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Integral kernel operators on spaces of analytic functions

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Dr. José Ángel Peláez Márquez

THESIS BY COMPILATION

This doctoral thesis has been is written in the format of collection of manuscripts following the regulation of the "Escuela de Doctorado (ED-UMA)" of the University of Málaga.

The four manuscripts that support this thesis are the following, in which the authors are listed alphabetically as usual and my contribution to the research has been proportional to the number of authors in each work:

(I) **Hilbert-type operator induced by radial weight**
J.A. Peláez and E. de la Rosa, J. Math. Anal. Appl. 495 (2021), no. 1, Paper No. 124689, 22 pp. DOI: `https://doi.org/10.1016/j.jmaa.2020.124689`. Journal impact factor: 1.417 (77/333 Mathematics).

(II) **Hilbert-type operator induced by radial weight on Hardy spaces.**
N. Merchán, J.A. Peláez and E. de la Rosa Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 118 (2024), no. 1, Paper No. 2, 36 pp. DOI `https://doi.org/10.1007/s13398-023-01500-z` Journal impact factor: 1.8 (39/489 Mathematics).

(III) **Bergman projection on Lebesgue space induced by doubling weight**
J.A. Peláez, E. de la Rosa and J. Rättyä, Results Math.79(2024), no.1, Paper No. 27, 19 pp. DOI: `https://doi.org/10.1007/s00025-023-02048-5`. Journal impact factor: 1.1 (98/489 Mathematics).

(IV) **Littlewood-Paley inequalities for fractional derivative on Bergman spaces**
J.A. Peláez and E. de la Rosa, Ann. Fenn. Math. 47 (2022), no. 2, 1109–1130. DOI: `https://doi.org/10.54330/afm.121831`. Journal impact factor: 0.9 (147/330 Mathematics)

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Contents

Resumen

Esta tesis, avalada por los trabajos (I), (II), (III) y (IV), presentada en la modalidad de Tesis por Compendio de Publicaciones, se sitúa en el contexto de la teoría de operadores concretos sobre espacios de funciones, habitualmente analíticas y definidas en el disco unidad del plano complejo $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Dentro de esta línea desempeña un papel fundamental el estudio de operadores integrales inducidos por núcleos, a menudo reproductores en algún sentido, ya que muchos de los operadores más importantes están definidos o se pueden representar de esta manera. A modo de ejemplo, basta citar la proyección de Bergman [7, 8, 12, 15, 46, 48] y el operador de Hilbert [5, 13, 14, 17, 29]. Esta línea de investigación es una de las más prolíficas en el área de Análisis Matemático del Departamento de Análisis Matemático, Estadística e Investigación Operativa y Matemática Aplicada de la Universidad de Málaga.

La influencia de la teoría de pesos en el desarrollo de esta tesis es evidente, ya que consideramos tanto espacios de funciones con pesos como operadores concretos inducidos por pesos. Concretamente, nos referimos como peso a una función $\omega : \mathbb{D} \to \mathbb{R}$ no negativa e integrable, y cuando el peso satisface $\omega(z) = \omega(|z|)$, $z \in \mathbb{D}$, se dice que es radial.

A la hora de abordar preguntas sobre operadores concretos inducidos por pesos o espacios de funciones analíticas con pesos, una de las principales consideraciones a tener en cuenta es si dicho peso es radial o no. Si bien los resultados que involucran pesos no necesariamente radiales son bastante habituales en la literatura sobre espacios de funciones medibles, tales resultados son mucho menos frecuentes en el contexto de funciones analíticas con las que nosotros trabajamos, debido en parte a la dificultad para obtener resultados de esa magnitud.

Uno de los espacios de funciones analíticas esencial en esta tesis es el espacio de Bergman pesado A^p_ω. Entre los obstáculos más determinantes para trabajar en espacios de Bergman inducidos por pesos no radiales se encuentra el hecho de que la teoría aún está en sus primeras etapas de desarrollo, ya que muchos de los pilares básicos y fundamentales para que la teoría pueda avanzar no son ciertos o no se conocen. Para ilustrarlo, basta citar que mientras que los polinomios son densos en el espacio de Bergman A^p_ω para ω radial, esto no es cierto en general pues existen pesos no radiales que inducen espacios de Bergman donde los polinomios no son densos [20]. A pesar de que se conocen condiciones suficientes útiles, obtener una descripción de los pesos ω tales que los polinomios son densos en A^p_ω es uno de los problemas abiertos de más interés en esta teoría [45, Chapter 1].

Los trabajos que avalan esta tesis se situarán en el contexto de la teoría de pesos radiales.

Esta tesis doctoral está estructurada en tres bloques; el primero de ellos, formado por el capítulo 1, constituye el marco teórico en el que se encuadra la tesis y la formulación de los problemas abordados; y el segundo, formado por los capítulos 2, 3 y 4, consiste en la colección de los trabajos que la avalan. El bloque final recoge conclusiones que se pueden deducir de los resultados que hemos obtenido, así como futuras líneas de investigación ligadas a los problemas que se tratan en la tesis.

En cuanto al marco teórico, en la sección 1.1 se introducirán los conceptos básicos relativos a los espacios de las funciones analíticas en los cuales estudiaremos la acción de los operadores. Por su parte, la sección 1.2 ofrece unas pinceladas sobre la teoría de pesos generales, y en concreto, de pesos radiales que satisfacen ciertas propiedades de duplicación.

Con el objetivo de resumir los contenidos de esta tesis, definimos las clases de pesos con las que trabajaremos. Decimos que un peso radial ω es doblante por arriba si verifica que existe $C = C(\omega) > 1$ tal que

$$\widehat{\omega}(r) \leqslant C\widehat{\omega}\left(\frac{1+r}{2}\right) \text{ para todo } 0 \leqslant r < 1,$$

donde $\widehat{\omega}(z) = \widehat{\omega}(|z|) = \int_{|z|}^1 \omega(s)ds$ denota la cola integral del peso. Denotaremos por $\widehat{\mathcal{D}}$ a esta clase de pesos radiales, la que, como veremos, surge de forma natural en el estudio de diversos problemas sobre teoría de operadores concretos en espacios de funciones analíticas. Un peso radial $\omega \in \check{\mathcal{D}}$ si existen $K = K(\omega) > 1$ y $C = C(\omega) > 1$ tales que

$$\widehat{\omega}(r) \geqslant C\widehat{\omega}\left(1 - \frac{1-r}{K}\right)$$

para todo $0 \leqslant r < 1$, o equivalentemente existen $K = K(\omega) > 1$ y $C = C(\omega) > 0$ tales que

$$\int_r^{1-\frac{1-r}{K}} \omega(s)ds \geqslant C\widehat{\omega}(r),\ 0 \leqslant r < 1.$$

Además, un peso ω se dice que pertenece a la clase $\mathcal{M}$ si existen constantes $C = C(\omega) > 1$ y $K = K(\omega) > 1$ tales que $\omega_x \geqslant C\omega_{Kx}$ para todo $x \geqslant 1$, donde $\omega_x = \int_0^1 s^x\omega(s)ds$, $x \geqslant 0$, denotan los momentos inducidos por el peso ω. Por último, decimos que el peso es doblante si $\omega \in \mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$.

Cabe destacar la relevancia de los núcleos reproductores de Bergman como objeto de estudio en esta tesis, ya que influyen notablemente en el núcleo de los operadores integrales que abordaremos, por lo cual se dedica la sección 1.3 a la introducción de los mismos y la discusión de algunas de sus propiedades. Su existencia se debe a que para pesos radiales la convergencia en el espacio de Hilbert A^2_ω implica la convergencia uniforme en subconjuntos compactos de $\mathbb{D}$, por lo que los evaluadores puntuales son funcionales lineales y continuos. Entonces el teorema de Representación de Riesz garantiza la existencia de núcleos reproductores $B_z^\omega \in A_\omega^2$ de manera que se satisface la siguiente fórmula de reproducción

$$f(z) = \langle f, B_z^\omega \rangle_{A_\omega^2} = \int_{\mathbb{D}} f(\zeta)\overline{B_z^\omega(\zeta)}\omega(\zeta)dA(\zeta), \quad f \in A_\omega^2.$$

Por otra parte, la teoría general sobre espacios de Hilbert asegura la acotación de la proyección ortogonal P_ω de L_ω^2 a A_ω^2, la cual usando la fórmula de reproducción anterior, y el hecho que P_ω es idempotente y autoadjunta, permite escribirla mediante la siguiente expresión

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B_z^\omega(\zeta)}\omega(\zeta)\, dA(\zeta), \quad z \in \mathbb{D}.$$

Este operador integral se denomina la *proyección de Bergman* inducida por el peso ω, y será uno de los operadores estudiados en esta tesis. El estudio de propiedades esenciales de proyecciones que van de espacios de funciones medibles a espacios de funciones analíticas es básico dentro de esta teoría y, en particular, encontrar proyecciones analíticas acotadas en espacios de Lebesgue L^p constituye uno de los pilares para el desarrollo de la misma. Su especial relevancia se debe no solo a la dificultad que conlleva abordarlo, sino también a las numerosas y fundamentales aplicaciones que tiene. A modo de ejemplo, aunque estos temas serán discutidos más adelante de forma extensa, basta mencionar que la existencia de proyecciones analíticas acotadas resulta de gran utilidad para establecer, entre otros resultados, relaciones de dualidad para espacios de funciones analíticas y obtener normas equivalentes en términos de las derivadas, conocidas como fórmulas de Littlewood-Paley. Dedicaremos la sección 1.5.2 a la introducción y planteamiento del problema abordado referente a la proyección de Bergman y el capítulo 3 a la discusión de los resultados obtenidos en el artículo (III).

El otro operador integral que se estudia en profundidad en esta tesis, cuyo núcleo viene definido en términos de núcleos reproductores de Bergman, está estrechamente relacionado con la matriz de Hilbert

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & . \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & . \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & . \\ . & . & . & . \end{pmatrix}.$$

Esta matriz induce el denominado como *operador de Hilbert* considerando su acción sobre la sucesión de los coeficientes de Taylor $\{\widehat{f}(k)\}_k$ de una función $f \in \mathcal{H}(\mathbb{D})$,

$$H(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1} \right) z^n,$$

siempre que el lado derecho tenga sentido y defina una función analítica. Basta suponer, por ejemplo, que f pertenece al espacio de Hardy H^1. El operador de Hilbert es un prototipo de operador de Hankel que ha atraído mucha atención durante los últimos años por parte de muchos investigadores. Se pueden encontrar en la literatura numerosos resultados sobre acotación, compacidad, cálculo de su norma y espectro [3, 5, 13, 14, 17] que muestran su relación con otras áreas del análisis tales como las funciones de Legendre de primer tipo, los operadores de composición o la proyección de Riesz, también conocida como proyección de Szegö.

En muchos de estos trabajos juega un papel fundamental la expresión integral del operador

$$H(f)(z) = \int_0^1 f(t) \frac{1}{1 - tz}\, dt,$$

que es válida para funciones $f \in H^1$. En la tesis consideraremos una generalización que consiste en reemplazar el núcleo $\frac{1}{1-tz}$ de la expresión integral del operador clásico de Hilbert por un núcleo más general. En concreto para un peso radial ω, consideramos el operador de tipo Hilbert H_ω definido como el operador integral

$$H_\omega(f)(z) = \int_0^1 f(t) \left(\frac{1}{z} \int_0^z B_t^\omega(\zeta) d\zeta \right) \omega(t) dt.$$

Cabe destacar que cuando $\omega = 1$ recuperamos la expresión integral del operador H. De forma más específica, el problema abordado en los trabajos (I) y (II) consiste en describir los pesos radiales ω tales que el operador H_ω mantiene ciertas características del operador clásico.

En esta tesis se dedica la sección 1.5.1 a la introducción y planteamiento del problema abordado en los trabajos (I) y (II), y el capítulo 2 se dedica a la exposición y discusión de los resultados obtenidos, así como la colección de ambos trabajos, donde pueden encontrarse también todas las pruebas.

Una de las primeras dificultades que encontramos al abordar la acotación de H_ω en espacios de funciones analíticas es que el núcleo del operador generalizado no es tan manejable como el del operador clásico. De hecho, bajo la única hipótesis de que el peso sea radial, los núcleos reproductores de Bergman B_z^ω pueden tener ceros [11, 51] y, hasta donde sabemos, no se conoce una fórmula explícita para su expresión. Esta falta de información sobre el núcleo plantea serias dificultades para resolver los problemas que se acaban de mencionar.

Entre otros resultados, se sabe que el operador de Hilbert no está acotado en el espacio de las funciones holomorfas y acotadas H^∞, pero sí que está acotado del espacio H^∞ al espacio de funciones analíticas de oscilación media acotada BMOA, y en consecuencia al espacio de Bloch $\mathcal{B}$ ([29]). Tras observar que el operador H_ω tampoco es acotado en H^∞ para ningún peso radial, en los trabajos (I) y (II) se caracterizan los pesos radiales tales que se satisface que $H_\omega : H^\infty \to \mathcal{B}$ y $H_\omega : H^\infty \to$ BMOA es acotado. Curiosamente, esto ocurre simultáneamente y la condición sobre el peso que describe este hecho es, precisamente, $\omega \in \widehat{\mathcal{D}}$. Entre las técnicas empleadas señalamos que se utilizan ciertas descripciones de la clase $\widehat{\mathcal{D}}$, algo que ocurre en muchos resultados de la tesis, y estimaciones puntuales para los núcleos reproductores.

Además, en cuanto a la acción del operador de Hilbert en espacios de Hardy H^p, $1 < p < \infty$, se puede encontrar en [14] que el operador es acotado en H^p si y solo si $1 < p < \infty$. En el artículo (II) se describen completamente los pesos radiales ω tales que H_ω es acotado en H^p, lo que ocurre si y solo si $\omega \in \widehat{\mathcal{D}}$ y la condición de tipo Muckenhoupt

$$M_p(\omega) = \sup_{0<r<1} \left(\int_0^r \frac{1}{\widehat{\omega}(t)^p} dt \right)^{\frac{1}{p}} \left(\int_r^1 \left(\frac{\widehat{\omega}(t)}{1-t} \right)^{p'} dt \right)^{\frac{1}{p'}} < \infty$$

se satisface. Merece la pena resaltar la gran variedad de técnicas empleadas en la prueba de este resultado, tales como las relaciones de inclusión entre el espacio de Hardy y otros espacios relacionados, descomposición de la norma en espacios de norma mixta [33], estimaciones de las medias integrales de los núcleos reproductores [46], y herramientas de análisis real tales como las desigualdades de Hardy clásicas con pesos [36].

A pesar de que el operador de Hilbert no es acotado en H^1 [14], en los trabajos (I) y (II) también se obtienen resultados sobre la acotación en H^1, ya que se describen todos los pesos radiales tales que $H_\omega : H^1 \to H^1$. Se demuestra que la acotación en H^1 es equivalente a que $\omega \in \widehat{\mathcal{D}}$ y

$$M_1(\omega) = \sup_{a\in[0,1)} \frac{\widehat{\omega}(a)}{1-a} \left(\int_0^a \frac{ds}{\widehat{\omega}(s)} \right) < \infty.$$

Con el objetivo de probar este resultado empleamos como herramientas fundamentales, además de algunas de las ya citadas anteriormente, diferentes descripciones geométricas de las medidas de Carleson en diversos espacios y propiedades básicas de una base universal de polinomios de Cesàro. Debido a la especial relevancia de esta herramienta en varios de los trabajos de esta tesis, se dedicará la sección 1.4 a la discusión de la misma.

Por último, merece la pena señalar que la condición $\omega \in \widehat{\mathcal{D}}$ y $M_p(\omega) < \infty$ son independientes, pues existen pesos que no pertenecen a $\widehat{\mathcal{D}}$ que verifican esa condición, por ejemplo, los pesos exponenciales. Y recíprocamente, también podemos encontrar pesos que pertenecen a $\widehat{\mathcal{D}}$ que no verifican la condición de Muckenhoupt, como es el caso de ciertos pesos estándar.

A fin de ampliar nuestro conocimiento sobre la acción del operador H_ω en los espacios de Hardy H^p, $1 \leqslant p < \infty$, se aborda el operador definido en un espacio más grande, el espacio de Lebesgue de funciones medibles en el radio $[0,1)$, denotado por $L^p_{[0,1)}$. Se demuestra que el operador $H_\omega : L^p_{[0,1)} \to H^p$ es acotado si y solo si $\omega \in \widehat{\mathcal{D}}$ y

$$m_p(\omega) = \sup_{0<r<1} \left(1 + \int_0^r \frac{1}{\widehat{\omega}(t)^p} dt \right)^{\frac{1}{p}} \left(\int_r^1 \omega(t)^{p'} dt \right)^{\frac{1}{p'}} < \infty.$$

En cuanto al caso $p = 1$ también se demuestra que $H_\omega : L^1_{[0,1)} \to H^1$ es acotado si y solo si $\omega \in \widehat{\mathcal{D}}$ y

$$m_1(\omega) = \operatorname*{ess\,sup}_{t\in[0,1)} \omega(t) \left(1 + \int_0^t \frac{ds}{\widehat{\omega}(s)}\right) < \infty.$$

Conviene señalar que estos dos últimos resultados, además de resolver la acotación del operador en un espacio relacionado, nos dan condiciones suficientes para garantizar la acotación de $H_\omega : H^p \to H^p$. Este hecho hace necesario establecer una comparativa entre las condiciones suficientes $M_p(\omega) < \infty$ y $m_p(\omega) < \infty$. Es claro, por la observación anterior, que $m_p(\omega) < \infty$ implica $M_p(\omega) < \infty$ si $\omega \in \widehat{\mathcal{D}}$, y además son equivalentes si el peso $\omega \in \widehat{\mathcal{D}}$ y verifica la estimación puntual

$$\omega(t) \lesssim \frac{\widehat{\omega}(t)}{1-t}, \quad t \in [0,1).$$

Además, construimos pesos $\omega \in \mathcal{D}$ que presentan un fuerte comportamiento oscilatorio y que verifican $M_p(\omega) < \infty$ pero $m_p(\omega) = \infty$.

A fin de concluir esta extensa y completa discusión sobre la acción del operador en espacios de Hardy, se puede observar que para cualquier peso radial ω, la acotación del operador H_ω en H^p implica la acotación en el caso $p = \infty$, esto es $H_\omega : H^\infty \to \mathrm{BMOA}$ acotado. Este hecho es un fenómeno más general, ya que demostramos que si $1 \leqslant q < p < \infty$ y H_ω es acotado en H^q entonces también lo está en H^p.

Por otro lado, se conocen resultados sobre la acción del operador clásico de Hilbert en espacios de Bergman [13, 29]. En el artículo (I) se considera este problema y se describen los pesos radiales ν, y los pesos $\omega \in \widehat{\mathcal{D}}$, tales que el operador H_ω es acotado desde un espacio de Lebesgue con peso $L^p_{\widehat{\nu},[0,1)}$ al espacio de Bergman A^p_ν. Debido a que el espacio de Bergman A^p_ν está contenido en el espacio de Lebesgue $L^p_{\widehat{\nu},[0,1)}$, el resultado probado nos aporta además condiciones suficientes para la acotación del operador en A^p_ν.

Finalmente merece la pena subrayar la conexión hallada en el trabajo (I) entre la acción del operador de tipo Hilbert H_ω y de la proyección de Bergman P_ω inducida por un peso radial en espacios de Lebesgue. En particular, la conexión hallada nos permite afirmar que si $1 < p < \infty$, entonces $P_\omega : L^p_{\widehat{\nu}} \to A^p_{\widehat{\nu}}$ y $H_\omega : L^p_{\widehat{\nu},[0,1)} \to A^p_\nu$ están simultáneamente acotados bajo ciertas condiciones de los pesos, concretamente siempre que $\omega \in \mathcal{D}$, $\nu \in \widehat{\mathcal{D}}$.

Esta conexión entre el operador de tipo Hilbert H_ω y la proyeccion de Bergman P_ω da valor al trabajo realizado sobre el operador H_ω y proporciona otra razón para el estudio de este nuevo operador; ya que no solo los problemas abordados son interesantes en sí mismos, sino que también nos podría permitir usar el operador H_ω como "toy model" para atacar algunas de las cuestiones que aún permanecen abiertas sobre la acción de la proyección de Bergman P_ω en espacios de Lebesgue. El artículo (III) se centra, precisamente, en una de estas cuestiones.

Merece la pena recalcar, como se comentó anteriormente, que el estudio de la proyección de Bergman en espacios de Lebesgue constituye un pilar fundamental en el desarrollo de la teoría de operadores sobre espacios de Bergman. Además del reto matemático que supone resolver los problemas abiertos existentes, la potencial aplicabilidad de los resultados obtenidos es otro de los motivos por el que este tema de investigación ha suscitado bastante interés entre muchos investigadores destacados en el área [7, 8, 10, 12, 15, 46, 48].

El problema de describir los pesos radiales ω tales la proyección de Bergman P_ω verifica la desigualdad

$$\|P_\omega(f)\|_{L^p_\omega} \leqslant C\|f\|_{L^p_\omega}, \quad f \in L^p_\omega, \quad 1 < p < \infty, \quad p \neq 2, \tag{1}$$

fue planteado por Dostanić en [15] y permancece abierto desde hace décadas. En su trabajo, demostró que la siguiente condición es necesaria para que la proyección esté acotada en el espacio de Lebesgue L^p_ω:

$$\omega_{np+1}^{\frac{1}{p}}\omega_{np'+1}^{\frac{1}{p'}} \lesssim \omega_{2n+1}, \quad n \in \mathbb{N}.$$

Se deduce de la desigualdad de Hölder que la desigualdad contraria $\omega_{np+1}^{\frac{1}{p}}\omega_{np'+1}^{\frac{1}{p'}} \geqslant \omega_{2n+1}$ es siempre cierta, luego la condición de Dostanić es una desigualdad de Hölder al revés en términos de momentos. Por otra parte, la mejor condición suficiente que se conoce para que (1) ocurra es $\omega \in \widehat{\mathcal{D}}$ y fue probada en [48, Theorem 7].

Con respecto al problema de dos pesos, quizás el resultado más conocido es el debido a Bekollé y Bonami [7, 8], quienes describieron todos los pesos arbitrarios ν, no necesariamente radiales, tales que la proyección P_α inducida por un peso estándar $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$, $\alpha > -1$, es acotada en L^p_ν; es decir, tales que se satisface la desigualdad $\|P_\alpha(f)\|_{L^p_\nu} \leqslant C\|f\|_{L^p_\nu}$ para toda $f \in L^p_\nu$.

Recordemos que los núcleos reproductores de Bergman que determinan las proyecciones de Bergman estándar P_α tienen propiedades que hacen más fácil su manejo, tales como una expresión explícita, el hecho de no tener ceros, etc. Sin embargo, en el caso de suponer únicamente que el peso sea radial, esto no es cierto.

Hasta donde sabemos, el problema de describir los pesos radiales ω, ν tales que $P_\omega : L^p_\nu \to L^p_\nu$ es acotado permanece abierto aún a día de hoy, incluso cuando ν es estándar. El tercer trabajo de esta tesis (III) se centra, precisamente, en estudiar dicho problema. En la sección 1.5.2 se ofrece una introducción al problema y el capítulo 3 se dedica a la discusión del mismo. En ese artículo, también estudiaremos la proyección maximal definida como

$$P^+_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)|B^\omega_z(\zeta)|\omega(\zeta)\, dA(\zeta), \quad z \in \mathbb{D}.$$

Lo primero que probamos en el artículo (III) es que la condición

$$D_p(\omega,\nu) = \sup_{n\in\mathbb{N}\cup\{0\}} \frac{(\nu_{np+1})^{\frac{1}{p}}\,(\sigma_{np'+1})^{\frac{1}{p'}}}{\omega_{2n+1}} < \infty$$

donde $\sigma = \left(\frac{\omega}{\nu^{\frac{1}{p}}}\right)^{p'}$, es necesaria para que la proyección P_ω esté acotada en el espacio de Lebesgue L^p_ν para cualesquiera pesos radiales. Observemos que esta condición generaliza la obtenida por Dostanić en el caso $\omega = \nu$. En cuanto a la dirección opuesta, probamos que la condición $D_p(\omega,\nu) < \infty$ implica la acotación de $P_\omega : L^p_\nu \to L^p_\nu$ para $\nu \in \mathcal{D}$. En conclusión, se demuestra en (III) que si $\nu \in \mathcal{D}$, entonces la condición $D_p(\omega,\nu) < \infty$ es equivalente a que $P_\omega : L^p_\nu \to L^p_\nu$ y $P^+_\omega : L^p_\nu \to L^p_\nu$ estén acotadas. La prueba de este resultado se basa fundamentalmente en obtener una mejora de [48, Teorema 13].

Otro problema de gran interés dentro de la teoría de funciones y la teoría de operadores en espacios de funciones analíticas es la obtención de normas equivalentes en términos de la derivada. Un caso particular, pero muy significativo dentro de esta temática es la obtención de las conocidas como fórmulas de Littlewood-Paley. Su valor se debe a que nos proporcionan normas equivalentes que las convierten en un ingrediente fundamental para abordar ciertos problemas concretos; por ejemplo, estas fórmulas permiten deshacerse de la integral al estudiar la acción de ciertos operadores integrales . Quizás una de sus aplicaciones más interesantes es que las fórmulas de Littlewood-Paley nos permiten demostrar la acotación de la proyección de Bergman en determinados espacios, y recíprocamente, las proyecciones de Bergman acotadas son utilizadas en ocasiones para probar fórmulas de tipo Littlewood-Paley.

La fórmula de Littlewood-Paley clásica asegura que para todo $n \in \mathbb{N}$ y $0 < p < \infty$

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np}\omega(z)\,dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \tag{2}$$

para ω un peso estándar $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$, $-1 < \alpha < \infty$. Peláez y Rättyä describieron todos los pesos radiales tales que se verifica la fórmula anterior, que son precisamente los pesos que pertenecen a $\mathcal{D}$ [48, Teorema 5]. Además probaron que la desigualdad

$$\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np}\omega(z)\,dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p \lesssim \|f\|^p_{A^p_\omega} \tag{3}$$

se satisface si y solo si $\omega \in \widehat{\mathcal{D}}$ [48, Teorema 6] y que una condición necesaria para que se satisfaga la desigualdad contraria es $\omega \in \mathcal{M}$.

Con el objetivo de establecer notación necesaria para formular el nuevo problema abordado, definimos el espacio de tipo Dirichlet $D^p_{\nu,k}$ de las funciones analíticas en el disco tales que $\|f\|^p_{D^p_{\nu,k}} = \sum_{j=0}^{k-1} |f^{(j)}(0)|^p + \int_{\mathbb{D}} |f^{(k)}(z)|^p (1-|z|)^{kp}\nu(z)\,dA(z) < \infty$. Observemos que cuando $\nu \in \mathcal{M}\backslash\widehat{\mathcal{D}}$, los espacios $D^p_{\nu,k}$ y A^p_ν son distintos, debido a los resultados expuestos anteriormente.

El nuevo problema que nos planteamos en (III) consiste en describir la acotación de la proyección de Bergman en el espacio de tipo Dirichlet $P_\omega : L^p_\nu \to D^p_{\nu,k}$ cuando $\omega \in \widehat{\mathcal{D}}$ y $\nu \in \mathcal{M}$. Bajo esas condiciones en los pesos, demostramos que son equivalentes la condición $D_p(\omega,\nu) < \infty$, que $P_\omega : L^p_\nu \to L^p_\nu$ esté acotada, y que $P_\omega : L^p_\nu \to D^p_{\nu,k}$ también lo esté.

La demostración de este resultado utiliza fuertemente las fórmulas de Littlewood-Paley [48, Teoremas 5 y 6], manteniendo el espíritu de conexión entre este tipo de fórmulas y la acotación de la proyección de Bergman. En las pruebas se emplea un tipo de test de Schur y las estimaciones de las medias integrales de las derivadas de los núcleos reproductores de Bergman obtenidas en [48].

Por último, señalamos que Dostanić demostró en [15] que la proyección P_ω no es acotada en L^p_ω en el caso de pesos exponenciales $\omega(z) = \exp\left(\frac{\alpha}{(1-|z|^2)^\beta}\right)$, $\alpha > 0$, $0 < \beta \leqslant 1$, a menos que $p = 2$. Este resultado ha sido generalizado por varios autores [10, 12, 57], y en particular, nosotros utilizamos la condición necesaria $D_p(\omega, \nu) < \infty$ para demostrar que para pesos exponenciales $\nu(r) = \exp\left(-\frac{\alpha}{(1-r^l)^\beta}\right)$ y $\omega(r) = \exp\left(-\frac{\widetilde{\alpha}}{(1-r^{\widetilde{l}})^{\widetilde{\beta}}}\right)$, con $0 < \alpha, \widetilde{\alpha}, l, \widetilde{l} < \infty$ y $0 < \beta, \widetilde{\beta} \leqslant 1$, la proyección $P_\omega : L^p_\nu \to L^p_\nu$ es acotada si y solo si $\beta = \widetilde{\beta}$ y $\widetilde{\alpha} = \frac{2\alpha}{p}\left(\frac{\widetilde{l}}{l}\right)^\beta$.

Finalmente, en el último trabajo de esta colección (IV), se afronta el problema de extender las fórmulas (2) y (3) al contexto de la derivada fraccionaria. Debido a un resultado clásico, la fórmula de Littlewood-Paley (2) se puede generalizar para la derivada fraccionaria cuando ω es estándar [9, 21],

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} |D^\beta(f)(z)|^p (1-|z|)^{\beta p} \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}), \quad 0 < p < \infty, \tag{4}$$

donde la derivada fraccionaria de orden $\beta > 0$ viene dada por

$$D^\beta(f)(z) = \frac{2}{\Gamma(\beta+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \widehat{f}(n) z^n,$$

y, como es usual, $\{\widehat{f}(n)\}_{n\in\mathbb{N}}$ denotan los coeficientes de Taylor de la función $f \in \mathcal{H}(\mathbb{D})$.

En el artículo (IV) se busca extender las fórmulas (2) y (3) para un operador más general que D^β, y describir los pesos radiales tales que se satisfacen dichas fórmulas. Para un peso radial μ definimos la derivada fraccionaria de f inducida por μ como

$$D^\mu(f)(z) = \sum_{n=0}^{\infty} \frac{\widehat{f}(n)}{\mu_{2n+1}} z^n, \ \ z \in \mathbb{D}.$$

Este operador está bien definido y cuando μ es estándar, $\mu(z) = \beta(1-|z|^2)^{\beta-1}$, $\beta > 0$, $D^\mu(f) = D^\beta(f)$ y $\widehat{\mu}(z) \asymp (1-|z|)^\beta$, por lo que una interpretación adecuada de la fórmula de Littlewood-Paley (4) para la derivada fraccionaria generalizada D^μ sería

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} |D^\mu(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}). \tag{5}$$

En el artículo (IV), dado un peso $\mu \in \mathcal{D}$, se describen los pesos radiales ω tales que se da (5), que precisamente son los pesos $\omega \in \mathcal{D}$. No sólo se describe la equivalencia sino que también se describen los pesos radiales que verifican la desigualdad

$$\int_{\mathbb{D}} |D^\mu(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z) \leqslant C\|f\|_{A^p_\omega}^p, \quad f \in \mathcal{H}(\mathbb{D}),$$

que en este caso son los pesos $\omega \in \widehat{\mathcal{D}}$. La prueba de la suficiencia del último resultado mencionado prácticamente se reduce a demostrar una desigualdad entre las medias integrales de la derivada fracionaria de $D^\mu f$ y las medias integrales de la propia función

$$M_p(r, D^\mu f) \leqslant C\frac{M_p(r,f)}{\widehat{\mu}\left(\frac{r}{\rho}\right)}, \quad 0 < r < \rho < 1,\ 0 < p < \infty.$$

Este tipo de estimaciones para las medias integrales de las derivadas de exponente natural son bien conocidas y pueden encontrarse en la literatura [41, Lemma 3.1]. Sin embargo, la prueba en el caso clásico es completamente diferente a la que se realiza para obtener la desigualdad de arriba. En concreto, para obtenerla recurrimos a las propiedades de una base universal de polinomios de Cesàro, introducidas por Jevtić y Pavlović [28] (ver sección 1.4).

Con respecto a la prueba de (5), una manera de interpretar dicha fórmula es considerar la equivalencia $\|f\|_{A^p_\omega}^p \asymp \|D^\mu f\|_{A^p_{\omega\widehat{\mu}^p}}^p$. La forma de proceder en la prueba de este resultado es construir normas ad-hoc para los espacios de Bergman A^p_ω y $A^p_{\omega\widehat{\mu}^p}$, en la línea de los resultados de descomposición de la norma [33, 38, 44]. Estos resultados solo son válidos para $1 < p < \infty$, debido en parte a que sus pruebas utilizan la acotación de la proyección de Riesz en $L^p(\partial\mathbb{D})$ para obtener estimaciones precisas de la norma H^p de polinomios de tipo bloque $\Delta_{n_1,n_2}z = \sum_{k=n_1}^{n_2} z^k$. En lugar de este tipo de polinomios, en el artículo construimos una base universal de polinomios de Cesàro que nos permita obtener resultados de descomposición de la norma tanto para A^p_ω como para $A^p_{\omega\widehat{\mu}^p}$ y además válidos para un rango mayor de valores de p, $0 < p < \infty$. En concreto, demostramos que podemos tomar el mismo $k > 1$ y la misma familia de polinomios de Cesàro $\{V_{n,k}\}_n$ para obtener las equivalencias $\|f\|_{A^p_\omega}^p \asymp \sum_{n=0}^{\infty} \omega_{k^n}\|V_{n,k} * f\|_{H^p}^p$ y $\|D^\mu(f)\|_{A^p_{\omega\widehat{\mu}^p}}^p \asymp \sum_{n=0}^{\infty} (\omega\widehat{\mu}^p)_{k^n}\|V_{n,k} * D^\mu(f)\|_{H^p}^p$ para todo $p \in (0,\infty)$, donde $*$ denota convolución.

En esta tesis, la sección 1.6 y el capítulo 4 reúnen la exposición del problema (5), los resultados obtenidos y las demostraciones de los mismos.

Por último, se dedica el capítulo 5 a la exposición de algunos problemas abiertos y posibles futuras líneas de investigación.

Abstract

This thesis, supported by the works (I), (II), (III) and (IV) and presented in the modality of Thesis by Compilation, is framed within the context of concrete operator theory on spaces of functions, usually spaces of analytic functions and defined on the unit disc of the complex plane $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In this research line, the study of integral operators induced by kernels, often reproducing in some sense, plays a fundamental role. This topic holds notable relevance within the theory, as many of the most important operators are defined or can be represented in this way. For instance, notable examples include the Bergman projection [7, 8, 12, 15, 46, 48] and the Hilbert operator [5, 13, 14, 17, 29] . This research line is one of the most prolific within Mathematical Analysis research area of the University of Málaga.

The theory of weights is vital in this thesis. The influence of weights on its development becomes evident when considering both weighted function spaces and concrete operators induced by weights. To be precise, a function $\omega : \mathbb{D} \to \mathbb{R}$ is a weight if it is non-negative and integrable. If the weight satisfies $\omega(z) = \omega(|z|)$, $z \in \mathbb{D}$, it is said to be a radial weight.

In order to study questions regarding weighted spaces of analytic functions or concrete operators induced by weights, one of the main considerations is whether the weight is radial or not. While results involving general weights are quite typical in the literature concerning spaces of measurable functions, such results are much less common in the setting of analytic functions we work with, primarily due to the extreme difficulty in attaining them.

One of the spaces of analytic functions which is essential in this thesis is the weighted Bergman space A^p_ω. Among the most significant obstacles when considering a non-radial weight is the fact that the theory of Bergman spaces induced by non-radial weights is in its early stages of development, and some basic and essential facts necessary for the progress of the theory are either unknown or do not hold. To illustrate this, it is enough to highlight that polynomials are dense in A^p_ω if the weight ω is radial, but this is not true in general because there exist non-radial weights inducing Bergman spaces A^p_ω such that the polynomials are not dense [20]. Even though sufficient conditions are known and useful, describing the weights ω for which polynomials are dense in A^p_ω remains one of the most intriguing open problems in this theory [45, Chapter 1]. The papers supporting this thesis are framed within the context of radial weights.

This thesis is structured in three sections. The first one, consisting of the chapter 1,

provides the theoretical framework where the thesis is framed and formulates the problems addressed, whereas the second, comprising chapters 2, 3 and 4, includes the collection of the works that support it. Finally, it is included a chapter about conclusions of the obtained results and future research on the topic.

Regarding the mathematical framework, section 1.1 introduces basic concepts about spaces of analytic functions where we address the action of concrete operators. On the other hand, section 1.2 provides an overview of general aspects already existing in the literature on classes of weights, specifically radial weights satisfying some doubling properties.

To summarize the results and contents of this thesis, let us define the classes of weights that we work with. A radial weight ω is an upper doubling weight if there exists $C = C(\omega) > 1$ such that

$$\widehat{\omega}(r) \leqslant C\widehat{\omega}\left(\frac{1+r}{2}\right) \quad \text{for all } 0 \leqslant r < 1,$$

where $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)ds$ denote the integral tail of the weight. We denote by $\widehat{\mathcal{D}}$ the class of these radial weights, which naturally appears in several problems concerning concrete operator theory in spaces of analytic functions. A radial weight $\omega \in \check{\mathcal{D}}$ if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\widehat{\omega}(r) \geqslant C\widehat{\omega}\left(1 - \frac{1-r}{K}\right)$$

for all $0 \leqslant r < 1$, or equivalently there exist $K = K(\omega) > 1$ and $C = C(\omega) > 0$ such that

$$\int_r^{1-\frac{1-r}{K}} \omega(s)ds \geqslant C\widehat{\omega}(r), \; 0 \leqslant r < 1.$$

In addition, a radial weight ω belongs to the class $\mathcal{M}$ if there exist constants $C = C(\omega) > 1$ and $K = K(\omega) > 1$ such that $\omega_x \geqslant C\omega_{Kx}$ for all $x \geqslant 1$, where $\omega_x = \int_0^1 s^x\omega(s)ds$, $x \geqslant 0$ denote the moments induced by the weight ω. Finally we say that ω is a radial doubling weight if $\omega \in \mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$.

It is worth highlighting the relevance of Bergman reproducing kernels as an object of study in this thesis, since they significantly influence the kernel of the integral operators that we address. Therefore, a meaningful knowledge of these kernels is vital to understand the behavior of the concrete operator we study, and that is the reason why section 1.3 is devoted to their introduction and the discussion of some of their properties. Their existence is due to the fact that for radial weights, convergence in the Hilbert space A^2_ω implies uniform convergence on compact subsets of $\mathbb{D}$, so the point evaluations are bounded linear functionals in A^2_ω. Thus, the Riesz Representation Theorem guarantees the existence of such reproducing kernels $B^\omega_z \in A^2_\omega$ so that the following reproduction formula is satisfied

$$f(z) = \langle f, B^\omega_z \rangle_{A^2_\omega} = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\omega(\zeta)dA(\zeta), \quad f \in A^2_\omega.$$

On the other hand, the general theory of Hilbert spaces ensures the boundedness of the orthogonal projection P_ω from L^2_ω to A^2_ω. By using the previous reproducing formula and the fact that P_ω is idempotent and self-adjoint, this projection can be expressed by

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\omega(\zeta)\, dA(\zeta), \quad z \in \mathbb{D}.$$

The above integral operator is called the Bergman projection induced by the weight ω, and it will be one of the operators studied in this thesis. The study of essential properties of projections from spaces of measurable functions to spaces of analytic functions is fundamental within this theory. In particular, bounded analytic projections in Lebesgue spaces L^p constitute a cornerstone for its development. Its special relevance and appeal are not only due to the mathematical difficulties it presents, but also to the numerous and fundamental applications in pivotal questions within operator theory. For example, although these topics will be discussed extensively later, it is enough to mention that the existence of bounded analytic projections are very useful to establish, among other results, duality relationships and to obtain equivalent norms in terms of derivatives, usually called Littlewood-Paley formulas. We will dedicate section 1.5.2 to introduce and pose the problem related to the Bergman projection which is studied in (III), and chapter 3 to discuss the results obtained about it.

The other integral operator deeply studied in this thesis, whose kernel is also defined in terms of Bergman reproducing kernels, is closely related to the Hilbert matrix

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & . \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & . \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & . \\ . & . & . & . \end{pmatrix}.$$

This matrix induces an operator by considering its action on the sequence of Taylor coefficients $\{\widehat{f}(k)\}_k$ of a function $f \in \mathcal{H}(\mathbb{D})$, called the Hilbert operator, which is given by

$$H(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1} \right) z^n,$$

whenever the right-hand side makes sense and defines an analytic function. It is sufficient to assume, for example, that f belongs to the Hardy space H^1.

The Hilbert operator is a prototype of Hankel operator, which has attracted considerable attention from many researchers during the last years. A good number of results regarding the boundedness, compactness, the operator norm and spectrum can be found in [3, 5, 13, 14, 17]. They show its link with different topics in mathematical analysis such as Legendre functions of the first kind, weighted composition operators, or the Riesz projection, also called the Szegö projection.

In many of these works, the following integral expression of the operator plays a key role

$$H(f)(z) = \int_0^1 f(t)\frac{1}{1-tz}\,dt.$$

We consider a generalization which consists of replacing the kernel $\frac{1}{1-tz}$ in the integral expression of the classical Hilbert operator with a more general kernel. Being precise, for a radial weight ω, we consider the Hilbert-type operator H_ω defined as the integral operator

$$H_\omega(f)(z) = \int_0^1 f(t)\left(\frac{1}{z}\int_0^z B_t^\omega(\zeta)d\zeta\right)\omega(t)dt.$$

Note that the choice $\omega = 1$ gives the integral representation of the classical operator H. The problem addressed in works (I) and (II) consists of describing the features of a radial weight ω so that H_ω keeps some of the nice properties of the classical Hilbert operator.

Section 1.5.1 is devoted to introducing the problem we deal with in both papers (I) and (II), and chapter 2 focuses on the exposition and discussion of the results proved and some conclusions, as well as the compilation of both works where all the proofs can also be found.

A primary challenge we face when studying the boundedness of H_ω in spaces of analytic functions is the complexity of its kernel compared to that of the classical operator. In fact, if ω is only assumed to be radial, the Bergman reproducing kernels B_z^ω may have zeros [11, 51], and as far as we know there do not exist neat formulas for their expressions. This lack of information presents notable obstacles in addressing the problems just mentioned.

Among other results for the Hilbert operator, it is known that the operator $H_\omega : H^\infty \to H^\infty$ is not bounded, but it is bounded from the space H^∞ to the space of analytic functions of bounded mean oscillation BMOA [29], and as a consequence, to the Bloch space $\mathcal{B}$. A calculation shows that H_ω is also not bounded in H^∞ for any radial weight, and the results in works (I) and (II) characterize the radial weights such that $H_\omega : H^\infty \to \mathcal{B}$ and $H_\omega : H^\infty \to$ BMOA are both bounded. Curiously, both are bounded simultaneously, and the condition on the weight that describes this fact is precisely $\omega \in \widehat{\mathcal{D}}$. As for the tools employed in the proof, we mainly use some descriptions of the class $\widehat{\mathcal{D}}$ which is a recurring ingredient in a good number of proofs in this thesis, as well as pointwise estimations for Bergman reproducing kernels.

In regards to the action of H_ω on Hardy spaces H^p, $1 < p < \infty$, we know that the classical Hilbert operator is bounded in H^p if and only if $1 < p < \infty$, and the result can be found in [14]. In paper (II) the radial weights ω such that H_ω is bounded in H^p, $1 < p < \infty$, are fully described. This happens if and only if $\omega \in \widehat{\mathcal{D}}$ and the Muckenhoupt-type condition

$$M_p(\omega) = \sup_{0<r<1}\left(\int_0^r \frac{1}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}}\left(\int_r^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}dt\right)^{\frac{1}{p'}} < \infty$$

holds. It is worth mentioning the variety of ingredients employed in the proof, such as some well-known inclusions between the Hardy space H^p and other related spaces, norm decomposition results for mixed norm spaces [33], precise estimates of the integral means of the reproducing kernels [46], and even real analysis tools such as classical weighted Hardy inequalities [36].

Despite classical Hilbert operator is not bounded in H^1 [14], papers (I) and (II) also present results regarding the boundedness of the operator H_ω in H^1. In paper (II) the radial weights for which $H_\omega : H^1 \to H^1$ is bounded are described. It is shown that the boundedness in H^1 is equivalent to $\omega \in \widehat{\mathcal{D}}$ and

$$M_1(\omega) = \sup_{a\in[0,1)} \frac{\widehat{\omega}(a)}{1-a}\left(\int_0^a \frac{ds}{\widehat{\omega}(s)}\right) < \infty.$$

In order to prove this result, we mainly use some of the aforementioned tools, different geometric descriptions of Carleson measures in some spaces and basic properties of an universal Cesàro basis of polynomials. Due to the extensive use of this ingredient in this thesis, section 1.4 will be dedicated to its discussion.

Finally, it is worth noting that the condition $M_p(\omega) < \infty$ and $\omega \in \widehat{\mathcal{D}}$ are independent, since there are weights that do not belong to $\widehat{\mathcal{D}}$ which satisfy this condition, for example, exponential weights. Conversely, we can also find upper radial doubling weights which do not satisfy the Muckenhoupt-type condition $M_p(\omega) < \infty$; this happens to some standard weights.

In order to learn about the action of the operator H_ω in Hardy spaces, we deal with the operator defined in a larger space, the Lebesgue space of measurable functions on the interval $[0,1)$, denoted by $L^p_{[0,1)}$, $1 \leqslant p < \infty$. This new problem is addressed in (I), although the result shown is finally improved in (II). It is proved that the operator $H_\omega : L^p_{[0,1)} \to H^p$ is bounded if and only if $\omega \in \widehat{\mathcal{D}}$ and

$$m_p(\omega) = \sup_{0<r<1}\left(1+\int_0^r \frac{1}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}}\left(\int_r^1 \omega(t)^{p'}\,dt\right)^{\frac{1}{p'}} < \infty.$$

For the case $p = 1$ it is also shown that $H_\omega : L^1_{[0,1)} \to H^1$ is bounded if and only if $\omega \in \widehat{\mathcal{D}}$ and

$$m_1(\omega) = \operatorname*{ess\,sup}_{t\in[0,1)} \omega(t)\left(1+\int_0^t \frac{ds}{\widehat{\omega}(s)}\right) < \infty.$$

It is important to note that these two last results, besides solving the boundedness of the operator in the Lebesgue space, provide sufficient conditions to guarantee the boundedness of $H_\omega : H^p \to H^p$. In view of the latest, we compare the conditions $M_p(\omega) < \infty$ and $m_p(\omega) < \infty$. From the previous remark, it is obvious that $m_p(\omega) < \infty$ implies $M_p(\omega) < \infty$ when $\omega \in \widehat{\mathcal{D}}$, and they are equivalent if the weight $\omega \in \widehat{\mathcal{D}}$ and satisfies the pointwise estimate

$$\omega(t) \lesssim \frac{\widehat{\omega}(t)}{1-t}, \quad t \in [0,1).$$

Moreover, we build weights $\omega \in \mathcal{D}$ with an strong oscillatory behavior which satisfy $M_p(\omega) < \infty$ but $m_p(\omega) = \infty$.

Bearing in mind the obtained results and in order to conclude this extensive discussion on the action of the operator in Hardy spaces, let us observe that for any radial weight ω the boundedness of H_ω in H^p implies the boundedness of $H_\omega : H^\infty \to \mathrm{BMOA}$. This is a particular case of a general phenomenon, since we prove that if $1 \leqslant q < p < \infty$ and H_ω is bounded in H^q then it is also bounded in H^p.

On the other hand, there are known results regarding the action of the classical Hilbert operator in Bergman spaces [13, 29]. In paper (I) we deal with this problem and describe when the operator H_ω is bounded from a weighted Lebesgue space $L^p_{\hat{\nu},[0,1)}$ to the Bergman space A^p_ν, for $\omega \in \widehat{\mathcal{D}}$ and ν a radial weight. Since the Bergman space A^p_ν is contained in the Lebesgue space $L^p_{\hat{\nu},[0,1)}$, this result provides sufficient conditions for the boundedness of the operator on A^p_ν.

Finally, it is worth highlighting the connection found in work (I) between the action of the Hilbert-type operator H_ω and the Bergman projection P_ω induced by a radial weight in Lebesgue spaces. In particular if $1 < p < \infty$, $P_\omega : L^p_{\hat{\nu}} \to A^p_{\hat{\nu}}$ and $H_\omega : L^p_{\hat{\nu},[0,1)} \to A^p_\nu$ are simultaneously bounded whenever $\omega \in \mathcal{D}$ and $\nu \in \widehat{\mathcal{D}}$.

This link between the Hilbert-type operator H_ω and the Bergman projection P_ω adds significant value to the work on the operator H_ω and provides another reason to study this new operator. Not only are the addressed problems interesting by themselves, but the operator H_ω could also be used as a "toy model" to deal with some of the unsolved questions regarding the action of the Bergman projection on Lebesgue spaces. Paper (III) focuses precisely on one of these open problems.

It is worth noting, as mentioned earlier, that the study of the Bergman projection in Lebesgue spaces is a cornerstone in the development of operator theory on Bergman spaces. Besides the mathematical challenge entailed in solving existing open problems, the potential applicability of the obtained results explains the considerable interest among many prominent researchers in this field of knowledge [7, 8, 10, 12, 15, 46, 48]

The problem of describing the radial weights ω such that Bergman projection P_ω satisfies the inequality

$$\|P_\omega(f)\|_{L^p_\omega} \leqslant C\|f\|_{L^p_\omega}, \quad f \in L^p_\omega, \quad 1 < p < \infty, p \neq 2 \tag{6}$$

was formally posed by Dostanić in [15] and remains open after decades. In this paper, he provided a necessary condition for P_ω to be bounded in L^p_ω:

$$\omega_{np+1}^{\frac{1}{p}} \omega_{np'+1}^{\frac{1}{p'}} \lesssim \omega_{2n+1}, \quad n \in \mathbb{N}.$$

Using Hölder's inequality it can be proved the converse inequality $\omega_{np+1}^{\frac{1}{p}}\omega_{np'+1}^{\frac{1}{p'}} \geqslant \omega_{2n+1}$, so Dostanić condition is a reverse Hölder condition in terms of the moments. On the other hand, the best known sufficient condition for (6) to hold is $\omega \in \widehat{\mathcal{D}}$ and it was proved in [48, Theorem 7].

Regarding the two- weight problem, possibly one of the most prominent and well-known results is due to Bekollé and Bonami [7, 8]. They describe the arbitrary weights ν such that the Bergman projection P_α induced by a standard weight $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$, $\alpha > -1$ is bounded on L^p_ν.

Let us recall that standard reproducing kernels have the explicit formula $(1-\bar{\zeta}z)^{-(2+\alpha)}$, $\alpha > -1$, while Bergman reproducing kernels induced by a radial weight ω do not have a neat formula. Therefore Bergman projections P_α are much more easier to deal with than P_ω.

As far as we know, the problem of describing radial weights ω, ν such that $P_\omega : L^p_\nu \to L^p_\nu$ is bounded remains open, even if ν is standard. The third work of this thesis (III) focuses precisely on studying this question. Section 1.5.2 provides an introduction to the problem, and chapter 3 is devoted to its discussion. In that paper, we also deal with the maximal projection defined as

$$P^+_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\,|B^\omega_z(\zeta)|\,\omega(\zeta)\,dA(\zeta), \quad z \in \mathbb{D}.$$

Firstly, in paper (III) we prove that condition

$$D_p(\omega,\nu) = \sup_{n\in\mathbb{N}\cup\{0\}} \frac{(\nu_{np+1})^{\frac{1}{p}}\,(\sigma_{np'+1})^{\frac{1}{p'}}}{\omega_{2n+1}} < \infty$$

is necessary for $P_\omega : L^p_\nu \to L^p_\nu$ to be bounded for any radial weights ω, ν, where $\sigma = \sigma_{\omega,\nu,p} = \left(\frac{\omega}{\nu^{\frac{1}{p}}}\right)^{p'} = \frac{\omega^{\frac{p}{p-1}}}{\nu^{\frac{1}{p-1}}}$. Note that the condition we have obtained as necessary generalizes the one Dostanić proved in the case $\omega = \nu$. In the opposite direction, we prove that the condition $D_p(\omega,\nu) < \infty$ implies the boundedness of $P_\omega : L^p_\nu \to L^p_\nu$ for $\nu \in \mathcal{D}$. In conclusion, in (III) it is shown that if $\nu \in \mathcal{D}$, then the condition $D_p(\omega,\nu) < \infty$ is equivalent to $P_\omega : L^p_\nu \to L^p_\nu$ or $P^+_\omega : L^p_\nu \to L^p_\nu$ is bounded. The proof of this result draws on an improvement of [48, Thorem 13] obtained in (III).

An important topic in function theory and operator theory in spaces of analytic functions is obtaining equivalent norms in terms of derivatives. A particular but significant case is obtaining the called Littlewood-Paley formulas. Their value lies in providing equivalent norms which makes them a fundamental ingredient to tackle with certain concrete problems. For example, these formulas allows to get rid of the integral when dealing with the action of certain integral operators. Moreover it is interesting to understand their notable link with the boundedness of the Bergman projection on L^p-spaces. The bounded Bergman projections on weighted Lebesgue spaces are often used to obtain this kind of inequalities, and reciprocally, Littlewood-Paley formulas are helpful to prove the boundedness of projections in given spaces.

The most known Littlewood-Paley formula asserts that for each $n \in \mathbb{N}$ and $0 < p < \infty$

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np}\omega(z)\,dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \tag{7}$$

for ω a standard radial weight $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$, $-1 < \alpha < \infty$. Peláez and Rättyä described the radial weights ω such that the above formula holds in [48, Theorem 5]. This class of weights is precisely $\omega \in \mathcal{D}$. Going further, they proved that

$$\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np}\omega(z)\,dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p \lesssim \|f\|_{A^p_\omega}^p \tag{8}$$

holds if and only if $\omega \in \widehat{\mathcal{D}}$ [48, Theorem 6] and a necessary condition for the converse inequality to hold is $\omega \in \mathcal{M}$.

We define the Dirichlet-type space $D^p_{\nu,k}$ which consists of the analytic functions on $\mathbb{D}$ such that $\|f\|_{D^p_{\nu,k}}^p = \sum_{j=0}^{k-1} |f^{(j)}(0)|^p + \int_{\mathbb{D}} |f^{(k)}(z)|^p(1-|z|)^{kp}\nu(z)\,dA(z) < \infty$. Bearing in mind the aforementioned results, observe that the spaces $D^p_{\nu,k}$ and A^p_ν are different for $\nu \in \mathcal{M}\backslash\widehat{\mathcal{D}}$.

Another goal of (III) consists of describing the boundedness of the Bergman projection $P_\omega : L^p_\nu \to D^p_{\nu,k}$ when $\omega \in \widehat{\mathcal{D}}$ and $\nu \in \mathcal{M}$. Under these hypothesis on the weights, we show that $D_p(\omega,\nu) < \infty$, $P_\omega : L^p_\nu \to L^p_\nu$ bounded and $P_\omega : L^p_\nu \to D^p_{\nu,k}$ bounded are equivalent. The proof of this theorem crucially relies on the Littlewood-Paley formulas [48, Theorems 5-6]. Moreover, key and notable ingredients such as a kind of Schur test and the integral means estimations of the derivatives of Bergman reproducing kernels are employed.

It is also worth mentioning that Dostanić proved in [15] that Bergman projection P_ω is not bounded on L^p_ω for exponential weights $\omega(z) = \exp\left(\frac{\alpha}{(1-|z|^2)^\beta}\right)$, $\alpha > 0$, $0 < \beta \leqslant 1$ and $p \neq 2$. This result has been generalized by several authors [10, 12, 57], and in particular, we show the usefulness of the necessary condition $D_p(\omega,\nu) < \infty$ in the setting of exponential weights $\nu(r) = \exp\left(-\frac{\alpha}{(1-r^l)^\beta}\right)$ and $\omega(r) = \exp\left(-\frac{\widetilde{\alpha}}{(1-r^{\widetilde{l}})^{\widetilde{\beta}}}\right)$, where $0 < \alpha, \widetilde{\alpha}, l, \widetilde{l} < \infty$ and $0 < \beta, \widetilde{\beta} \leqslant 1$. In fact, it is proved that the Bergman projection $P_\omega : L^p_\nu \to L^p_\nu$ is bounded if and only if $\beta = \widetilde{\beta}$ and $\widetilde{\alpha} = \frac{2\alpha}{p}\left(\frac{\widetilde{l}}{l}\right)^\beta$.

In the last paper of this collection (IV), our main aim consists of extending formulas (7) and (8) to the context of the fractional derivative. Due to a classic result, the Littlewood-Paley formula (7) can be generalized for the classical fractional derivative when ω is standard [9, 21],

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |D^\beta(f)(z)|^p (1-|z|)^{\beta p}\omega(z)\,dA(z), \quad f \in \mathcal{H}(\mathbb{D}), \quad 0 < p < \infty, \tag{9}$$

where the fractional derivative of order $\beta > 0$ is given by

$$D^{\beta}(f)(z) = \frac{2}{\Gamma(\beta+1)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \widehat{f}(n) z^n,$$

and as usual $\{\widehat{f}(n)\}_{n\in\mathbb{N}}$ denotes the sequence of Taylor coefficients of the function $f \in \mathcal{H}(\mathbb{D})$.

The primary purpose of the paper (IV) is twofold: extending the Littlewood-Paley formulas (7) and (8) by replacing the higher order derivative $f^{(n)}$ with a more general fractional derivative than D^{β}, and describing the radial weights such that the arising formulas hold.

For a radial weight μ, we define the fractional derivative of f induced by μ

$$D^{\mu}(f)(z) = \sum_{n=0}^{\infty} \frac{\widehat{f}(n)}{\mu_{2n+1}} z^n, \ z \in \mathbb{D}.$$

This operator is well-defined on $\mathcal{H}(\mathbb{D})$, and for the standard weight $\mu(z) = \beta(1-|z|^2)^{\beta-1}$, $\beta > 0$, $D^{\mu}(f) = D^{\beta}(f)$ and $\widehat{\mu}(z) \asymp (1-|z|)^{\beta}$. So, an appropriate interpretation of the Littlewood-Paley formula (9) for the generalized fractional derivative D^{μ} is

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |D^{\mu}(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}). \tag{10}$$

In paper (IV), our main result describes the radial weights ω such that the above formula holds for a given $\mu \in \mathcal{D}$, and such condition on the weight is $\omega \in \mathcal{D}$. Going further, not only the equivalence is described but also it is shown that the following inequality

$$\int_{\mathbb{D}} |D^{\mu}(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z) \leqslant C\|f\|_{A^p_\omega}^p, \quad f \in \mathcal{H}(\mathbb{D}),$$

holds if and only if $\omega \in \widehat{\mathcal{D}}$. The proof of the sufficiency of the last mentioned result boils down to prove an inequality between the integral means of the fractional derivative $D^{\mu}(f)$ and the integral means of the function f:

$$M_p(r, D^{\mu}f) \leqslant C \frac{M_p(r,f)}{\widehat{\mu}\left(\frac{r}{\rho}\right)}, \quad 0 < r < \rho < 1, \, 0 < p < \infty.$$

This kind of estimates are well-known for the integral means of derivatives with natural exponent, and they can be found in the literature [41, Lemma 3.1]. However, the proof of the classical case is different from the proof of the above inequality. Being precise, we appeal to the smooth properties of an universal Cesàro basis of polynomials introduced by Jevtić and Pavlović [28] (see section 1.4).

Concerning the proof of (10), a possible reformulation is the equivalence $\|f\|_{A^p_\omega}^p \asymp \|D^{\mu}f\|_{A^p_{\omega\widehat{\mu}^p}}^p$. Our way to prove this formula consists of constructing ad-hoc norms for the

weighted Bergman spaces A^p_ω and $A^p_{\omega\widehat{\mu}^p}$, in the spirit of the decomposition results [33, 38, 44]. The results in [33, 38, 44] are valid for $1 < p < \infty$, partially because their proofs employ the boundedness of the Riesz projection on $L^p(\partial\mathbb{D})$ to obtain precise estimates of the H^p-norm of polynomials of the type $\Delta_{n_1,n_2} z = \sum_{k=n_1}^{n_2} z^k$. Instead of this kind of polynomials, we construct an universal Cesàro basis of polynomials in order to obtain a norm decomposition result valid for both A^p_ω and $A^p_{\omega\widehat{\mu}^p}$, and $0 < p < \infty$. Specifically, we prove that there exist $k > 1$ and an universal Cesàro basis of polynomials $\{V_{n,k}\}_{n=0}^\infty$ so that the following equivalences $\|f\|^p_{A^p_\omega} \asymp \sum_{n=0}^\infty \omega_{k^n} \|V_{n,k} * f\|^p_{H^p}$ and $\|D^\mu(f)\|^p_{A^p_{\omega\widehat{\mu}^p}} \asymp \sum_{n=0}^\infty (\omega\widehat{\mu}^p)_{k^n} \|V_{n,k} * f\|^p_{H^p}$ hold for any $p \in (0,\infty)$, where $*$ denotes the convolution.

In this thesis, section 1.6 and chapter 4 gather the exposition of the problem (10), the obtained results, and their proofs. Finally, chapter 5 is dedicated to present some open problems and possible future research.

Part I

Mathematical setting

Chapter 1

Introduction

1.1 Spaces of analytic functions

Let $\mathcal{H}(\mathbb{D})$ be the space of analytic functions in the unit disc of the complex plane $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $0 < p \leqslant \infty$, the Hardy space H^p consists of $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

and

$$M_\infty(r, f) = \max_{0\leqslant\theta\leqslant 2\pi} |f(re^{i\theta})|.$$

The Hardy spaces H^p are closely related to other classical spaces of analytic functions. Specifically, we define the Dirichlet-type space D^p_{p-1} of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{D^p_{p-1}} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|)^{p-1} dA(z) < \infty,$$

and the Hardy-Littlewood space $HL(p)$ which consists of the $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{HL(p)} = \sum_{n=0}^{\infty} |\widehat{f}(n)|^p (n+1)^{p-2} < \infty.$$

Observe that the above expressions for $\|\cdot\|_{H^p}$, $\|\cdot\|_{HL(p)}$ and $\|\cdot\|_{D^p_{p-1}}$ are norms if $p \geqslant 1$ and quasinorms if $0 < p < 1$. It is well known that $H^2 = HL(2) = D^2_1$. Moreover, these spaces satisfy the well-known inclusions [19, 21, 31]

$$D^p_{p-1} \subsetneq H^p \subsetneq HL(p), \quad 0 < p < 2, \tag{1.1.1}$$

and

$$HL(p) \subsetneq H^p \subsetneq D^p_{p-1}, \quad 2 < p < \infty, \tag{1.1.2}$$

Furthermore, if we restrict to the $f \in \mathcal{H}(\mathbb{D})$ such that its sequence of Taylor coefficients is positive and decreasing, then $\|f\|_{H^p} \asymp \|f\|_{HL(p)} \asymp \|f\|_{D^p_{p-1}}$ [39, Proof of Theorem A]. Let us see that the insluions in (1.1.1) and (1.1.2) are strict. Bearing in mind that a lacunary series $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^{n_k}$, with $\frac{n_{k+1}}{n_k} \geqslant c > 1$ belongs to H^p if and only if it belongs to H^2 [60, Th. 8.20 p. 215, Vol I], and that it belongs to D^p_{p-1} if and only if $\sum_{n=0}^{\infty} |\widehat{f}(n)|^p < \infty$ [25], it is easy to provide some simple examples. Indeed, on the one hand if $0 < p < 2$, $f(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\frac{1}{p}}} z^{2^n} \in H^p \backslash D^p_{p-1}$ and $g(z) = \sum_{n=0}^{\infty} z^{2^n} \in HL(p) \backslash H^p$. On the other hand if $2 < p < \infty$, $f(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\frac{1}{2}}} z^{2^n} \in D^p_{p-1} \backslash H^p$ and $g(z) = \sum_{n=0}^{\infty} 2^{-n(1-\frac{2}{p})} z^{2^n} \in H^p \backslash HL(p)$.

Moreover, we define the space

$$H(\infty, p) = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|^p_{H(\infty,p)} = \int_0^1 M^p_\infty(r, f)\, dr < \infty \right\}.$$

This space also satisfies the following inclusions, which will be of particular interest in works (I) and (II) (see [23, Lemma 4] and [53, p. 127])

$$H^p \subsetneq H(\infty, p), \quad D^p_{p-1} \subsetneq H(\infty, p), \quad 0 < p < \infty. \tag{1.1.3}$$

and the following one proved in [34, Lemma 6]

$$HL(p) \subsetneq H(\infty, p), \quad 1 < p < \infty. \tag{1.1.4}$$

As for the strict embeddings in (1.1.3), $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ belongs to $H(\infty, p)$ but neither to H^p nor to D^p_{p-1}, $0 < p < \infty$. Regarding (1.1.4), the previous function does not belong to $HL(p)$ for $p > 2$, so it would serve as a counterexample for that range of p-values. However, if we consider $g(z) = \sum_{n=0}^{\infty} 2^{-n(1-\frac{2}{p})} z^{2^n} \in H(\infty, p) \backslash HL(p)$ for $p > 1$. Other classical spaces of analytic functions that appear in this thesis are BMOA and the Bloch space $\mathcal{B}$. On the one hand, we recall that the space BMOA consists of those functions in the Hardy space H^1 that have bounded mean oscillation on the boundary of $\mathbb{D}$, which can be equipped with the Garsia norm

$$\|f\|^2_{\mathrm{BMOA}} = |f(0)|^2 + \sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|^2_{H^2},$$

where φ_a is the involutive automorphism $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ [24]. On the other hand, the Bloch space $\mathcal{B}$ is the space of the analytic functions on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)\, |f'(z)| < \infty.$$

We also consider the space $HL(\infty)$ of the $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{HL(\infty)} = \sup_{n\in\mathbb{N}\cup\{0\}} (n+1)\left|\widehat{f}(n)\right| < \infty.$$

It is worth mentioning that the following chain of inclusions hold, and the proofs can be found in [24]

$$HL(\infty) \subsetneq \text{BMOA} \subsetneq \mathcal{B}. \tag{1.1.5}$$

Further, the inclusions are strict. Indeed, the function $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ is a Bloch function but it has radial limit almost nowhere, so $f \notin$ BMOA [24]. The function $f(z) = \sum_{n=0}^{\infty} 2^{-n} log(n+2) z^{2^n} \in \text{BMOA}\setminus HL(\infty)$ [34, Lemma 32].

For a non negative integrable function $\omega : \mathbb{D} \to \mathbb{R}$ and $0 < p < \infty$, the Lebesgue space L^p_ω consists of complex-valued measurable functions f on $\mathbb{D}$ such that

$$\|f\|^p_{L^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty,$$

where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized area measure on $\mathbb{D}$. The corresponding weighted Bergman space is $A^p_\omega = L^p_\omega \cap \mathcal{H}(\mathbb{D})$. As usual, we write A^p_α for the classical weighted Bergman spaces induced by the standard weights $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$, $\alpha > -1$.

1.2 Radial weights

A function $\omega : \mathbb{D} \to \mathbb{R}$ is a weight if it is non negative and integrable. If the weight satisfies $\omega(z) = \omega(|z|)$, $z \in \mathbb{D}$, it is said to be a radial weight.

In order to study questions on weighted spaces of analytic functions or concrete operators induced by weights, one of the main factors is whether the weight is radial or not. While results involving general weights are quite typical in the literature concerning spaces of measurable functions, such results are much less common in the setting of analytic functions we work with, primarily due to the extreme difficulty in attaining them. Another major obstacle could be that the theory of Bergman spaces induced by non-radial weights is barely developed, and some basic and essential facts for the progress of the theory are either unknown or do not hold. For example, polynomials are dense in A^p_ω if the weight is radial, but this is not true in general; the weight $\omega(z) = \exp\left(\frac{1-|z|^2}{|1-z|^2}\right)$ satisfies that polynomials are not dense in A^p_ω [20]. Despite these challenges, there have been some advances in the theory of non-radial weights [27, 50], specially for the Bekollé-Bonami class [1, 4, 7, 8, 54].

In all the research papers that collect this thesis, we work in the setting of radial doubling weights. This class of weights stem from the work of Lusky [32], however its connection with plenty of problems on the weighted theory of analytic functions becomes apparent in recent papers by Peláez and Rättyä [45, 46, 48].

Throughout this thesis we denote the integral tail of the weight by $\widehat{\omega}(z) = \int_{|z|}^{1} \omega(s)ds$, and we assume $\widehat{\omega}(z) > 0$ for all $z \in \mathbb{D}$, otherwise $A^p_\omega = \mathcal{H}(\mathbb{D})$. Also, denote the moments of the weight by $\omega_x = \int_0^1 s^x \omega(s)ds$, $x \geqslant 0$.

In this section, we will introduce the definitions, basic properties, and characterizations of the doubling weight classes which will be useful throughout the rest of this thesis. Additionally, we will provide examples to illustrate some of these properties.

A radial weight ω belongs to the class $\widehat{\mathcal{D}}$ if there exists $C = C(\omega) > 1$ such that

$$\widehat{\omega}(r) \leqslant C\widehat{\omega}\left(\frac{1+r}{2}\right) \quad \text{for all } 0 \leqslant r < 1.$$

When ω satisfies the above condition we say that is an upper doubling weight.

Regarding concrete examples, standard weights $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$, $\alpha > -1$ and $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha \left(\log \frac{e}{1-|z|}\right)^\gamma$, $\alpha > -1$, $\gamma \in \mathbb{R}$, belong to $\widehat{\mathcal{D}}$. Moreover, the increasing weight $v_\beta(z) = \frac{1}{(1-|z|)\left(\log \frac{e}{1-|z|}\right)^\beta}$, $\beta > 1$, is also an upper doubling weight.

The condition that defines the class $\widehat{\mathcal{D}}$ implies a restriction on the decay of the weight, specifically, if the weight ω belongs to $\widehat{\mathcal{D}}$, ω cannot decrease very rapidly. This property is illustrated by the fact that exponential weights $\omega(z) = \exp\left(-\frac{1}{(1-|z|^2)^\alpha}\right)$, $\alpha > 0$ do not belong to the class $\widehat{\mathcal{D}}$.

The proper behavior of weights in $\widehat{\mathcal{D}}$ can be disrupted in several ways. In fact, observe that definition does not imply the necessity of smoothness or differentiability of the weight, actually doubling weights may admit a strong oscillatory behavior; for instance, $\omega(r) = \left|\sin\left(\log \frac{1}{1-r}\right)\right| v_\beta(r) + 1 \in \widehat{\mathcal{D}}$. Moreover, it can be equal zero on sets of positive measure arbitrarily close to 1. For example, define $r_n = 1 - \frac{1}{2^n}$, $n \in \mathbb{N} \cup \{0\}$, then $\omega(t) = \sum_{n=0}^{\infty} \chi_{[r_{2n}, r_{2n+1})}(t) \in \widehat{\mathcal{D}}$. Going further, for each prefixed $r > 0$, the quantity $\omega(\Delta(z,r))$ may equal to zero for some z arbitrarily close to the boundary if $\omega \in \widehat{\mathcal{D}}$ [49, Proposition 3], where $\Delta(z,r)$ denotes the hyperbolic disc of center z and radio r.

Now, we gather the following characterizations of the class $\widehat{\mathcal{D}}$, which will provide us very useful information about the doubling condition in terms of moments and integral tails. The result and its proof can be found in [42, Lemma 2.1] and [48, Theorem 6]. Let us denote $\omega_{[\beta]}(s) = (1-s)^\beta \omega(s)$, $\beta > 0$.

Theorem A. *Let ω be a radial weight on $\mathbb{D}$. Then, the following statements are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;

(ii) There exist $C = C(\omega) \geqslant 1$ and $\alpha_0 = \alpha_0(\omega) > 0$ such that

$$\widehat{\omega}(r) \leqslant C\left(\frac{1-r}{1-t}\right)^\alpha \widehat{\omega}(t), \quad 0 \leqslant r \leqslant t < 1;$$

for all $\alpha \geqslant \alpha_0$*;*

(iii)

$$\int_0^1 s^x \omega(s)ds \asymp \widehat{\omega}\left(1-\frac{1}{x}\right), \quad x \in [1,\infty);$$

(iv) There exist $C = C(\omega) > 0$ *and* $\alpha = \alpha(\omega) > 0$ *such that*

$$\omega_x \leqslant C\left(\frac{y}{x}\right)^\alpha \omega_y, \quad 0 < x \leqslant y < \infty;$$

(v) $\sup_{n\in\mathbb{N}} \frac{\omega_n}{\omega_{2n}} < \infty$;

(vi) For some (equivalently for each) $\beta > 0$ *there exists a constant* $C = C(\omega,\beta) > 0$ *such that* $x^\beta(\omega_{[\beta]})_x \leqslant C\omega_x$, $0 < x < \infty$.

A radial weight $\omega \in \check{\mathcal{D}}$ if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\widehat{\omega}(r) \geqslant C\widehat{\omega}\left(1-\frac{1-r}{K}\right)$$

for all $0 \leqslant r < 1$, or equivalently there exist $K = K(\omega) > 1$ and $C = C(\omega) > 0$ such that

$$\int_r^{1-\frac{1-r}{K}} \omega(s)ds \geqslant C\widehat{\omega}(r),\ 0 \leqslant r < 1. \tag{1.2.1}$$

Moreover, a radial weight $\omega \in \mathcal{M}$ if there exist constants $C = C(\omega) > 1$ and $K = K(\omega) > 1$ such that $\omega_x \geqslant C\omega_{Kx}$ for all $x \geqslant 1$. We set $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$ and when $\omega \in \mathcal{D}$ we say that ω is a doubling weight.

Peláez and Rättyä showed that the classes $\check{\mathcal{D}}$ and $\mathcal{M}$ are closely related, although class $\mathcal{M}$ seems to be the one that appears more naturally in operator theory problems in weighted Bergman spaces. They recently proved that $\check{\mathcal{D}} \subset \mathcal{M}$ [48, Proof of Theorem 3] and further $\check{\mathcal{D}} \subsetneq \mathcal{M}$ [48, Proposition 14]. However, [48, Theorem 3] shows that $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}} = \widehat{\mathcal{D}} \cap \mathcal{M}$. These embeddings show that the class $\mathcal{M}$ allows more irregularities in the weight than $\check{\mathcal{D}}$. This is because when expressed in terms of moments, they "absorb" some irregularities which are not permitted in the class $\check{\mathcal{D}}$. However, when both intersect with $\widehat{\mathcal{D}}$, they give rise to the same class of weights because these kinds of irregularities are also not permitted in $\widehat{\mathcal{D}}$.

Both conditions, $\check{\mathcal{D}}$ and $\mathcal{M}$, are the natural converses of the class $\widehat{\mathcal{D}}$ in a certain sense, one in terms of integral tails and the other one in terms of moments. The following two characterizations confirm this fact (see Lemma A(ii) and (vi)).

Lemma B. *[48, (2.27)] Let* ω *be a radial weight. The following statements are equivalent:*

(i) $\omega \in \check{\mathcal{D}}$;

(ii) There exist $C = C(\omega) > 0$ *and* $\alpha_0 = \alpha_0(\omega) > 0$ *such that*

$$\widehat{\omega}(s) \leqslant C \left(\frac{1-s}{1-t} \right)^{\alpha} \widehat{\omega}(t), \quad 0 \leqslant t \leqslant s < 1$$

for all $0 < \alpha \leqslant \alpha_0$.

Lemma C. *[48, (2.16)] Let* ω *be a radial weight. The following statements are equivalent:*

(i) $\omega \in \mathcal{M}$;

(ii) For each $\beta > 0$, *there is* $C = C(\beta, \omega)$ *such that*

$$\omega_x \leqslant C x^{\beta} \left(\omega_{[\beta]} \right)_x, \quad x \geqslant 1.$$

Here are some notable examples of the class $\mathcal{D}$. For instance, each standard radial weight belongs to $\mathcal{D}$ and $\omega(z) = (1-|z|^2)^{\alpha} \left(\log \frac{e}{1-|z|^2} \right)^{\beta} \in \mathcal{D}$, with $\alpha > -1$ and $\beta \in \mathbb{R}$. As for the class $\check{\mathcal{D}} \backslash \mathcal{D}$, it contains weights that tend to zero exponentially, for example $\omega(z) = \exp\left(-\frac{1}{1-|z|} \right)$. A typical example of a weight belonging to $\widehat{\mathcal{D}} \backslash \mathcal{D}$ is the increasing weight $\omega(z) = (1-|z|^2)^{-1} \left(\log \frac{e}{1-|z|^2} \right)^{-\beta}$, where $\beta > 1$.

An interesting property of standard Bergman spaces A^p_α is the called transition phenomenon from A^p_α to H^p. This property arises from the fact that in a certain sense, the Hardy space H^p is the limit of A^p_α when $\alpha \to -1$. However, this estimation is very rough because some of the A^p_α properties do not transfer to the H^p space. Going further, in this sense some weighted Bergman spaces A^p_ω can be very small compared to any standard Bergman space A^p_α, so that actually lies closer to H^p than any A^p_α, and the transition phenomenon appears in this context. Specifically if the weight ω belongs to the subclass of $\widehat{\mathcal{D}}$ which consists of the weights $\lim_{r \to 1^-} \frac{(1-r)\omega(r)}{\widehat{\omega}(r)} = 0$ [45, Section 1.4 (iii)], then

$$H^p \subset A^p_\omega \subset \left(\bigcap_{\alpha > -1} A^p_\alpha \right).$$

Regarding $\omega \in \mathcal{D}$, each weighted Bergman space induced by doubling weight lies between two standard Bergman spaces, and its behavior in many respects is very similar to the standard Bergman space.

The differences between Bergman spaces induced by weights in $\widehat{\mathcal{D}}$ or $\mathcal{D}$ becomes evident in different topics of the theory. Roughly speaking, the function theory and the operator theory on A^p_ω induced by $\omega \in \widehat{\mathcal{D}}$ is considerably more complicated than when $\omega \in \mathcal{D}$. While A^p_ω, $\omega \in \mathcal{D}$, can be treated in a similar way than standard weighted Bergman spaces, the study of Bergman spaces induced by upper doubling weights requires different and more technical methods, which sometimes come from H^p-theory and harmonic analysis. For example, if $\omega \in \mathcal{D}$ there exists a Littlewood-Paley formula, which plays a significant role to address several problems in operator theory and spaces of analytic functions (see section 1.6). Indeed, if $\omega \in \mathcal{D}$, for each $n \in \mathbb{N}$ and $0 < p < \infty$,

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np} \omega(z)\, dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}). \tag{1.2.2}$$

[48, Theorem 5]. However, this is not true in general for upper doubling weights, $p \neq 2$. Indeed, Peláez and Rättyä proved in [45, Proposition 4.3] that for fixed $p \neq 2$ there is a weight $\omega \in \widehat{\mathcal{D}}$ such that for any function $\varphi : [0,1) \to (0,\infty)$ the formula

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |f'(z)|^p \varphi(|z|)^p \omega(z) dA(z)$$

is not valid for all $f \in \mathcal{H}(\mathbb{D})$.

Another evidence that shows the aforementioned differences is the following. If $\omega \in \mathcal{D}$, then $\|f\|_{A^p_\omega} \asymp \|f\|_{A^p_{\widetilde{\omega}}}$, where $\widetilde{\omega}$ is the regularized weight $\widetilde{\omega}(r) = \frac{\widehat{\omega}(r)}{1-r}$. However, as far we know, such a result cannot be obtained for A^p_ω, $\omega \in \widehat{\mathcal{D}}$.

Moreover, while weights $\omega \in \mathcal{D}$ satisfy that $\omega_{[\gamma]} \in \mathcal{D}$ for all $\gamma > 0$, this fact does not remain true for weights in $\widehat{\mathcal{D}}$ [47].

Finally, in regards to operator theory, if $\omega \in \mathcal{D}$ then the Bergman projection $P^+_\omega : L^p_\omega \to L^p_\omega$ is bounded. However, if $\omega \in \widehat{\mathcal{D}} \setminus \mathcal{D}$, $P_\omega : L^p_\omega \to L^p_\omega$ is bounded but $P^+_\omega : L^p_\omega \to L^p_\omega$ is not [48, Theorem 7 and 9].

The study of the intrinsic nature of these class of weights entails a considerable difficulty, which has led to a deep research for years, collected in works such as [42, 45, 46, 48].

1.3 Bergman reproducing kernels

For any radial weight ω, the norm convergence in A^2_ω implies the uniform convergence on compact subsets of $\mathbb{D}$, and hence the point evaluations L_z are bounded linear functionals on A^2_ω. Therefore, by Riesz representation theorem, there exist Bergman reproducing kernels $B^\omega_z \in A^2_\omega$ such that

$$L_z(f) = f(z) = \langle f, B^\omega_z \rangle_{A^2_\omega} = \int_{\mathbb{D}} f(\zeta) \overline{B^\omega_z(\zeta)} \omega(\zeta) dA(\zeta), \quad f \in A^2_\omega. \tag{1.3.1}$$

In both papers (I) and (II), we tackle the boundedness of an integral operator whose kernel explicitly depends on a Bergman reproducing kernel. In addition, in paper (III), we study the Bergman projection, so a meaningful knowledge of these kernels is vital to understand the behavior of the operators. It is known that in the case of standard weights $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$ the classical Bergman kernel B_z^α has a neat expression $B_z^\alpha(\zeta) = \frac{1}{(1-\overline{z}\zeta)^{2+\alpha}}$, $z, \zeta \in \mathbb{D}$, they have no zeros and its modulus is essentially constant on bounded hyperbolic regions. However, in general for a radial weight ω, Bergman reproducing kernel B_z^ω may have zeros [11, 51], and the lack of such a neat explicit expression for them is one of the main difficulties in studying the aforementioned operators in (I), (II) and (III). It is well-known that $B_z^\omega(\zeta) = \sum \overline{e_n(z)} e_n(\zeta)$ for each orthonormal basis $\{e_n\}$ of A^2_ω, and therefore we can use the formula

$$B_z^\omega(\zeta) = \sum_{n=0}^{\infty} \frac{(\overline{z}\zeta)^n}{2\omega_{2n+1}}, \quad z, \zeta \in \mathbb{D},$$

where we have considered as basis $\{e_n\}_n$ the normalized monomials.

Despite these obstacles, some progress has been made about meaningful facts concerning these kernels by assuming certain conditions on the inducing weight. In fact, by simply assuming that $\omega \in \widehat{\mathcal{D}}$, Peláez and Rättyä were able to estimate the integral means of Bergman reproducing kernels and their derivatives in [46, Theorem 1], and this is one of the main results we use in works (I), (II) and (III). Precisely, for $0 < q < \infty$, $N \in \mathbb{N}$,

$$M_q^q(r, (B_\zeta^\omega)^{(N)}) \asymp 1 + \int_0^{r|\zeta|} \frac{dx}{\widehat{\omega}(x)^q (1-x)^{q(N+1)}}, \quad 0 \leqslant r < 1, \quad \zeta \in \mathbb{D}. \tag{1.3.2}$$

The proof of this result employs many tools, including the universal Cesàro basis of polynomials which will be discussed in the next section.

1.4 Universal Cesàro basis of polynomials and norm decomposition

The appropriate choice of the norm is a key to understand the action of a concrete operator on a space of functions. In the literature, we can find many results on norm decomposition into blocks for several spaces, which often allow us to deal with the sequence of the coefficients of the function to estimate its norm (see [33, Theorem 2.1], [44, Theorem 4]). In paper (III), we appeal to an universal Cesàro basis of polynomials instead of the block-type polynomials $\Delta_{n_1,n_2} z = \sum_{n=n_1}^{n_2} z^n$ that are usually used in norm decomposition results. This technique not only enables us to obtain a new norm decomposition ad-hoc to study Littlewood-Paley formulas for a generalized fractional derivative D^μ, valid for $0 < p < \infty$, but also provides us an effective tool to estimate the integral means of $D^\mu(f)$. Moreover, in papers (I) and (II), we use a classical decomposition result of mixed norm spaces [33, Theorem 2.1], as well as Universal Cesàro basis of polynomials in order to estimate the norm in the Dirichlet-type space D_0^1 of a concrete test function.

The Hadamard product of a polynomial $W(z) = \sum_{k\in J} b_k z^k$, where J denotes a finite subset of $\mathbb{N}$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{D})$ is

$$(W * f)(z) = \sum_{k\in J} a_k b_k z^k, \quad z \in \mathbb{D}.$$

For a given C^∞-function $\Phi : \mathbb{R} \to \mathbb{C}$ with compact support, set

$$A_{\Phi,m} = \max_{x\in\mathbb{R}} |\Phi(x)| + m \max_{x\in\mathbb{R}} |\Phi^{(m)}(x)|, \quad m \in \mathbb{N} \cup \{0\},$$

and define the polynomials

$$W_n^\Phi(z) = \sum_{k\in\mathbb{Z}} \Phi\left(\frac{k}{n}\right) z^k, \quad n \in \mathbb{N}.$$

The next result can be found in [38, pp. 111–113].

Theorem D. *Let $\Phi : \mathbb{R} \to \mathbb{C}$ be a compactly supported C^∞-function. Then for each $0 < p < \infty$ and $m \in \mathbb{N}$ with $mp > 1$, there exists a constant $C = C(p) > 0$ such that*

$$\|W_n^\Phi * f\|_{H^p} \leqslant C A_{\Phi,m} \|f\|_{H^p}$$

for all $f \in H^p$ and $n \in \mathbb{N}$.

The previous theorem justifies why $\{W_n^\Phi\}_{n\in\mathbb{N}}$ is usually called universal Cesàro basis of polynomials for H^p for any $0 < p < \infty$. Specifically, we consider the following particular family of polynomials $\{W_n^\Phi\}_{n\in\mathbb{N}}$:

Proposition 1.4.1. *Let $k \in \mathbb{N}$, $k > 1$ and $\Psi : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that $\Psi \equiv 1$ on $(-\infty, 1]$, $\Psi \equiv 0$ on $[k, \infty)$ and Ψ is decreasing and positive on $(1, k)$. Set $\psi(t) = \Psi\left(\frac{t}{k}\right) - \Psi(t)$ for all $t \in \mathbb{R}$. Let $V_{0,k}(z) = \sum_{j=0}^{k-1} \Psi(j) z^j$ and*

$$V_{n,k}(z) = W_{k^{n-1}}^\psi(z) = \sum_{j=0}^{\infty} \psi\left(\frac{j}{k^{n-1}}\right) z^j = \sum_{j=k^{n-1}}^{k^{n+1}-1} \psi\left(\frac{j}{k^{n-1}}\right) z^j, \quad n \in \mathbb{N}.$$

Then,

$$f(z) = \sum_{n=0}^{\infty} (V_{n,k} * f)(z), \quad z \in \mathbb{D}, \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.4.1}$$

and for each $0 < p < \infty$ there exists a constant $C = C(p, \Psi, k) > 0$ such that

$$\|V_{n,k} * f\|_{H^p} \leqslant C \|f\|_{H^p}, \quad f \in H^p, \quad n \in \mathbb{N}. \tag{1.4.2}$$

If $k = 2$ we simply denote $V_{n,2} = V_n$. A proof of Proposition 1.4.1 for this choice appears in [28, p. 175–177] or [40, p. 143–144].

1.5 Integral operators

1.5.1 Hilbert-type operator

The main objective of this section is to introduce the problem related to the classical Hilbert operator. The Hilbert matrix is the infinite matrix

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & . \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & . \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & . \\ . & . & . & . \end{pmatrix}.$$

It can be viewed as an operator on spaces of analytic functions, called the Hilbert operator, by its action on the Taylor coefficients

$$\widehat{f}(k) \mapsto \sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1}, \quad n \in \mathbb{N} \cup \{0\}.$$

That is, if $f(z) = \sum_{k=0}^{\infty} \widehat{f}(k) z^k \in \mathcal{H}(\mathbb{D})$ the Hilbert operator is given by

$$H(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1} \right) z^n, \qquad (1.5.1)$$

whenever the right-hand side makes sense. As a result of Hardy inequality [19, p. 48], if $f \in H^1$, the power series is well-defined and defines an analytic function on $\mathbb{D}$, so $H(f)$ is well defined.

The Hilbert operator is a prototype of Hankel operator, which have garnered significant attention in recent years within operator theory. It is shown in [14] that the Hilbert operator H is bounded on Hardy spaces H^p if and only if $1 < p < \infty$. It is worth mentioning that H is not bounded in H^1 although it is well-defined in this case. Moreover, the norm of the operator on Hardy spaces has been computed in [17]. Regarding Bergman spaces, Diamantopoulos proved in [13] that $H : A^p \to A^p$ if and only if $2 < p < \infty$. In fact, the operator is not even well-defined in A^2 [17], and in [29] some results on boundedness in H^1 and A^2 were obtained. Specifically, authors found some subspaces of H^1 and A^2 such that H maps them to H^1 and A^2.

As for the case $p = \infty$, the classical Hilbert operator H is not bounded in H^∞. However, it is bounded from H^∞ to BMOA [29]. Readers may consult [5, Chapter 14] for further information about Hilbert operator.

It is worth underlying that the following integral representation for the Hilbert operator

$$H(f)(z) = \int_0^1 f(t) \frac{1}{1-tz} \, dt, \quad f \in H^1, \qquad (1.5.2)$$

plays a key role in many of the proofs of the results just mentioned.

Going further, different generalizations of the Hilbert operator have arisen during the last decades [22, 23, 44]. Bearing in mind the formula (1.5.2), we are interested in replacing the kernel $\frac{1}{1-tz}$ of the integral representation with a more general kernel. For a radial weight ω, we consider the Hilbert-type operator

$$H_\omega(f)(z) = \int_0^1 f(t) \left(\frac{1}{z} \int_0^z B_t^\omega(\zeta) d\zeta \right) \omega(t) dt,$$

where recall that $\{B_z^\omega\}_{z\in\mathbb{D}} \subset A_\omega^2$ are the Bergman reproducing kernels of A_ω^2. The choice $\omega = 1$ gives the integral representation (1.5.2) of the classical Hilbert operator.

Once the generalized Hilbert operator has been defined, the next natural step is to describe the features of a radial weight ω so that H_ω has some of the neat properties of the classical Hilbert operator. This constitutes the main aim of works (I) and (II).

1.5.2 Bergman projection induced by radial weight

In this section, we introduce the Bergman projection induced by a radial weight and pose the addressed problem in (III). The boundedness of projections on L^p-spaces stands as a captivating subject that has garnered substantial attention over recent decades. Its appeal arises not only from the mathematical difficulties it presents but also from its many applications in pivotal questions within operator theory such us duality relationships or Littlewood-Paley inequalities for weighted Bergman spaces.

Since A_ω^2 is a closed subspace of L_ω^2, the orthogonal projection P_ω is bounded from L_ω^2 to A_ω^2 and is given by

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta) \overline{B_z^\omega(\zeta)} \omega(\zeta)\, dA(\zeta), \quad z \in \mathbb{D}.$$

Indeed, bearing in mind the reproduction formula for functions in A_ω^2 (1.3.1) and the fact that P_ω is self-adjoint and idempotent,

$$P_\omega(f)(z) = \langle P_\omega(f), B_z^\omega \rangle_{A_\omega^2} = \langle f, P_\omega(B_z^\omega) \rangle_{A_\omega^2} = \langle f, B_z^\omega \rangle_{A_\omega^2}, \quad f \in L_\omega^2.$$

If ω is standard, $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$ with $\alpha > -1$, the corresponding Bergman projection is denoted by P_α. Moreover, we will deal with the maximal Bergman projection, which is defined by

$$P_\omega^+(f)(z) = \int_{\mathbb{D}} f(\zeta) |B_z^\omega(\zeta)| \omega(\zeta)\, dA(\zeta), \quad z \in \mathbb{D}.$$

The problem of describing the radial weights ω such that P_ω is bounded on the Lebesgue space L_ω^p, that is, describing the radial weights ω such that

$$\|P_\omega(f)\|_{L_\omega^p} \leqslant C \|f\|_{L_\omega^p}, \quad f \in L_\omega^p, \quad 1 < p < \infty \tag{1.5.3}$$

was formally posed by Dostanić in [15, Problem 2.5 a)]. Its many applications and links with other topics have made this problem an interesting and challenging subject of study, which remains open after decades. Several authors have worked on it, and some results have been obtained [12, 15, 16, 46, 48, 57]. Dostanić provided a necessary condition for (1.5.3) to hold:

$$\omega_{np+1}^{\frac{1}{p}}\omega_{np'+1}^{\frac{1}{p'}} \lesssim \omega_{2n+1}, \quad n \in \mathbb{N}. \tag{1.5.4}$$

Observe that the converse inequality $\omega_{2n+1} \leqslant \omega_{np+1}^{\frac{1}{p}}\omega_{np'+1}^{\frac{1}{p'}}$ holds for any radial weight and $1 < p < \infty$, so Dostanić condition is a reverse Hölder condition in terms of moments. In addition, he proved in [15, Theorem 2.1] that Bergman projections are not bounded on L^p_ω for exponential weights $\omega(z) = \exp\left(\frac{A}{(1-|z|^2)^\alpha}\right)$, $A > 0$ and $0 < \alpha \leqslant 1$ unless $p = 2$ (see [10, 12, 27, 57] for related results).

In the opposite direction, Peláez and Rättyä recently gave a sufficient condition for (1.5.3). If $\omega \in \widehat{\mathcal{D}}$, then $P_\omega : L^p_\omega \to L^p_\omega$ is bounded [48, Theorem 7]. A simple observation yields that weights belonging $\widehat{\mathcal{D}}$ satisfy (1.5.4), but unfortunately it is not known if Dostanić's condition implies $\omega \in \widehat{\mathcal{D}}$. The proof of this result draws on the fact that for any radial weight ω and $1 < p < \infty$, the Bergman projection P_ω is bounded in $L^{p'}_\omega$ if and only if $(A^p_\omega)^\star \simeq A^{p'}_\omega$ with equivalence of norms under the A^2_ω-pairing. This fact evidences the connection between duality relationships and the bounded Bergman projections.

In addition, they proved that if $\omega \in \widehat{\mathcal{D}}$, $P^+_\omega : L^p_\omega \to L^p_\omega$ is bounded if and only if $\omega \in \mathcal{D}$ [48, Theorem 9]. The last two theorems shed some light on the differences between the action of the Bergman projection P_ω and the action of the maximal projection P^+_ω on L^p_ω, as it shows that kernel cancellation plays a key role in this problem. In fact, if $\omega \in \widehat{\mathcal{D}} \setminus \mathcal{D}$, $P_\omega : L^p_\omega \to L^p_\omega$ is bounded but $P^+_\omega : L^p_\omega \to L^p_\omega$ is not.

Regarding the two-weight problem, the aim is to describe the radial weights ω, ν such that the following inequality holds:

$$\|P_\omega(f)\|_{L^p_\nu} \leqslant C\|f\|_{L^p_\nu}, \quad f \in L^p_\nu. \tag{1.5.5}$$

A folklore result due to Forelli and Rudin completely solves the particular case when both ω and ν are standard weights. Indeed, if $1 \leqslant p < \infty$, α, $\gamma > -1$, $\omega(z) = (\gamma+1)(1-|z|^2)^\gamma$ and $\nu(z) = (\alpha+1)(1-|z|^2)^\alpha$, it is well known that $P_\gamma : L^p_\alpha \to L^p_\alpha$ is bounded if and only if $p(\gamma+1) > \alpha+1$ [59]. Going further, Bekollé and Bonami [7, 8] described the inequality (1.5.5) for ν being an arbitrary weight (not necessarily radial) and the inducing weight ω being standard $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$, $\alpha > -1$. Being precise, they proved that for a weight ν, $p > 1$ and $\alpha > -1$,the standard projection is bounded $P_\alpha : L^p_\nu \to L^p_\nu$ if and only if $\frac{\nu}{(1-|z|^2)^\alpha}$ belongs to the Bekollé class $B_p(\alpha)$, which is the Bergman-space analogue of the well- known Muckenhoupt class A_p from harmonic analysis.

Recall that a weight ν on $\mathbb{D}$ belongs to the Bekollé class $B_p(\alpha)$, $p>1$ and $\alpha>-1$ if

$$\sup \frac{\left(\int_{S(\theta,h)} \nu dA_\alpha\right)\left(\int_{S(\theta,h)} \nu^{-\frac{p'}{p}} dA_\alpha\right)^{\frac{p}{p'}}}{(A_\alpha(S(\theta,h)))^p} < \infty$$

where the supremum runs over all the Carleson squares

$$S(\theta,h)=\{z=re^{i\alpha}: 1-h<r<1,\ |\theta-\alpha|<h/2\},\quad \theta\in[0,2\pi],\quad h\in(0,1),$$

and A_α denote the measure given by $dA_\alpha=(\alpha+1)(1-|z|^2)^\alpha dA$.

It is worth noting that the aforementioned results offer complete descriptions for (1.5.5) to hold in the standard Bergman projection case. When the weight inducing the projection is standard, Bergman reproducing kernels have neat properties that are not generally satisfied when the weight is only assumed to be radial (see section 1.3). Some advances have been made regarding the two-weight problem by assuming some conditions on the weights [46, 48]. However, as far as we know, the problem of describing the radial weights ω, ν such that (1.5.5) holds remains an open problem, even for ν a standard weight.

1.6 Littlewood-Paley inequalities and fractional derivative induced by radial weight

In this section, we introduce some Littlewood-Paley formulas, which allow to establish equivalent norms in Bergman spaces in terms of derivatives. We refer to a Littlewood-Paley formula as an equivalence of the form

$$\|f\|_{A^p_\omega}^p \asymp \sum_{j=0}^{n-1}|f^{(j)}(0)|^p+\int_{\mathbb{D}}|f^{(n)}(z)|^p W_{n,p}(z)\,dA(z),$$

where $n\in\mathbb{N}$ and $W_{n,p}$ is a weight that depends on n and p. It is worth underlying that equivalent norms in terms of the derivatives plays a significant role in addressing several problems in operator theory and spaces of analytic functions; for instance, to study the boundedness and compactness of concrete operators such as composition operators [55] or Volterra-type operators

$$T_g(f)(z)=\int_0^z f(\xi)g'(\xi)\,d\xi$$

[1, 2, 45], or even as a fundamental tool to estimate the norm of some functions. Moreover, it is interesting to observe the notable link between the boundedness of the Bergman projection on L^p-spaces and Littlewood-Paley formulas. Bounded Bergman projections on weighted Lebesgue spaces are often used to obtain this kind of inequalities (see [1, 59]), and reciprocally, Littlewood-Paley formulas are helpful to prove the boundedness of projections in given spaces [46, 48].

If ω is only assumed to be radial, there is no any Littlewood-Paley formula for A^p_ω, $p \neq 2$. Indeed, Peláez and Rättyä proved in [45, Proposition 4.3] that for fixed $p \neq 2$ there is a weight $\omega \in \widehat{\mathcal{D}}$ such that for any function $\varphi : [0,1) \to (0,\infty)$ the formula

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} |f'(z)|^p \varphi(|z|)^p \omega(z) dA(z)$$

cannot be valid for all $f \in \mathcal{H}(\mathbb{D})$. As a consequence, if we only assume that ω is radial and we look for an equivalent norm for the weighted Bergman space in terms of the derivative of the function, we are forced to consider formulas inherited from equivalent norms in H^p. In fact, the next result [45, Theorem 4.2] follows from well-known Hardy-Stein formula:

$$\|f\|^p_{A^p_\omega} = p^2 \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \omega^\star(z) dA(z) + \omega(\mathbb{D})|f(0)|^p,$$

where $\omega^\star(z) = \int_{|z|}^1 \omega(s) \log \frac{s}{|z|} s ds$. Moreover, using Calderon's formula for Hardy spaces it can be proved that

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} \left(\int_{\Gamma(u)} |f^{(n)}(z)|^2 \left(1 - \left|\frac{z}{u}\right|\right)^{2n-2} dA(z) \right)^{\frac{p}{2}} \omega(u) dA(u) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p,$$

where the lens-type region $\Gamma(u)$ is defined by

$$\Gamma(u) = \left\{ z \in \mathbb{D} : |\theta - \arg z| < \frac{1}{2}\left(1 - \frac{|z|}{r}\right) \right\},\ u = re^{i\theta} \in \overline{\mathbb{D}} \backslash \{0\}.$$

Both formulas above are not the Littlewood-Paley type, and as a result, we must assume extra hypotheses about the weight. The most-known Littlewood-Paley formula states that for each $n \in \mathbb{N}$ and $0 < p < \infty$

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np} \omega(z)\, dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.6.1}$$

for ω a standard radial weight $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$, $-1 < \alpha < \infty$.

Several generalizations of the Littlewood-Paley formulas have arisen for different classes of weights [1, 4, 6, 41, 50, 56], but the problem of describing the radial weights for which (1.6.1) holds, despite remaining open for years, has been recently solved by Peláez and Rättyä in [48, Theorem 5]. They proved that (1.6.1) holds for a radial weight ω if and only if $\omega \in \mathcal{D}$.

Going further, the inequality

$$\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np} \omega(z)\, dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p \lesssim \|f\|^p_{A^p_\omega}, \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.6.2}$$

is valid for each pair (n,p), $n \in \mathbb{N}$ and $0<p<\infty$, if and only if $\omega \in \widehat{\mathcal{D}}$ [48, Theorem 6].

However, as for the converse inequality

$$\|f\|_{A^p_\omega}^p \lesssim \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np}\omega(z)\,dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.6.3}$$

they showed that $\omega \in \mathcal{M}$ is necessary for (1.6.3) to hold, and they managed to prove that $\omega \in \check{\mathcal{D}}$ is a sufficient condition, but the problem of describing the radial weights for which the inequality (1.6.3) holds remains open.

On the other hand, (1.6.1) can be generalized to fractional derivatives when ω is a standard weight. Indeed, for $f(z)=\sum_{n=0}^\infty \widehat{f}(n)z^n \in \mathcal{H}(\mathbb{D})$ and $\beta>0$, consider the operator

$$D^\beta(f)(z) = \frac{2}{\Gamma(\beta+1)} \sum_{n=0}^\infty \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \widehat{f}(n) z^n, \quad z \in \mathbb{D}, \tag{1.6.4}$$

which basically coincides with the fractional derivative of order $\beta>0$ introduced by Hardy and Littlewood in [26, p. 409]. The differences between (1.6.4) and [26, (3.13)] are in the multiplicative factor $\frac{2}{\Gamma(\beta+1)}$ and the inessential factor z^β. A folklore result states that

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |D^\beta(f)(z)|^p (1-|z|)^{\beta p}\omega(z)\,dA(z), \quad f \in \mathcal{H}(\mathbb{D}), \tag{1.6.5}$$

for any $\beta, p>0$ and any standard radial weight ω (see [9, Theorem A] for the range $p \geqslant 1$). Moreover, Flett proved in [21, Theorem 6] that (1.6.5) remains true for any $\beta, p>0$ if $D^\beta f$ is replaced with the multiplier transformation $f^{[\beta]}(z)=\sum_{n=0}^\infty (n+1)^\beta \widehat{f}(n) z^n$, which may also be regarded as fractional derivative of order $\beta>0$.

For a radial weight μ, we define the fractional derivative of f induced by μ

$$D^\mu(f)(z) = \sum_{n=0}^\infty \frac{\widehat{f}(n)}{\mu_{2n+1}} z^n, \ z \in \mathbb{D}.$$

It is clear that $D^\mu(f)$ is a polynomial if f is a polynomial and it follows from the inequality

$$\mu_{2n+1} \geqslant \epsilon^{n+\frac{1}{2}} \widehat{\mu}(\sqrt{\epsilon}), \quad 0<\epsilon<1, \quad n \in \mathbb{N},$$

that $D^\mu(f) \in \mathcal{H}(\mathbb{D})$ for each $f \in \mathcal{H}(\mathbb{D})$. See [52, 58] for related definitions or reformulations of classical and generalized fractional derivative. Observe that if μ is the standard weight $\mu(z)=\beta(1-|z|^2)^{\beta-1}$, $\beta>0$, $D^\mu(f)$ is nothing but $D^\beta(f)$ (see (1.6.4)).

The primary purpose of the paper (IV) is twofold: extending the Littlewood-Paley formulas (1.6.1) and (1.6.2) replacing the higher order derivative $f^{(n)}$ with the fractional derivative $D^\mu(f)$ induced by $\mu \in \mathcal{D}$, and describing the radial weights such that the arising formulas hold. With this aim, observe that $\widehat{\mu}(z) \asymp (1-|z|)^\beta$ when $\mu(z) = \beta(1-|z|^2)^{\beta-1}$, $\beta > 0$, so an appropriate interpretation of the Littlewood-Paley estimate (1.6.5) is

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |D^\mu(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}),$$

when μ is a standard weight.

The motivation for establishing these equivalences in terms of fractional derivatives is not limited to merely generalizing known results such as (1.6.1), (1.6.2) and (1.6.5). In fact, recent research has shown that certain fractional derivatives have turned out useful in proving significant theorems in the operator theory. For instance, certain fractional derivatives induced by radial weights have been employed to show that $P_\omega : L^\infty \to \mathcal{B}$ is bounded and onto if $\omega \in \mathcal{D}$ [52, Theorem 5]. It is also worth mentioning that these fractional derivatives have also been used to describe the boundedness of the small Hankel operator h_f^ν on weighted Bergman spaces induced by doubling weights [18].

Part II

Collection of manuscripts

Chapter 2

Hilbert-type operator induced by radial weight.

2.1 Summary of (I) and (II).

These two papers, the first one accepted for publication the 14^{th} October 2020 in *Journal of Mathematical Analysis and Applications* and the second one the 31^{th} August 2023 in *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.*, are focused on the action of Hilbert-type operator H_ω introduced in section 1.5.1

$$H_\omega(f)(z) = \int_0^1 f(t)\left(\frac{1}{z}\int_0^z B_t^\omega(\zeta)d\zeta\right)\omega(t)dt = \int_0^1 f(t)K_t^\omega(z)\,\omega(t)dt$$

in classical function spaces, where K_t^ω denotes the kernel of H_ω as integral operator, given by $K_t^\omega(z) = \frac{1}{z}\int_0^z B_t^\omega(\zeta)d\zeta$. We will also deal with the sublinear Hilbert-type operator

$$\widetilde{H_\omega}(f)(z) = \int_0^1 |f(t)|\left(\frac{1}{z}\int_0^z B_t^\omega(\zeta)d\zeta\right)\omega(t)\,dt.$$

Let us begin by establishing the results for Hardy spaces. First of all, regarding the case $p=\infty$, it is well-known that the classical Hilbert operator H is not bounded on H^∞. A simple computation shows that this is a general phenomenon rather than a particular case, because for any radial weight ω, H_ω is not bounded in H^∞. Indeed, if ω is a radial weight and $x \in (0,1)$

$$H_\omega(1)(x) = \sum_{n=0}^\infty \frac{\omega_n}{2\omega_{2n+1}(n+1)}x^n \geqslant \frac{1}{2x}\sum_{n=0}^\infty \frac{x^{n+1}}{n+1} = \frac{1}{2x}\log\left(\frac{1}{1-x}\right),$$

so H_ω is not bounded on H^∞. However, the classical Hilbert operator H is bounded from H^∞ to the space BMOA [29, Theorem 1.2]. So, it is natural wondering about the radial weights such that $H_\omega : H^\infty \to \mathrm{BMOA}$ is bounded. The next result answers this question. The full version of this result can be found in (II), although the proof of $(iii) \Leftrightarrow (iv)$ can be found in (I).

Theorem 2.1.1. *Let ω be a radial weight and let $T \in \{H_\omega, \widetilde{H_\omega}\}$. Then, the following statements are equivalent:*

(i) $T : H^\infty \to HL(\infty)$ *is bounded;*

(ii) $T : H^\infty \to \mathrm{BMOA}$ *is bounded;*

(iii) $T : H^\infty \to \mathcal{B}$ *is bounded;*

(iv) $\omega \in \widehat{\mathcal{D}}$.

Let us recall that the Hilbert operator H is bounded on Hardy spaces H^p if and only if $1 < p < \infty$ [14].

Going further, S. Li and S. Stević adressed the following standard weighted Hilbert operator in [30]

$$H_\beta(f)(z) = \int_0^1 \frac{f(t)}{(1-tz)^{\beta+1}}(1-t)^\beta dt,\ \beta > -1.$$

They proved the next result regarding the boundedness of H_β on H^p.

Theorem E. *(i) If $2 \leqslant p < \infty$ and $\beta + \frac{1}{p'} > 0$ then*

$$\|H_\beta(f)\|_{H^p} \lesssim \frac{\Gamma(\frac{1}{p})\Gamma(\frac{1}{p'}+\beta)}{\Gamma(1+\beta)}\|f\|_{H^p}.$$

(ii) If $1 < p < 2$ and $\beta + \frac{1}{p'} > 0$, then for each $f \in H^p$ with $f(0) = 0$,

$$\|H_\beta(f)\|_{H^p} \lesssim \frac{\Gamma(\frac{1}{p})\Gamma(\frac{1}{p'}+\beta)}{\Gamma(1+\beta)}\|f\|_{H^p}.$$

The next natural step is to study the action of H_ω between Hardy spaces H^p, for $1 < p < \infty$. The following result gives a complete answer to the boundedness question and it can be found in (II), although we appeal for some approaches obtained in (I) for its proof.

Theorem 2.1.2. *Let ω be a radial weight and $1 < p < \infty$. Let $X_p, Y_p \in \{H(\infty,p), H^p, D^p_{p-1}, HL(p)\}$ and $T \in \{H_\omega, \widetilde{H_\omega}\}$. Then the following statements are equivalent:*

(i) $T : X_p \to Y_p$ *is bounded;*

(ii) $\omega \in \mathcal{D}$ *and* $M_p(\omega) = \sup_{N\in\mathbb{N}} \left(\sum_{n=0}^{N} \frac{1}{(n+1)^2 \omega_{2n+1}^p} \right)^{\frac{1}{p}} \left(\sum_{n=N}^{\infty} \omega_{2n+1}^{p'} (n+1)^{p'-2} \right)^{\frac{1}{p'}} < \infty;$

(iii) $\omega \in \widehat{\mathcal{D}}$ *and* $M_p(\omega) = \sup_{N\in\mathbb{N}} \left(\sum_{n=0}^{N} \frac{1}{(n+1)^2 \omega_{2n+1}^p} \right)^{\frac{1}{p}} \left(\sum_{n=N}^{\infty} \omega_{2n+1}^{p'} (n+1)^{p'-2} \right)^{\frac{1}{p'}} < \infty;$

(iv) $\omega \in \widehat{\mathcal{D}}$ *and* $M_{p,c}(\omega) = \sup_{0<r<1} \left(\int_0^r \frac{1}{\widehat{\omega}(t)^p} dt \right)^{\frac{1}{p}} \left(\int_r^1 \left(\frac{\widehat{\omega}(t)}{1-t} \right)^{p'} dt \right)^{\frac{1}{p'}} < \infty.$

The standard choice $\omega(z) = (\beta+1)(1-|z|)^\beta$, $\beta > -1$ shows that $H_\beta : H^p \to H^p$ is bounded if and only if $\beta > -\frac{1}{p'}$, so this theorem extends the known result stated in Theorem E, as well as the sufficient condition in the one in [14] for the choice $\omega = 1$.

The proof of (i)⇒(iii) of Theorem 2.1.2 has two steps. Firstly, we prove that $\omega \in \widehat{\mathcal{D}}$ is necessary for $H_\omega : X_p \to Y_p$ to be bounded, and later on the condition $M_p(\omega) < \infty$ is obtained by employing the polynomials $f_{N,M}(z) = \sum_{k=N}^{M} \omega_{2k}^{\alpha} (k+1)^\beta z^k$, $N, M \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$ as test functions. Then, we can deduce that any radial weight ω satisfying the condition $M_p(\omega) < \infty$ belongs to $\mathcal{M}$. This proves (ii)⇔(iii). For the proof of (iii)⇔(iv) the main tool employed is a known description of the class $\widehat{\mathcal{D}}$ [42, Lemma 2.1].

Finally, we prove (iv)⇒(i) which is the most involved implication in the proof of Theorem 2.1.2. In order to obtain it, we merge techniques coming from complex and harmonic analysis. Specifically, we use that $\omega \in \mathcal{D}$ if and only if the regularized weight $\widetilde{\omega}(t) = \frac{\widehat{\omega}(t)}{1-t} \in \mathcal{D}$. Moreover we employ decomposition norm theorems for mixed norm spaces [33, Theorem 2.1], the well-known inclusions between the spaces previously discussed in (1.1.1), (1.1.2) and (1.1.3), and precise estimates of the integral means of order p of the derivative of the kernels $G_u^\omega = \frac{d}{dz} K_u^\omega$ where $K_u^\omega(z) = \frac{1}{z} \int_0^z B_u^\omega(z)\, du$, valid for $\omega \in \widehat{\mathcal{D}}$ (see [46])

$$M_q^q(r, G_\zeta^\omega) \asymp |\zeta|^q M_q^q(r, B_\zeta^\omega) \asymp |\zeta|^q + |\zeta|^q \int_0^{r|\zeta|} \frac{dx}{\widehat{\omega}(x)^q (1-x)^q}, \quad 0 \leqslant r < 1, \quad \zeta \in \mathbb{D}.$$

In addition, it is worth emphasizing that one of the key steps in the proof is obtaining the following approach,

$$\|H_\omega(f)\|_{H^p} \asymp \|H_\omega(f)\|_{D^p_{p-1}} \asymp \|H_\omega(f)\|_{HL(p)}$$

for any non-negative $f \in L^1_{\omega,[0,1)}$. This can be found in work (I) and will be used repeatedly in the proofs of the results in this discussion. Regarding harmonic analysis results we draw on, the previous ingredients lead us to a couple of classical weighted inequalities for Hardy operators (see [36]).

Observe that both, the discrete condition $M_p(\omega) < \infty$ and its continuous version $M_{p,c}(\omega) < \infty$, are used in the proof of Theorem 2.1.2. The first one follows from (i), and the condition $M_{p,c}(\omega) < \infty$ is employed to prove that $T : X_p \to Y_p$ is bounded.

As for the case $p = 1$ we obtain the following result.

Theorem 2.1.3. *Let ω be a radial weight, $X_1, Y_1 \in \{H(\infty,1), H^1, D^1_0, HL(1)\}$ and $T \in \{H_\omega, \widetilde{H_\omega}\}$. Then the following statements are equivalent:*

(i) $T : X_1 \to Y_1$ *is bounded;*

(ii) $\omega \in \widehat{\mathcal{D}}$ *and the measure μ_ω defined as* $d\mu_\omega(z) = \omega(z)\left(\int_0^{|z|} \frac{ds}{\widehat{\omega}(s)}\right) \chi_{[0,1)}(z)\, dA(z)$ *is a* 1*-Carleson measure for* X_1*;*

(iii) $\omega \in \widehat{\mathcal{D}}$ *and satisfies the condition*

$$M_{1,c}(\omega) = \sup_{a\in[0,1)} \frac{1}{1-a}\int_a^1 \omega(t)\left(\int_0^t \frac{ds}{\widehat{\omega}(s)}\right) dt < \infty;$$

(iv) $\omega \in \mathcal{D}$ *and satisfies the condition* $M_{1,c}(\omega) < \infty$*;*

(v) $\omega \in \widehat{\mathcal{D}}$ *and satisfies the condition* $M_{1,d}(\omega) = \sup_{a\in[0,1)} \frac{\widehat{\omega}(a)}{1-a}\left(\int_0^a \frac{ds}{\widehat{\omega}(s)}\right) < \infty$*;*

(vi) $\omega \in \widehat{\mathcal{D}}$ *and satisfies the condition*

$$M_1(\omega) = \sup_{N\in\mathbb{N}} (N+1)\omega_{2N} \sum_{k=0}^{N} \frac{1}{(k+1)^2\omega_{2k}} < \infty.$$

We recall that given a Banach space (or a complete metric space) X of analytic functions on $\mathbb{D}$, a positive Borel measure μ on $\mathbb{D}$ is called a q-Carleson measure for X if the identity operator $I_d : X \to L^q(\mu)$ is bounded. The origin of this concept dates back to Lennart Carleson, who provided a geometric description of p-Carleson measures for Hardy spaces H^p, $0 < p < \infty$, in route of the Corona Theorem [19, Chapter 9]. These measures are called classical Carleson measures.

The proof of Theorem 2.1.3 uses characterizations of Carleson measures for X_1-spaces, universal Cesàro basis of polynomials (see section 1.4), estimates for the integral means of the kernels

$$M_1(\rho, K_t^\omega) \asymp 1 + \int_0^{\rho t} \frac{ds}{\widehat{\omega}(s)}, \quad 0 \leqslant t, \rho < 1,$$

and pointwise estimations for the kernels on the interval $[0,1)$

$$K_s^\omega(t) \asymp 1 + \int_0^{st} \frac{1}{\widehat{\omega}(x)(1-x)} dx, \quad 0 \leqslant t, s < 1,$$

both valid for $\omega \in \widehat{\mathcal{D}}$.

Concerning the classes of radial weights $\widehat{\mathcal{D}}$ and $M_{p,c} = \{\omega : M_{p,c}(\omega) < \infty\}$, $1 \leqslant p < \infty$, a standard weight, $\omega(z) = (1-|z|)^\beta$, $\beta > -1$, satisfies the condition $M_{p,c}(\omega) < \infty$ if and only if $\beta > \frac{1}{p} - 1$, so in this case $H_\omega : H^p \to H^p$ is bounded if and only if $\beta > \frac{1}{p} - 1$. Moreover, a calculation shows that the exponential type weight $\omega(r) = \exp\left(-\frac{1}{1-r}\right) \in M_{p,c}$ for any $p \in [1,\infty)$, but $\omega \notin \widehat{\mathcal{D}}$, see [56, Example 3.2] for further details. So, $\widehat{\mathcal{D}}$ and $M_{p,c}$ are not included in each other.

To proceed, we establish the following notation. For a radial weight ω and $0 < p < \infty$, let $L^p_{\omega,[0,1)}$ be the Lebesgue space of measurable functions such that $\|f\|^p_{L^p_{\omega,[0,1)}} = \int_0^1 |f(t)|^p \omega(t)\,dt < \infty$. If $\omega = 1$ we simply write $L^p_{[0,1)}$ instead of $L^p_{[0,1),1}$.

In paper (I), a description of the upper doubling weights $\omega \in \widehat{\mathcal{D}}$ such that $H_\omega : L^p_{[0,1)} \to H^p$ is bounded is obtained. Therefore, by joining (1.1.3) and this result, we get a sufficient condition for the boundedness of $H_\omega : H^p \to H^p$, $1 < p < \infty$. Later on, in work (II) we achieve the following improvement of the just mentioned result by using Theorem 2.1.2.

Theorem 2.1.4. *Let ω be a radial weight. Let $Y_p \in \{H(\infty,p), H^p, D^p_{p-1}, HL(p)\}$, $1 < p < \infty$ and $T \in \{H_\omega, \widetilde{H_\omega}\}$. Then the following statements are equivalent:*

(i) $T : L^p_{[0,1)} \to Y_p$ is bounded;

(ii) $\omega \in \mathcal{D}$ and satisfies the condition

$$m_p(\omega) = \sup_{0<r<1} \left(1 + \int_0^r \frac{1}{\widehat{\omega}(t)^p} dt\right)^{\frac{1}{p}} \left(\int_r^1 \omega(t)^{p'}\,dt\right)^{\frac{1}{p'}} < \infty;$$

(iii) $\omega \in \widehat{\mathcal{D}}$ and satisfies the condition $m_p(\omega) < \infty$.

Regarding the case $p = 1$, we obtain the following analogous result.

Theorem 2.1.5. *Let ω be a radial weight, let $Y_1 \in \{H(\infty,1), H^1, D^1_0, HL(1)\}$ and let $T \in \{H_\omega, \widetilde{H_\omega}\}$. Then the following statements are equivalent:*

(i) $T : L^1_{[0,1)} \to Y_1$ is bounded;

(ii) $\omega \in \mathcal{D}$ and $m_1(\omega) = \operatorname{ess\,sup}_{t \in [0,1)} \omega(t) \left(1 + \int_0^t \frac{ds}{\widehat{\omega}(s)}\right) < \infty$;

(iii) $\omega \in \widehat{\mathcal{D}}$ and $m_1(\omega) = \operatorname{ess\,sup}_{t \in [0,1)} \omega(t) \left(1 + \int_0^t \frac{ds}{\widehat{\omega}(s)}\right) < \infty$.

Based on the above findings, we compare the conditions $M_{p,c}(\omega) < \infty$ and $m_p(\omega) < \infty$ to establish a connection between the boundedness of $T : X_p \to Y_p$ and the boundedness of $T : L^p_{[0,1)} \to Y_p$, where $X_p, Y_p \in \{H(\infty,p), H^p, D^p_{p-1}, HL(p)\}$ and $T \in \{H_\omega, \widetilde{H_\omega}\}$ for $1 \leqslant p < \infty$. This comparison will provide deeper insights into the behavior of the operator T under different weight conditions. Bearing in mind (1.1.3), it is clear that the condition $m_p(\omega) < \infty$ implies that $M_{p,c}(\omega) < \infty$, for any weight $\omega \in \widehat{\mathcal{D}}$. Moreover, a simple observation yields $M_{p,c}(\omega) < \infty$ if and only if

$$\sup_{0<r<1}\left(1+\int_0^r \frac{1}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}}\left(\int_r^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}dt\right)^{\frac{1}{p'}} < \infty, \quad \text{when } 1<p<\infty$$

and $\sup_{a\in[0,1)} \frac{\widehat{\omega}(a)}{1-a}\left(1+\int_0^a \frac{ds}{\widehat{\omega}(s)}\right) < \infty$ if and only if $M_{1,d}(\omega) < \infty$. So, the conditions $M_{p,c}(\omega) < \infty$ and $m_p(\omega) < \infty$, are equivalent for any $1 \leqslant p < \infty$ whenever ω satisfies the pointwise inequality

$$\omega(t) \lesssim \frac{\widehat{\omega}(t)}{1-t}, \quad t \in [0,1), \tag{2.1.1}$$

and $\omega \in \widehat{\mathcal{D}}$. The condition (2.1.1) implies restrictions on the decay and on the regularity and smoothness of the weight; in fact, if ω satisfies (2.1.1) then ω cannot decrease rapidly or exhibit strong oscillations. For instance, the exponential type weight $\omega(r) = \exp\left(-\frac{1}{1-r}\right)$, which is a prototype of rapidly decreasing weight (see [37]), has the property

$$\widehat{\omega}(r) \asymp \omega(r)(1-r)^2, \quad 0 \leqslant r < 1,$$

so it does not satisfy (2.1.1) and it does not belong to $\widehat{\mathcal{D}}$. Conversely, any regular $\mathcal{R}$ or rapidly increasing weight $\mathcal{I}$, which are large subclasses of $\widehat{\mathcal{D}}$ (see [45, Section 1.2]), satisfies (2.1.1). However, we construct in paper (II) weights $\omega \in \mathcal{D}$ with pronounced oscillatory behaviour so that $M_{p,c}(\omega) < \infty$ and $m_p(\omega) = \infty$, and consequently they do not satisfy (2.1.1).

Bearing in mind Theorems 2.1.1, 2.1.2 and 2.1.3, we deduce that if $T : X_p \to Y_p$ is bounded then $T : H^\infty \to HL(\infty)$, where as usual $X_p, Y_p \in \{H(\infty,p), H^p, D^p_{p-1}, HL(p)\}$, $1 \leqslant p < \infty$, and $T \in \{H_\omega, \widetilde{H_\omega}\}$. We prove that this result is a general phenomenon for Hilbert-type operators and parameters $1 \leqslant q < p$.

Theorem 2.1.6. *Let ω be radial weight, $T \in \{H_\omega, \widetilde{H_\omega}\}$ and $1 \leqslant q < p < \infty$. Further, let $X_q, Y_q \in \{H^q, D^q_{q-1}, HL(q), H(\infty,q)\}$ and $X_p, Y_p \in \{H^p, D^p_{p-1}, HL(p), H(\infty,p)\}$. If $T : X_q \to Y_q$ is bounded, then $T : X_p \to Y_p$ is bounded.*

We also deal with the compactness of H_ω in paper (II), and we show that that there does not exist radial weights ω such that $H_\omega : X_p \to Y_p$ is compact, where $X_p, Y_p \in \{H^p, D^p_{p-1}, HL(p), H(\infty,p)\}$ and $1 \leqslant p < \infty$, neither radial weights such that $H_\omega : H^\infty \to \mathcal{B}$ is compact.

After the extensive and comprehensive description of the boundedness of the operator H_ω in Hardy spaces, we will briefly discuss the problem solved in (I) regarding the action of the H_ω operator from the weighted Lebesgue space $L^p_{\widehat{\nu},[0,1)}$ to Bergman space A^p_ν induced by radial weight ν.

Firstly, observe that by an integration of (1.1.3), we can obtain the following inequality

$$\int_0^1 |f(t)|^p \widehat{\nu}(t)\,dt \leqslant \int_0^1 M_\infty^p(t,f)\widehat{\nu}(t)\,dt \leqslant \frac{\pi}{2}\|f\|^p_{A^p_\nu}, \quad f \in \mathcal{H}(\mathbb{D}), \tag{2.1.2}$$

which plays a key role in this problem because it allows us to provide a sufficient condition for the boundedness of the operator $H_\omega : A^p_\nu \to A^p_\nu$. The result we prove in (I) is the following:

Theorem 2.1.7. *Let $1 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ and ν a radial weight on $\mathbb{D}$. Then, $H_\omega : L^p_{\widehat{\nu},[0,1)} \to A^p_\nu$ is bounded if and only if*

$$\sup_{0<r<1} \left(1 + \int_0^r \frac{\widehat{\nu}(t)}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}} \left(\int_r^1 \left(\frac{\omega(t)}{\widehat{\nu}(t)^{\frac{1}{p}}}\right)^{p'} dt\right)^{\frac{1}{p'}} < \infty. \tag{2.1.3}$$

In particular if (2.1.3) *holds, $H_\omega : A^p_\nu \to A^p_\nu$ is bounded.*

It is worth mentioning that Theorem 2.1.7 generalizes and improves [44, Theorem 2], because we do not need to assume any integrability condition on the weights similar to [44, (1.2)] neither assuming that ν is a regular weight [44, (2.1)]. The proof of the sufficiency in Theorem 2.1.7 draws on the inequality

$$\|f\|^p_{A^p_\nu} \lesssim |f(0)|^p \widehat{\nu}(0) + \int_0^1 M_q^p(t,f')(1-t)^{p\left(1-\frac{1}{q}\right)}\widehat{\nu}(t)dt, \quad f \in \mathcal{H}(\mathbb{D}), \tag{2.1.4}$$

which is valid for any $1 < q < p < \infty$ and any radial weight ν. This inequality is inherited from [33, Corollary 3.1] through integration.

The inequality (2.1.4) allows to get rid of the integral in the expression of K_t^ω and to employ estimates for the Bergman reproducing kernels previously obtained in [46, Theorem 1] for upper doubling weights. After employing all the previous tools, the problem boils down to describing a couple of weighted classical Hardy inequalities [36], one of them follows from the Muckenhoupt-type condition (2.1.3) and the other one holds whenever

$$\sup_{0<r<1} (1-r)^{\frac{1}{p}} \widehat{\nu}(r)^{\frac{1}{p}} \left(\int_0^r \left(\frac{\omega(t)}{\widehat{\nu}(t)^{\frac{1}{p}}}\right)^{p'} \frac{1}{\widehat{\omega}(t)^{p'}}dt\right)^{\frac{1}{p'}} < \infty,$$

which can be deduced from (2.1.3).

Finally, we highlight a connection between the actions of the Hilbert-type operator H_ω and the Bergman projection P_ω on Lebesgue spaces. While the generalization is reasonable as such, this connection also justifies the study of this new operator H_ω, since it could even be used as a toy model to study open problems related to the Bergman projection.

Theorem 2.1.8. *Let $1 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ and ν a radial weight. Then, the following statements are equivalent:*

(i) $P_\omega : L^p_{\widehat{\nu}} \to A^p_{\widehat{\nu}}$ *is bounded;*

(ii) $P^+_\omega : L^p_{\widehat{\nu}} \to L^p_{\widehat{\nu}}$ *is bounded;*

(iii) $\omega \in \mathcal{D}$, $\nu \in \widehat{\mathcal{D}}$ *and* $H_\omega : L^p_{\widehat{\nu},[0,1)} \to A^p_\nu$ *is bounded;*

(iv) $\omega \in \mathcal{D}$ *and* $\nu \in \widehat{\mathcal{D}}$ *and*

$$A_p(\omega,\widehat{\nu}) = \sup_{0\leqslant r<1} \frac{\widehat{\widehat{\nu}}(r)^{\frac{1}{p}} \left(\int_r^1 \left(\frac{\omega(s)}{\widehat{\nu}(s)^{\frac{1}{p}}} \right)^{p'} ds \right)^{\frac{1}{p'}}}{\widehat{\omega}(r)} < \infty;$$

(v) $\omega \in \mathcal{D}$ *and* $\nu \in \widehat{\mathcal{D}}$ *and*

$$\sup_{0<r<1} \left(1 + \int_0^r \frac{\widehat{\nu}(t)}{\widehat{\omega}(t)^p} dt \right)^{\frac{1}{p}} \left(\int_r^1 \left(\frac{\omega(t)}{\widehat{\nu}(t)^{\frac{1}{p}}} \right)^{p'} dt \right)^{\frac{1}{p'}} < \infty.$$

In particular, Theorem 2.1.8 says that $P_\omega : L^p_{\widehat{\nu}} \to A^p_{\widehat{\nu}}$ and $H_\omega : L^p_{\widehat{\nu},[0,1)} \to A^p_\nu$ are simultaneously bounded whenever $\omega \in \mathcal{D}$, $\nu \in \widehat{\mathcal{D}}$ and $1 < p < \infty$.

- **Hilbert-type operator induced by radial weight**
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Hilbert-type operator induced by radial weight ☆

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A B S T R A C T

We consider the Hilbert-type operator defined by

$$H_\omega(f)(z) = \int_0^1 f(t)\left(\frac{1}{z}\int_0^z B_t^\omega(u)\,du\right)\omega(t)dt,$$

where $\{B_\zeta^\omega\}_{\zeta\in\mathbb{D}}$ are the reproducing kernels of the Bergman space A^2_ω induced by a radial weight ω in the unit disc $\mathbb{D}$. We prove that H_ω is bounded from H^∞ to the Bloch space if and only if ω belongs to the class $\widehat{\mathcal{D}}$, which consists of radial weights ω satisfying the doubling condition $\sup_{0\leqslant r<1}\frac{\int_r^1 \omega(s)\,ds}{\int_{\frac{1+r}{2}}^1 \omega(s)\,ds}<\infty$. Further, we describe the weights $\omega\in\widehat{\mathcal{D}}$ such that H_ω is bounded on the Hardy space H^1, and we show that for any $\omega\in\widehat{\mathcal{D}}$ and $p\in(1,\infty)$, $H_\omega : L^p_{[0,1)}\to H^p$ is bounded if and only if the Muckenhoupt type condition

$$\sup_{0<r<1}\left(1+\int_0^r \frac{1}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}}\left(\int_r^1 \omega(t)^{p'}\,dt\right)^{\frac{1}{p'}}<\infty,$$

holds. Moreover, we address the analogous question about the action of H_ω on weighted Bergman spaces A^p_ν.

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ORIGINAL PAPER

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Hilbert-type operator induced by radial weight on Hardy spaces

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Abstract

We consider the Hilbert-type operator defined by

$$H_\omega(f)(z)=\int_0^1 f(t)\left(\frac{1}{z}\int_0^z B_t^\omega(u)\,du\right)\omega(t)dt,$$

where $\{B_\zeta^\omega\}_{\zeta\in\mathbb{D}}$ are the reproducing kernels of the Bergman space A^2_ω induced by a radial weight ω in the unit disc $\mathbb{D}$. We prove that H_ω is bounded on the Hardy space H^p, $1<p<\infty$, if and only if

$$\sup_{0\le r<1}\frac{\widehat{\omega}(r)}{\widehat{\omega}\left(\frac{1+r}{2}\right)}<\infty, \qquad (\dagger)$$

and

$$\sup_{0<r<1}\left(\int_0^r\frac{1}{\widehat{\omega}(t)^p}dt\right)^{\frac{1}{p}}\left(\int_r^1\left(\frac{\widehat{\omega}(t)}{1-t}\right)^{p'}dt\right)^{\frac{1}{p'}}<\infty,$$

where $\widehat{\omega}(r)=\int_r^1\omega(s)\,ds$. We also prove that $H_\omega: H^1\to H^1$ is bounded if and only if ($\dagger$) holds and

$$\sup_{r\in[0,1)}\frac{\widehat{\omega}(r)}{1-r}\left(\int_0^r\frac{ds}{\widehat{\omega}(s)}\right)<\infty.$$

As for the case $p=\infty$, H_ω is bounded from H^∞ to BMOA, or to the Bloch space, if and only if ($\dagger$) holds. In addition, we prove that there does not exist radial weights ω such that $H_\omega: H^p\to H^p$, $1\le p<\infty$, is compact and we consider the action of H_ω on some spaces of analytic functions closely related to Hardy spaces.

Keywords Hilbert operator · Hardy space · Bergman reproducing kernel · Radial weight

Mathematics Subject Classification 47G10 · 30H10

Chapter 3

Bergman projection on Lebesgue space induced by doubling weight

3.1 Summary of (III) .

This paper, accepted for publication the 28^{th} November 2023 in *Results in Mathematics*, is focused on the boundedness on Lebesgue spaces of the Bergman projection

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B_z^\omega(\zeta)}\omega(\zeta)\,dA(\zeta), \quad z \in \mathbb{D},$$

induced by a radial weight ω. Specifically, we are interested in the study of the radial weights ω, ν such that the inequality

$$\|P_\omega(f)\|_{L^p_\nu} \leqslant C\|f\|_{L^p_\nu}, \quad f \in L^p_\nu,$$

holds. As we introduced in Section 1.5.2, this is an open problem, even when ν is standard.

Our first result provides a useful necessary condition for $P_\omega : L^p_\nu \to L^p_\nu$ to be bounded for any radial weights ω, ν. From now, we write

$$\sigma = \sigma_{\omega,\nu,p} = \left(\frac{\omega}{\nu^{\frac{1}{p}}}\right)^{p'} = \frac{\omega^{\frac{p}{p-1}}}{\nu^{\frac{1}{p-1}}}$$

for short.

Proposition 3.1.1. *Let ω and ν be radial weights and $1 < p < \infty$. If $P_\omega : L^p_\nu \to L^p_\nu$ is bounded, then*

$$D_p(\omega,\nu) = \sup_{n\in\mathbb{N}\cup\{0\}} \frac{(\nu_{np+1})^{\frac{1}{p}}\,(\sigma_{np'+1})^{\frac{1}{p'}}}{\omega_{2n+1}} \leqslant \|P_\omega\|_{L^p_\nu\to L^p_\nu} < \infty. \tag{3.1.1}$$

The condition $D_p(\omega,\nu)<\infty$ generalizes the one proved by Dostanić in the case $\omega=\nu$ [15]. As discussed in Section 1.5.2, Dostanić posed the problem of describing the radial weights for which P_ω is bounded in L^p_ω and demonstrated the necessity of the condition

$$D_p(\omega,\omega)=\sup_{n\in\mathbb{N}\cup\{0\}}\frac{(\omega_{np+1})^{\frac{1}{p}}(\omega_{np'+1})^{\frac{1}{p'}}}{\omega_{2n+1}}<\infty. \tag{3.1.2}$$

It is not known whether or not the condition (3.1.1) is sufficient to guarantee the boundedness of $P_\omega : L^p_\nu \to L^p_\nu$, even for ν a standard weight. In this paper we will show that the condition $D_p(\omega,\nu)<\infty$ describes the boundedness of $P_\omega : L^p_\nu \to L^p_\nu$ for $\nu\in\mathcal{D}$. Therefore, we solve the problem for a large class of weights ν which contains the standard weights.

Apart from the moment condition $D_p(\omega,\nu)<\infty$, we naturally face the requirement of the finiteness of the quantity

$$A_p(\omega,\nu)=\sup_{0\leqslant r<1}\frac{\left(\int_r^1\nu(t)t\,dt\right)^{\frac{1}{p}}\left(\int_r^1\sigma(t)t\,dt\right)^{\frac{1}{p'}}}{\int_r^1\omega(t)t\,dt}.$$

If r is uniformly bounded away from one, then the quotient in the supremum above is certainly uniformly bounded, in particular, it equals to $\|\nu\|_{L^1}^{\frac{1}{p}}\|\sigma\|_{L^1}^{\frac{1}{p'}}/\|\omega\|_{L^1}\in(0,\infty)$ for $r=0$ as all the weights involved are non-trivial radial weights. Therefore the boundedness of $A_p(\omega,\nu)$ is equivalent to the fact that $\sigma\in L^1$ and

$$\limsup_{r\to1^-}\frac{\widehat{\nu}(r)^{\frac{1}{p}}\widehat{\sigma}(r)^{\frac{1}{p'}}}{\widehat{\omega}(r)}=\limsup_{r\to1^-}\frac{\left(\int_r^1\nu(t)t\,dt\right)^{\frac{1}{p}}\left(\int_r^1\sigma(t)t\,dt\right)^{\frac{1}{p'}}}{\int_r^1\omega(t)t\,dt}<\infty.$$

The next theorem is our first main result.

Theorem 3.1.1. *Let ω be a radial weight, $\nu\in\mathcal{D}$ and $1<p<\infty$. Then the following statements are equivalent:*

(i) $P_\omega : L^p_\nu\to L^p_\nu$ *is bounded;*

(ii) $P^+_\omega : L^p_\nu\to L^p_\nu$ *is bounded;*

(iii) $D_p(\omega,\nu)<\infty$*;*

(iv) $A_p(\omega,\nu)<\infty$ *and* $\omega\in\widehat{\mathcal{D}}$.

The condition $A_p(\omega,\nu)<\infty$ had been previously introduced in this context, as Peláez and Rättyä proved in [48]. However the condition $D_p(\omega,\nu)<\infty$ seems to be new. To be precise, the next result follows from [48, Theorem 13]. Here and from now on

$$M_p(\omega,\nu)=\sup_{0\leqslant r<1}\left(\int_0^r\frac{\nu(s)s}{\left(\int_s^1\omega(t)t\,dt\right)^p}\,ds+1\right)^{\frac{1}{p}}\left(\int_r^1\sigma(t)t\,dt\right)^{\frac{1}{p'}}.$$

Theorem F. *Let $1 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$, and let ν be a radial weight. Then the following statements are equivalent:*

(i) $P_\omega^+ : L_\nu^p \to L_\nu^p$ *is bounded;*

(ii) $A_p(\omega, \nu) < \infty$ *and* $\omega, \nu \in \mathcal{D}$*;*

(iii) $M_p(\omega, \nu) < \infty$ *and* $\omega, \nu \in \mathcal{D}$*.*

In order to prove Theorem 3.1.1, we aim to attain an improvement of Theorem F, by relaxing hypotheses which guarantee the boundedness of $P_\omega^+ : L_\nu^p \to L_\nu^p$. In fact, several properties on the weights can be deduced from the conditions $A_p(\omega, \nu) < \infty$, $D_p(\omega, \nu) < \infty$ and $M_p(\omega, \nu) < \infty$. These results allow to assume strictly weaker hypotheses in Theorem 3.1.2 (below) than in Theorem F.

Theorem 3.1.2. *Let $1 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$, and let ν be a radial weight. Then the following statements are equivalent:*

(i) $P_\omega^+ : L_\nu^p \to L_\nu^p$ *is bounded;*

(ii) $D_p(\omega, \nu) < \infty$ *and* $\nu \in \mathcal{M}$*;*

(iii) $A_p(\omega, \nu) < \infty$ *and* $\nu \in \mathcal{M}$*;*

(iv) $M_p(\omega, \nu) < \infty$ *and* $\nu \in \mathcal{M}$*.*

As we introduced in section 1.6, it is a remarkable fact the relation between Littlewood-Paley formulas and the boundedness of the Bergman projection on L^p spaces. Indeed, bounded Bergman projections on weighted Lebesgue spaces are often used to obtain this kind of inequalities and reciprocally, Littlewood-Paley formulas are helpful to prove the boundedness of projections in some spaces of functions. Now, we pose a new problem in this line of thought which requires the introduction of some extra notation.

For a radial weight ν, $k \in \mathbb{N}$ and $0 < p < \infty$, the Dirichlet space $D_{\nu,k}^p$ consists of the analytic functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{D_{\nu,k}^p}^p = \sum_{j=0}^{k-1} |f^{(j)}(0)|^p + \int_{\mathbb{D}} |f^{(k)}(z)|^p (1-|z|)^{kp} \nu(z)\, dA(z) < \infty.$$

As we discussed in Section 1.6, the identity $D_{\nu,k}^p = A_\nu^p$ holds for $k \in \mathbb{N}$ if and only if $\nu \in \mathcal{D}$ and a necessary condition (and also sufficient for even p's) for the embedding $D_{\nu,k}^p \subset A_\nu^p$ is $\nu \in \mathcal{M}$ [48]. Therefore the spaces $D_{\nu,k}^p$ and A_ν^p are different for $\nu \in \mathcal{M} \backslash \widehat{\mathcal{D}}$.

Our next result describes the boundedness of the Bergman projection $P_\omega : L_\nu^p \to D_{\nu,k}^p$ when $\omega \in \widehat{\mathcal{D}}$ and $\nu \in \mathcal{M}$. The theorem, whose proof crucially relies on the aforementioned

Littlewood-Paley formulas, also provides sufficient conditions for $P_\omega : L^p_\nu \to L^p_\nu$ to be bounded.

For $k \in \mathbb{N}$ and a radial weight ω we denote

$$T^+_{\omega,k}(f)(z) = (1-|z|)^k \int_{\mathbb{D}} f(\zeta) \left|(B^\omega_\zeta)^{(k)}(z)\right| \omega(\zeta)\, dA(\zeta), \quad z \in \mathbb{D}.$$

Theorem 3.1.3. *Let $1 < p < \infty$, $k \in \mathbb{N}$, $\omega \in \widehat{\mathcal{D}}$ and $\nu \in \mathcal{M}$. Then the following statements are equivalent:*

(i) $T^+_{\omega,k} : L^p_\nu \to L^p_\nu$ *is bounded;*

(ii) $P_\omega : L^p_\nu \to D^p_{\nu,k}$ *is bounded;*

(iii) $P_\omega : L^p_\nu \to L^p_\nu$ *is bounded;*

(iv) $D_p(\omega, \nu) < \infty$*;*

(v) $A_p(\omega, \nu) < \infty$*.*

As for the proof of Theorem 3.1.3 let us mention that the proof of (ii)⇒(iii) is based in the fact that the inequality

$$\|f\|_{D^p_{\nu,k}} \leqslant C\|f\|_{A^p_\nu}, \quad f \in \mathcal{H}(\mathbb{D}),$$

holds if and only if $\nu \in \widehat{\mathcal{D}}$ [48, Theorem 6]. On the other hand, the most involved implication (v)⇒(i) consists in proving that $T^+_{\omega,k} : L^p_\nu \to L^p_\nu$ is bounded under the hypothesis $\omega \in \widehat{\mathcal{D}}$ and $A_p(\omega, \nu) < \infty$. The proof of this statement draws on a kind of Schur test and precise asymptotic estimates of the integral means of the kth-derivative of the Bergman reproducing kernel B^ω_z obtained in [46, Theorem 1], which have been already used several times throughout this thesis.

As we already mentioned, the boundedness of Bergman projections on Lebesgue spaces L^p_ν is really useful to tackle natural questions on operator theory on weighted Bergman spaces A^p_ν. The following interesting result shows an indirect and curious way to obtain this kind of information.

Theorem 3.1.4. *Let $1 < p < \infty$, $k \in \mathbb{N}$, and let ω and ν be radial weights. If $P_\omega : L^p_\nu \to D^p_{\nu,k}$ is bounded, then $P_\nu : L^p_\nu \to D^p_{\nu,k}$ and $P_\nu : L^p_\nu \to L^p_\nu$ are bounded.*

The essence of Theorem 3.1.4 lies in the observation that if there exists a radial weight ω such that $P_\omega : L^p_\nu \to D^p_{\nu,k}$ is bounded, then it necessarily follows that P_ν is bounded on L^p_ν. In other words, in this setting it is the choice $\omega = \nu$ that makes the inequality $\|P_\omega(f)\|_{L^p_\nu} \lesssim \|f\|_{L^p_\nu}$ easiest to achieve.

Our last main result concerns exponential weights. The proof is based on a proper application of Proposition 3.1.1.

Theorem 3.1.5. *Let* $1 < p < \infty$*, and let*

$$\nu(r) = \exp\left(-\frac{\alpha}{(1-r^l)^\beta}\right) \quad \textit{and} \quad \omega(r) = \exp\left(-\frac{\widetilde{\alpha}}{(1-r^{\widetilde{l}})^{\widetilde{\beta}}}\right), \quad 0 \leqslant r < 1,$$

where $0 < \alpha, \widetilde{\alpha}, l, \widetilde{l} < \infty$ *and* $0 < \beta, \widetilde{\beta} \leqslant 1$*. Then* $P_\omega : L^p_\nu \to L^p_\nu$ *is bounded if and only if* $\beta = \widetilde{\beta}$ *and* $\widetilde{\alpha} = \frac{2\alpha}{p}\left(\frac{\widetilde{l}}{l}\right)^\beta$.

The statement in Theorem 3.1.5 is essentially known [10, 12, 57]. Our contribution consists in completing the picture for any $0 < l, \widetilde{l} < \infty$ and showing the usefulness of the condition $D_p(\omega, \nu) < \infty$ in the setting of exponential weights.

- **Bergman projection on Lebesgue space induced by doubling weight**
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Results in Mathematics

Bergman Projection on Lebesgue Space Induced by Doubling Weight

José Ángel Peláez, Elena de la Rosa, and Jouni Rättyä

Abstract. Let ω and ν be radial weights on the unit disc of the complex plane, and denote $\sigma=\omega^{p'}\nu^{-\frac{p'}{p}}$ and $\omega_x=\int_0^1 s^x\omega(s)\,ds$ for all $1\le x<\infty$. Consider the one-weight inequality

$$\|P_\omega(f)\|_{L^p_\nu}\le C\|f\|_{L^p_\nu},\quad 1<p<\infty, \qquad (\dagger)$$

for the Bergman projection P_ω induced by ω. It is shown that the moment condition

$$D_p(\omega,\nu)=\sup_{n\in\mathbb{N}\cup\{0\}}\frac{(\nu_{np+1})^{\frac{1}{p}}(\sigma_{np'+1})^{\frac{1}{p'}}}{\omega_{2n+1}}<\infty$$

is necessary for $(\dagger)$ to hold. Further, $D_p(\omega,\nu)<\infty$ is also sufficient for $(\dagger)$ if ν admits the doubling properties $\sup_{0\le r<1}\frac{\int_r^1\nu(s)s\,ds}{\int_{\frac{1+r}{2}}^1\nu(s)s\,ds}<\infty$ and $\sup_{0\le r<1}\frac{\int_r^1\nu(s)s\,ds}{\int_r^{1-\frac{1-r}{K}}\nu(s)s\,ds}<\infty$ for some $K>1$. In addition, an analogous result for the one weight inequality $\|P_\omega(f)\|_{D^p_{\nu,k}}\le C\|f\|_{L^p_\nu}$, where

$$\|f\|^p_{D^p_{\nu,k}}=\sum_{j=0}^{k-1}|f^{(j)}(0)|^p+\int_{\mathbb{D}}|f^{(k)}(z)|^p(1-|z|)^{kp}\nu(z)\,dA(z)<\infty,\quad k\in\mathbb{N},$$

is established. The inequality $(\dagger)$ is further studied by using the necessary condition $D_p(\omega,\nu)<\infty$ in the case of the exponential type weights $\nu(r)=\exp\left(-\frac{\alpha}{(1-r^l)^\beta}\right)$ and $\omega(r)=\exp\left(-\frac{\tilde{\alpha}}{(1-r^{\tilde{l}})^{\tilde{\beta}}}\right)$, where $0<\alpha,\tilde{\alpha},l,\tilde{l}<\infty$ and $0<\beta,\tilde{\beta}\le 1$.

Mathematics Subject Classification. 30H20, 47G10.

Keywords. Bergman projection, doubling weight, Dirichlet type space, exponential weight.

Chapter 4

Littlewood-Paley inequalities for fractional derivatives on Bergman spaces

4.1 Summary of (IV) .

In this paper, published the 17^{th} September 2022 in *Annales Fennici Mathematici*, we tackle the problem of extending the Littlewood-Paley formulas (1.6.1) and (1.6.2) by replacing the higher order derivative $f^{(n)}$ with the fractional derivative $D^{\mu}(f)$ induced by $\mu \in \mathcal{D}$,

$$D^{\mu}(f)(z) = \sum_{n=0}^{\infty} \frac{\widehat{f}(n)}{\mu_{2n+1}} z^n, \; z \in \mathbb{D},$$

and describing the radial weights such that the appropriate formulas hold. Bearing in mind that $\widehat{\mu}(z) \asymp (1-|z|)^{\beta}$ when $\mu(z) = \beta(1-|z|^2)^{\beta-1}$, $\beta > 0$, and $D^{\beta} = D^{\mu}$ for such μ, a suitable interpretation of the Littlewood-Paley estimate (1.6.5) is

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |D^{\mu}(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}).$$

Our main result describes the radial weights such that the formula above holds.

Theorem 4.1.1. *Let ω be a radial weight, $0 < p < \infty$ and $\mu \in \mathcal{D}$. Then*

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |D^{\mu}(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}), \tag{4.1.1}$$

if and only if $\omega \in \mathcal{D}$.

In particular, as a byproduct of Theorem 4.1.1 we obtain a proof of the folklore result

$$\|f\|_{A^p_\alpha}^p \asymp \int_{\mathbb{D}} |D^{\beta}(f)(z)|^p (1-|z|)^{\beta p+\alpha}\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}),$$

for any $\beta, p > 0$, and $\alpha > -1$. En route to the proof of Theorem 4.1.1 we will establish the following result, which generalizes [48, Theorem 6] to the setting of fractional derivatives induced by radial doubling weights.

Theorem 4.1.2. *Let ω be a radial weight, $0 < p < \infty$ and $\mu \in \mathcal{D}$. Then, there exists a constant $C = C(\omega, \mu, p) > 0$ such that*

$$\int_{\mathbb{D}} |D^{\mu}(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z) \leqslant C \|f\|^p_{A^p_\omega}, \quad f \in \mathcal{H}(\mathbb{D}), \tag{4.1.2}$$

if and only if $\omega \in \widehat{\mathcal{D}}$.

The proof of (4.1.2) of is strongly based on the following inequality between the integral means of order p of $D^\mu f$ and f,

$$M_p(r, D^\mu f) \leqslant C \frac{M_p(r, f)}{\widehat{\mu}\left(\frac{r}{\rho}\right)}, \; 0 < r < \rho < 1, \; 0 < p < \infty. \tag{4.1.3}$$

The inequality (4.1.3) is a natural extension of [41, Lemma 3.1], however, the proofs are completely different by the lack of classical tools for the fractional derivative induced by a radial weight that are available for the classical derivative f'. Being precise, while the proof of the result [41, Lemma 3.1] relies on the Cauchy formula for f', Minkowski's inequality for the case $p \geqslant 1$ and factorization results of H^p functions when $0 < p < 1$, the proof of (4.1.3) is strongly based on the smooth properties of an universal Cesàro basis of polynomials introduced by Jevtić and Pavlović (see section 1.4).

Conversely, we employ some ideas from the proof of [48, Theorem 6] to prove the other implication of Theorem 4.1.2. In particular, the proof shows that (4.1.2) holds if and only if the inequality there holds for all monomial.

As for the proof of Theorem 4.1.1 we show that the inequality

$$\|f\|^p_{A^p_\omega} \lesssim \int_{\mathbb{D}} |D^{\mu}(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}),$$

implies that $\omega \in \mathcal{M}$, and together with Theorem 4.1.2 yields that $\omega \in \mathcal{D}$ when (4.1.1) holds.

Concerning the proof of the sufficiency in Theorem 4.1.1, the formula (4.1.1) can be understood as

$$\|f\|^p_{A^p_\omega} \asymp \|D^\mu f\|^p_{A^p_{\omega \widehat{\mu}^p}}.$$

Therefore, we construct ad hoc norms for the weighted Bergman spaces A^p_ω and $A^p_{\omega\widehat{\mu}^p}$, in the spirit of the decomposition results [33, Theorem 2.1], [38, Theorem 7.5.8], [44, Theorem 4]. These results are valid for $1 < p < \infty$ and their proofs employ the boundedness of the Riesz projection on $L^p(\partial\mathbb{D})$ to deal with the H^p-norm of polynomials of the type

$\Delta_{n_1,n_2} z = \sum_{k=n_1}^{n_2} z^k$. The main difference between them and our result is that we use universal Cesàro basis of polynomials instead of block-type polynomials Δ_{n_1,n_2}, and the smoothness of these polynomials allows us to achieve the decomposition result for $0 < p < \infty$.

Specifically, we show that if $\omega \in \check{\mathcal{D}}$ and by definition there exist $k = k(\omega) > 1$ and $C = C(\omega) > 0$ such that $C \int_{1-\frac{1-r}{k}}^{1} \omega(s)\, ds \leqslant \int_{r}^{1-\frac{1-r}{k}} \omega(s) ds$, $0 \leqslant r < 1$ then $C \int_{1-\frac{1-r}{k}}^{1} \omega(s)\widehat{\mu}^p(s)\, ds \leqslant \int_{r}^{1-\frac{1-r}{k}} \omega(s)\widehat{\mu}^p(s)\, ds$, $0 \leqslant r < 1$. This means that $\omega\widehat{\mu}^p \in \check{\mathcal{D}}$ with the same $k = k(\omega) > 1$ as ω in the definition. Then we prove the following decomposition result:

Proposition 4.1.1. *Let $0 < p < \infty$, $\eta \in \mathcal{D}$ and $k = k(\eta) > 1$, $k \in \mathbb{N}$ such that* (1.2.1) *holds for η and k. If $\{V_{n,k}\}_{n=0}^{\infty}$ is a sequence of polynomials considered in Proposition 1.4.1, then there are constants $C_1 = C_1(p, \eta) > 0$ and $C_2 = C_2(p, \eta) > 0$ such that*

$$C_1 \sum_{n=0}^{\infty} \eta_{k^n} \|V_{n,k} * f\|_{H^p}^p \leqslant \|f\|_{A^p_\eta}^p \leqslant C_2 \sum_{n=0}^{\infty} \eta_{k^n} \|V_{n,k} * f\|_{H^p}^p, \quad f \in \mathcal{H}(\mathbb{D}).$$

As a consequence, we can assert that there exist $k > 1$ and a shared universal Cesàro basis of polynomials $\{V_{n,k}\}_{n=0}^{\infty}$ so that

$$\|f\|_{A^p_\omega}^p \asymp \sum_{n=0}^{\infty} \omega_{k^n} \|V_{n,k} * f\|_{H^p}^p$$

and

$$\|D^\mu(f)\|_{A^p_{\omega\widehat{\mu}^p}}^p \asymp \sum_{n=0}^{\infty} (\omega\widehat{\mu}^p)_{k^n} \|V_{n,k} * D^\mu(f)\|_{H^p}^p$$

for any $p \in (0, \infty)$, which allows to estimate the corresponding norms dealing with blocks of the coefficients of the functions.

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Littlewood–Paley inequalities for fractional derivative on Bergman spaces

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Abstract. For any pair (n,p), $n \in \mathbb{N}$ and $0 < p < \infty$, it has been recently proved by Peláez and Rättyä (2021) that a radial weight ω on the unit disc of the complex plane $\mathbb{D}$ satisfies the Littlewood–Paley equivalence

$$\int_{\mathbb{D}} |f(z)|^p \,\omega(z)\,dA(z) \asymp \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np}\omega(z)\,dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p,$$

for any analytic function f in $\mathbb{D}$, if and only if $\omega \in \mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$. A radial weight ω belongs to the class $\widehat{\mathcal{D}}$ if $\sup_{0\le r<1} \frac{\int_r^1 \omega(s)\,ds}{\int_{\frac{1+r}{2}}^1 \omega(s)\,ds} < \infty$, and $\omega \in \check{\mathcal{D}}$ if there exists $k > 1$ such that $\inf_{0\le r<1} \frac{\int_r^1 \omega(s)\,ds}{\int_{1-\frac{1-r}{k}}^1 \omega(s)\,ds} > 1$.

In this paper we extend this result to the setting of fractional derivatives. Being precise, for an analytic function $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ we consider the fractional derivative $D^\mu(f)(z) = \sum_{n=0}^{\infty} \frac{\widehat{f}(n)}{\mu_{2n+1}} z^n$ induced by a radial weight $\mu \in \mathcal{D}$ where $\mu_{2n+1} = \int_0^1 r^{2n+1}\mu(r)\,dr$. Then, we prove that for any $p \in (0,\infty)$, the Littlewood–Paley equivalence

$$\int_{\mathbb{D}} |f(z)|^p \omega(z)\,dA(z) \asymp \int_{\mathbb{D}} |D^\mu(f)(z)|^p \left[\int_{|z|}^1 \mu(s)\,ds\right]^p \omega(z)\,dA(z)$$

holds for any analytic function f in $\mathbb{D}$ if and only if $\omega \in \mathcal{D}$. We also prove that for any $p \in (0,\infty)$, the inequality

$$\int_{\mathbb{D}} |D^\mu(f)(z)|^p \left[\int_{|z|}^1 \mu(s)\,ds\right]^p \omega(z)\,dA(z) \lesssim \int_{\mathbb{D}} |f(z)|^p \omega(z)\,dA(z)$$

holds for any analytic function f in $\mathbb{D}$ if and only if $\omega \in \widehat{\mathcal{D}}$.

Chapter 5

Conclusions and future research

5.1 Hilbert type-operators. Papers (I)-(II)

5.1.1 Conclusions

The results about the boundedness and compactness of the Hilbert-type operators

$$H_\omega(f)(z) = \int_0^1 f(t) \left(\frac{1}{z} \int_0^z B_t^\omega(\zeta) d\zeta \right) \omega(t) dt$$

on Hardy and weighted Bergman spaces, point out that the action of the classical Hilbert operator [13, 14, 17, 29] on these spaces is a particular case of a general phenomenon which occurs for operators H_ω induced by upper radial doubling weights ω. As for the proofs, they show that merging techniques coming from different areas may be necessary and appropriate to tackle problems on operator theory. For instance, we use tools on function theory, functional analysis, harmonic analysis and obviously complex analysis.

5.1.2 Future research

When studying the action of the Hilbert operators H_ω, we focused mainly on its boundedness on Hardy and Bergman spaces, but there are a good number of open interesting questions which we did not face because of the lack of time to think of them or ideas to solve them. Here we list a few:

(i) Theorems 1-2 of (II) imply that $\omega \in \widehat{\mathcal{D}}$ is a necessary condition for the boundedness of $H_\omega : H^p \to H^p$, $1 \leqslant p < \infty$. Does it remain true for the boundedness $H_\omega : A^p_\alpha \to A^p_\alpha$? Indeed, going further: Can be removed the hypothesis $\omega \in \widehat{\mathcal{D}}$ in Theorem 4 of (I)? Can it be deduced from the boundedness of $H_\omega : A^p_\alpha \to A^p_\alpha$?

(ii) By and large, given three radial weights ν, η, ω and $0 < p, q < \infty$ we are interested in studying the operator H_ω from A^p_ν to A^q_η. In particular, it would be meaningful to find/describe ν, η, ω such that $H_\omega : A^2_\nu \to A^2_\eta$ is compact (if they exist) and study the Schatten classes for this operator.

(iii) It does not make sense to study the classical Hilbert operator on Hardy spaces H^p if $0 < p < 1$ because it is not well-defined on them. However, there are radial weights ω such that $H_\omega(f) \in \mathcal{H}(\mathbb{D})$ for any $f \in H^p$, $0 < p < 1$. Therefore a natural question is study its boundedness on H^p, $0 < p < 1$.

(iv) In [3] it is described the point spectrum of the classical Hilbert operator on Hardy spaces H^p, $1 < p < \infty$. We are interested in extending these results for the Hilbert-type operators H_ω.

5.2 Bergman projection induced by radial weight

5.2.1 Conclusions

In the paper (III) it is considered the boundedness of the Bergman projection induced by a radial weight

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\omega(\zeta)\, dA(\zeta), \quad z \in \mathbb{D},$$

on a Lebesgue space L^p_ν, $1 < p < \infty$, induced by a radial weight. The techniques employed in (III) allow to obtain the necessary condition (3.1.2) which turns to be sufficient when ν is a radial doubling weight. Therefore, the tools used in (III) are adequate to solve a natural question for the Bergman projection P_ω, whose solution extends the classical result for standard weights [59] to a much more general setting. However this approach presents clear limitations because it does not fit in the setting of non-radial weights and even for radial weights $\omega \notin \mathcal{D}$ or $\nu \notin \mathcal{D}$. Consequently, there is a lot to be done in this research line, and in particular, it would be very interesting to obtain some new meaningful results in the framework of non-radial weights.

5.2.2 Future research

Here we highlight some meaningful open problems within this research line:

(i) Dostanić problem [15]: Describe the radial weights ω such that

$$\|P_\omega(f)\|_{L^p_\omega} \leqslant C\|f\|_{L^p_\omega}, \quad f \in L^p_\omega, \quad 1 < p < \infty.$$

(ii) It would also be of interest to extend Theorem 2 in (III) as follows: Given $\nu, \eta \in \mathcal{D}$ and $0 < p, q < \infty$, describe the radial weights ω such that $P_\omega : L^p_\nu \to L^q_\eta$ is bounded or compact.

(iii) If $\omega \in \mathcal{D}$ and $1 < p < \infty$, find a large class of non-radial weights ν such that $P_\omega : L^p_\nu \to L^p_\nu$ is bounded.

5.3 Littlewood-Paley formulas and fractional derivatives

5.3.1 Conclusions

The main theorems of the paper (IV) indicate that the results on the Littlewood-Paley formula

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np} \omega(z)\, dA(z) + \sum_{j=1}^{n-1} |f^{(j)}(0)|^p, \quad f \in \mathcal{H}(\mathbb{D}), \quad 0 < p < \infty$$

obtained in [48, Theorem 5] remain true if the quantity $|f^{(n)}(z)|^p(1-|z|)^{np}$ is replaced by $|D^\mu(f)(z)|^p \widehat{\mu}(z)^p$, where $\mu \in \mathcal{D}$. Moreover, there are important technical differences between the proofs in [48] and those in (IV) which are partially based in the lack of classical tools to deal with the generalized fractional derivative D^μ.

5.3.2 Future research

Regarding future research related to paper (IV) and the previous paper [48], it is worth exploring the following directions:

(i) For any radial weight μ, the fractional derivative $D^\mu(f)$ of $f \in \mathcal{H}(\mathbb{D})$ is also an analytic function in $\mathbb{D}$, so it makes sense wondering about the hypothesis $\mu \in \mathcal{D}$ in Theorems 1-2 of (IV). For instance, given $\omega \in \mathcal{D}$: Which are the radial weight μ such that the Littlewood-Paley formula

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |D^\mu(f)(z)|^p \widehat{\mu}(z)^p \omega(z)\, dA(z), \quad f \in \mathcal{H}(\mathbb{D}), \tag{5.3.1}$$

holds?

(ii) The radial weights such that the reverse inequality

$$\|f\|_{A^p_\omega}^p \lesssim \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p \omega(z)\, dA(z) + |f'(0)|^p$$

holds are not yet described [48]. Therefore, making progress on this inequality would also be an interesting aim worth pursuing. After obtaining such a characterization for the particular case of the derivative with a natural exponent, we would try to extend this result to the fractional derivative D^μ in the same spirit of Theorem 1 and 2 in (IV).

(iii) A further problem is to describe, for each $\alpha > 1$, the radial weights ω such that

$$\int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{\alpha p} \omega(z)\, dA(z).$$

A prototype is the exponential weight $\omega(s) = \exp\left(-\frac{1}{(1-s)^{\alpha-1}}\right)$ [41].

(iv) In a joint work with A. Moreno and J.A. Peláez not included in this thesis [35], a Bloch description in terms of the generalized fractional derivative D^μ is obtained. This also constitutes a natural extension of the Littlewood-Paley formula (5.3.1) to the case $p = \infty$. Being precise we show that the equivalence

$$\|f\|_{\mathcal{B}} \asymp \sup_{z\in\mathbb{D}} \widehat{\mu}(z)|D^\mu(f)(z)|, \quad f \in \mathcal{H}(\mathbb{D}),$$

holds if and only if $\mu \in \mathcal{D}$.

Consequently, we wonder if it would be possible to describe the BMOA space in terms of the generalized fractional derivative D^μ. For instance, for $\mu \in \mathcal{D}$: Is it true that $f \in$ BMOA if and only if

$$\sup_{a\in\mathbb{D}} \frac{\int_{S(a)} |D^\mu(f)(z)|^2 \widehat{\mu}(z) dA(z)}{1-a} < \infty?$$

Here $S(a)$ denotes the Carleson square

$$S(a) = \left\{ re^{i\theta} \in \mathbb{D} : |a| < r < 1, |\arg(ae^{-i\theta})| < \frac{1-|a|}{2} \right\}.$$

5.4 Further future work

Maybe one of the most natural questions is whether the results obtained in this thesis can be extended to a larger class of weights maybe including non-radial weights. In order to get this aim, a first step is getting a good understanding of essential properties of non radial weights. An example of such a preliminary question to study in this line of research is the following: it is known that for a radial weight ω, $\frac{\omega(z)}{(1-|z|)^\gamma}$ satisfies the Bekollé-Bonami condition $B_p(\gamma)$ [8], $p > 1$ and $\gamma > -1$, if and only if

$$\sup_{n\in\mathbb{N}} \frac{(\omega_{np})^{\frac{1}{p}} (\sigma_{np'})^{\frac{1}{p'}}}{\int_0^1 r^{2n+1}(1-r^2)^\gamma dr} < \infty,$$

where $\sigma(r) = \omega(r)^{-\frac{p'}{p}}(1-r^2)^{\gamma p'}$, see Lemma 9 of (III).

Therefore a natural question is whether or not this description of Bekollé-Bonami type weights remain true for general weights. That is, if ω is a weight we define the moments of ω by

$$\omega_x = \int_{\mathbb{D}} |z|^x \omega(z)dA(z), \quad x \geqslant 0,$$

and we wonder if the condition

$$\sup_{n\in\mathbb{N}} \frac{(\omega_{np})^{\frac{1}{p}}(\sigma_{np'})^{\frac{1}{p'}}}{\int_0^1 r^{2n+1}(1-r^2)^\gamma dr} < \infty$$

is equivalent to $\frac{\omega(z)}{(1-|z|)^\gamma} \in B_p(\gamma)$.

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