\mathrm{k1}}, \mathrm{c}_{\mathrm{k2}}, \dots, \mathrm{c}_{\mathrm{kn}})$ shows the k-th row of $C \in \Phi$. We assume that $\mathbf{0} \notin \Phi^{\mathrm{k}}$ for k =1, …, p.

Let $\mathrm{X}_{\mathrm{E}}(C) = \{\mathbf{x} \in \mathrm{X}: \nexists \mathbf{y} \in X, C\mathbf{y} \geq C\mathbf{x}, C\mathbf{y} \neq C\mathbf{x}\}$ be the set of efficient solutions of IMOLP (22). From $\mathrm{X}_{\mathrm{E}}(C)$ two types of efficient solutions associated with problem (22) can be obtained: “possibly efficient solutions” and “necessarily efficient solutions”.

Definition 5. A solution is necessarily efficient of problem (22) if and only if it is efficient for any $C \in \Phi$. The necessarily efficient solution set is denoted by N_{E}.

Definition 6. A solution is possibly efficient of problem (22) if and only if it is efficient for at least one $C \in \Phi$. The possibly efficient solution set is denoted by P_{E}.

These solution sets can be defined, respectively as:
$\mathrm{N}_{\mathrm{E}} = \bigcap_{C \in \Phi} X_E(C)$ (23)
where $X_E(C)$ is the efficient solution set for each $C \in \Phi$.
$\mathrm{P}_{\mathrm{E}} = \bigcup_{C \in \Phi} X_E(C)$ (24)
Hence, $\mathrm{N}_{\mathrm{E}} \subseteq \mathrm{P}_{\mathrm{E}}$ and a necessarily efficient solution is also a possibly efficient solution. Then we call a solution a “strictly possibly efficient solution”, belonging to the solution set P_{SE}, if it is not necessarily efficient but possibly efficient.

Let
$\mathrm{RN}(\Phi) = \{\boldsymbol{c}: \forall\, C \in \Phi\, \exists\, \mathbf{v} > \mathbf{0}, \boldsymbol{c} = \mathbf{v}^{\mathrm{T}} C\} = \bigcap_{C \in \Phi} R(C)$ (25)
and
$\mathrm{RP}(\Phi) = \{\boldsymbol{c}: \exists\, \mathbf{v} > \mathbf{0}\, \exists\, C \in \Phi, \boldsymbol{c} = \mathbf{v}^{\mathrm{T}} C\} = \bigcup_{C \in \Phi} R(C)$ (26)
where $R(C) = \{\mathbf{v}^{\mathrm{T}} C: \mathbf{v} > \mathbf{0}\}$.

Consider $E(C)$, a set of p × n matrices where **x** is efficient, i.e.,
$E(C) = \{C: \nexists\, \mathbf{x}^* \in \mathrm{X}: C\mathbf{x}^* \geq C\mathbf{x}, C\mathbf{x}^* \neq C\mathbf{x}\}$. (27)

If RN(Φ) is not empty, then as suggested by (Inuiguchi and Sakawa, 1996) and proved in (Inuiguchi, 2013):
$\mathbf{x} \in \mathrm{N}_{\mathrm{E}} \Leftrightarrow \mathrm{RN}(\Phi) \cap \mathrm{P}(\mathbf{x}) \neq \emptyset$ (28)
$\mathbf{x} \in \mathrm{P}_{\mathrm{E}} \Leftrightarrow \mathrm{RP}(\Phi) \cap \mathrm{P}(\mathbf{x}) \neq \emptyset$ (29)

Consider, for the k-th objective function, a closed and convex hyperbox Φ^{k} with 2^{n} extreme points (where n is the number of interval coefficients and, without loss of generality, all the coefficients are defined as intervals).

Let $\hat{\mathrm{F}}$ be a matrix of $\mathrm{p} \times 2^{\mathrm{n}}$ objective functions, since (22) has p objective interval functions. Namely, $\hat{\mathrm{F}}$ is a $(\mathrm{p} \times 2^{\mathrm{n}}) \times \mathrm{n}$ matrix.
A solution $\mathbf{x}^*$ is weakly efficient with respect to $\hat{\mathrm{F}}$ if and only if there exists a $(\mathrm{p} \times 2^{\mathrm{n}})$-dimensional weight vector $\hat{\mathbf{w}} \in \mathrm{R}^{\mathrm{p}\times 2^{\mathrm{n}}}$, $\hat{\mathbf{w}} \geq \mathbf{0}$, $\sum_{\mathrm{t}=1}^{\mathrm{p}\times 2^{\mathrm{n}}} \hat{w}_{\mathrm{t}} = 1$, such that there is an $\mathbf{x}^*$ that optimizes the following LP problem:

$$\max \hat{\mathrm{z}}(\mathbf{x}) = \hat{\mathbf{w}}^{\mathrm{T}} \hat{\mathrm{F}}\, \mathbf{x},$$
$$\mathrm{s.t.}\, \mathbf{x} \in \mathrm{X}. \tag{30}$$

Let $\mathrm{X}(\hat{\mathbf{w}}^{\mathrm{T}}\hat{\mathrm{F}})$ be the set of optimal solutions to problem (30) then, we have
$$\mathrm{P_E} = \bigcup \left\{ \mathrm{X}(\hat{\mathbf{w}}^{\mathrm{T}}\hat{\mathrm{F}}) : \hat{\mathbf{w}} > \mathbf{0}, \sum_{\mathrm{t}=1}^{\mathrm{p}\times 2^{\mathrm{n}}} \hat{\mathrm{w}}_{\mathrm{t}} = 1 \right\}. \tag{31}$$

From (28) and (29), consider the following solution sets for Problem (22) (Inuiguchi, 2013):

$$\mathrm{PW_E} = \bigcup \left\{ \mathrm{X}(\mathbf{v}^{\mathrm{T}} C) : \mathbf{v} = (\mathrm{v}_1, \dots, \mathrm{v}_{\mathrm{p}})^{\mathrm{T}} > \mathbf{0}, C \in \Phi \right\}, \tag{32}$$

$$\mathrm{NW_E} = \bigcup \left\{ \bigcap \{ \mathrm{X}(\mathbf{v}^{\mathrm{T}} C) : C \in \Phi \} : \mathbf{v} = (\mathrm{v}_1, \dots, \mathrm{v}_{\mathrm{p}})^{\mathrm{T}} > \mathbf{0} \right\} \tag{33}$$

Inuiguchi (2013) established the relation between solution sets $\mathrm{PW_E}$ and $\mathrm{P_E}$ and $\mathrm{NW_E}$ and $\mathrm{N_E}$, respectively, as

$$\mathrm{P_E} = \mathrm{PW_E}, \tag{34}$$
$$\mathrm{N_E} \supseteq \mathrm{NW_E} \tag{35}$$

Inuiguchi (2013) concluded that the weighted sum method does not perform well for the characterization of necessarily efficient solutions. This result is different from the conventional multiple objective linear programming. Similar results are also obtained with reference point methods (Wierzbicki, 1980).
We note that

$$\mathrm{NW_E} = \bigcup \left\{ N_0(\mathbf{v}^{\mathrm{T}}\Phi) : \mathbf{v} = (\mathrm{v}_1, \dots, \mathrm{v}_{\mathrm{p}})^{\mathrm{T}} > \mathbf{0} \right\},$$

Where $\mathbf{v}^{\mathrm{T}}\Phi = \{\mathbf{v}^{\mathrm{T}} C : C \in \Phi\}$. Namely, (35) implies

$$\exists \mathbf{v} = (\mathrm{v}_1, \dots, \mathrm{v}_{\mathrm{p}})^{\mathrm{T}} > \mathbf{0},\ \mathbf{v}^{\mathrm{T}}\Phi \subseteq \mathrm{P}(\mathbf{x}) \text{ implies } \mathbf{x} \in \mathrm{N_E}. \tag{36}$$

Consider the following set of objective function coefficients (Inuiguchi, 2013):

$$\Lambda(\Phi) = \bigcap \{ \Lambda : \Lambda \text{ is a convex cone, and } \forall C \in \Phi, \mathrm{R}(C) \cap \Lambda \neq \emptyset \} \tag{37}$$

When $\Lambda(\Phi)$ is not empty, we obtain (Inuiguchi, 2013):
$$\mathbf{x} \in \mathrm{N_E} \Leftrightarrow \Lambda(\Phi) \subseteq \mathrm{P}(\mathbf{x}) \tag{38}$$

Consider Uni $= \{ \boldsymbol{c} = (\mathrm{c}_1, \mathrm{c}_2, \dots, \mathrm{c}_{\mathrm{n}})^{\mathrm{T}} : \sum_{\mathrm{j}=1}^{\mathrm{n}} |c_{\mathrm{j}}| = 1 \}$.
The relations between $\mathrm{RN}(\Phi)$ and $\Lambda(\Phi)$ are established and proven in (Inuiguchi, 2013), i.e.:

Theorem 6
i) $\mathrm{RN}(\Phi) = \varnothing$ if $\Lambda(\Phi) \cap \mathrm{Uni}$ is neither an empty set nor a singleton.
ii) $\Lambda(\Phi) = \varnothing$ if $\mathrm{RN}(\Phi) \cap \mathrm{Uni}$ is neither an empty set nor a singleton.
iii) $\mathrm{RN}(\Phi) \cap \mathrm{Uni}$ is a singleton if and only if $\Lambda(\Phi) \cap \mathrm{Uni}$ is a singleton, and additionally we have $\Lambda(\Phi) = \mathrm{RN}(\Phi)$.

We note that the assertions in Inuiguchi (2013) corresponding to Theorem 6 are wrong but proof implies Theorem 6. The same mistake is found in Inuiguchi et al. (2016).

Consider the conic hull $\mathrm{Cone}\,(\Phi^k)$ of Φ^k defined by $\mathrm{Cone}\,(\Phi^k) = \{\boldsymbol{\lambda}^T \mathbf{c}_k : \boldsymbol{\lambda} \geq 0, \mathbf{c}_k \in \Phi^k\}$, k=1, 2, ..., p and $\mathbf{c}_k$ is an n-dimensional vector representing the coefficients of k-th objective function of $C\mathbf{x}$ with $C \in \Phi$.

Then, we have the following properties:

Theorem 7. We obtain the following relations:

$$\Lambda(\Phi) - \{\mathbf{0}\} = \bigcap_{k=1,2,\ldots,p} \mathrm{Cone}(\Phi^k) - \{\mathbf{0}\}, \tag{39}$$

$$\mathrm{RP}(\Phi) \supseteq \bigcup_{k=1,2,\ldots,p} \mathrm{Int}(\mathrm{Cone}(\Phi^k)), \tag{40}$$

Where $\mathrm{Int}(Z)$ is the interior of a set $Z \subseteq \mathbf{R}^p$.

Proof: Consider any $C \in \Phi$. Let $\mathbf{v}^{\varepsilon} = (v_1^{\varepsilon}, v_2^{\varepsilon}, .., v_p^{\varepsilon})^T > \mathbf{0}$ be a p-dimensional vector defined by $v_k^{\varepsilon} = 1 - (p-1)\varepsilon$, $v_j^{\varepsilon} = \varepsilon$, $j \neq k$ with a very small positive number ε. Then, $\mathbf{v}^{\varepsilon T} C \in \mathrm{R}(C)$ approaches to the k-th row $\mathbf{c}_k \in \Phi^k$ of C as $\varepsilon > 0$ approaches to zero. This implies $\mathrm{Cl}(\mathrm{R}(C)) \cap \mathrm{Cone}(\Phi^k) \neq \emptyset$ because Φ^k is closed, where $\mathrm{Cl}(\mathrm{R}(C))$ is the closure of $\mathrm{R}(C)$. Therefore, we obtain that for each $k \in \{1, \ldots, p\}$, for any convex cone Λ such that $\mathrm{Int}(\Lambda) \supseteq \mathrm{Cone}(\Phi^k) - \{\mathbf{0}\}$, $\forall C \in \Phi$, $\mathrm{R}(C) \cap \Lambda \neq \emptyset$ is satisfied because we assume $\mathbf{0} \notin \Phi^k$, k=1, …, p. Moreover, because $\mathrm{Cone}(\Phi^k) - \{\mathbf{0}\}$ is a convex cone, we have

$$\mathrm{Cone}(\Phi^k) - \{\mathbf{0}\} = \bigcap \{\Lambda : \Lambda \text{ is a convex cone such that } \mathrm{Int}(\Lambda) \supseteq \mathrm{Cone}(\Phi^k) - \{\mathbf{0}\}\}.$$

Accordingly, for each $k \in \{1, \ldots, p\}$,

$$\mathrm{Cone}(\Phi^k) \supseteq \mathrm{Cone}(\Phi^k) - \{\mathbf{0}\}$$
$$\supseteq \bigcap \{\Lambda : \Lambda \text{ is a convex cone such that } \forall C \in \Phi,\ \mathrm{R}(C) \cap \Lambda \neq \emptyset\} = \Lambda(\Phi).$$

Hence, we obtain $\Lambda(\Phi) \subseteq \bigcap_{k=1,2,\ldots,p} \mathrm{Cone}(\Phi^k)$. This implies $\Lambda(\Phi) - \{\mathbf{0}\} \subseteq \bigcap_{k=1,2,\ldots,p} \mathrm{Cone}(\Phi^k) - \{\mathbf{0}\}$.
On the other hand, let us assume that $\bigcap_{k=1,2,\ldots,p} \mathrm{Cone}(\Phi^k) - \{\mathbf{0}\} \neq \emptyset$. Consider $\boldsymbol{v}_1 \in (\bigcap_{k=1,2,\ldots,p} \mathrm{Cone}(\Phi^k) - \{\mathbf{0}\}) \cap \mathrm{Uni}$. For $\boldsymbol{v}_1$, we obtain $C_1 = (\boldsymbol{v}_1^1, \boldsymbol{v}_1^2, \ldots, \boldsymbol{v}_1^p)^T \in \Phi$ by selecting vector $\boldsymbol{v}_1^k \in \Phi^k$ such that there exists $\mu_1^k > 0$ satisfying $\boldsymbol{v}_1 = \mu_1^k \boldsymbol{v}_1^k$, k=1, 2, …, p. For C_1, we have $R(C_1) = \{\mu \boldsymbol{v}_1 \mid \mu > 0\}$. Then we know that a convex cone Λ should satisfy $R(C_1) \cap \Lambda \neq \varnothing$, or equivalently, $R(C_1) \subseteq \Lambda$ to satisfy $\forall C \in \Phi, \mathrm{R}(C) \cap \Lambda \neq \emptyset$. This implies $\boldsymbol{v}_1 \in \Lambda(\Phi)$. Therefore, we obtain that $\bigcap_{k=1,2,\ldots,p} \mathrm{Cone}(\Phi^k) - \{\mathbf{0}\} \subseteq \Lambda(\Phi)$, i.e., $\bigcap_{k=1,2,\ldots,p} \mathrm{Cone}(\Phi^k) - \{\mathbf{0}\} \subseteq \Lambda(\Phi) - \{\mathbf{0}\}$ when $\bigcap_{k=1,2,\ldots,p} \mathrm{Cone}(\Phi^k) - \{\mathbf{0}\} \neq \emptyset$. Hence, we obtain (39).
(40) can be proved from the fact that for any $C \in \Phi$ such that its k-th row $\mathbf{c}_k^T$ is included in $\mathrm{Int}(\Phi^k)$, $\mathbf{v}^T C$ is included in $\mathrm{Int}(\Phi^k)$ by selecting $\mathbf{v} = (v_1, v_2, \ldots, v_p)^T > \mathbf{0}$ such that $v_k >> v_j$, $j \neq k$ and $\sum_{j=1}^{p} v_j = 1$.

From Theorem 7, we know that $\Lambda(\Phi) \neq \emptyset$ can be confirmed by the consistency of the following linear system of inequalities with variable vectors $\boldsymbol{\lambda}$ and $\boldsymbol{v} = \left(v_1 , v_2 , \dots , v_n \right)^{\mathrm{T}}$:

$$\begin{aligned}
&\lambda_k \boldsymbol{c}_k^{\mathrm{L}} \leq \boldsymbol{v} \leq \lambda_k \boldsymbol{c}_k^{\mathrm{U}}, \mathrm{k} = 1, 2, \dots, \mathrm{p}, \\
&\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{\mathrm{p}})^{\mathrm{T}} \geq \mathbf{0}, \\
&\textstyle\sum_{j=1}^{n} v_j = 1, v_j \geq 0,\ \mathrm{j} = 1, 2, \dots, \mathrm{n}.
\end{aligned} \tag{41}$$

The existence and uniqueness of the solution to the above system of inequalities can be examined by a linear programming technique. Moreover, from Theorem 6, we know also the existence of $\mathrm{RN}(\Phi)$ because the consistency of the above system of inequalities implies the existence of $\Lambda(\Phi) \cap \mathrm{Uni}$.

We note that constraints $\sum_{j=1}^{n} v_j = 1,\ v_j \geq 0,\ \mathrm{j} = 1, 2, \dots, \mathrm{n}$ of (41) guarantee that $= \left(v_1 , v_2 , \dots , v_n \right)^{\mathrm{T}} \neq \mathbf{0}$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{\mathrm{p}})^{\mathrm{T}} \geq \mathbf{0}$. The set of solutions satisfying (41) is closed so that a linear programming technique is easily applied.

When $\mathrm{RN}(\Phi) \neq \emptyset$ is not empty, the necessary efficiency can be tested by the possible optimality with objective coefficient vector set $\mathrm{RN}(\Phi)$. When $\Lambda(\Phi)$ has non-zero elements, the necessary efficiency can be tested by the necessary optimality with objective coefficient vector set $\Lambda(\Phi)$. We note that $\Lambda(\Phi) \subseteq \mathrm{Cone}(\Phi^{\mathrm{k}})$, k=1, 2,…, p and testing $\Lambda(\Phi) \subseteq \mathrm{P}(\mathbf{x})$ is a much weaker condition than $\mathbf{v}^{\mathrm{T}}\Phi \subseteq \mathrm{P}(\mathbf{x})$ for any $\mathbf{v} = (\mathrm{v}_1, \dots, \mathrm{v}_{\mathrm{p}})^{\mathrm{T}} > \mathbf{0}$.

Let $\vartheta^{\mathrm{k}}\,(\boldsymbol{c}'_{\mathrm{k}}\!: \mathbf{c}_{k}^{\mathrm{C}}) = \{\mathbf{c} = (\mathrm{c}_1, \mathrm{c}_2, \dots, \mathrm{c}_{\mathrm{n}})^{\mathrm{T}}\!: \mathrm{c}_{\mathrm{kj}}^{\mathrm{C}} - \tau_{\mathrm{kj}}^{\mathrm{L}}\mathrm{c}'_{\mathrm{kj}} \leq \mathrm{c}_{\mathrm{kj}} \leq \mathrm{c}_{\mathrm{kj}}^{\mathrm{C}} + \tau_{\mathrm{kj}}^{\mathrm{U}}\mathrm{c}'_{\mathrm{kj}}, \mathrm{j} = 1,2, \dots, \mathrm{n}\}$, where $\tau_{\mathrm{kj}}^{\mathrm{L}}$ and $\tau_{\mathrm{kj}}^{\mathrm{U}}$ are the lower and upper tolerance bounds of objective function coefficient c_{kj} obtained by Filippi’s method with respect to objective function coefficient vector of $\mathbf{x}$ to the following problem:

$$\begin{aligned}
&\max \mathrm{z}(\mathbf{x}) = (\mathbf{c}_{\mathrm{k}}^{\mathrm{C}})^{\mathrm{T}}\, \mathbf{x}, \\
&\mathrm{s.t.}\, \bar{\mathrm{A}}\bar{\mathbf{x}} = \mathbf{b}, \bar{\mathbf{x}} = (\mathbf{x}^{\mathrm{T}}, \mathbf{s})^{\mathrm{T}} \geq \mathbf{0}.
\end{aligned} \tag{42}$$

We note that we can extend theorems in what follows to cases using tolerance bounds with respect to any objective function coefficient vector $\mathbf{v}^{\mathrm{T}}C, C \in \Phi, \mathbf{v} > \mathbf{0}$. For the sake of simplicity, we describe them with respect to an objective function coefficient vector $\mathbf{c}_{\mathrm{k}}^{\mathrm{C}}$.

Theorem 8. Let $\mathbf{x}^*$ be an optimal basic solution to the LP problem (42) which is not degenerate for some $\mathrm{k} \in \{1, \dots, \mathrm{p}\}$. If $\exists \mathbf{v} = (\mathrm{v}_1, \dots, \mathrm{v}_{\mathrm{p}})^{\mathrm{T}} > \mathbf{0}$, $\mathbf{v}^{\mathrm{T}}\Phi \subseteq \vartheta^{\mathrm{k}}\,(\boldsymbol{c}'_{\mathrm{k}}\!: \mathbf{c}_{k}^{\mathrm{C}})$ then $\mathbf{x}^* \in \mathrm{N}_{\mathrm{E}}$.

Proof: It is straightforwardly obtained from (42) and $\vartheta^{\mathrm{k}}\,(\boldsymbol{c}'_{\mathrm{k}}\!: \mathbf{c}_{k}^{\mathrm{C}}) \subseteq \mathrm{P}(\mathbf{x}^*)$.

Theorem 9. Let $\bar{\mathbf{x}}^*$ be an optimal basic solution to an LP problem (42) which is neither degenerate nor a unique optimal solution to that LP problem. Let $\mathbf{x}^*$ be the optimal solution to problem $\max\limits_{x \in X} \boldsymbol{c}_k^{\mathrm{C}^{\mathrm{T}}} \mathbf{x}$ corresponding to $\bar{\mathbf{x}}^*$. Assume $\Lambda(\Phi) \neq \emptyset$ and consider $\widehat{\Lambda}$ such that $\mathrm{Cone}(\widehat{\Lambda}) \supseteq \Lambda(\Phi)$. Then if $\widehat{\Lambda} \cap \mathrm{P}(\mathbf{x}^*) = \emptyset$, $\mathbf{x}^*$ is strictly possibly efficient to IMOLP problem (22), i.e. $\mathbf{x}^* \in \mathrm{P}_{\mathrm{SE}}$.

Proof: $\widehat{\Lambda} \cap \mathrm{P}(\mathbf{x}^*) = \varnothing$ and $\widehat{\Lambda} \supseteq \Lambda(\Phi)$ imply $\Lambda(\Phi) \not\subset \mathrm{P}(\mathbf{x}^*)$. From the contraposition of (38), we conclude that $\mathbf{x}^* \notin \mathrm{N_E}$.

When $\Lambda(\Phi) \neq \varnothing$, we may define $\widehat{\Lambda} = \Phi^{\mathrm{k}}$ for any $\mathrm{k} \in \{1,2, \ldots, \mathrm{p}\}$.

Example 2. Now, consider the following interval MOLP problem:

$\max\, [1, 2.5]\mathrm{x}_1 + [3, 4]\mathrm{x}_2,$
$\max\, [2, 3]\mathrm{x}_1 + [1.5, 2.5]\mathrm{x}_2,$
$\mathrm{s.t.}\, 3\mathrm{x}_1 + 4\mathrm{x}_2 \leq 42,$
$3\mathrm{x}_1 + \mathrm{x}_2 \leq 24,$
$\mathrm{x}_1 \geq 0, 0 \leq \mathrm{x}_2 \leq 9.$

In this case, $\Phi = \{C = (c_{\mathrm{ij}}): 1 \leq c_{11} \leq 2.5, 3 \leq c_{12} \leq 4, 2 \leq c_{21} \leq 3, 1.5 \leq c_{22} \leq 2.5\}$ and $\mathrm{RP}(\Phi) = \{\boldsymbol{c}: \boldsymbol{c} = \mathrm{k}_1(1, 4) + \mathrm{k}_2(3, 1.5), \mathrm{k}_1 > 0,\ \mathrm{k}_2 > 0\}$.
Solving Problem (22) when $c_{11}^{\mathrm{C}} = 1.75, c_{12}^{\mathrm{C}} = 3.5$, we obtain $(2, 9)^{\mathrm{T}}$ as an optimal solution. The set according to which $(2, 9)^{\mathrm{T}}$ is the solution that maximizes the linear objective function $\boldsymbol{c}\mathbf{y}$ with $\mathbf{y} \in \{\mathbf{x}: 3\mathrm{x}_1 + 4\mathrm{x}_2 \leq 42, 3\mathrm{x}_1 + \mathrm{x}_2 \leq 24, \mathrm{x}_1 \geq 0, 0 \leq \mathrm{x}_2 \leq 9\}$ can be represented by: $\mathrm{P}((2, 9)^{\mathrm{T}}) = \{\boldsymbol{c}: \boldsymbol{c} = \mathrm{k}_1(3, 4) + \mathrm{k}_2(0, 1), \mathrm{k}_1 > 0,\ \mathrm{k}_2 > 0\}$.

Through the observation of Figure 5 it is possible to conclude that $\mathrm{RP}(\Phi) \cap \mathrm{P}((2, 9)^{\mathrm{T}}) \neq \varnothing$. Hence, solution $(2, 9)^{\mathrm{T}}$ is efficient for at least one $C \in \Phi$, being a possibly efficient solution. In addition, according to the Filippi's tolerance approach solution $(2, 9)^{\mathrm{T}}$ is optimal within $\vartheta^1\, (\boldsymbol{c}_1': \mathbf{c}_1^{\mathrm{C}}) = \{c_{1\mathrm{j}}: 0 \leq c_{11} \leq 2.1, 2.8 \leq c_{12} \leq +\infty\}$. Therefore, since $\Lambda(\Phi) \not\subset \mathrm{P}(\mathbf{x}^*)$, then $\mathbf{x}^*$ is strictly possibly efficient to IMOLP problem (22). Figure 5 also corroborates Theorem 9. Finally, Theorem 6 is also illustrated in this example because we have $\Lambda(\Phi) \cap$ Uni is neither empty set nor a singleton and thus $\mathrm{RN}(\Phi) = \varnothing$.

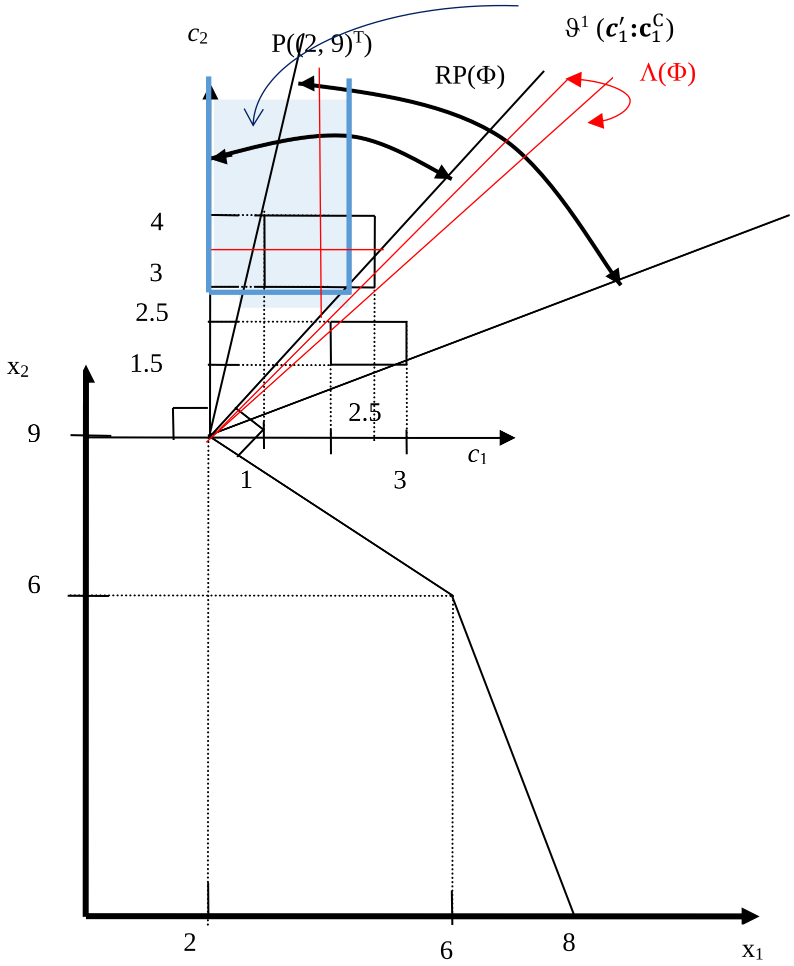

Figure 5. An illustrative example of a possibly efficient solution which corroborates Theorem 9.

Example 3. Consider the following interval MOLP problem:

$\max\,[6,9]x_1 + [4.5,8]x_2,$
$\max\,[-4.5,-1.5]x_1 + [7.5,10.5]x_2,$
$\text{s.t.}\ 3x_1 + 4x_2 \leq 42,$
$3x_1 + x_2 \leq 24,$
$x_1 \geq 0, 0 \leq x_2 \leq 9.$

Solving Problem (22) when $c_{11}^{C} = 7.5, c_{12}^{C} = 6.25$, we obtain $(6,6)^{T}$ as an optimal solution.

As shown in Figure 6, $\mathrm{RN}(\Phi)\cap\mathrm{P}((6,6)^{T}) = \varnothing$ and thus $(6,6)^{T}$ is not a necessarily efficient solution although it is a unique necessarily optimal solution to the ILP with the first objective function of Example 3. Indeed, $(6,6)^{T}$ is not efficient with respect to $\begin{bmatrix} 6 & 8 \\ -4 & 8 \end{bmatrix} \in \Phi$.

Furthermore, according to the Filippi's approach solution $(6,6)^{T}$ is optimal within $\vartheta^{1}\,(\boldsymbol{c}_1' : \mathbf{c}_1^{C}) = \{c_{1j}: 5.769 \leq c_{11} \leq 10.714, 3.571 \leq c_{12} \leq 7.692\}$.

On the other hand, Theorem 6 is also illustrated since $\mathrm{RN}(\Phi)\cap$ Uni is neither an empty set nor a singleton and thus $\Lambda(\Phi) = \varnothing$.

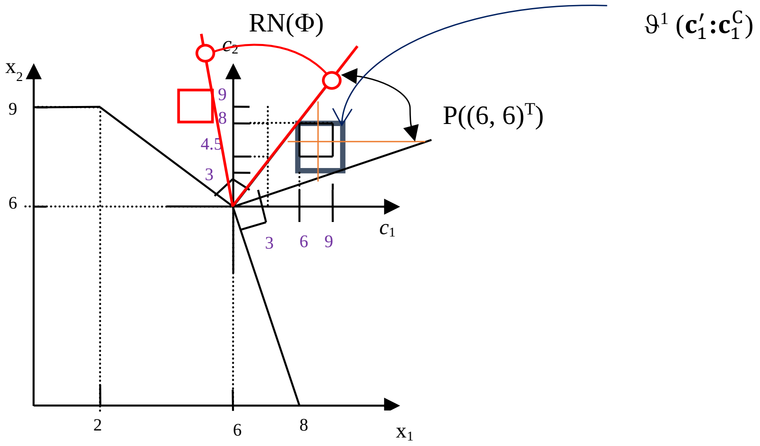

Figure 6. A necessarily optimal solution which is not a necessarily efficient solution.

Therefore, a unique necessarily optimal solution might not correspond to a necessarily efficient solution.

Figures 5 and 6 illustrate possibly efficient solutions; however, from the point of view of robustness, a possibly efficient solution to the IMOLP problem which is also a necessarily optimal solution for at least one of its objective functions is better than an efficient solution to the IMOLP which is only a possibly optimal solution to one of its objective functions.

Finally, Figure 6 also demonstrates that $\Phi^{\mathrm{k}} \subseteq \vartheta^{\mathrm{k}}(\boldsymbol{c}_{\mathrm{k}}^{\prime}:\mathbf{c}_k^{\mathrm{C}})$ is not sufficient for $\mathbf{x}^* \in \mathrm{N_E}$.

Theorem 10. Let $\bar{\mathbf{x}}^*$ be an optimal basic solution to problem (42) for some $\mathrm{k} \in \{1, \dots, \mathrm{p}\}$ which is not degenerate. Let $\mathbf{x}^*$ be the optimal solution to problem $\max\limits_{x \in X} \boldsymbol{c}_k^{\mathrm{C}^{\mathrm{T}}} \mathbf{x}$ corresponding to $\bar{\mathbf{x}}^*$. If $\Lambda(\Phi) \neq \varnothing$ and $\Lambda(\Phi) \subseteq \mathrm{cone}(\vartheta^{\mathrm{k}}(\boldsymbol{c}_{\mathrm{k}}^{\prime}:\mathbf{c}_k^{\mathrm{C}}))$ holds, then $\mathbf{x}^* \in \mathrm{N_E}$.

Proof: Since $\vartheta^{\mathrm{k}}(\boldsymbol{c}_{\mathrm{k}}^{\prime}:\mathbf{c}_k^{\mathrm{C}}) \subseteq \mathrm{P}(\mathbf{x}^*)$ holds and $\mathrm{P}(\mathbf{x}^*)$ is a convex cone, we have $\mathrm{cone}\left(\vartheta^{\mathrm{k}}(\boldsymbol{c}_{\mathrm{k}}^{\prime}:\mathbf{c}_k^{\mathrm{C}})\right) \subseteq \mathrm{P}(\mathbf{x}^*)$. Accordingly, $\Lambda(\Phi) \subseteq \mathrm{cone}(\vartheta^{\mathrm{k}}(\boldsymbol{c}_{\mathrm{k}}^{\prime}:\mathbf{c}_k^{\mathrm{C}}))$ implies $\Lambda(\Phi) \subseteq \mathrm{P}(\mathbf{x}^*)$. When $\Lambda(\Phi) \neq \varnothing$, from (38), we have $\mathbf{x}^* \in \mathrm{N_E} \Leftrightarrow \Lambda(\Phi) \subseteq \mathrm{P}(\mathbf{x}^*)$. Hence, we obtain $\mathbf{x}^* \in \mathrm{N_E}$.

Theorem 11. Let $\bar{\mathbf{x}}^*$ be an optimal basic solution to problem (42) for some $\mathrm{k} \in \{1, \dots, \mathrm{p}\}$ which is not degenerate. Let $\mathbf{x}^*$ be the optimal solution to problem $\max\limits_{x \in X} \boldsymbol{c}_k^{\mathrm{C}^{\mathrm{T}}} \mathbf{x}$ corresponding to $\bar{\mathbf{x}}^*$. If $\Phi^{\mathrm{k}} \subseteq \mathrm{Int}\,(\vartheta^{\mathrm{k}}(\boldsymbol{c}_{\mathrm{k}}^{\prime}:\mathbf{c}_k^{\mathrm{C}}))$, then $\mathbf{x}^* \in \mathrm{N_E}$.

Proof: Since we have $\vartheta^{\mathrm{k}}(\boldsymbol{c}_{\mathrm{k}}^{\prime}:\mathbf{c}_k^{\mathrm{C}}) \subseteq \mathrm{P}(\mathbf{x}^*)$ by the definition of $\vartheta^{\mathrm{k}}(\boldsymbol{c}_{\mathrm{k}}^{\prime}:\mathbf{c}_k^{\mathrm{C}})$, $\Phi^{\mathrm{k}} \subseteq \mathrm{Int}(\vartheta^{\mathrm{k}}(\boldsymbol{c}_{\mathrm{k}}^{\prime}:\mathbf{c}_k^{\mathrm{C}}))$ implies $\Phi^{\mathrm{k}} \subseteq \mathrm{Int}(\mathrm{P}(\mathbf{x}^*))$. For $\mathbf{v} = (\mathrm{v}_1, \dots, \mathrm{v}_{\mathrm{p}})^{\mathrm{T}} > \mathbf{0}$ such that $\mathrm{v_k} >> \mathrm{v_j}$, for all $\mathrm{j} \neq \mathrm{k}$ and $\sum_{\mathrm{j}=1}^{\mathrm{p}} \mathrm{v_j} = 1$, we have $\mathbf{v}^{\mathrm{T}}\Phi \cong \Phi^{\mathrm{k}}$. Therefore, we can have $\mathbf{v}^{\mathrm{T}}\Phi \subseteq \mathrm{P}(\mathbf{x}^*)$. From (36), we obtain $\mathbf{x}^* \in \mathrm{N_E}$.

Example 4. Consider the following interval MOLP problem:

$\max\ [6, 9]\mathrm{x}_1 + [4, 5]\mathrm{x}_2,$

$\max\ [2,\ 4]x_1\ +\ [-1, 2.5]x_2,$
$\text{s.t.}\ 3x_1\ +\ 4x_2 \leq 42,$
$3x_1\ +\ x_2 \leq 24,$
$x_1 \geq 0, 0 \leq x_2 \leq 9.$

The interval coefficients of the first and second objective functions are depicted in Figure 7.
Solving Problem (22) when $c_{11}^{C} = 7.5, c_{12}^{C} = 4.5,$ we obtain $(6, 6)^{T}$ as an optimal solution.
Futhermore, according to the Filippi's approach solution $(6, 6)^{T}$ is optimal within
$\vartheta^{1}\ (\boldsymbol{c}_1' : \mathbf{c}_1^{C}) = \{c_{1j}: 4.655\ \leq c_{11} \leq\ 9.643, 3.214 \leq c_{12} \leq\ 6.207\}.$

Since $\Lambda(\Phi) \neq \emptyset$ and $\Lambda(\Phi) \subseteq \text{cone}(\vartheta^{k}(\mathbf{c}_1' : \mathbf{c}_1^{C})) = P(6,6)^{T}$, this solution is a necessarily efficient solution. Figure 7 corroborates Theorem 10. Additionally, since $\Phi^{k} \subseteq \text{Int}\ (\vartheta^{k}(\boldsymbol{c}_{k}' : \mathbf{c}_{k}^{C}))$ Theorem 11 is also illustrated. Then again, Theorem 6 is also depicted since $\Lambda(\Phi) \cap \text{Uni}\ \neq \emptyset$ neither an empty set nor a singleton and thus $\text{RN}(\Phi) = \emptyset$.

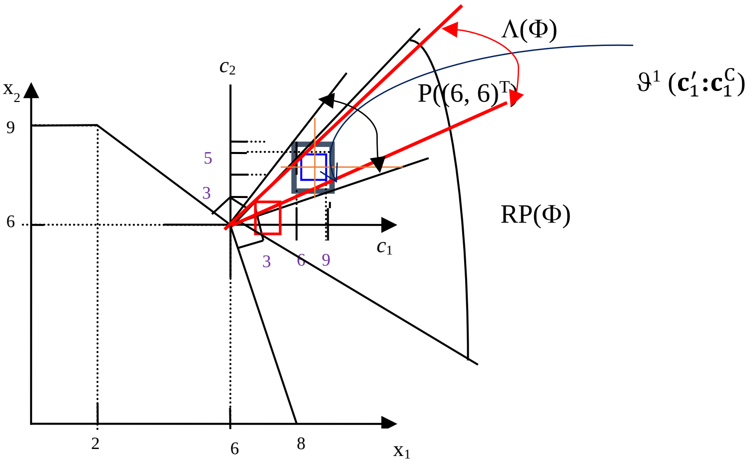

Figure 7. A necessarily efficient solution with $\Lambda(\Phi) \neq \emptyset$.

5. A new approach to obtain possibly efficient and necessarily efficient solutions

Additive and multiplicative tolerances can be used as a sufficient condition in testing the necessary efficiency in interval MOLP (Hladík, 2008; Hladík, 2012). Checking possible efficiency can be done effectively (in polynomial time); nevertheless, necessary efficiency is hard to check (Oliveira and Antunes, 2007); (Hladík, 2012). Based on those observations, we consider the following procedure for checking the sufficient conditions of necessary efficiency/strict possible efficiency of basic solutions investigated in this paper, where the sufficient conditions can be checked by linear programming techniques although we do not write the details here. We note that the conditions with RN(Φ) are discarded because their tests may require a computationally heavy task.

In this section, we consider nondegenerate optimal basic solutions to problem (42) and describe an algorithm for testing their necessary efficiency in problem (22).

An algorithm for testing necessary efficiency in problem (22) of nondegenerate optimal basic solutions to problem (42)

Step 1:
Obtain basic optimal solutions to problem (42) which are not degenerate for each $k \in \{1, \dots, p\}$. Set each individual optimal solution as $\bar{\mathbf{x}}^k$ for $k \in \{1, \dots, p\}$. Let $\mathbf{x}^k$ be the optimal solution to problem $\max_{x \in X} \mathbf{c}_k^{C^T} \mathbf{x}$ corresponding to $\bar{\mathbf{x}}^k$. Set $k = 1$, we apply the following procedure for each $k \in \{1, \dots, p\}$.

Step 2:
Let $\breve{N}_E = \varnothing$ and $\breve{P}_{SE} = \varnothing$. Check whether $\Lambda(\Phi)$ is empty or not (from the consistency of the linear system of inequalities (41)).

Step 3:
If k>p, terminate the procedure obtaining $\breve{N}_E$ and $\breve{P}_{SE}$ as the set of found necessarily efficient solutions and the set of found strictly possibly optimal solutions, respectively. Otherwise, specify $\boldsymbol{c}'_k = (c'_{k1}, c'_{k2}, \dots, c'_{kn})^T$ and obtain the intervals $\vartheta^k = \{c_{kj}: c_{kj}^C - \tau_{kj}^L c'_{kj} \le c_{kj} \le c_{kj}^C + \tau_{kju}^U c'_{kj}, \text{for all j}\}$ for each solution $\mathbf{x}^k$. If $\Lambda(\Phi) \neq \varnothing$, go to Step 4; otherwise, go to Step 5.

Step 4:
Check sufficient conditions for $\Lambda(\Phi) \subseteq P(\mathbf{x}^k)$, i.e., $\mathbf{x}^k \in N_E$, such as $\Lambda(\Phi) \subseteq \vartheta^k$. If one of the sufficient conditions for $\Lambda(\Phi) \subseteq P(\mathbf{x}^k)$ is satisfied, we know $\mathbf{x}^k$ is a necessarily efficient solution, then update $\breve{N}_E = \breve{N}_E \cup \{\mathbf{x}^k\}$. Otherwise, check sufficient conditions for $\Lambda(\Phi) \not\subset P(\mathbf{x}^k)$ such as $\widehat{\Lambda} \cap P(\mathbf{x}^k) = \varnothing$ with some $\widehat{\Lambda}$ such that $\mathrm{Cone}(\widehat{\Lambda}) \supseteq \mathrm{Cone}(\Lambda(\Phi))$ (Theorem 9) and $\Phi^k \not\subset \overline{\left[c_j^{Lsens}, c_j^{Usens}\right]}$ which implies $\Phi^k \not\subset P(\mathbf{x}^k)$ (Corollary 3 and Theorem 9). If one of the sufficient conditions for $\Lambda(\Phi) \not\subset P(\mathbf{x}^k)$ is satisfied, we know $\mathbf{x}^k$ is not a necessarily efficient solution but a possibly optimal solution and update $\breve{P}_{SE} = \breve{P}_{SE} \cup \{\mathbf{x}^k\}$. Return to Step 3.

Step 5:
Check sufficient conditions for $\mathbf{x}^k \in N_E$ such as $\exists \mathbf{v} = (v_1, \dots, v_p)^T > \mathbf{0}$, $\mathbf{v}^T \Phi \subseteq P(\mathbf{x}^k)$ or $\exists \mathbf{v} = (v_1, \dots, v_p)^T > \mathbf{0}$, $\mathbf{v}^T \Phi \subseteq \vartheta^k(\boldsymbol{c}'_k : \mathbf{c}_k^C)$ (Theorem 8) and $\Phi^k \subseteq \mathrm{Int}\,(\vartheta^k(\boldsymbol{c}'_k : \mathbf{c}_k^C))$ (Theorem 11). If one of the sufficient conditions for $\mathbf{x}^k \in N_E$ is satisfied, we know $\mathbf{x}^k$ is a necessarily efficient solution and update $\breve{N}_E = \breve{N}_E \cup \{\mathbf{x}^k\}$. Otherwise, check sufficient conditions for $\mathbf{x}^k \notin N_E$ such as $\exists C \in \Phi$, $\nexists \mathbf{v} = (v_1, \dots, v_p)^T > \mathbf{0}$, $\mathbf{v}^T C \not\in P(\mathbf{x}^k)$. If one of the sufficient conditions for $\mathbf{x}^k \notin N_E$ is satisfied, we know $\mathbf{x}^k$ is not a necessarily efficient solution but a possibly optimal solution and update $\breve{P}_{SE} = \breve{P}_{SE} \cup \{\mathbf{x}^k\}$. Return to Step 1 and set k:=k+1.

In the proposed approach, we assume non degeneracy of $\mathbf{x}^k$. However, if $\mathbf{x}^k$ is degenerate, we can consider its several optimal bases and look for individual tolerances such that at least one of the optimal bases remains optimal, possibly enlarging the tolerances.
Among the several conditions which appear in Step 5, two conditions, in particular, $\exists \mathbf{v} = (v_1, \dots, v_p)^T > \mathbf{0}$, $\mathbf{v}^T \Phi \subseteq \vartheta^k(\boldsymbol{c}'_k : \mathbf{c}_k^C)$ and $\Phi^k \subseteq \mathrm{Int}\,(\vartheta^k(\boldsymbol{c}'_k : \mathbf{c}_k^C))$, can be tested easily after the calculation of $\vartheta^k(\boldsymbol{c}'_k : \mathbf{c}_k^C)$. The latter can be confirmed by the satisfaction of $c_{kj}^C - \tau_{kj}^L c'_{kj} < c_{kj}^L, c_{kj}^U < c_{kj}^C + \tau_{kj}^U c'_{kj}, j = 1, 2, \dots, n$. On the other hand, the former can be examined by a consistency test of the following linear inequalities:

$$\mathrm{c}^{\mathrm{C}}_{\mathrm{kj}} - \tau^{\mathrm{L}}_{\mathrm{kj}}\mathrm{c}'_{\mathrm{kj}} \le \sum_{l=1}^{p} v_l \mathrm{c}^{\mathrm{L}}_{\mathrm{lj}}, \sum_{l=1}^{p} v_l \mathrm{c}^{\mathrm{U}}_{\mathrm{lj}} \le \mathrm{c}^{\mathrm{C}}_{\mathrm{kj}} + \tau^{\mathrm{U}}_{\mathrm{kj}}\mathrm{c}'_{\mathrm{kj}}, \mathrm{j} = 1, 2, \dots, \mathrm{n},$$
$$v_l > 0, l = 1, 2, \dots, \mathrm{p}. \tag{43}$$

By the procedure proposed above, we obtain a set of necessarily optimal solutions as $\breve{\mathrm{N}}_{\mathrm{E}}$ and a set of strictly possibly efficient solutions $\breve{\mathrm{P}}_{\mathrm{SE}}$.
We do not further describe how we can computationally check the conditions employed in the procedure above because it is beyond the scope of this paper.

Example 5. Consider the following interval MOLP problem:
$\max\, [2, 3]\mathrm{x}_1 + [1.5, 2.5]\mathrm{x}_2,$
$\max\, [3, 4]\mathrm{x}_1 + [0.5, 0.8]\mathrm{x}_2,$
$\mathrm{s.t.}\, 3\mathrm{x}_1 + 4\mathrm{x}_2 \le 42,$
$3\mathrm{x}_1 + \mathrm{x}_2 \le 24,$
$\mathrm{x}_1 \ge 0, 0 \le \mathrm{x}_2 \le 9.$

Step 1:
For k =1
Obtain the optimal solution to:
$\max\, 2.5\mathrm{x}_1 + 2\mathrm{x}_2,$
$\mathrm{s.t.}\, 3\mathrm{x}_1 + 4\mathrm{x}_2 \le 42,$
$3\mathrm{x}_1 + \mathrm{x}_2 \le 24,$
$\mathrm{x}_1 \ge 0, 0 \le \mathrm{x}_2 \le 9.$
The optimal solution is $\mathbf{x}^1 = (6, 6)^{\mathrm{T}}$.
For k =2
Obtain the optimal solution to:
$\max\, 3.5\mathrm{x}_1 + 0.65\mathrm{x}_2,$
$\mathrm{s.t.}\, 3\mathrm{x}_1 + 4\mathrm{x}_2 \le 42,$
$3\mathrm{x}_1 + \mathrm{x}_2 \le 24,$
$\mathrm{x}_1 \ge 0, 0 \le \mathrm{x}_2 \le 9.$
The optimal solution is $\mathbf{x}^2 = (8, 0)^{\mathrm{T}}$.

Step 2:
Let $\breve{\mathrm{N}}_{\mathrm{E}} = \emptyset$ and $\breve{\mathrm{P}}_{\mathrm{SE}} = \emptyset$.
Check whether $\Lambda(\Phi)$ is empty or not (from the consistency of the linear system of inequalities (41)). Since the system is inconsistent, $\Lambda(\Phi) = \emptyset$.

Step 3:
For k=1, according to Filippi's method solution $\mathbf{x}^1 = (6, 6)^{\mathrm{T}}$ is optimal within $\vartheta^1\, (\boldsymbol{c}'_1 : \mathbf{c}^{\mathrm{C}}_1) = \{c_{1\mathrm{j}}: 1.9 \le c_{11} \le 3.5, 1.2 \le c_{12} \le 2.5\}$. Since $\Lambda(\Phi) = \emptyset$, go to Step 5.

Step 5:
Since $\Phi^1 \subseteq \mathrm{Int}\ (\vartheta^1)$, then $\mathbf{x}^1$ is necessarily efficient (Theorem 11) and update $\breve{\mathrm{N}}_{\mathrm{E}} = \breve{\mathrm{N}}_{\mathrm{E}} \cup \{\mathbf{x}^1\}$. Go to Step 3.

Step 3:
For k = 2, according to the tolerance region obtained using Filippi's method, solution $\mathbf{x}^2 = (8, 0)^{\mathrm{T}}$ is optimal within $\vartheta^2\, (\boldsymbol{c}'_2 : \mathbf{c}^{\mathrm{C}}_2) = \{c_{2\mathrm{j}}: 2 \le c_{11} \le +\infty, -\infty \le c_{12} \le 0.8333\}$. Go to Step 5.

Step 5:

Since $\Phi^2 \subseteq \mathrm{Int}\,(\vartheta^2)$, then $\mathbf{x}^2$ is necessarily efficient (Theorem 11) and update $\breve{\mathrm{N}}_{\mathrm{E}} = \{\mathbf{x}^1\} \cup \{\mathbf{x}^2\}$ – Figure 8. Return to Step 3.

Step 3:

Since k=2, we terminate the procedure obtaining $\breve{\mathrm{N}}_{\mathrm{E}} = \{\mathbf{x}^1, \mathbf{x}^2\}$ as the set of found of necessarily efficient solutions.

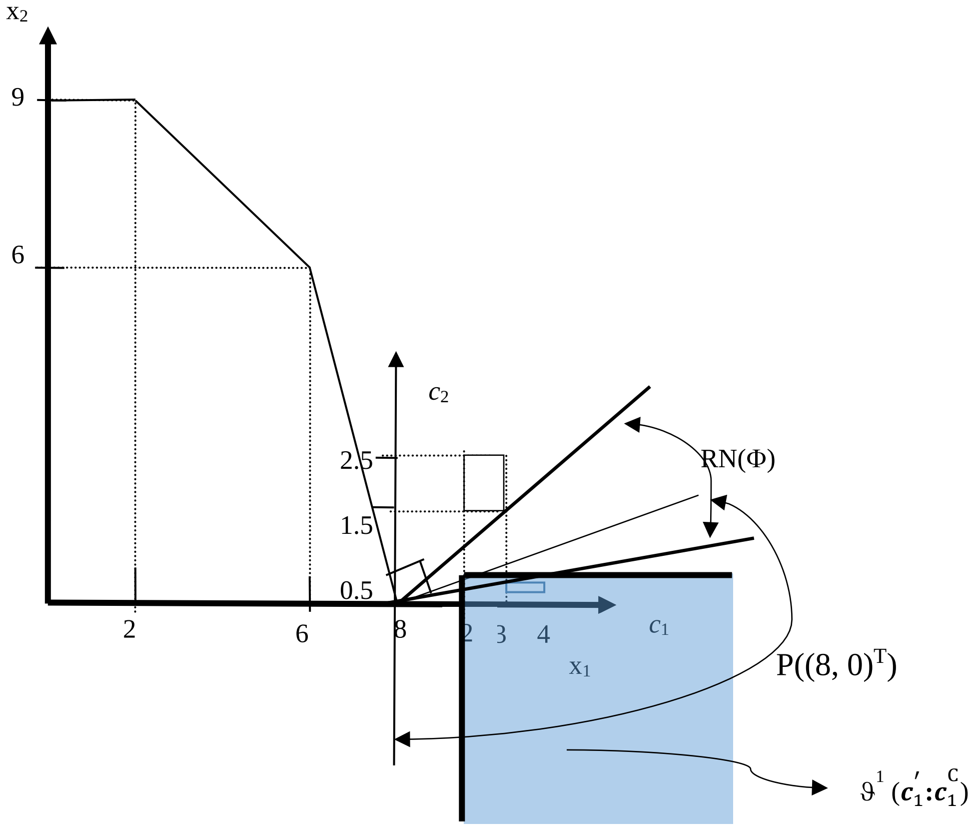

Figure 8. A necessarily efficient solution (Example 5).

For this problem, $\mathrm{N}_{\mathrm{E}} \supseteq \{\mathbf{x}^1, \mathbf{x}^2\}$. From the point of view of robustness, solutions $\mathbf{x}^1$ and $\mathbf{x}^2$ have equally good properties against the uncertainty.

Consider the interval MOLP problem of Example 2:

$\max\,[1, 2.5]\mathrm{x}_1 + [3, 4]\mathrm{x}_2,$

$\max\,[2, 3]\mathrm{x}_1 + [1.5, 2.5]\mathrm{x}_2,$

$\mathrm{s.t.}\,3\mathrm{x}_1 + 4\mathrm{x}_2 \le 42,$

$3\mathrm{x}_1 + \mathrm{x}_2 \le 24,$

$\mathrm{x}_1 \ge 0, 0 \le \mathrm{x}_2 \le 9.$

Step 1:

For k =1

Obtain the optimal solution to:

$\max\,1.75\mathrm{x}_1 + 3.5\mathrm{x}_2,$

$\mathrm{s.t.}\,3\mathrm{x}_1 + 4\mathrm{x}_2 \le 42,$

$3\mathrm{x}_1 + \mathrm{x}_2 \le 24,$

$\mathrm{x}_1 \ge 0, 0 \le \mathrm{x}_2 \le 9.$

The optimal solution is $\mathbf{x}^1 = (2, 9)^\mathrm{T}$
For k =2
Obtain the optimal solution to:
$\max 2.5\mathrm{x}_1 + 2\mathrm{x}_2,$
$\mathrm{s.t.}\, 3\mathrm{x}_1 + 4\mathrm{x}_2 \leq 42,$
$3\mathrm{x}_1 + \mathrm{x}_2 \leq 24,$
$\mathrm{x}_1 \geq 0, 0 \leq \mathrm{x}_2 \leq 9.$
The optimal solution is $\mathbf{x}^2 = (6, 6)^\mathrm{T}$.

Step 2:
Let $\breve{\mathrm{N}}_\mathrm{E} = \varnothing$ and $\breve{\mathrm{P}}_\mathrm{SE} = \varnothing$.
Check whether $\Lambda(\Phi)$ is empty or not (from the consistency of the linear system of inequalities (41)). Since the system is consistent, with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^\mathrm{T} = (0.18, 0.22)^\mathrm{T} \geq \mathbf{0}$, $\boldsymbol{v} = (v_1, v_2)^\mathrm{T} = (0.45, 0.55)^\mathrm{T}$, then $\Lambda(\Phi) \neq \varnothing$.

Step 3:
For k =1, according to the tolerance region obtained using Filippi's method, solution $\mathbf{x}^1 = (2, 9)^\mathrm{T}$ is optimal within $\vartheta^1\ (\boldsymbol{c}_1^\prime\text{:}\, \mathbf{c}_1^\mathrm{C}) = \{c_{1\mathrm{j}}\text{: } 0 \leq c_{11} \leq 2.1,\ 2.8 \leq c_{12} \leq +\infty\}$. Since $\Lambda(\Phi) \neq \varnothing$ go to step 4.

Step 4:
Since $\Phi^1 \not\subset \mathrm{P}(\mathbf{x}^1)$, solution $\mathbf{x}^1 = (2, 9)^\mathrm{T}$ is strictly possibly efficient (Corollary 3 and Theorem 9) and update $\breve{\mathrm{P}}_\mathrm{SE} = \breve{\mathrm{P}}_\mathrm{SE} \cup \{\mathbf{x}^1\}$. Go to Step 3.

Step 3:
For k =2, according to Filippi's tolerance approach solution $\mathbf{x}^2 = (6, 6)^\mathrm{T}$ is optimal within $\vartheta^2\ (\boldsymbol{c}_2^\prime\text{:}\, \mathbf{c}_2^\mathrm{C}) = \{c_{2\mathrm{j}}\text{: } 1.9 \leq c_{21} \leq 3.5,\ 1.2 \leq c_{22} \leq 2.5\}$. Since $\Lambda(\Phi) \neq \varnothing$ go to step 4.

Step 4:
Since $\Phi^2 \subseteq \mathrm{Int}\,(\vartheta^2)$, then $\mathbf{x}^2$ is necessarily efficient (Theorem 11) and update $\breve{\mathrm{N}}_\mathrm{E} = \breve{\mathrm{N}}_\mathrm{E} \cup \{\mathbf{x}^2\}$ – Figure 8. Return to Step 3.

Step 3:
Since k=2, we terminate the procedure obtaining $\breve{\mathrm{P}}_\mathrm{SE} = \{\mathbf{x}^1\}$ and $\breve{\mathrm{N}}_\mathrm{E} = \{\mathbf{x}^2\}$ as the set of found possibly and necessarily efficient solutions, respectively.
For this new problem, $\mathrm{P}_\mathrm{SE} \supseteq \{\mathbf{x}^1\}$ and $\mathrm{N}_\mathrm{E} \supseteq \{\mathbf{x}^2\}$. From the point of view of robustness $\mathbf{x}^2$ is better than $\mathbf{x}^1$.
Moreover, in this case, since $\mathrm{RN}(\Phi) \cap$ Uni is a singleton, $\Lambda(\Phi) \cap$ Uni is a singleton, and additionally we have $\Lambda(\Phi) = \mathrm{RN}(\Phi)$.

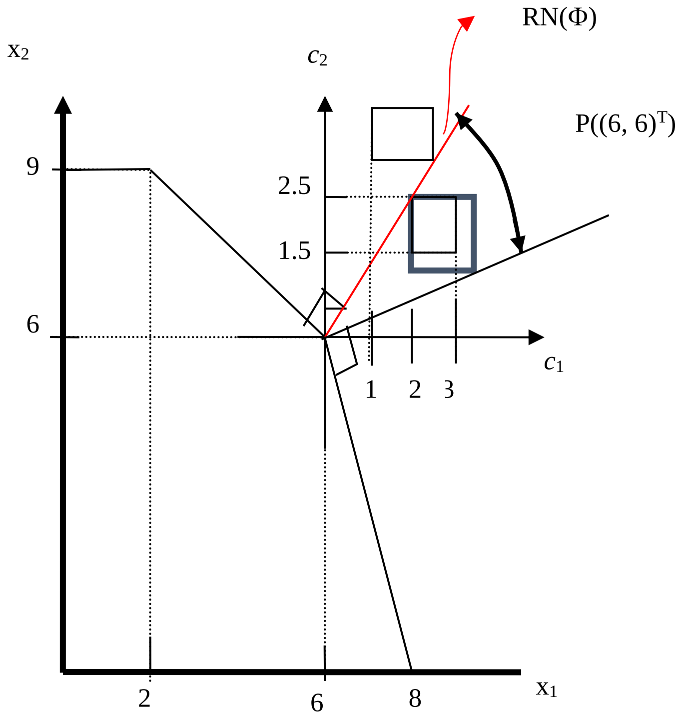

Figure 9. A necessarily efficient solution with $\Lambda(\Phi) = \mathbf{RN}(\Phi)$ (Example 3).

6. Conclusions

This paper establishes sufficient conditions for testing necessarily and possibly efficiency/optimality by combining the results obtained in Inuiguchi (2013) and Inuiguchi et al. (2016) with the tolerance region obtained using Filippi's method and sensitivity analysis. The validity and usefulness of the obtained conditions are confirmed by numerical examples. The examples allow illustrating how these new necessarily and possibly efficiency/optimality conditions bring new insights into this field of research. Moreover, based on these conditions a conceivable procedure for testing necessary efficiency and strict possible efficiency is given. By the procedure, some of nondegenerate basic solutions optimizing the centre values of individual interval objective functions are classified into a set of necessarily efficient solutions and a set of strictly possibly efficient solutions efficiently, because we do not employ a branch and bound method for testing necessary efficiency through necessary and sufficient conditions as used in some previous methods (Bitran, 1981).

The proposed procedure can be extended to accommodate all possibly efficient basic solutions. Moreover, the conditions shown in this paper or their sufficient conditions can be tested by linear programming techniques. Those investigations improve the computational aspects of the proposed procedure and they are a part of future topics of this paper. On the other hand, the tolerance region obtained for a nondegenerate basic solution can be used to measure the potential proximity of the solution to necessary efficiency. Using this measure, we may compare possibly efficient solutions in terms of robustness.

Acknowledgements

This research was partly supported by the Spanish Ministry of Economy and Competitiveness (project ECO2017-88883-R) and by the Fundação para a Ciência e a Tecnologia (FCT) under project grant UID/Multi/00308/2019. This work has been also

partly supported by the Consejería de Innovación, Ciencia y Empresa de la Junta de Andalucía (PAI group SEJ-532). Carla Oliveira Henriques also acknowledges the training received from the University of Malaga PhD Programme in Economy and Business [Programa de Doctorado en Economía y Empresa de la Universidad de Malaga].
José Rui Figueira acknowledges the support from the FCT grant SFRH/BSAB/139892/2018 under POCH Program.

References

Antunes, C. A., & Clímaco, J. (2000). Decision aid in the optimization of the interval objective function. In *Decision making: Recent developments and worldwide applications* (pp. 251-261). Springer, Boston, MA. doi: 10.1007/978-1-4757-4919-9_17

Ben-Tal, A,, Ghaoui, L. E., & Nemirovski, A. (2009). Robust Optimization, Princeton University Press; Princeton and Oxford.

Bitran, G. R. (1980). Linear multiple objective problems with interval coefficients. *Management Science*, 26(7), 694-706. doi: 10.1287/mnsc.26.7.694.

Borgonovo, E., Buzzard, G. T., & Wendell, R. E. (2018). A global tolerance approach to sensitivity analysis in linear programming. *European Journal of Operational Research*, 267(1), 321-337. doi: 10.1016/j.ejor.2017.11.034

Filippi, C. (2005). A fresh view on the tolerance approach to sensitivity analysis in linear programming. *European Journal of Operational Research*, *167*(1), 1-19. doi: 10.1016/j.ejor.2004.01.050.

Gal, T. (1995). *Postoptimal Analyses, Parametric Programming and Related Topics: Degeneracy, Multicriteria Decision Making, Redundancy*. 2nd ed. Edited by Walter de Gruyter: Berlin; New York.

Garajová, E., Hladík, M., & Rada, M. (2018). Interval linear programming under transformations: optimal solutions and optimal value range. *Central European Journal of Operations Research*, 1-14 [article in press]. doi: 10.1007/s10100-018-0580-5.

Hladík, M. (2008). Computing the tolerances in multiobjective linear programming. Optimization Methods & Software, 23(5), 731-739. doi: 10.1080/10556780802264204

Hladik, M. (2011). Tolerance analysis in linear systems and linear programming. *Optimization Methods and Software*, 26(3), 381-396. doi: 10.1080/10556788.2011.556635.

Hladík, M. (2012). Complexity of necessary efficiency in interval linear programming and multiobjective linear programming. *Optimization Letters*, *6*(5), 893-899. doi: 10.1007/s11590-011-0315-1.

Ida, M. (1999). Necessary efficient test in interval multiobjective linear programming. In *Proceedings of the eighth international fuzzy systems association world congress* (pp. 500-504).

Ida, M. (2005). Efficient solution generation for multiple objective linear programming based on extreme ray generation method. *European Journal of Operational Research, 160*(1), 242-251. doi: 10.1016/j.ejor.2003.08.039.

Inuiguchi, M. (2013) Necessary efficiency is partitioned into possible and necessary optimalities. In *Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting, IFSA/NAFIPS 2013*, pp. 209–214. doi: 10.1109/IFSA-NAFIPS.2013.6608401.

Inuiguchi, M., Kato, K., & Katagiri, H. (2016). Fuzzy multi-criteria optimization: possibilistic and fuzzy/stochastic approaches. In *Multiple Criteria Decision Analysis* (pp. 851-902). Springer, New York, NY. doi: 10.1007/978-1-4939-3094-4.

Inuiguchi, M., & Kume, Y. (1989). A discrimination method of possibly efficient extreme points for interval multiobjective linear programming problems. *Transactions of the Society of Instrument and Control Engineers, 25*(7), 824-826. doi: 10.9746/sicetr1965.25.824

Inuiguchi, M., & Sakawa, M. (1994). Possible and necessary optimality tests in possibilistic linear programming problems. *Fuzzy Sets and Systems, 67*(1), 29-46. doi: 10.1016/0165-0114(94)90206-2.

Inuiguchi, M., & Sakawa, M. (1995). Minimax regret solution to linear programming problems with an interval objective function. *European Journal of Operational Research*, *86*(3), 526-536. doi: 10.1016/0377-2217(94)00092-Q.

Inuiguchi, M., & Sakawa, M. (1996). Possible and necessary efficiency in possibilistic multiobjective linear programming problems and possible efficiency test. *Fuzzy Sets and Systems*, *78*(2), 231-241. doi: 10.1016/0165-0114(95)00169-7.

Inuiguchi, M., & Sakawa, M. (1997). An achievement rate approach to linear programming problems with an interval objective function. *Journal of the Operational Research Society, 48*(1), 25-33. doi: 10.1057/palgrave.jors.2600322.

Mausser, H. E., & Laguna, M. (1999). Minimising the maximum relative regret for linear programmes with interval objective function coefficients. *Journal of the Operational Research Society*, *50*(10), 1063-1070. doi: 10.1057/palgrave.jors.2600789.

Oliveira, C., & Antunes, C. H. (2007). Multiple objective linear programming models with interval coefficients–an illustrated overview. *European Journal of Operational Research*, *181*(3), 1434-1463. doi: 10.1016/j.ejor.2005.12.042.

Oliveira, C., Antunes, C. H., & Barrico, C. (2014). An enumerative algorithm for computing all possibly optimal solutions to an interval LP. *Top*, *22*(2), 530-542. doi: 10.1007/s11750-012-0268-2.

Rivaz, S., & Yaghoobi, M. A. (2013). Minimax regret solution to multiobjective linear programming problems with interval objective functions coefficients. *Central European Journal of Operations Research*, *21*(3), 625-649. doi: 10.1007/s10100-012-0252-9.

Rivaz, S., & Yaghoobi, M. A. (2015). Some results in interval multiobjective linear programming for recognizing different solutions. *Opsearch*, *52*(1), 75-85. doi: 10.1007/s12597-013-0167-9.

Rivaz, S., Yaghoobi, M. A., & Hladík, M. (2016). Using modified maximum regret for finding a necessarily efficient solution in an interval MOLP problem. *Fuzzy Optimization and Decision Making*, 15(3), 237-253.

Rivaz, S., & Yaghoobi, M. A. (2015). Weighted sum of maximum regrets in an interval MOLP problem. *International Transactions in Operational Research*, *25*(5), 1659-1676. doi: 10.1111/itor.12216.

Steuer, R. E. (1981). Algorithms for linear programming problems with interval objective function coefficients. *Mathematics of Operations Research*, *6*(3), 333-348. doi: 10.1287/moor.6.3.333.

Wang, H. F., & Wang, M. L. (2001). Decision analysis of the interval-valued multiobjective linear programming problems. In *Multiple criteria decision making in the new millennium* (pp. 210-218). Springer, Berlin, Heidelberg. doi: 10.1007/978-3-642-56680-6_19

Wendell, R. E. (1982). A preview of a tolerance approach to sensitivity analysis in linear programming. *Discrete Mathematics*, *38*(1), 121-124. doi: 10.1016/0012-365X(82)90178-9.

Wendell, R. E. (1984). Using bounds on the data in linear programming: The tolerance approach to sensitivity analysis. *Mathematical Programming*, *29*(3), 304-322. doi: 10.1007/BF02591999.

Wendell, R. E. (1985). The tolerance approach to sensitivity analysis in linear programming. *Management Science*, *31*(5), 564-578. doi: 10.1287/mnsc.31.5.564.

Wendell, R. E. (1992). Sensitivity Analysis Revisited and Extended. *Decision Sciences*, *23*(5), 1127-1142. doi: 10.1111/j.1540-5915.1992.tb00439.x.

Wendell, R. E. (1997). Linear programming 3: The tolerance approach. In *Advances in Sensitivity Analysis and Parametric Programming* (pp. 137-157). Springer, Boston, MA. doi: 10.1007/978-1-4615-6103-3_5.

Wendell, R. E. (2004). Tolerance sensitivity and optimality bounds in linear programming. *Management Science*, *50*(6), 797-803. doi: 10.1287/mnsc.1030.0221.

Wendell, R. E., & Chen, W. (2010). Tolerance sensitivity analysis: Thirty years later. Croatian Operational Research Review, 1(1), 12-21. https://hrcak.srce.hr/93431.

Wierzbicki, A. P. (1980). The use of reference objectives in multiobjective optimization. In *Multiple criteria decision making theory and application* (pp. 468-486). Springer, Berlin, Heidelberg. doi: 10.1007/978-3-642-48782-8_32.

Wondolowski Jr, F. R. (1991). A generalization of Wendell's tolerance approach to sensitivity analysis in linear programming. *Decision Sciences*, *22*(4), 792-811. doi:

10.1111/j.1540-5915.1991.tb00365.x.