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# The boundary value contact problem of electroelasticity for piecewise-homogeneous piezo-electric plate with elastic inclusion and cut

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**Abstract**

In this paper a contact problem of the theory of electroelasticity for piecewise-homogeneous plate of piezo-electric material with infinite cut and elastic finite inclusion of variable bending rigidity is considered. By using methods of the theory of analytic function, the problem is reduced to a system of singular integro-differential equations with fixed singularity. Using an integral transformation, a Riemann problem is obtained, the solution of which is presented in explicit form.

**Keywords**

asymptotic estimates, contact problem, elastic inclusion, integral transformation, integro-differential equations, piezo-electric material, Riemann problem

## 1. Introduction

Exact or approximate solutions of static contact problems for different domains, reinforced with elastic mountings, thin inclusions, or patches of variable stiffness were obtained earlier, and the behavior of the contact stresses at the ends of the contact line have been investigated as a function of the law of variation of the geometrical and physical parameters of these components [1–10]. Inhomogeneity problems are addressed elsewhere [11–19]. The first fundamental problem for a piecewise-homogeneous plane was solved when a crack of finite length arrives at the interface of two bodies at the right angle [20], and also a similar problem for a piecewise-homogeneous plane when acted upon symmetrical normal stresses at the crack sides [21, 22], as well as the contact problems for piecewise-homogeneous planes with a semi-infinite and finite inclusion [23, 24].

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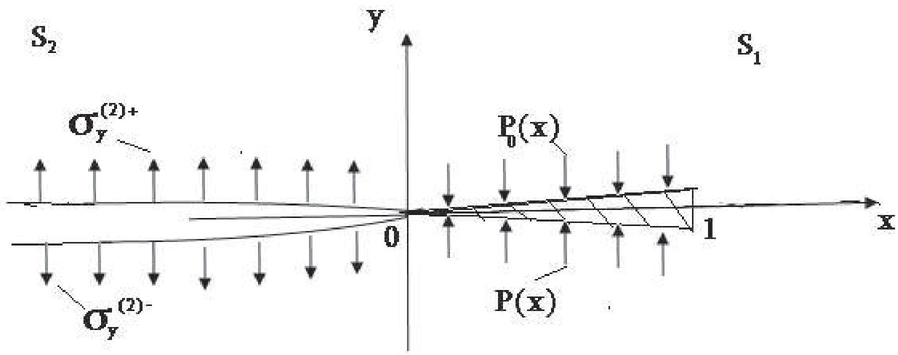


**Figure 1.** Statement of the problem.

## 2. Problem statement and its solution

We will consider a piecewise-homogeneous plate of piezo-electric material, weakened with infinite crack and reinforced with a finite inclusion(beam) as an electrode by a normal force of intensity $p_0(x)$. Let us assume that $p_0(x)$ is a bounded function on the segment. The normal stresses $q_0(x)$ and the electric potential are given at the edges of the crack.

*The problem* consists of determining the expansion of the cut and the jump $p(x)$ of normal contact stresses along the contact line and of establishing their behavior in the neighborhood of the ends of the inclusion. It is formulated as follows: suppose an elastic body $S$ occupies the plane of complex variable $z = x + iy$, which contains, along the section $l_1 = (0, 1)$ an elastic inclusion and an infinite cut along the half-axis $l_2 = (-\infty, 0)$ and consists of two half-planes of dissimilar piezo-electric materials

$$S^{(1)} = \{z|\mathrm{Re}\, z > 0, \quad z \notin [0, 1]\}, \qquad S^{(2)} = \{z|\mathrm{Re}\, z > 0, \quad z \notin (-\infty, 0)\},$$

joined along the $OY$ axis. Quantities and functions, referred to as the half-plane $S^{(k)}$, will be denoted by the subscript $k$ ($k = 1, 2$), while the boundary values of the other functions on the upper and lower sides of the patch will by denoted by a plus or minus sign, respectively.

In conditions of plane deformation, the homogeneous fields of mechanical and electrical stresses act on plate at infinity: $\sigma_{11}^{\infty}$, $\sigma_{33}^{\infty}$, $\tau_{13}^{\infty}$, $E_1^{\infty} = E_3^{\infty} = 0$. At the boundary of the inclusion electrical field's potential is $\varphi_1^+ = \varphi_1^- = 0$ and at the boundary of the crack $\sigma_y^{(2)+}(x) = \sigma_y^{(2)-}(x) = q_0(x)$, $\tau_{xy}^{(2)+}(x) = \tau_{xy}^{(2)-}(x) = 0$, $\varphi_2^+ = \varphi_2^- = \varphi_2(x)$. (see Figure 1)

According to the equilibrium equation of inclusions elements and Hooke's law, we have

$$\frac{d^2}{dx^2} D(x) \frac{d^2 v^{(1)}(x)}{dx^2} = p_0(x) - p(x), \qquad 0 < x < 1, \tag{1}$$

and the equilibrium equation of the inclusion has the form

$$\int_0^1 [\, p(t) - p_0(t)]\, dt = 0, \qquad \int_0^1 t[\, p(t) - p_0(t)]\, dt = 0, \tag{2}$$

where $v^{(1)}(x)$ is the vertical displacement of inclusion points and $p(x)$ is the jump of tangential contact stresses to be determined. $D(x) = E_1(x)h_1^3(x)/(1 - \nu_1^2)$ with $E_1(x)$, $h_1(x)$, $\nu_1$ the modulus of elasticity , thickness, and Poisson's coefficient of the inclusions material, respectively.

At the boundary of the crack we have

$$\sigma_y^{(2)+}(x) + \sigma_y^{(2)-}(x) = 2q_0(x), \qquad x < 0. \tag{3}$$

In the interface of two materials we have

$$\begin{aligned} &\sigma_x^{(1)} = \sigma_x^{(2)}, \quad \tau_{xy}^{(1)} = \tau_{xy}^{(2)}, \quad u^{(1)} = u^{(2)}, \\ &\nu^{(1)} = \nu^{(2)}, \quad E_y^{(1)} = E_y^{(2)}, \quad D_x^{(1)} = D_x^{(2)}, \end{aligned} \tag{4}$$

where $\sigma_x^{(j)}$, $\tau_{xy}^{(j)}$ are stress components; $u^{(j)}$, $\nu^{(j)}$ are displacement components, $E_y^{(j)}$ and $D_x^{(j)}$ are components of vectors of electrical stress and of electrical inductive $(j = 1, 2)$.

In the plane *XOY* for stress function $\varphi_1^{(j)}$ and electrical field's potential $\varphi_2^{(j)}$ we obtain the system of differential equations [25]:

$$l_{11}^{(j)}\varphi_1^{(j)} + l_{12}^{(j)}\varphi_2^{(j)} = 0, \qquad l_{12}^{(j)}\varphi_1^{(j)} + l_{22}^{(j)}\varphi_2^{(j)} = 0, \tag{5}$$

where

$$\begin{aligned} &l_{11}^{(j)} = a_{10}^{(j)}\partial_1^4 + a_{12}^{(j)}\partial_1^2\partial_2^2 + a_{14}^{(j)}\partial_2^4, && \partial_1 = \frac{\partial}{\partial x}, && \partial_2 = \frac{\partial}{\partial y} \\ &l_{12}^{(j)} = l_{21}^{(j)} = a_{21}^{(j)}\partial_1^2\partial_2 + a_{23}^{(j)}\partial_2^3, && l_{22}^{(j)} = a_{20}^{(j)}\partial_1^2 + a_{22}^{(j)}\partial_2^2, && a_{10}^{(j)} = s_{33}^{(j)} - (s_{13}^{(j)})^2(s_{11}^{(j)})^{-1} \\ &a_{12}^{(j)} = s_{44}^{(j)} + 2s_{13}^{(j)}(1 - s_{12}^{(j)}(s_{11}^{(j)})^{-1}), && a_{14}^{(j)} = s_{11}^{(j)} - (s_{12}^{(j)})^2(s_{11}^{(j)})^{-1}, && \\ &a_{21}^{(j)} = s_{13}^{(j)}d_{13}^{(j)}(s_{11}^{(j)})^{-1} - d_{33}^{(j)} + d_{15}^{(j)}, && a_{23}^{(j)} = d_{13}^{(j)}\left(s_{12}^{(j)}(s_{11}^{(j)})^{-1} - 1\right), && \\ &a_{20}^{(j)} = \epsilon_{11}^{(j)}, && a_{22}^{(j)} = \epsilon_{33}^{(j)} - (d_{13}^{(j)})^2(s_{11}^{(j)})^{-1}, && j = 1, 2, \end{aligned}$$

where $s_{nk}^{(j)}$, $d_{nk}^{(j)}$, $\epsilon_{nk}^{(j)}$ are elastic tractability, piezo-electric modules, and dielectric constants, respectively.

General solutions of equation (5) is represented using three analytical functions:

$$\begin{aligned} &\varphi_1^{(j)} = 2\mathrm{Re}\sum_{k=1}^{3}\gamma_k^{(j)}\int\Phi_k^{(j)}(z_k^{(j)})\,dz_k^{(j)}, \quad \varphi_2^{(j)} = -2\mathrm{Re}\sum_{k=1}^{3}\lambda_k^{(j)}\Phi_k^{(j)}(z_k^{(j)}) \\ &z_k^{(j)} = x + \mu_k^{(j)}y, \qquad \mu_{3+k}^{(j)} = \overline{\mu}_k^{(j)}, \qquad \gamma_k^{(j)} = a_{20}^{(j)} + a_{22}^{(j)}(\mu_k^{(j)})^2, \\ &\lambda_k^{(j)} = a_{21}^{(j)}\mu_k^{(j)} + a_{23}^{(j)}(\mu_k^{(j)})^3. \end{aligned} \tag{6}$$

$\mu_k^{(j)}$ are roots of characteristic equations:

$$c_0^{(j)}(\mu^{(j)})^6 + c_1^{(j)}(\mu^{(j)})^4 + c_2^{(j)}(\mu^{(j)})^2 + c_3^{(j)} = 0, \qquad k = 1, 2, 3, \quad j = 1, 2,$$

where

$$\begin{aligned} &c_0^{(j)} = a_{14}^{(j)}a_{22}^{(j)} - (a_{23}^{(j)})^2, \qquad c_1^{(j)} = a_{12}^{(j)}a_{22}^{(j)} + a_{14}^{(j)}a_{20}^{(j)} - 2a_{21}^{(j)}a_{23}^{(j)}, \quad c_2^{(j)} = a_{10}^{(j)}a_{22}^{(j)} + a_{12}^{(j)}a_{20}^{(j)} - a_{21}^{(j)}, \\ &c_3^{(j)} = a_{10}^{(j)}a_{20}^{(j)}, \qquad \mathrm{Im}\,\mu_k^{(j)} \neq 0 \end{aligned}$$

Using equation (6), we obtain representation for stress component, displacements, vectors of electrical stress, and of electrical inductive:

$$\sigma_x^{(j)} = 2\mathrm{Re}\sum_{k=1}^{3}\gamma_k^{(j)}(\mu_k^{(j)})^2\Phi'^{(j)}_k(z_k^{(j)}),\quad \sigma_y^{(j)} = 2\mathrm{Re}\sum_{k=1}^{3}\gamma_k^{(j)}\Phi'^{(j)}_k(z_k^{(j)}),$$
$$\tau_{xy}^{(j)} = -2\mathrm{Re}\sum_{k=1}^{3}\gamma_k^{(j)}\mu_k^{(j)}\Phi'^{(j)}_k(z_k^{(j)}),\quad u^{(j)} = 2\mathrm{Re}\sum_{k=1}^{3}p_k^{(j)}\Phi_k^{(j)}(z_k^{(j)}),$$
$$\nu^{(j)} = 2\mathrm{Re}\sum_{k=1}^{3}q_k^{(j)}\Phi_k^{(j)}(z_k^{(j)}),\quad E_x^{(j)} = 2\mathrm{Re}\sum_{k=1}^{3}\lambda_k^{(j)}\Phi'^{(j)}_k(z_k^{(j)}),$$
$$E_y^{(j)} = 2\mathrm{Re}\sum_{k=1}^{3}\lambda_k^{(j)}\mu_k^{(j)}\Phi'^{(j)}_k(z_k^{(j)}),\quad D_x^{(j)} = 2\mathrm{Re}\sum_{k=1}^{3}r_k^{(j)}\mu_k^{(j)}\Phi'^{(j)}_k(z_k^{(j)}),$$
$$D_y^{(j)} = -\mathrm{Re}\sum_{k=1}^{3}r_k^{(j)}\Phi'^{(j)}_k(z_k^{(j)}),$$

where

$$p_k^{(j)} = a_{14}^{(j)}\gamma_k^{(j)}(\mu_k^{(j)})^2 + \frac{1}{2}(a_{12}^{(j)} - s_{44}^{(j)})\gamma_k^{(j)} - a_{23}^{(j)}\lambda_k^{(j)}\mu_k^{(j)},$$
$$q_k^{(j)} = \frac{1}{2}(a_{12}^{(j)} - s_{44}^{(j)})\gamma_k^{(j)}\mu_k^{(j)} + a_{10}^{(j)}\gamma_k^{(j)}(\mu_k^{(j)})^{-1} - (a_{21}^{(j)} - d_{15}^{(j)})\lambda_k^{(j)},$$
$$r_k^{(j)} = a_{20}^{(j)}\lambda_k^{(j)}(\mu_k^{(j)})^{-1} - d_{15}^{(j)}\gamma_k^{(j)}.$$

Introducing the notation $H_k^{(j)}(x) = [\Phi'^{(j)}_k(x)]^+ - [\Phi'^{(j)}_k(x)]^-$, $(k = 1,2,3,\ j = 1,2)$, the boundary value conditions

$$\sigma_y^{(1)+} - \sigma_y^{(1)-} = p(x),\qquad \tau_{xy}^{(1)+} - \tau_{xy}^{(1)-} = 0$$
$$\left(\frac{\partial u^{(1)}}{\partial x}\right)^+ - \left(\frac{\partial u^{(1)}}{\partial x}\right)^- = 0,\qquad \left(\frac{\partial \nu^{(1)}}{\partial x}\right)^+ - \left(\frac{\partial \nu^{(1)}}{\partial x}\right)^- = 0,$$
$$E_x^{(1)+}(x) = E_x^{(1)-}(x),\qquad D_y^{(1)+}(x) = D_y^{(1)-}(x),\qquad x \in l_1$$

$$\sigma_y^{(2)+} - \sigma_y^{(2)-} = 0,\qquad \tau_{xy}^{(2)+} - \tau_{xy}^{(2)-} = 0$$
$$\left(\frac{\partial u^{(2)}}{\partial x}\right)^+ - \left(\frac{\partial u^{(2)}}{\partial x}\right)^- = 0,\qquad \left(\frac{\partial \nu^{(2)}}{\partial x}\right)^+ - \left(\frac{\partial \nu^{(2)}}{\partial x}\right)^- = 2\left(\frac{\partial \nu}{\partial x}\right)^+ \equiv 2f(x),$$
$$E_x^{(2)+}(x) = E_x^{(2)-}(x) = -\frac{\partial\varphi_2}{\partial x},\qquad D_y^{(2)+}(x) = D_y^{(2)-}(x),\qquad x \in l_2$$

can be represented using three analytical functions :

$$\mathrm{Re}\sum_{k=1}^{3}r_k^{(1)}H_k^{(1)}(x) = 0,\qquad \mathrm{Re}\sum_{k=1}^{3}\gamma_k^{(1)}H_k^{(1)}(x) = \frac{p(x)}{2},\qquad \mathrm{Re}\sum_{k=1}^{3}p_k^{(1)}H_k^{(1)}(x) = 0,\qquad x \in l_1 \tag{7a}$$

$$\mathrm{Re}\sum_{k=1}^{3}\gamma_k^{(1)}\mu_k^{(1)}H_k^{(1)}(x) = 0,\qquad \mathrm{Re}\sum_{k=1}^{3}\lambda_k^{(1)}H_k^{(1)}(x) = 0,\qquad \mathrm{Re}\sum_{k=1}^{3}q_k^{(1)}H_k^{(1)}(x) = 0,\qquad x \in l_1 \tag{7b}$$

$$\mathrm{Re}\sum_{k=1}^{3}r_k^{(2)}H_k^{(2)}(x) = 0,\qquad \mathrm{Re}\sum_{k=1}^{3}\gamma_k^{(2)}H_k^{(2)}(x) = 0,\qquad \mathrm{Re}\sum_{k=1}^{3}p_k^{(2)}H_k^{(2)}(x) = 0,\qquad x \in l_2 \tag{8a}$$

$$\operatorname{Re}\sum_{k=1}^{3}\gamma_k^{(2)}\mu_k^{(2)}H_k^{(2)}(x)=0, \qquad \operatorname{Re}\sum_{k=1}^{3}\lambda_k^{(2)}H_k^{(2)}(x)=0, \qquad \operatorname{Re}\sum_{k=1}^{3}q_k^{(2)}H_k^{(2)}(x)=f(x), \qquad x\in l_2 \tag{8b}$$

Without loss of generality, assume that $\mu_k^{(j)}=i\beta_k^{(j)}$, $k=1,2,3, j=1,2$ [26,27].

Then, one obtains

$$[\Phi'^{(1)}_k(x)]^+-[\Phi'^{(1)}_k(x)]^-=\frac{\Delta_{2k}^{(1)}}{2\Delta_0^{(1)}}p(x), \qquad k=1,2,3, \quad 0<x<1 \tag{9}$$

$$[\Phi'^{(2)}_k(x)]^+-[\Phi'^{(2)}_k(x)]^-=\frac{\Delta_{3k}^{(2)}}{\Delta_0^{(2)}}if(x), \qquad k=1,2,3, \quad x<0 \tag{10}$$

where $\Delta_0^{(1)}\neq 0$ is the determinant of equation (7a), $\Delta_{2k}^{(1)}$ is corresponding algebraic additions, $-i\Delta_0^{(2)}\neq 0$ is the determinant of equation (8b), and $\Delta_{3k}^{(2)}$ is corresponding algebraic additions.

The solutions of the problems of linear conjugation with boundary conditions (9) and (10) are represented in the form

$$\begin{aligned}\Phi'^{(1)}_k(z_k^{(1)})&=\frac{\Delta_{2k}^{(1)}}{4\pi i\Delta_0^{(1)}}\int_0^1\frac{p(t)\,dt}{t-z_k^{(1)}}+W_k^{(1)}(z_k^{(1)})\equiv\Psi_k^{(1)}(z_k^{(1)})+W_k^{(1)}(z_k^{(1)}), \quad z_k^{(1)}\in S_k^{(1)}\\ \Phi'^{(2)}_k(z_k^{(2)})&=\frac{\Delta_{3k}^{(2)}}{2\pi\Delta_0^{(2)}}\int_{-\infty}^0\frac{f(t)\,dt}{t-z_k^{(2)}}+W_k^{(2)}(z_k^{(2)})\equiv\Psi_k^{(2)}(z_k^{(2)})+W_k^{(2)}(z_k^{(2)}), \quad z_k^{(2)}\in S_k^{(2)}\end{aligned} \tag{11}$$

where $W_k^{(j)}(z_k^{(j)})$ are analytic functions in the half-plates $S_k^{(j)}$, $k=1,2,3, j=1,2$.

To determinate the analytic functions $W_k^{(1)}(z_k^{(1)})$ we obtain the following from equation (4) (the boundary condition on the interface of two materials):

$$\sum_{k=1}^{3}\gamma_k^{(1)}(\beta_k^{(1)})^2M_k(t_k^{(1)})=\sum_{k=1}^{3}\gamma_k^{(2)}(\beta_k^{(2)})^2M_k(t_k^{(2)}), \quad \sum_{k=1}^{3}\gamma_k^{(1)}i\beta_k^{(1)}\widetilde{M}_k(t_k^{(1)})=\sum_{k=1}^{3}\gamma_k^{(2)}i\beta_k^{(2)}\widetilde{M}_k(t_k^{(2)}),$$
$$\sum_{k=1}^{3}\lambda_k^{(1)}i\beta_k^{(1)}M_k(t_k^{(1)})=\sum_{k=1}^{3}\lambda_k^{(2)}i\beta_k^{(2)}M_k(t_k^{(2)}), \qquad \sum_{k=1}^{3}p_k^{(1)}i\beta_k^{(1)}\widetilde{M}_k(t_k^{(1)})=\sum_{k=1}^{3}p_k^{(2)}i\beta_k^{(2)}\widetilde{M}_k(t_k^{(2)}),$$
$$\sum_{k=1}^{3}q_k^{(1)}i\beta_k^{(1)}M_k(t_k^{(1)})=\sum_{k=1}^{3}q_k^{(2)}i\beta_k^{(2)}M_k(t_k^{(2)}), \qquad \sum_{k=1}^{3}r_k^{(1)}i\beta_k^{(1)}\widetilde{M}_k(t_k^{(1)})=\sum_{k=1}^{3}r_k^{(2)}i\beta_k^{(2)}\widetilde{M}_k(t_k^{(2)}),$$

where

$$\begin{aligned}M_k(t_k^{(j)})&=W_k^{(j)}(t_k^{(1)})+\overline{W}_k^{(j)}(\bar{t}_k^{(j)})+\Psi_k^{(j)}(t_k^{(j)})+\overline{\Psi}_k^{(j)}(\bar{t}_k^{(j)})\\ \widetilde{M}_k(t_k^{(j)})&=W_k^{(j)}(t_k^{(j)})-\overline{W}_k^{(j)}(\bar{t}_k^{(j)})+\Psi_k^{(j)}(t_k^{(j)})-\overline{\Psi}_k^{(j)}(\bar{t}_k^{(j)})\\ t_k^{(j)}&=i\beta_k^{(j)}y, \qquad k=1,2,3, \quad j=1,2.\end{aligned}$$

After multiplication of the obtained expressions by $\frac{1}{2\pi i}\frac{dt}{t-z}$, $t=iy$, $z=x+iy$ and integrating along axis $OY$, by using Cauchy's theorem and formula, we obtain the system of algebraic equations with respect to $W_k^{(1)}(\beta_k^{(1)}z)$, $\overline{W}_k^{(2)}(-\beta_k^{(2)}z)$ $(k=1,2,3)$:

$$\begin{aligned}&\sum_{k=1}^{3}[\gamma_k^{(1)}(\beta_k^{(1)})^2W_k^{(1)}(\beta_k^{(1)}z)-\gamma_k^{(2)}(\beta_k^{(2)})^2\overline{W}_k^{(2)}(-\beta_k^{(2)}z)]\\ &\quad=-\sum_{k=1}^{3}\gamma_k^{(1)}(\beta_k^{(1)})^2\overline{\Psi}_k^{(1)}(-\beta_k^{(1)}z)+\sum_{k=1}^{3}\gamma_k^{(2)}(\beta_k^{(2)})^2\overline{\Psi}_k^{(2)}(\beta_k^{(2)}z)\end{aligned}$$

$$\sum_{k=1}^{3}[\gamma_k^{(1)} i\beta_k^{(1)} W_k^{(1)}(\beta_k^{(1)}z) + \gamma_k^{(2)} i\beta_k^{(2)} \overline{W}_k^{(2)}(-\beta_k^{(2)}z)]$$
$$= -\sum_{k=1}^{3}\gamma_k^{(1)} i\beta_k^{(1)} \overline{\Psi}_k^{(1)}(-\beta_k^{(1)}z) + \sum_{k=1}^{3}\gamma_k^{(2)} i\beta_k^{(2)} \Psi_k^{(2)}(\beta_k^{(2)}z)$$
$$\sum_{k=1}^{3}[\lambda_k^{(1)} i\beta_k^{(1)} W_k^{(1)}(\beta_k^{(1)}z) - \lambda_k^{(2)} i\beta_k^{(2)} \overline{W}_k^{(2)}(-\beta_k^{(2)}z)]$$
$$= -\sum_{k=1}^{3}\lambda_k^{(1)} i\beta_k^{(1)} \overline{\Psi}_k^{(1)}(-\beta_k^{(1)}z) + \sum_{k=1}^{3}\lambda_k^{(2)} i\beta_k^{(2)} \Psi_k^{(2)}(\beta_k^{(2)}z)$$
$$\sum_{k=1}^{3}[p_k^{(1)} i\beta_k^{(1)} W_k^{(1)}(\beta_k^{(1)}z) - p_k^{(2)} i\beta_k^{(2)} \overline{W}_k^{(2)}(-\beta_k^{(2)}z)]$$
$$= \sum_{k=1}^{3}p_k^{(1)} i\beta_k^{(1)} \overline{\Psi}_k^{(1)}(-\beta_k^{(1)}z) + \sum_{k=1}^{3}p_k^{(2)} i\beta_k^{(2)} \Psi_k^{(2)}(\beta_k^{(2)}z)$$
$$\sum_{k=1}^{3}[q_k^{(1)} i\beta_k^{(1)} W_k^{(1)}(\beta_k^{(1)}z) - q_k^{(2)} i\beta_k^{(2)} \overline{W}_k^{(2)}(-\beta_k^{(2)}z)]$$
$$= -\sum_{k=1}^{3}q_k^{(1)} i\beta_k^{(1)} \overline{\Psi}_k^{(1)}(-\beta_k^{(1)}z) + \sum_{k=1}^{3}q_k^{(2)} i\beta_k^{(2)} \Psi_k^{(2)}(\beta_k^{(2)}z)$$
$$\sum_{k=1}^{3}[r_k^{(1)} i\beta_k^{(1)} W_k^{(1)}(\beta_k^{(1)}z) + r_k^{(2)} i\beta_k^{(2)} \overline{W}_k^{(2)}(-\beta_k^{(2)}z)]$$
$$= \sum_{k=1}^{3}r_k^{(1)} i\beta_k^{(1)} \overline{\Psi}_k^{(1)}(-\beta_k^{(1)}z) + \sum_{k=1}^{3}r_k^{(2)} i\beta_k^{(2)} \Psi_k^{(2)}(\beta_k^{(2)}z).$$

Solving this system, we obtain:

$$\begin{aligned}
W_1^{(1)}(z_1^{(1)}) &= \sum_{k=1}^{3} A_k^{(1)}\overline{\Psi}_k^{(1)}\left(-\frac{\beta_k^{(1)}}{\beta_1^{(1)}}z_1^{(1)}\right) + \sum_{k=1}^{3} A_k^{(2)}\Psi_k^{(2)}\left(-\frac{\beta_k^{(2)}}{\beta_1^{(1)}}z_1^{(1)}\right)\\
W_2^{(1)}(z_2^{(1)}) &= \sum_{k=1}^{3} B_k^{(1)}\overline{\Psi}_k^{(1)}\left(-\frac{\beta_k^{(1)}}{\beta_2^{(1)}}z_2^{(1)}\right) + \sum_{k=1}^{3} B_k^{(2)}\Psi_k^{(2)}\left(-\frac{\beta_k^{(2)}}{\beta_2^{(1)}}z_2^{(1)}\right)\\
W_3^{(1)}(z_3^{(1)}) &= \sum_{k=1}^{3} C_k^{(1)}\overline{\Psi}_k^{(1)}\left(-\frac{\beta_k^{(1)}}{\beta_3^{(1)}}z_3^{(1)}\right) + \sum_{k=1}^{3} C_k^{(2)}\Psi_k^{(2)}\left(-\frac{\beta_k^{(2)}}{\beta_3^{(1)}}z_3^{(1)}\right)\\
\overline{W}_1^{(2)}(-z_1^{(2)}) &= \sum_{k=1}^{3} D_k^{(1)}\overline{\Psi}_k^{(1)}\left(-\frac{\beta_k^{(1)}}{\beta_1^{(2)}}z_1^{(2)}\right) + \sum_{k=1}^{3} D_k^{(2)}\Psi_k^{(2)}\left(\frac{\beta_k^{(2)}}{\beta_1^{(2)}}z_1^{(2)}\right)\\
\overline{W}_2^{(2)}(-z_2^{(2)}) &= \sum_{k=1}^{3} E_k^{(1)}\overline{\Psi}_k^{(1)}\left(-\frac{\beta_k^{(1)}}{\beta_2^{(2)}}z_2^{(2)}\right) + \sum_{k=1}^{3} E_k^{(2)}\Psi_k^{(2)}\left(\frac{\beta_k^{(2)}}{\beta_2^{(2)}}z_2^{(2)}\right)\\
\overline{W}_3^{(2)}(-z_3^{(2)}) &= \sum_{k=1}^{3} F_k^{(1)}\overline{\Psi}_k^{(1)}\left(-\frac{\beta_k^{(1)}}{\beta_3^{(2)}}z_3^{(2)}\right) + \sum_{k=1}^{3} F_k^{(2)}\Psi_k^{(2)}\left(\frac{\beta_k^{(2)}}{\beta_3^{(2)}}z_3^{(2)}\right),
\end{aligned} \tag{12}$$

where

$$A_k^{(j)} = \Big((-1)^j\gamma_k^{(j)}(\beta_k^{(j)})^2\widetilde{A}_{11} + i\gamma_k^{(j)}\beta_k^{(j)}\widetilde{A}_{21} + (-1)^j i\lambda_k^{(j)}\beta_k^{(j)}\widetilde{A}_{31} + p_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{41}$$
$$+(-1)^{(j)}q_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{51} + r_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{61}\Big)/\widetilde{\Delta}$$
$$B_k^{(j)} = \Big(\gamma_k^{(j)}(\beta_k^{(j)})^2\widetilde{A}_{12} + i\gamma_k^{(j)}\beta_k^{(j)}\widetilde{A}_{22} + i\lambda_k^{(j)}\beta_k^{(j)}\widetilde{A}_{32} + p_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{42} + q_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{52} + r_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{62}\Big)/\widetilde{\Delta}$$
$$C_k^{(j)} = \Big((-1)^j\gamma_k^{(j)}(\beta_k^{(j)})^2\widetilde{A}_{13} + i\gamma_k^{(j)}\beta_k^{(j)}\widetilde{A}_{23} + (-1)^j i\lambda_k^{(j)}\beta_k^{(j)}\widetilde{A}_{33} + p_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{43} + (-1)^j q_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{53}$$
$$+r_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{63}\Big)/\widetilde{\Delta}$$
$$D_k^{(j)} = \Big(\gamma_k^{(j)}(\beta_k^{(j)})^2\widetilde{A}_{14} + i\gamma_k^{(j)}\beta_k^{(j)}\widetilde{A}_{24} + i\lambda_k^{(j)}\beta_k^{(j)}\widetilde{A}_{34} + p_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{44} + q_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{54} + r_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{64}\Big)/\widetilde{\Delta}$$
$$E_k^{(j)} = \Big((-1)^j\gamma_k^{(j)}(\beta_k^{(j)})^2\widetilde{A}_{15} + i\gamma_k^{(j)}\beta_k^{(j)}\widetilde{A}_{25} + (-1)^j i\lambda_k^{(j)}\beta_k^{(j)}\widetilde{A}_{35} + p_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{45} + (-1)^j q_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{55}$$
$$+r_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{65}\Big)/\widetilde{\Delta}$$
$$F_k^{(j)} = \Big(\gamma_k^{(j)}(\beta_k^{(j)})^2\widetilde{A}_{16} + i\gamma_k^{(j)}\beta_k^{(j)}\widetilde{A}_{26} + i\lambda_k^{(j)}\beta_k^{(j)}\widetilde{A}_{36} + p_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{46} + q_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{56} + r_k^{(j)}i\beta_k^{(j)}\widetilde{A}_{66}\Big)/\widetilde{\Delta},$$
$$k = 1,2,3, \quad j = 1,2$$

$$\widetilde{\Delta} = -\prod_{k=1}^{3}\beta_k^{(1)}\beta_k^{(2)}\begin{vmatrix} -i\gamma_1^{(1)}\beta_1^{(1)} & -i\gamma_2^{(1)}\beta_2^{(1)} & -i\gamma_3^{(1)}\beta_3^{(1)} & i\beta_1^{(2)}\gamma_1^{(2)} & i\beta_2^{(2)}\gamma_2^{(2)} & i\beta_3^{(2)}\gamma_3^{(2)} \\ \gamma_1^{(1)} & \gamma_2^{(1)} & \gamma_3^{(1)} & \gamma_1^{(2)} & \gamma_2^{(2)} & \gamma_3^{(2)} \\ \lambda_1^{(1)} & \lambda_2^{(1)} & \lambda_3^{(1)} & -\lambda_1^{(2)} & -\lambda_2^{(2)} & -\lambda_3^{(2)} \\ p_1^{(1)} & p_2^{(1)} & p_3^{(1)} & p_1^{(2)} & p_2^{(2)} & p_3^{(2)} \\ q_1^{(1)} & q_2^{(1)} & q_3^{(1)} & -q_1^{(2)} & -q_2^{(2)} & -q_3^{(2)} \\ r_1^{(1)} & r_2^{(1)} & r_3^{(1)} & r_1^{(2)} & r_2^{(2)} & r_3^{(2)} \end{vmatrix} \neq 0,$$

where $\widetilde{A}_{ij}$ are corresponding algebraic additions.

Since Re $\widetilde{\Delta} = 0$, we have $\mathrm{Im}\{A_k, B_k, C_k\} = 0$, $k = 1,2,3$.

On the bases of conditions (2)–(3) and equations (11)–(12), we obtain the following system of singular integro-differential equations:

$$\frac{d^2}{dx^2}D(x)\frac{d}{dx}\left(\lambda_1\int_0^1\frac{p(t)dt}{t-x} + \lambda_2\int_0^1\frac{p(t)dt}{t+x} + \int_0^1 R_1(t,x)p(t)dt + \int_{-\infty}^0 R_2(t,x)f(t)dt\right) = p_0(x) - p(x), \quad x \in l_1 \tag{13}$$

$$\lambda_3\int_{-\infty}^0\frac{f(t)dt}{t-x} + \lambda_4\int_{-\infty}^0\frac{f(t)dt}{t+x} + \int_0^1 R_3(t,x)p(t)dt + \int_{-\infty}^0 R_4(t,x)f(t)dt = q_0(x), \qquad x \in l_2, \tag{14}$$

where

$$R_1(t,x) = \sum_{m\neq n=1}^{3}\frac{\omega_{mn}}{\beta_m^{(1)}t + \beta_n^{(1)}x}, \qquad R_2(t,x) = \sum_{m\neq n=1}^{3}\frac{\alpha_{mn}}{\beta_m^{(1)}t - \beta_n^{(2)}x},$$

$$\lambda_1 = \sum_{k=1}^{3}\frac{iq_k^{(1)}\Delta_k^{(1)}}{2\pi\,\Delta_0^{(1)}}, \qquad \lambda_2 = i\frac{q_1^{(1)}A_1^{(1)}\Delta_1^{(1)} + q_2^{(1)}B_2^{(1)}\Delta_2^{(1)} + q_3^{(1)}C_3^{(1)}\Delta_3^{(1)}}{2\pi\,\Delta_0^{(1)}}, \qquad \omega_{12} = \beta_1^{(1)}\frac{iq_1^{(1)}A_2^{(1)}\Delta_2^{(1)}}{2\pi\,\Delta_0^{(1)}},$$

$$\omega_{21}=\beta_2^{(1)}\frac{iq_2^{(1)}B_1^{(1)}\Delta_1^{(1)}}{2\pi\Delta_0^{(1)}},\qquad \omega_{13}=\beta_1^{(1)}\frac{iq_1^{(1)}A_3^{(1)}\Delta_3^{(1)}}{2\pi\Delta_0^{(1)}},\qquad \omega_{31}=\beta_3^{(1)}\frac{iq_3^{(1)}C_1^{(1)}\Delta_1^{(1)}}{2\pi\Delta_0^{(1)}},$$

$$\omega_{23}=\beta_2^{(1)}\frac{iq_2^{(1)}B_3^{(1)}\Delta_3^{(1)}}{2\pi\Delta_0^{(1)}},\qquad \omega_{32}=\beta_3^{(1)}\frac{iq_3^{(1)}C_2^{(1)}\Delta_2^{(1)}}{2\pi\Delta_0^{(1)}},$$

$$\alpha_{1n}=\frac{-2iq_1^{(1)}A_1^{(n)}\Delta_n^{(2)}\beta_1^{(1)}}{\pi\Delta_0^{(2)}},\quad \alpha_{2n}=\frac{-2iq_2^{(1)}B_1^{(n)}\Delta_n^{(2)}\beta_2^{(1)}}{\pi\Delta_0^{(2)}},\quad \alpha_{3n}=\frac{-2iq_3^{(1)}C_1^{(n)}\Delta_n^{(2)}\beta_3^{(1)}}{\pi\Delta_0^{(2)}},\qquad n=1,2,3.$$

$$R_3(t,x)=\sum_{m\neq n=1}^{3}\frac{r_{mn}}{\beta_m^{(2)}t-\beta_n^{(1)}x},\qquad R_4(t,x)=\sum_{m\neq n=1}^{3}\frac{q_{mn}}{\beta_m^{(2)}t+\beta_n^{(2)}x},$$

$$\lambda_3=-2\sum_{k=1}^{3}\frac{\gamma_k^{(2)}\Delta_k^{(2)}}{\pi\Delta_0^{(2)}},\qquad \lambda_4=-2\frac{\gamma_1^{(2)}D_1^{(2)}\Delta_1^{(2)}+\gamma_2^{(2)}E_2^{(2)}\Delta_2^{(2)}+\gamma_3^{(2)}F_3^{(2)}\Delta_3^{(2)}}{\pi\Delta_0^{(2)}},$$

$$r_{1n}=\frac{\gamma_1^{(2)}D_n^{(1)}\Delta_n^{(2)}\beta_1^{(2)}}{2\pi\Delta_0^{(1)}},\quad r_{2n}=\frac{\gamma_2^{(2)}E_n^{(1)}\Delta_n^{(1)}\beta_2^{(2)}}{2\pi\Delta_0^{(1)}},\quad r_{3n}=\frac{\gamma_3^{(2)}F_n^{(1)}\Delta_n^{(1)}\beta_3^{(2)}}{2\pi\Delta_0^{(1)}},\qquad n=1,2,3.$$

$$q_{12}=\beta_1^{(2)}\frac{\gamma_1^{(2)}D_2^{(2)}\Delta_2^{(1)}}{2\pi\Delta_0^{(2)}},\quad q_{21}=-2\beta_2^{(2)}\frac{\gamma_2^{(2)}E_1^{(2)}\Delta_1^{(2)}}{\pi\Delta_0^{(2)}},\quad q_{13}=-2\beta_1^{(2)}\frac{\gamma_1^{(2)}D_3^{(2)}\Delta_3^{(2)}}{\pi\Delta_0^{(2)}},$$

$$q_{31}=-2\beta_3^{(2)}\frac{\gamma_3^{(2)}F_1^{(2)}\Delta_1^{(2)}}{\pi\Delta_0^{(2)}},\quad q_{23}=-2\beta_2^{(2)}\frac{\gamma_2^{(2)}E_3^{(2)}\Delta_3^{(2)}}{\pi\Delta_0^{(2)}},\quad q_{32}=-2\beta_3^{(2)}\frac{\gamma_3^{(2)}F_2^{(2)}\Delta_2^{(2)}}{\pi\Delta_0^{(2)}}.$$

Introducing the notations

$$p^-(t)=\begin{cases}p(t), & 0<t<1\\ 0, & t>1\end{cases},\qquad \psi(t)=f(-t),\qquad p_0^-(t)=\begin{cases}p_0(t), & 0<t<1\\ 0, & t>1\end{cases},$$

$$F^+(t)=\begin{cases}0, & 0<t<1\\ \nu_1(t), & t>1\end{cases},\qquad D^-(t)=\begin{cases}D(t), & 0<t<1\\ 0, & t>1\end{cases},$$

$$K_1(t,x)=\frac{\lambda_1}{t-x}+\frac{\lambda_2}{t+x}+R_1(t,x),\qquad K_2(t,x)=\frac{\lambda_3}{t-x}+\frac{\lambda_4}{t+x}+R_4(t,x),$$

we have the system of integral equations

$$\frac{d}{dx}\left(\int_0^\infty K_1(t,x)p^-(t)dt+\int_0^\infty R_2(-t,x)\psi(t)dt\right)=\frac{1}{D^-(x)}\int_0^x dt\int_0^t[\,p_0^-(\tau)-p^-(\tau)]d\tau+F^+(x),\qquad x>0 \tag{15}$$

$$\int_0^\infty K_2(-t,-x)\psi(t)dt+\int_0^\infty R_3(t,-x)p^-(t)dt=q_0(-x),\qquad x>0 \tag{16}$$

To solve equations (15) and (16), when $D(x)=h_0x^3$, $x\in(0,1)$ (for example, when $h_1(x)=h_1x$, $E_1(x)=E_1=$ const), making the substitution $t=e^\zeta$, $x=e^\xi$ with notation $\varphi(x)=\int_0^x dt\int_0^t[\,p_0^-(\tau)-p^-(\tau)]d\tau$ and using generalized Fourier transform [28], we obtain the system

$$\begin{aligned}G_1(s)F^-(s)+G_2(s)\Phi(s-2i)&=\Psi^+(s)+P(s)\\ G_3(s)F^-(s)+G_4(s)\Phi(s-2i)&=Q(s)\end{aligned}\qquad -\infty<s<\infty, \tag{17}$$

where

$$F^-(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\varphi(e^{\zeta})e^{i\zeta s}d\zeta,\qquad \Phi(s)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\psi(e^{\zeta})e^{i\zeta s}d\zeta,$$

$$\Psi^+(s)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{3\xi}F^+(e^{\zeta})e^{i\zeta s}d\zeta,\qquad P(s)=-\frac{G_1(s)-h_0^{-1}}{s(s-i)}P_1(s-2i),\qquad P_1(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}p_0^-(e^{\zeta})e^{i\zeta s}d\zeta,$$

$$Q(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}q_0(-e^{\zeta})e^{2\xi}e^{i\zeta s}d\zeta+\frac{iG_3(s)}{s(s-i)}P_1(s-2i),$$

$$G_1(s)=-\sqrt{\frac{\pi}{2}}s(s-2i)(s-i)\left[\lambda_1\mathrm{cth}\lambda s+\frac{\lambda_2}{\mathrm{sh}\pi s}+\sum_{m\neq n=1}^{3}\frac{\omega_{mn}}{\mathrm{sh}\pi s}\frac{\beta_m^{(1)}}{\beta_n^{(1)2}}\exp\left(is\ln\frac{\beta_m^{(1)}}{\beta_n^{(1)}}\right)\right]-\frac{1}{h_0},$$

$$G_2(s)=\sqrt{\frac{\pi}{2}}(s-2i)\sum_{m\neq n=1}^{3}\frac{\alpha_{mn}}{\mathrm{sh}\pi s}\frac{\beta_m^{(1)}}{\beta_n^{(2)2}}\exp\left(is\frac{\beta_m^{(1)}}{\beta_n^{(2)}}\right),$$

$$G_3(s)=\sqrt{\frac{\pi}{2}}s(s-i)\sum_{m\neq n=1}^{3}\frac{r_{mn}}{\mathrm{sh}\pi s}\frac{\beta_m^{(2)}}{\beta_n^{(1)2}}\exp\left(is\ln\frac{\beta_m^{(2)}}{\beta_n^{(1)}}\right),$$

$$G_4(s)=i\sqrt{\frac{\pi}{2}}\left[\lambda_3\mathrm{cth}\lambda s+\frac{\lambda_4}{\mathrm{sh}\pi s}+\sum_{m\neq n=1}^{3}\frac{q_{mn}}{\mathrm{sh}\pi s}\frac{\beta_m^{(2)}}{\beta_n^{(2)2}}\exp\left(is\ln\frac{\beta_m^{(2)}}{\beta_n^{(2)}}\right)\right].$$

Excluding from equation (17) function $\Phi(s)$, we obtain the Riemann problem:

$$\frac{\Psi^+(s)}{\sqrt{s+i}}=\frac{G(s)}{\sqrt{1+s^2}}F^-(s)\sqrt{s-i}+\frac{H(s)}{\sqrt{s+i}}$$
$$G(s)=\frac{G_1(s)G_4(s)-G_2(s)G_3(s)}{G_4(s)},\qquad H(s)=\frac{Q(s)G_2(s)-P(s)G_4(s)}{G_4(s)}. \tag{18}$$

By virtue of functions $\Psi^+(s)$ and $F^-(s)$ definition, they will be boundary values of the functions which are holomorphic in the upper and lower half-planes, respectively.

The problem can be formulated as follows: it is required to obtain the function $\Psi^+(z)$, holomorphic in the half-plane $\mathrm{Im}\,z>0$ and which vanishes at infinity, and the function $F^-(z)$, holomorphic in the half-plane $\mathrm{Im}\,z<1$ (with the exception of a finite number of zeros of function $G(z)$) which vanishes at infinity and are continuous on the real axis by equation (18).

Since $\mathrm{Re}\,G_0(s)>0$ and $G_0(\infty)=G_0(-\infty)=1$, we have $\mathrm{Ind}G_0(s)=0$, $G_0(s)=G(s)/\sqrt{1+s^2}$.

The solution of equation (18) has the following form [29]:

$$F^-(z)=\frac{\widetilde{X}(z)}{\sqrt{z-i}},\quad \mathrm{Im}\,z\leq 0;\qquad \Psi^+(z)=\widetilde{X}(z)\sqrt{z+i},\quad \mathrm{Im}\,z>0$$
$$F^-(z)=(\Psi^+(z)-H(z))G^{-1}(z),\qquad 0<\mathrm{Im}\,z<1, \tag{19}$$

where

$$\widetilde{X}(z)=X(z)\left\{-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(t)dt}{X^+(t)\sqrt{t+i}(t-z)}\right\},\qquad X(z)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln G_0(t)dt}{t-z}\right\}$$

(here the integral should be understood in the sense of the Cauchy principal value).

Using the formula $\varphi''(x)=\frac{\varphi_0''(\ln x)-\varphi_0'(\ln x)}{x^2}$ and applying the inverse transformation $\varphi_0'(\ln x)=-\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}s\Phi^-(s)e^{-is\ln x}ds$, $\varphi_0''(\ln x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}s^2\Phi^-(s)e^{-is\ln x}ds$.

We will investigate the behavior of the function $p_0(x)-p(x)=\varphi''(x)$ in the neighborhood of the points $z=0$ and $z=1$.

We obtain by an inverse transformation: $p^0(x)-p(x)=O(1)$, $x\to 1-$.

The poles of the function $F^-(z)$ in the domain $D_0 = \{z : 0 < \mathrm{Im}\, z < 1\}$ may be zeros of the function $G(z)$. It can be shown that the function $G(z)$ has no zeros in the strip $0 < \mathrm{Im}\, z < 3/2$. Then, applying Cauchy's theorem to the functions $e^{-i\xi z} iz\Phi^-(z)$, $e^{-i\xi z} z^2\Phi^-(z)$, we obtain the following estimate:

$$p^0(x) - p(x) = O(x^{\delta-2}), \qquad x \to 0+, \quad \delta > \frac{3}{2}.$$

Since $\psi(t) = f(-t)$, crack opening behavior has the form

$$f(x) = O(x^{-1/2+\omega}), \qquad x \to 0-, \quad 0 < \omega < 1/2.$$

## 3. Conclusions

In this paper we consider a piecewise-homogeneous anisotropic plate of piezo-electric material, weakened by a crack that goes out at the interface of two materials. The crack propagation is delayed by the inclusion of an elastic non-homogeneous beam.

The resulting boundary-value contact problem is reduced to a system of singular integro-differential equations, which is reduced to the Riemann boundary value problem by the use of integral transformations.

The main result of this paper is that the solution of the problem was obtained in an explicit form. Also, on the basis of an asymptotic analysis, it turned out that the normal contact stress along the contact line of the inclusion with the plate is bounded at one end of the inclusion. At the other end (cusped and coming out at the interface of the two materials) of the inclusion, the normal contact stress admits the singularity with order less than 1/2. The order of the singularity can also be decreased by choosing the geometric and physical parameters of the problem. At the end of the crack, the singularity of the crack opening function is also decreased under the action of the inclusion.

The obtained results are significant in the problems of fracture mechanics and in those of stress concentration. These results can be successfully applied in geological and geophysical problems, particularly in the tasks of reinforcement of constructions and rocks and in delaying landslide processes.

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