



Cesàro-like operators acting on spaces of analytic functions

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Abstract

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$. If μ is a finite positive Borel measure on the interval $[0, 1)$ and f is an analytic function in $\mathbb{D}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$), we define

$$\mathcal{C}_{\mu}(f)(z) = \sum_{n=0}^{\infty} \mu_n \left(\sum_{k=0}^n a_k \right) z^n, \quad z \in \mathbb{D},$$

where, for $n \geq 0$, μ_n denotes the n -th moment of the measure μ , that is, $\mu_n = \int_{[0,1)} t^n d\mu(t)$. In this way, $\mathcal{C}_{\mu}$ becomes a linear operator defined on the space $\text{Hol}(\mathbb{D})$ of all analytic functions in $\mathbb{D}$. We study the action of the operators $\mathcal{C}_{\mu}$ on distinct spaces of analytic functions in $\mathbb{D}$, such as the Hardy spaces H^p , the weighted Bergman spaces A_{α}^p , $BMOA$, and the Bloch space $\mathcal{B}$.

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1 Introduction and main results

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$ and let $\text{Hol}(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence in compact subsets.

If $0 < r < 1$ and $f \in \text{Hol}(\mathbb{D})$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

For $0 < p \leq \infty$, the Hardy space H^p consists of those $f \in \text{Hol}(\mathbb{D})$ such that

$$\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0 < r < 1} M_p(r, f) < \infty.$$

We refer to [13] for the notation and results regarding Hardy spaces.

Let dA denote the area measure on $\mathbb{D}$, normalized so that the area of $\mathbb{D}$ is 1. Thus $dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta$. For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space A_α^p consists of those $f \in \text{Hol}(\mathbb{D})$ such that

$$\|f\|_{A_\alpha^p} \stackrel{\text{def}}{=} \left(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \right)^{1/p} < \infty,$$

where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$. We refer to [14,25,39] for the notation and results about Bergman spaces.

The space $BMOA$ consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\partial\mathbb{D}$. We refer to [16] for the theory of $BMOA$ -functions.

Finally, we recall that a function $f \in \text{Hol}(\mathbb{D})$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

The space of all Bloch functions is denoted by $\mathcal{B}$. A classical reference for the theory of Bloch functions is [2]. Let us recall that

$$H^\infty \subsetneq BMOA \subsetneq \mathcal{B}, \quad BMOA \subsetneq \bigcap_{0 < p < \infty} H^p, \quad \mathcal{B} \subsetneq A_\alpha^p \quad (p > 0, \alpha > -1).$$

The Cesàro operator $\mathcal{C}$ is defined over the space of all complex sequences as follows: If $(a) = \{a_k\}_{k=0}^\infty$ is a sequence of complex numbers then

$$\mathcal{C}((a)) = \left\{ \frac{1}{n+1} \sum_{k=0}^n a_k \right\}_{n=0}^\infty.$$

The operator $\mathcal{C}$ is known to be bounded from ℓ^p to ℓ^p for $1 < p < \infty$. In fact, the sharp inequalities

$$\|\mathcal{C}((a))\|_p \leq \frac{p}{p-1} \|a\|_p, \quad (a) \in \ell^p, \quad 1 < p < \infty,$$

were proved by Hardy [21] and Landau [29] (see also [24, Theorem 326, p.239]).

Identifying any given function $f \in \text{Hol}(\mathbb{D})$ with the sequence $\{a_k\}_{k=0}^\infty$ of its Taylor coefficients, the Cesàro operator $\mathcal{C}$ becomes a linear operator from $\text{Hol}(\mathbb{D})$ into itself as follows:

If $f \in \text{Hol}(\mathbb{D})$, $f(z) = \sum_{k=0}^\infty a_k z^k$ ($z \in \mathbb{D}$), then

$$\mathcal{C}(f)(z) = \sum_{n=0}^\infty \left(\frac{1}{n+1} \sum_{k=0}^n a_k \right) z^n, \quad z \in \mathbb{D}.$$

The Cesàro operator is bounded on H^p for $0 < p < \infty$. For $1 < p < \infty$, this follows from a result of Hardy on Fourier series [22] together with the M. Riesz's theorem on the conjugate function [13, Theorem 4.1]. Siskakis [33] used semigroups of composition operators to give an alternative proof of this result and to extend it to $p = 1$. A direct proof of the boundedness on H^1 was given by Siskakis in [34]. Miao [31] dealt with the case $0 < p < 1$. Stempak [36] gave a proof valid for $0 < p \leq 2$ and Andersen [1] provided another proof valid for all $p < \infty$.

In this paper we associate to every positive finite Borel measure on $[0, 1)$ a certain operator $\mathcal{C}_\mu$ acting on $\text{Hol}(\mathbb{D})$ which is a natural generalization of the classical Cesàro operator $\mathcal{C}$.

If μ is a positive finite Borel measure on $[0, 1)$ and n is a non-negative integer, we let μ_n denote the moment of order n of μ , that is,

$$\mu_n = \int_{[0,1)} t^n d\mu(t), \quad n = 0, 1, 2, \dots$$

If $f \in \text{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^\infty a_n z^n$ ($z \in \mathbb{D}$), we define $\mathcal{C}_\mu(f)$ as follows

$$\mathcal{C}_\mu(f)(z) = \sum_{n=0}^\infty \left(\mu_n \sum_{k=0}^n a_k \right) z^n, \quad z \in \mathbb{D}.$$

It is clear that $\mathcal{C}_\mu$ is a well defined linear operator $\mathcal{C}_\mu : \text{Hol}(\mathbb{D}) \rightarrow \text{Hol}(\mathbb{D})$. When μ is the Lebesgue measure on $[0, 1)$, the operator $\mathcal{C}_\mu$ reduces to the classical Cesàro operator $\mathcal{C}$.

Our main objective in this work is to characterize those positive finite Borel measures μ on $[0, 1)$ such that the operator $\mathcal{C}_\mu$ is bounded or compact on classical subspaces of $\text{Hol}(\mathbb{D})$ such as the Hardy spaces H^p , the weighted Bergman spaces A_α^p , and the spaces $BMOA$ and $\mathcal{B}$.

Measures of Carleson type will play a basic role in the sequel. If $I \subset \partial\mathbb{D}$ is an interval, $|I|$ will denote the length of I . The *Carleson square* $S(I)$ is defined as

$$S(I) = \left\{ r e^{it} : e^{it} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1 \right\}.$$

If $s > 0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s -Carleson measure if there exists a positive constant C such that

$$\mu(S(I)) \leq C|I|^s, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

If μ satisfies $\mu(S(I)) = o(|I|^s)$, as $|I| \rightarrow 0$, then we say that μ is a *vanishing s -Carleson measure*.

A 1-Carleson measure, respectively, a vanishing 1-Carleson measure, will be simply called a Carleson measure, respectively, a vanishing Carleson measure.

We recall that Carleson [7] proved that $H^p \subset L^p(d\mu)$ ($0 < p < \infty$), if and only if μ is a Carleson measure (see [13, Chapter 9]).

Following [38], if μ is a positive Borel measure on $\mathbb{D}$, $0 \leq \alpha < \infty$, and $0 < s < \infty$, we say that μ is an α -logarithmic s -Carleson measure if there exists a positive constant C such that

$$\frac{\mu(S(I)) \left(\log \frac{2}{|I|} \right)^\alpha}{|I|^s} \leq C, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

If $\mu(S(I)) \left(\log \frac{2}{|I|} \right)^\alpha = o(|I|^s)$, as $|I| \rightarrow 0$, we say that μ is a *vanishing α -logarithmic s -Carleson measure*.

A measure μ on $[0, 1)$ can be seen as a measure on $\mathbb{D}$ with support contained in the radius $[0, 1)$. In this way, a positive Borel measure μ on $[0, 1)$ is an s -Carleson measure if and only if there exists a positive constant C such that

$$\mu([t, 1)) \leq C(1-t)^s, \quad 0 \leq t < 1,$$

and we have similar statements for vanishing s -Carleson measures, for α -logarithmic s -Carleson measures, and for vanishing α -logarithmic s -Carleson measures.

Among other, we shall prove the following results.

Theorem 1 *Suppose that $1 \leq p < \infty$ and let μ be a positive finite Borel measure on $[0, 1)$. Then the following conditions are equivalent.*

- (i) *The measure μ is a Carleson measure.*
- (ii) *The operator C_μ is bounded from H^p into itself.*

Theorem 2 *Suppose that $1 \leq p < \infty$ and let μ be a positive finite Borel measure on $[0, 1)$. Then the following conditions are equivalent.*

- (i) *The measure μ is a vanishing Carleson measure.*
- (ii) *The operator C_μ is compact from H^p into itself.*

Danikas and Siskakis [12] observed that $\mathcal{C}(H^\infty) \not\subset H^\infty$ and $\mathcal{C}(BMOA) \not\subset BMOA$ and studied the action of the Cesàro operator on these spaces. We will devote Sects. 3.3 and 5 to study the Cesàro-like operators $\mathcal{C}_\mu$ acting on these spaces. Let us just mention here the following result.

Theorem 3 *Let μ be a positive finite Borel measure on $[0, 1)$. Then the following conditions are equivalent.*

- (i) *The measure μ is a 1-logarithmic 1-Carleson measure.*
- (ii) *The operator $\mathcal{C}_\mu$ is bounded from $BMOA$ into itself.*
- (iii) *The operator $\mathcal{C}_\mu$ is bounded from the Bloch space $\mathcal{B}$ into itself.*

Section 3 will be devoted to present the proofs of Theorem 1 and Theorem 2 as well as some further results concerning the action of the operators $\mathcal{C}_\mu$ on Hardy spaces. Section 4 will deal with the action of the operators $\mathcal{C}_\mu$ on Bergman spaces and, as we have already mentioned, Sect. 5 will be devoted to study the operators $\mathcal{C}_\mu$ acting on $BMOA$, the Bloch space, and some related spaces. In particular, Sect. 5 will include a proof of Theorem 3 and the substitute of this result concerning compactness.

In Sect. 2 we shall give two alternative representations of the operator $\mathcal{C}_\mu$, one of them is an integral representation and the other one involves the convolution with a fixed analytic function in $\mathbb{D}$. We shall also introduce a related operator which will be denoted T_μ and which will play a basic role in the proofs of some of our results.

Throughout the paper, if μ is a finite positive Borel measure on $[0, 1)$, for $n \geq 0$, μ_n will denote the moment of order n of μ . Also, we shall be using the convention that $C = C(p, \alpha, q, \beta, \dots)$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta, \dots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Furthermore, for two real-valued functions K_1, K_2 we write $K_1 \lesssim K_2$, or $K_1 \gtrsim K_2$, if there exists a positive constant C independent of the arguments such that $K_1 \leq CK_2$, respectively $K_1 \geq CK_2$. If we have $K_1 \lesssim K_2$ and $K_1 \gtrsim K_2$ simultaneously, then we say that K_1 and K_2 are equivalent and we write $K_1 \asymp K_2$.

Let us close this section noticing that, since the subspaces X of $\text{Hol}(\mathbb{D})$ we shall be dealing with are Banach spaces continuously embedded in $\text{Hol}(\mathbb{D})$, to prove that the operator $\mathcal{C}_\mu$ (or T_μ , to be defined below) is bounded on X it suffices to show that it maps X into X by appealing to the closed graph theorem.

2 Alternative representations of $\mathcal{C}_\mu$ and a related operator

A simple calculation with power series gives the following integral representation of the operators $\mathcal{C}_\mu$.

Proposition 1 *If μ is a positive finite Borel measure on $[0, 1)$ and $f \in \text{Hol}(\mathbb{D})$ then*

$$\mathcal{C}_\mu(f)(z) = \int_{[0,1)} \frac{f(tz)}{1-tz} d\mu(t), \quad z \in \mathbb{D}. \quad (1)$$

Next we shall give another expression for $\mathcal{C}_\mu(f)$ involving the convolution of analytic functions. If f and g are two analytic functions in the unit disc,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

the convolution $f \star g$ of f and g is defined by

$$f \star g(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

Lemma 1 *Let μ be a positive finite Borel measure on $[0, 1)$ and set*

$$F(z) = \sum_{n=0}^{\infty} \mu_n z^n, \quad z \in \mathbb{D}.$$

If $f \in \text{Hol}(\mathbb{D})$ and

$$g(z) = \frac{f(z)}{1-z}, \quad z \in \mathbb{D},$$

then $\mathcal{C}_\mu(f) = F \star g$.

The proof is elementary and will be omitted.

The following result regarding the radial measures μ we are considering will be used in our work.

Lemma 2 *Let μ be a finite positive Borel measure on the interval $[0, 1)$ and, for $n \geq 0$, let μ_n denote the moment of order n of μ .*

- (i) μ is a Carleson measure if and only if $\mu_n = O(\frac{1}{n})$.
- (ii) μ is a vanishing Carleson measure if and only if $\mu_n = o(\frac{1}{n})$.
- (iii) μ is a 1-logarithmic 1-Carleson measure if and only if $\mu_n = O(\frac{1}{n \log n})$.
- (iv) μ is a vanishing 1 logarithmic 1-Carleson measure if and only if $\mu_n = o(\frac{1}{n \log n})$.

Proof (i) is Proposition 8 of [8] and (ii) follows with a similar argument. Lemma 2.7 of [19] gives one implication of (iii) and the other one follows from the simple inequality

$$\mu \left(\left[1 - \frac{1}{n}, 1 \right) \right) \lesssim \int_{[1-\frac{1}{n}, 1)} t^n d\mu(t) \leq \mu_n.$$

Finally, (iv) can be proved with an argument similar the the one used to prove (iii). $\square$

Now we define a new operator operator T_μ associated to μ which will be important in our work because it will become the adjoint of C_μ in distinct instances.

If μ is a finite positive Borel measure on $[0, 1]$ and $f \in \text{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$) we set

$$T_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=n}^{\infty} \mu_k a_k \right) z^n,$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

Clearly, the operator T_μ is not defined over the whole space $\text{Hol}(\mathbb{D})$. We have the following result.

Proposition 2 *Let μ is a finite positive Borel measure on $[0, 1]$.*

- (a) *If P is a polynomial then $T_\mu(P)$ is well defined and it also a polynomial.*
 (b) *If μ is a Carleson measure then T_μ is well defined on H^1 .*

Proof (a) is clear. To prove (b) we use the fact that if μ is a Carleson measure then $\mu_n = O(n^{-1})$ (see Lemma 2). This and Hardy's inequality [13, p. 48] shows that if $f \in H^1$, $f(z) = \sum_{k=1}^{\infty} a_k z^k$, then there exists $C > 0$ such that

$$\sum_{k=n}^{\infty} \mu_k |a_k| \leq C \sum_{k=n}^{\infty} \frac{|a_k|}{k+1} \leq C\pi \|f\|_{H^1}$$

for all n . Clearly, this implies (b). $\square$

It is well known that, for $1 < p < \infty$, the dual of H^p is identifiable with H^q , $\frac{1}{p} + \frac{1}{q} = 1$, with the pairing

$$\langle f, g \rangle_{H^p} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in H^q$ (see [13, Theorem 7.3]).

Similarly, if $1 < p < \infty$ and $\alpha > 1$, the dual of A_α^p is identifiable with A_α^q with the pairing

$$\langle f, g \rangle_{p,\alpha} = \int_{\mathbb{D}} f(z) \overline{g(z)} dA_\alpha(z) = \sum_{n=0}^{\infty} c_{n,\alpha} a_n \overline{b_n},$$

where

$$c_{n,\alpha} = \frac{n! \Gamma(2+\alpha)}{\Gamma(n+2+\alpha)}, \quad n = 0, 1, 2, \dots,$$

and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A_\alpha^p$, $g(z) = \sum_{n=0}^{\infty} b_n z^n \in A_\alpha^q$ (see [25, Theorem 1.16 and p. 5]). A simple calculation gives the following result.

Proposition 3 *Let μ be a positive finite Borel measure on $[0, 1)$.*

(i) *If $1 < p < \infty$, $f \in H^p$, and g is a polynomial then*

$$\langle C_\mu(f), g \rangle_{H^p} = \langle f, T_\mu(g) \rangle_{H^p}.$$

(ii) *If $1 < p < \infty$, $\alpha > -1$, $f \in A_\alpha^p$, and g is a polynomial then*

$$\langle C_\mu(f), g \rangle_{p,\alpha} = \langle f, T_\mu(g) \rangle_{p,\alpha}.$$

Proposition 3, together with the fact that the polynomials are dense in all the spaces H^p ($p < \infty$) and A_α^p ($p < \infty$, $\alpha > -1$), readily implies the following result.

Proposition 4 *Suppose that $1 < p < \infty$ and let μ be a positive finite Borel measure on $[0, 1)$. Let q be the conjugate exponent of p , that is, $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *If C_μ is a bounded operator from H^p into itself, then there exists a positive constant C such that*

$$\|T_\mu(P)\|_{H^q} \leq C \|P\|_{H^q}$$

for every polynomial P . Consequently, T_μ extends to a bounded linear operator from H^q into itself. This extension, which will be also denoted by T_μ , is the adjoint of C_μ .

(ii) *Suppose that $\alpha > -1$. If C_μ is a bounded operator from A_α^p into itself, then there exists a positive constant C such that*

$$\|T_\mu(P)\|_{A_\alpha^q} \leq C \|P\|_{A_\alpha^q}$$

for every polynomial P . Consequently, T_μ extends to a bounded linear operator from A_α^q into itself. This extension, which will be also denoted by T_μ , is the adjoint of C_μ .

3 The operators C_μ acting on Hardy spaces

In this section we shall study the action of the operators C_μ on Hardy spaces.

We shall use complex interpolation to prove some of our results. Let us refer to [39, Chapter 2] for the terminology and basic results concerning complex interpolation.

If X_0 and X_1 are two compatible Banach spaces then, for $0 < \theta < 1$, $(X_0, X_1)_\theta$ stands for the space obtained by the complex method of interpolation of Calderón [5]. It is well known (see [6,26,32]) that if $1 \leq p_0, p_1 \leq \infty$, $0 < \theta < 1$, and $1/p = (1 - \theta)/p_0 + \theta/p_1$, then

$$(H^{p_0}, H^{p_1})_\theta = H^p. \quad (2)$$

In particular,

$$H^p = (H^2, H^1)_\theta, \quad \text{if } 1 < p < 2 \text{ and } \theta = \frac{2}{p} - 1. \quad (3)$$

3.1 Proof of Theorem 1

We shall split it in several cases.

Proof of the implication (i) $\Rightarrow$ (ii) when $p = 1$. Assume that μ is a Carleson measure and take $f \in H^1$. Set

$$g(z) = \frac{f(z)}{1-z}, \quad z \in \mathbb{D},$$

and

$$t_k = 1 - \frac{1}{2^k}, \quad k = 0, 1, 2, \dots$$

Using the integral representation on $\mathcal{C}_\mu$, we see that, for $0 < r < 1$,

$$\begin{aligned} M_1(r, \mathcal{C}_\mu(f)) &= \frac{1}{2\pi} \int_0^{2\pi} \left| \int_{[0,1)} \frac{f(re^{i\theta})}{1-re^{i\theta}} d\mu(t) \right| d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \int_{[0,1)} |g(re^{i\theta})| d\mu(t) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \int_{[t_{k-1}, t_k)} |g(re^{i\theta})| d\mu(t) \right) d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \left[\sup_{0 \leq t \leq t_k} |g(re^{i\theta})| \right] \right) \mu([t_{k-1}, t_k]) d\theta. \end{aligned}$$

Since μ is a Carleson measure, $\mu([t_{k-1}, t_k]) \lesssim \frac{1}{2^k}$. Using this, the Hardy-Littlewood maximal theorem [13, Theorem 1.9], the fact that integral means $M_1(s, g)$ increase with s , and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} M_1(r, \mathcal{C}_\mu(f)) &\leq \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\frac{1}{2\pi} \int_0^{2\pi} \left[\sup_{0 \leq t \leq t_k} |g(re^{i\theta})| \right] d\theta \right) \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{2^k} M_1(rt_k, g) \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{2^k} 2^k \int_{t_k}^{t_{k+1}} M_1(rt, g) dt \end{aligned}$$

$$\begin{aligned}
&\lesssim \int_0^r \frac{1}{2\pi} \int_0^{2\pi} |g(te^{i\theta})| d\theta dt \\
&\lesssim \int_0^r \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(te^{i\theta})}{1-te^{i\theta}} \right| d\theta dt \\
&\lesssim \int_0^r M_2(t, f) \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1-te^{i\theta}|^2} d\theta \right)^{1/2} dt \\
&\lesssim \int_0^r M_2(t, f)(1-t)^{-1/2} dt.
\end{aligned}$$

Making the change of variables $t = rs$ in the last integral and setting $f_r(z) = f(rz)$ ($z \in \mathbb{D}$), it follows that

$$\begin{aligned}
M_1(r, \mathcal{C}_\mu(f)) &\lesssim \int_0^1 M_2(sr, f)(1-sr)^{-1/2} ds \\
&= \int_0^1 M_2(s, f_r)(1-sr)^{-1/2} ds \leq \int_0^1 M_2(s, f_r)(1-s)^{-1/2} ds.
\end{aligned}$$

Using a result of Hardy and Littlewood [23] (see also [34]) we see that

$$\int_0^1 M_2(s, f_r)(1-s)^{-1/2} dt \lesssim \|f_r\|_{H^1}.$$

Then it follows that

$$M_1(r, \mathcal{C}_\mu(f)) \lesssim M_1(r, f). \quad (4)$$

This implies that $\mathcal{C}_\mu(f) \in H^1$ and that $\|\mathcal{C}_\mu(f)\|_{H^1} \lesssim \|f\|_{H^1}$. $\square$

Proof of the implication (i) $\Rightarrow$ (ii) when $p = 2$. Assume that μ is a Carleson measure and take $f \in H^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$). Using [8, Proposition 1] we see that $|\mu_n| \lesssim \frac{1}{n+1}$. Using this, the definition of $\mathcal{C}_\mu(f)$, and the fact that the Cesàro operator is bounded on H^2 , it follows that

$$\begin{aligned}
\|\mathcal{C}_\mu(f)\|_{H^2}^2 &= \sum_{n=0}^{\infty} \mu_n^2 \left| \sum_{k=0}^n a_k \right|^2 \lesssim \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \left| \sum_{k=0}^n a_k \right|^2 \\
&= \|\mathcal{C}(f)\|_{H^2}^2 \lesssim \|f\|_{H^2}^2.
\end{aligned}$$

$\square$

Proof of the implication (i) $\Rightarrow$ (ii) for $1 < p < 2$. Since (i) $\Rightarrow$ (ii) when $p = 1$ and $p = 2$, the fact that (i) $\Rightarrow$ (ii) when $1 < p < 2$ follows using (3) and Theorem 2.4 of [39]. $\square$

To prove the remaining case, that is, the implication (i) $\Rightarrow$ (ii) for $2 < p < \infty$ we shall use ideas of Andersen [1]. Actually, our next argument works for $1 < p < \infty$.

Proof of the implication (i) $\Rightarrow$ (ii) for $1 < p < \infty$. Assume that μ is a Carleson measure, $1 < p < \infty$, and $f \in H^p$.

For $0 < r < 1$, set

$$K_{r,\mu}(\theta, \varphi) = \int_{[0,1)} \frac{(1+t)(1-t)}{|1-te^{i\varphi}|^2(1-tre^{i\theta})} d\mu(t), \quad \theta, \varphi \in [-\pi, \pi].$$

Arguing just as in [1, p. 621], using Fubini's theorem, we have that

$$\mathcal{C}_\mu(f)(re^{i\theta}) = \int_{-\pi}^{\pi} K_{r,\mu}(\theta, \varphi) f(re^{i(\theta+\varphi)}) d\varphi. \quad (5)$$

Now, letting $\{t_k\}_{k=0}^\infty$ be as above, using the fact that $\mu([t_k, t_{k+1})) \lesssim \frac{1}{2^k}$, and simple estimates, we obtain

$$\begin{aligned} |K_{r,\mu}(\theta, \varphi)| &\leq 2 \int_{[0,1)} \frac{1-t}{|1-te^{i\varphi}|^2|1-tre^{i\theta}|} d\mu(t) \\ &\lesssim \int_{[0,1)} \frac{1-t}{[(1-t)^2 + \varphi^2][(1-t)^2 + \theta^2]^{1/2}} d\mu(t) \\ &\lesssim \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \frac{1-t}{[(1-t)^2 + \varphi^2][(1-t)^2 + \theta^2]^{1/2}} d\mu(t) \\ &\lesssim \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\frac{1}{2^k}}{\left[\left(\frac{1}{2^{k+1}}\right)^2 + \varphi^2\right] \left[\left(\frac{1}{2^{k+1}}\right)^2 + \theta^2\right]^{1/2}} \\ &\lesssim \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \frac{\frac{1}{2^k}}{\left[\left(\frac{1}{2^{k+1}}\right)^2 + \varphi^2\right] \left[\left(\frac{1}{2^{k+1}}\right)^2 + \theta^2\right]^{1/2}} dt \\ &\lesssim \int_0^1 \frac{1-t}{[(1-t)^2 + \varphi^2][(1-t)^2 + \theta^2]^{1/2}} dt \\ &= \int_0^1 \frac{x}{[x^2 + \varphi^2][x^2 + \theta^2]^{1/2}} dx \end{aligned}$$

Then, using Lemma 2.1 of [1], we see that for all $\theta, \varphi \in (-\pi, \pi) \setminus \{0\}$ and $r \in (0, 1)$, we have

$$|K_{r,\mu}(\theta, \varphi)| \lesssim \frac{H(\varphi/\theta)}{|\theta|},$$

where

$$H(s) = \frac{\log(2 + 1/|s|)}{1 + |s|}, \quad s \neq 0.$$

Using this and (5) it follows that

$$|\mathcal{C}_\mu(f)(re^{i\theta})| \lesssim \int_{-\pi}^{\pi} \frac{H(\varphi/\theta)}{|\theta|} |f(re^{i(\theta+\varphi)})| d\varphi, \quad \theta \in (-\pi, \pi) \setminus \{0\}, \quad 0 < r < 1.$$

Then the argument in p. 622 of [1] yields that

$$M_p(r, \mathcal{C}_\mu(f)) \lesssim M_p(r, f) \quad (6)$$

and, hence $\|\mathcal{C}_\mu(f)\|_{H^p} \lesssim \|f\|_{H^p}$. $\square$

Proof of the implication (ii) $\Rightarrow$ (i) for $1 \leq p \leq 2$. Suppose that $1 \leq p \leq 2$ and that $\mathcal{C}_\mu$ is bounded on H^p . Recall that, for $\alpha > 0$,

$$\frac{1}{(1-z)^\alpha} = \sum_{n=0}^{\infty} a_n(\alpha) z^n, \quad z \in \mathbb{D}$$

where

$$a_n(\alpha) \asymp n^{\alpha-1}. \quad (7)$$

For $0 < a < 1$, set

$$f_a(z) = \left(\frac{1-a^2}{(1-az)^2} \right)^{1/p} = (1-a^2)^{1/p} \sum_{n=0}^{\infty} a_n(2/p) a^n z^n, \quad z \in \mathbb{D}.$$

We have that

$$f_a \in H^p \text{ and } \|f_a\|_{H^p} = 1, \quad 0 < a < 1.$$

Since $\mathcal{C}_\mu$ is bounded on H^p , we have

$$\|\mathcal{C}_\mu(f_a)\|_{H^p}^p \lesssim 1. \quad (8)$$

Now

$$\mathcal{C}_\mu(f_a)(z) = (1-a^2)^{1/p} \sum_{n=0}^{\infty} \mu_n \left(\sum_{k=0}^n a_k(2/p) a^k \right) z^n, \quad z \in \mathbb{D}.$$

Using the fact that $1 \leq p \leq 2$, [13, Theorem 6.2], (7), and the fact that the sequence $\{\mu_n\}$ is decreasing, we obtain

$$\begin{aligned}
& (1-a)\mu_N^p \sum_{n=0}^N (n+1)^{p-2} \left(\sum_{k=0}^n k^{\frac{2}{p}-1} a^k \right)^p \\
& \leq (1-a) \sum_{n=0}^N (n+1)^{p-2} \mu_n^p \left(\sum_{k=0}^n k^{\frac{2}{p}-1} a^k \right)^p \\
& \leq (1-a) \sum_{n=0}^{\infty} (n+1)^{p-2} \mu_n^p \left(\sum_{k=0}^n k^{\frac{2}{p}-1} a^k \right)^p \\
& \asymp (1-a^2) \sum_{n=0}^{\infty} (n+1)^{p-2} \mu_n^p \left(\sum_{k=0}^n a_k (2/p) a^k \right)^p \\
& \lesssim \|C_\mu(f_a)\|_{H^p}^p,
\end{aligned}$$

for every positive integer N and every $a \in (0, 1)$. Taking $a = 1 - \frac{1}{N}$ and using the fact that C_μ is bounded on H^p , we obtain

$$\frac{\mu_N^p}{N} \sum_{n=0}^N (n+1)^{p-2} \left(\sum_{k=0}^n k^{\frac{2}{p}-1} \right)^p \asymp \mu_N^p N^p \lesssim \|C_\mu(f_a)\|_{H^p}^p \lesssim \|f_a\|_{H^p}^p. \quad (9)$$

This and (8) imply that $\mu_N \lesssim \frac{1}{N}$. Using again Lemma 2, this yields that μ is a Carleson measure. $\square$

Proof of the implication (ii) $\Rightarrow$ (i) for $2 \leq p < \infty$. Suppose that $2 < p < \infty$ and that C_μ is a bounded operator on H^p . Let q be the conjugate exponent of p , that is, $\frac{1}{p} + \frac{1}{q} = 1$. Bearing in mind Proposition 2 and Proposition 3, we see that the operator T_μ , initially defined over polynomials, extends to a bounded operator on H^q .

For $0 < a < 1$ and $N \in \mathbb{N}$, set

$$\begin{aligned}
f_a(z) &= \left(\frac{1-a^2}{(1-az)^2} \right)^{1/q} = (1-a^2)^{1/q} \sum_{n=0}^{\infty} a_n(2/q) a^n z^n, \quad z \in \mathbb{D}, \\
f_{a,N}(z) &= (1-a^2)^{1/q} \sum_{n=0}^N a_n(2/q) a^n z^n, \quad z \in \mathbb{D}.
\end{aligned}$$

We have that for all $a \in (0, 1)$, $f_a \in H^q$ and $\|f_a\|_{H^q} = 1$. Since T_μ is bounded on H^q , it follows that

$$\|T_\mu(f_a)\|_{H^q} \lesssim 1 \quad (10)$$

Also, for every a , $f_{a,N} \rightarrow f_a$, as $N \rightarrow \infty$ in H^q and uniformly on compact subsets of $\mathbb{D}$. Now, $T_\mu(f_{a,N})(z) = (1-a^2)^{1/q} \sum_{n=0}^N \left(\sum_{k=n}^N \mu_k a_k(2/q) a^k \right) z^n$ ($z \in \mathbb{D}$) and then, using that $1 < q < 2$ and [13, Theorem 6.2], we have that

$$(1-a) \sum_{n=1}^N (n+1)^{q-2} \left(\sum_{k=n}^N \mu_k a_k (2/q) a^k \right)^q \lesssim \|T_\mu(f_{a,N})\|_{H^q}^q.$$

Letting N tend to ∞ , we obtain

$$(1-a) \sum_{n=1}^{\infty} (n+1)^{q-2} \left(\sum_{k=n}^{\infty} \mu_k a_k (2/q) a^k \right)^q \lesssim \|T_\mu(f_a)\|_{H^q}^q.$$

Taking $a = 1 - \frac{1}{N}$ and letting $[N/2]$ denote the largest integer less than or equal to $N/2$, we obtain

$$\begin{aligned} \|T_\mu(f_a)\|_{H^q}^q &\gtrsim (1-a) \sum_{n=1}^N (n+1)^{q-2} \left(\sum_{k=n}^N \mu_k a_k (2/q) a^k \right)^q \\ &\gtrsim \frac{\mu_N^q}{N} \sum_{n=1}^N n^{q-2} \left(\sum_{k=n}^N k^{\frac{2}{q}-1} \right)^q \gtrsim \frac{\mu_N^q}{N} \sum_{n=1}^{[N/2]} n^{q-2} \left(\sum_{k=[N/2]}^N k^{\frac{2}{q}-1} \right)^q \\ &\asymp \frac{\mu_N^q}{N} \sum_{n=1}^{[N/2]} n^{q-2} \left(N^{2/q} \right)^q \asymp \mu_N^q N^q. \end{aligned} \quad (11)$$

Using (10), it follows that $\mu_N \lesssim \frac{1}{N}$ and then Lemma 2 implies that μ is a Carleson measure. $\square$

3.2 Proof of Theorem 2

Proof Let us start with the implication (ii) $\Rightarrow$ (i). We shall consider the cases $1 \leq p \leq 2$ and $2 < p < \infty$ separately.

Suppose first that $1 \leq p \leq 2$ and $\mathcal{C}_\mu$ is compact from H^p into itself. As in the proof of Theorem 1, for $0 < a < 1$, set

$$f_a(z) = \left(\frac{1-a^2}{(1-az)^2} \right)^{1/p}, \quad z \in \mathbb{D}.$$

We have that $\|f_a\|_{H^p} = 1$ for all a and, also, $f_a \rightarrow 0$, as $a \rightarrow 1$, uniformly on compact subsets of $\mathbb{D}$. Hence, $\|\mathcal{C}_\mu(f_a)\|_{H^p} \rightarrow 0$, as $a \rightarrow 1$. But in the course of the proof of the implication (ii) $\Rightarrow$ (i) of Theorem 1, we obtained that $\mu_N N \lesssim \|\mathcal{C}_\mu(f_a)\|_{H^p}$ for $a = 1 - \frac{1}{N}$ (see (9)). Then it follows that $\mu_N = o(\frac{1}{N})$ and this implies that μ is a vanishing Carleson measure.

Suppose now that $2 < p < \infty$ and $\mathcal{C}_\mu$ is compact from H^p into itself. By Theorem 1, μ is a Carleson measure and then it follows that the operator T_μ is well defined on H^q ($\frac{1}{p} + \frac{1}{q} = 1$) and it is the adjoint of $\mathcal{C}_\mu$. For $0 < a < 1$, set $f_a(z) = \left(\frac{1-a^2}{(1-az)^2} \right)^{1/q}$, ($z \in \mathbb{D}$). We have that $\|f_a\|_{H^q} = 1$ for all a and, also, $f_a \rightarrow 0$, as $a \rightarrow 1$, uniformly

on compact subsets of $\mathbb{D}$. By Schauder's theorem [10, p. 174], T_μ is a compact operator from H^q into itself and, hence, $\|T_\mu(f_a)\|_{H^q} \rightarrow 0$. In the course of the proof of the implication (ii) $\Rightarrow$ (i) of Theorem 1, we obtained that $\mu_N N \lesssim \|T_\mu(f_a)\|_{H^q}$ for $a = 1 - \frac{1}{N}$ (see (11)). Then it follows that $\mu_N = o\left(\frac{1}{N}\right)$ and, hence, μ is a vanishing Carleson measure.

To prove the other implication we shall consider the cases $p = 2$, $p = 1$, $1 < p < 2$, and $2 < p < \infty$ separately.

Let us start with the case $p = 2$. So assume that μ is a vanishing Carleson measure and let $\{f_n\}$ be a sequence of functions in H^2 with $\|f_n\|_{H^2} \leq 1$, for all n , and such that $f_n \rightarrow 0$, uniformly on compact subsets of $\mathbb{D}$.

Since μ is a vanishing Carleson measure $\mu_k = o\left(\frac{1}{k}\right)$, as $k \rightarrow \infty$. Say

$$\mu_k = \frac{\varepsilon_k}{k+1}, \quad k = 0, 1, 2, \dots$$

Then $\{\varepsilon_k\} \rightarrow 0$. Say that, for every n ,

$$f_n(z) = \sum_{k=0}^{\infty} a_k^{(n)} z^k, \quad z \in \mathbb{D}.$$

Since the Cesàro operator $\mathcal{C}$ is bounded on H^2 , there exists $M > 0$ such that

$$\|\mathcal{C}(f_n)\|_{H^2}^2 \leq M, \quad \text{for all } n. \quad (12)$$

Take $\varepsilon > 0$ and next take a natural number N such that

$$k \geq N \Rightarrow \varepsilon_k^2 < \frac{\varepsilon}{2M}.$$

We have

$$\begin{aligned} \|\mathcal{C}_\mu(f_n)\|_{H^2}^2 &= \sum_{k=0}^{\infty} \mu_k^2 \left| \sum_{j=0}^k a_j^{(n)} \right|^2 \\ &= \sum_{k=0}^N \mu_k^2 \left| \sum_{j=0}^k a_j^{(n)} \right|^2 + \sum_{k=N+1}^{\infty} \frac{\varepsilon_k^2}{(k+1)^2} \left| \sum_{j=0}^k a_j^{(n)} \right|^2 \\ &\leq \sum_{k=0}^N \mu_k^2 \left| \sum_{j=0}^k a_j^{(n)} \right|^2 + \frac{\varepsilon}{2M} \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} \left| \sum_{j=0}^k a_j^{(n)} \right|^2 \\ &= \sum_{k=0}^N \mu_k^2 \left| \sum_{j=0}^k a_j^{(n)} \right|^2 + \frac{\varepsilon}{2M} \|\mathcal{C}(f_n)\|_{H^2}^2 \end{aligned}$$

$$\leq \sum_{k=0}^N \mu_k^2 \left| \sum_{j=0}^k a_j^{(n)} \right|^2 + \frac{\varepsilon}{2}.$$

Now, since $f_n \rightarrow 0$, uniformly on compact subsets of $\mathbb{D}$, it follows that

$$\sum_{k=0}^N \mu_k^2 \left| \sum_{j=0}^k a_j^{(n)} \right|^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Then it follows that there exist $n_0 \in \mathbb{N}$ such that $\|\mathcal{C}_\mu(f_n)\|_{H^2}^2 < \varepsilon$ for all $n \geq n_0$. So, we have proved that $\|\mathcal{C}_\mu(f_n)\|_{H^2}^2 \rightarrow 0$. The compactness of $\mathcal{C}_\mu$ on H^2 follows.

Let us move to the case $p = 1$. Assume that μ is a vanishing Carleson measure and let $\{f_n\}$ be a sequence of functions in H^1 with $\|f_n\|_{H^1} \leq 1$, for all n , and such that $f_n \rightarrow 0$, uniformly on compact subsets of $\mathbb{D}$.

Set

$$g_n(z) = \frac{f_n(z)}{1-z}, \quad z \in \mathbb{D}, \quad n \in \mathbb{N},$$

and

$$t_k = 1 - \frac{1}{2^k}, \quad k = 0, 1, 2, \dots$$

As in the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 1 when $p = 1$ we see that, for $0 < r < 1$ and $n \in \mathbb{N}$,

$$M_1(r, \mathcal{C}_\mu(f_n)) \leq \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \left[\sup_{0 \leq t \leq t_k} |g_n(rte^{i\theta})| \right] \right) \mu([t_{k-1}, t_k]) d\theta$$

and, hence,

$$\|\mathcal{C}_\mu(f_n)\|_{H^1} \leq \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \left[\sup_{0 \leq t \leq t_k} |g_n(te^{i\theta})| \right] \right) \mu([t_{k-1}, t_k]) d\theta. \quad (13)$$

Since μ is a vanishing Carleson measure $\mu([t_{k-1}, t_k]) = o(2^{-k})$ and, hence, we have

$$\mu([t_{k-1}, t_k]) = \frac{\varepsilon_k}{2^k}, \quad \text{where } \varepsilon_k \geq 0 \text{ and } \{\varepsilon_k\} \rightarrow 0.$$

On the other hand, looking at the proof of Theorem 1, we see that there exists $C > 0$ such that

$$\sum_{k=1}^{\infty} \int_{t_k}^{t_{k+1}} M_1(t, g_n) dt \leq C \|f_n\|_{H^1} \leq C, \quad n \in \mathbb{N}. \quad (14)$$

Take $\varepsilon > 0$ and then take $N \in \mathbb{N}$ so that $\varepsilon_k \leq \frac{\varepsilon}{2CK}$, for all $k \geq N$, where K is the constant in the Hardy-Littlewood maximal estimate

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\sup_{0 < t < 1} |F(te^{i\theta})| \right] d\theta \leq K \|F\|_{H^1}.$$

Using (13) we see that

$$\|\mathcal{C}_\mu(f_n)\|_{H^1} \leq I(n) + II(n),$$

where

$$\begin{aligned} I(n) &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=1}^N \left[\sup_{0 \leq t \leq t_k} |g_n(te^{i\theta})| \right] \right) \mu([t_{k-1}, t_k]) d\theta, \\ II(n) &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=N+1}^{\infty} \left[\sup_{0 \leq t \leq t_k} |g_n(te^{i\theta})| \right] \right) \mu([t_{k-1}, t_k]) d\theta. \end{aligned}$$

Using (14), we obtain

$$\begin{aligned} II(n) &\leq \sum_{k=N+1}^{\infty} \frac{\varepsilon_k}{2^k} \frac{1}{2\pi} \int_0^{2\pi} \left[\sup_{0 \leq t \leq t_k} |g_n(te^{i\theta})| \right] d\theta \\ &\leq \frac{\varepsilon}{2C} \sum_{k=1}^{\infty} \frac{1}{2^k} M_1(t_k, g_n) \\ &\leq \frac{\varepsilon}{2C} \sum_{k=1}^{\infty} \frac{1}{2^k} \int_{t_k}^{t_{k+1}} M_1(t, g_n) dt \\ &\leq \frac{\varepsilon}{2}. \end{aligned}$$

Since $f_n \rightarrow 0$, uniformly on compact subsets of $\mathbb{D}$, it is clear that $I(n) \rightarrow 0$, as $n \rightarrow \infty$. Then it follows that there exists $n_0 \in \mathbb{N}$ such that $\|\mathcal{C}_\mu(f_n)\|_{H^1} < \varepsilon$ whenever $n \geq n_0$. Thus, we have shown that $\|\mathcal{C}_\mu(f_n)\|_{H^1} \rightarrow 0$, as $n \rightarrow \infty$ and the compactness of $\mathcal{C}_\mu$ on H^1 follows.

To deal with the cases $1 < p < 2$ and $2 < p < \infty$, we use again complex interpolation.

Suppose first that $1 < p < 2$ and μ is a vanishing Carleson measure. Recall that

$$H^p = (H^2, H^1)_\theta, \text{ with } \theta = \frac{2}{p} - 1.$$

We have also that if $2 < s < \infty$ then

$$H^2 = (H^s, H^1)_\alpha$$

for a certain $\alpha \in (0, 1)$, namely, $\alpha = (\frac{1}{2} - \frac{1}{s}) / (1 - \frac{1}{s})$. Since H^2 is reflexive, and $\mathcal{C}_\mu$ is compact from H^2 into H^2 and from H^1 into H^1 , Theorem 10 of [11] gives that and $\mathcal{C}_\mu$ is compact from H^p into H^p .

Suppose now that $2 < p < \infty$ and μ is a vanishing Carleson measure. Let q be conjugate exponent of p . Take q_1 with $1 < q_1 < q < 2$. We have that T_μ is compact from H^2 into itself and continuous from H^{q_1} into H^{q_1} . Also, $H^q = (H^2, H^{q_1})_\theta$ for a certain $\theta \in (0, 1)$. Then, Theorem 10 of [11] gives that and T_μ is compact from H^q into H^q and, hence, $\mathcal{C}_\mu$ is compact from H^p into itself. $\square$

3.3 The operators $\mathcal{C}_\mu$ acting on H^∞

For the constant function 1 we have

$$\mathcal{C}(1)(z) = \frac{1}{z} \log \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{z^n}{n+1}, \quad z \in \mathbb{D}.$$

Consequently, $\mathcal{C}(H^\infty) \not\subset H^\infty$.

If μ is positive finite Borel measure on $[0, 1)$ then

$$\mathcal{C}_\mu(1)(z) = \int_{[0,1)} \frac{d\mu(t)}{1-tz} = \sum_{n=0}^{\infty} \mu_n z^n, \quad z \in \mathbb{D}.$$

So, it follows that

$$\mathcal{C}_\mu(1) \in H^\infty \Leftrightarrow \int_{[0,1)} \frac{d\mu(t)}{1-t} < \infty \Leftrightarrow \sum_{n=0}^{\infty} \mu_n < \infty.$$

This easily implies the following result.

Theorem 4 *Let μ be positive finite Borel measure on $[0, 1)$. Then the following conditions are equivalent.*

- (i) $\mathcal{C}_\mu$ is a bounded operator from H^∞ into itself.
- (ii) $\int_{[0,1)} \frac{d\mu(t)}{1-t} < \infty$.
- (iii) $\sum_{n=0}^{\infty} \mu_n < \infty$.

Danikas and Siskakis [12] proved that

$$\mathcal{C}(H^\infty) \subset BMOA \subset \mathcal{B}.$$

We extend this result obtaining a characterization of those positive finite Borel measure μ on $[0, 1)$ for which $\mathcal{C}_\mu(H^\infty) \subset \mathcal{B}$.

Theorem 5 *Let μ be positive finite Borel measure on $[0, 1)$. Then the following conditions are equivalent*

- (i) $\mathcal{C}_\mu$ is a bounded operator from H^∞ into the Bloch space $\mathcal{B}$.
(ii) μ is a Carleson measure.

Proof Let us start with the implication (i) $\Rightarrow$ (ii). So, assume that $\mathcal{C}_\mu(H^\infty) \subset \mathcal{B}$. Then $\mathcal{C}_\mu(1) \in \mathcal{B}$, but, as we have seen above

$$\mathcal{C}_\mu(1)(z) = \sum_{n=0}^{\infty} \mu_n z^n, \quad z \in \mathbb{D},$$

and then, using the fact that the sequence $\{\mu_n\}$ is a decreasing sequence of nonnegative numbers and Lemma B, we see that $\mu_n = O(\frac{1}{n})$ which is equivalent to saying that μ is a Carleson measure.

Let us turn now to prove the other implication. So, assume that μ is a Carleson measure and take $f \in H^\infty$. Using the integral representation of $\mathcal{C}_\mu$ we see that

$$\mathcal{C}_\mu(f)'(z) = \int_{[0,1)} \frac{tf'(tz)}{1-tz} d\mu(t) + \int_{[0,1)} \frac{tf(tz)}{(1-tz)^2} d\mu(t), \quad z \in \mathbb{D}.$$

Hence, using that $f \in H^\infty \subset \mathcal{B}$, we obtain

$$\begin{aligned} |\mathcal{C}_\mu(f)'(z)| &\leq \int_{[0,1)} \frac{|f'(tz)|}{|1-tz|} d\mu(t) + \int_{[0,1)} \frac{|f(tz)|}{|1-tz|^2} d\mu(t) \\ &\lesssim \int_{[0,1)} \frac{d\mu(t)}{(1-|tz|)^2}, \quad z \in \mathbb{D}. \end{aligned} \quad (15)$$

Take $z \in \mathbb{D}$ and set $r = |z|$. Set also

$$\phi(t) = \mu([0, t)) - \mu([0, 1)) = -\mu([t, 1)), \quad 0 \leq t < 1.$$

Integrating by parts and using the fact that μ is a Carleson measure, we obtain

$$\begin{aligned} \int_{[0,1)} \frac{d\mu(t)}{(1-|tz|)^2} &= \int_{[0,1)} \frac{d\mu(t)}{(1-tr)^2} = \mu([0, 1)) + 2r \int_0^1 \frac{\mu([t, 1))}{(1-tr)^3} dt \\ &\lesssim \mu([0, 1)) + \int_0^1 \frac{1-t}{(1-tr)^3} dt \\ &= \mu([0, 1)) + \int_0^r \frac{1-t}{(1-tr)^3} dt + \int_r^1 \frac{1-t}{(1-tr)^3} dt \\ &\lesssim \mu([0, 1)) + \int_0^r \frac{1}{(1-t)^2} dt + \frac{1}{(1-r)^3} \int_r^1 (1-t) dt \\ &\lesssim \frac{1}{1-r}. \end{aligned}$$

This and (15) yield that $\mathcal{C}_\mu(f) \in \mathcal{B}$. □

It is natural to ask whether or not μ being a Carleson measure implies that $C_\mu(H^\infty) \subset BMOA$. We do not know the answer to this question.

4 The operators C_μ acting on Bergman spaces

The boundedness of the Cesàro operator on Bergman spaces was studied in [1] and [35] where the following result was proved.

Theorem A *If $p > 0$ and $\alpha > -1$, then the Cesàro operator is bounded from A_α^p into itself.*

In the course of our proof of Theorem 1, we proved that if μ is a Carleson measure, $1 \leq p < \infty$, and $f \in H^p$, then $M_p(r, C_\mu(f)) \lesssim M_p(r, f)$ (see (4) and (6)). This readily yields that if μ is a Carleson measure, $1 \leq p < \infty$, and $\alpha > -1$, then C_μ is bounded from A_α^p into itself.

For $p > 1$ we shall give a different proof of this result and we shall also prove that the converse is true. Hence, our work in particular will lead to a new proof of the boundedness of the classical Cesàro operator on the spaces A_α^p ($1 < p < \infty$, $\alpha > -1$).

Theorem 6 *Suppose that $1 < p < \infty$ and $\alpha > -1$. Let μ be a positive finite Borel measure on $[0, 1)$. Then the following conditions are equivalent.*

- (i) *The measure μ is a Carleson measure.*
- (ii) *The operator C_μ is bounded from A_α^p into itself.*

Let us collect several results which will be needed in the proof of Theorem 6.

Let us start recalling the given $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, the mean Lipschitz space Λ_α^p consists of those functions f analytic in $\mathbb{D}$ having a non-tangential limit almost everywhere for which $\omega_p(\delta, f) = O(\delta^\alpha)$, as $\delta \rightarrow 0$, where $\omega_p(\cdot, f)$ is the integral modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f . A classical result of Hardy and Littlewood [23] (see also Chapter 5 of [13]) asserts that for $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, we have that $\Lambda_\alpha^p \subset H^p$ and

$$\Lambda_\alpha^p = \left\{ f \text{ analytic in } \mathbb{D} : M_p(r, f') = O\left(\frac{1}{(1-r)^{1-\alpha}}\right), \quad \text{as } r \rightarrow 1 \right\}.$$

The space Λ_α^p is a Banach space with the norm $\|\cdot\|_{p,\alpha}$ given by

$$\|f\|_{p,\alpha} = |f(0)| + \sup_{0 \leq r < 1} (1-r)^{1-\alpha} M_p(r, f').$$

Of special interest are the spaces $\Lambda_{1/p}^p$ since they lie in the border of continuity. Indeed, if $1 < p < \infty$ and $\alpha > \frac{1}{p}$ then each $f \in \Lambda_\alpha^p$ has a continuous extension to the closed unit disc. This is not true for $\alpha = \frac{1}{p}$. This follows easily noticing that the function $f(z) = \log(1-z)$ belongs to $\Lambda_{1/p}^p$ for all $p \in (1, \infty)$. Cima and Petersen proved

in [9] that $\Lambda_{1/2}^2 \subset BMOA$ and this result was generalized by Bourdon, Shapiro and Sledd who proved in [4] that

$$\Lambda_{1/p}^p \subset BMOA, \quad 1 < p < \infty.$$

This was shown to be sharp in a very strong sense in [3].

The following result of Merchán [30, Lemma 1] (see also [18, Theorem 2] and [17, Theorem 2]) will be needed in our work.

Lemma B *Let $f \in \text{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$). Suppose that $1 < p < \infty$ and that the sequence $\{a_n\}$ is a decreasing sequence of nonnegative numbers. If $1 < p < \infty$ and X is a subspace of $\text{Hol}(\mathbb{D})$ with $\Lambda_{1/p}^p \subset X \subset \mathcal{B}$, then*

$$f \in X \Leftrightarrow a_n = O\left(\frac{1}{n}\right).$$

We shall also use some results on pointwise multipliers and coefficient multipliers of Bergman spaces and Hardy spaces.

Let us start recalling that for $g \in \text{Hol}(\mathbb{D})$, the multiplication operator M_g is defined by

$$M_g(f)(z) \stackrel{\text{def}}{=} g(z)f(z), \quad f \in \text{Hol}(\mathbb{D}), \quad z \in \mathbb{D}.$$

If X and Y are two spaces of analytic functions in $\mathbb{D}$ (which will always be assumed to be Banach or F -spaces continuously embedded in $\text{Hol}(\mathbb{D})$) and $g \in \text{Hol}(\mathbb{D})$ then g is said to be a *pointwise multiplier* from X to Y if $M_g(X) \subset Y$. The space of all multipliers from X to Y will be denoted by $M(X, Y)$. Using the closed graph theorem we see that if $g \in M(X, Y)$ then M_g is a bounded operator from X into Y . The following result is a particular case of Theorem C of [37].

Theorem C *Suppose that $1 < p < \infty$ and $\alpha > -1$. Then*

$$M\left(A_{\alpha}^p, A_{\alpha}^{p/(p+1)}\right) = A_{\alpha}^1.$$

If X and Y are two spaces of analytic functions in $\mathbb{D}$, a function $F \in \text{Hol}(\mathbb{D})$ is said to be a *coefficient multiplier* (or a *convolution multiplier*) from X to Y if

$$f \in X \Rightarrow F \star f \in Y.$$

The following result is due to Duren and Shields, it is a particular case of [15, Theorem 4].

Theorem D *Suppose that $1 < p < \infty$ and $F \in \text{Hol}(\mathbb{D})$. Let m be a positive integer such that $(m+1)^{-1} \leq \frac{p}{p+1} < m^{-1}$. Then F is a coefficient multiplier from $H^{p/(p+1)}$ to H^p if and only if the $(m+1)$ -th derivative $F^{(m+1)}$ of F satisfies*

$$M_p\left(r, F^{(m+1)}\right) = O\left((1-r)^{\frac{1}{p}-1-m}\right).$$

We can now proceed to prove Theorem 6.

Proof of the implication (i) $\Rightarrow$ (ii) in Theorem 6. Assume that μ is a Carleson measure and set

$$F(z) = \sum_{n=0}^{\infty} \mu_n z^n, \quad z \in \mathbb{D}.$$

Since μ is a Carleson measure $\mu_n = O\left(\frac{1}{n}\right)$. This, the simple fact that $\{\mu_n\}$ is a decreasing sequence of nonnegative numbers, and Lemma B imply that $F \in \Lambda_{1/p}^p$ and, hence

$$M_p(r, F') = O\left((1-r)^{\frac{1}{p}-1}\right).$$

Using [13, Theorem 5.5], we see that this implies

$$M_p(r, F^{(m+1)}) = O\left((1-r)^{\frac{1}{p}-1-m}\right), \quad m = 1, 2, 3, \dots,$$

and then Theorem D gives that F is a coefficient multiplier from $H^{p/(p+1)}$ into H^p . Trivially, this implies that

$$F \text{ is also a coefficient multiplier from } A_{\alpha}^{p/(p+1)} \text{ into } A_{\alpha}^p. \quad (16)$$

Take $f \in A_{\alpha}^p$. We have to prove that $C_{\mu}(f) \in A_{\alpha}^p$. Set $g(z) = \frac{f(z)}{1-z}$ ($z \in \mathbb{D}$). A simple computation shows that $\frac{1}{1-z} \in A_{\alpha}^1$. Then, using Theorem C we deduce that $g \in A_{\alpha}^{p/(p+1)}$. This and (16) imply that $F \star g \in A_{\alpha}^p$. By Lemma 1 this is equivalent to saying that $C_{\mu}(f) \in A_{\alpha}^p$. $\square$

Proof of the implication (ii) $\Rightarrow$ (i) in Theorem 6. Suppose that C_{μ} is a bounded operator on A_{α}^p . Let q be the exponent conjugate to p , that is, $\frac{1}{p} + \frac{1}{q} = 1$. Let T_{μ} be the adjoint of C_{μ} , it is a bounded operator on A_{α}^q .

For $0 < b < 1$, set

$$f_b(z) = \frac{(1-b)^{1-\frac{1}{q}}}{(1-bz)^{1+\frac{\alpha+1}{q}}} = \sum_{k=0}^{\infty} a_{k,b} z^k, \quad z \in \mathbb{D}.$$

Using [39, Lemma 3.10], we see that

$$\|f_b\|_{A_{\alpha}^q}^q \asymp 1. \quad (17)$$

Also,

$$a_{k,b} \asymp (1-b)^{1-\frac{1}{q}} k^{(\alpha+1)/q} b^k.$$

For $N \in \mathbb{N}$, set

$$f_{b,N}(z) = \sum_{k=0}^N a_{k,b} z^k, \quad z \in \mathbb{D}.$$

Bearing in mind Proposition 2 and Proposition 3, we see that

$$T_\mu(f_{b,N})(z) = \sum_{n=0}^N \left(\sum_{k=n}^N \mu_k a_{k,b} \right) z^n.$$

Since the coefficients $a_{k,b}$ are nonnegative, it follows that the sequence of the Taylor coefficients of $T_\mu(f_{b,N})$ is a decreasing sequence of nonnegative numbers, then (see, e. g., [20, Proposition 1])

$$\begin{aligned} \|T_\mu(f_{b,N})\|_{A_\alpha^q}^q &\gtrsim \sum_{n=1}^N n^{q-\alpha-3} \left(\sum_{k=n}^N \mu_k a_{k,b} \right)^q \\ &\gtrsim (1-b)^{q-1} \sum_{n=1}^N n^{q-\alpha-3} \left(\sum_{k=n}^N k^{\frac{\alpha+1}{q}} b^k \int_{[b,1)} t^k d\mu(t) \right)^q \\ &\gtrsim (1-b)^{q-1} \mu([b,1))^q \sum_{n=1}^N n^{q-\alpha-3} \left(\sum_{k=n}^N k^{\frac{\alpha+1}{q}} b^{2k} \right)^q. \end{aligned}$$

Since $f_{b,N} \rightarrow f_b$ in A_α^q as $N \rightarrow \infty$, using the fact that T_μ is bounded on A_α^q , (17), and simple estimations, we deduce that

$$\begin{aligned} 1 &\gtrsim (1-b)^{q-1} \mu([b,1))^q \sum_{n=1}^{\infty} n^{q-\alpha-3} \left(\sum_{k=n}^{\infty} k^{\frac{\alpha+1}{q}} b^{2k} \right)^q \\ &\gtrsim (1-b)^{q-1} \mu([b,1))^q \sum_{n=1}^{\infty} n^{q-\alpha-3} n^{\alpha+1} \left(\sum_{k=n}^{\infty} b^{2k} \right)^q \\ &\asymp (1-b)^{q-1} \mu([b,1))^q \sum_{n=1}^{\infty} n^{q-2} \frac{b^{2nq}}{(1-b)^q} \\ &\asymp \left(\frac{\mu([b,1))}{1-b} \right)^q. \end{aligned}$$

Hence, μ is a Carleson measure. $\square$

5 The operators $\mathcal{C}_\mu$ acting on $BMOA$ and on the Bloch space

Let λ be defined by $\lambda(z) = \log \frac{1}{1-z}$ ($z \in \mathbb{D}$). Then $\lambda \in BMOA$. In fact, it is true that $\lambda \in \Lambda_{1/p}^p$ for all $p > 1$. Danikas and Siskakis [12] observed that $\mathcal{C}(\lambda) \notin BMOA$. This implies that the Cesàro operator does not map $BMOA$ into itself. Our Theorem 3 includes a characterization of those μ so that $\mathcal{C}_\mu$ maps $BMOA$ into itself.

Since $\Lambda_{1/2}^2 \subset BMOA \subset \mathcal{B}$, Theorem 3 follows from the following result.

Theorem 7 *Let μ be a positive finite Borel measure on $[0, 1)$ and let X and Y be two Banach subspaces of $\text{Hol}(\mathbb{D})$ with $\Lambda_{1/2}^2 \subset X \subset \mathcal{B}$ and $\Lambda_{1/2}^2 \subset Y \subset \mathcal{B}$. Then the following conditions are equivalent.*

- (i) *The measure μ is a 1-logarithmic 1-Carleson measure.*
- (ii) *The operator $\mathcal{C}_\mu$ is bounded from X into Y .*

Proof Let us start showing that (i) $\Rightarrow$ (ii). So assume that μ is a 1-logarithmic 1-Carleson measure and take $f \in X$. We recall that μ being a 1-logarithmic 1-Carleson measure is equivalent to

$$\mu_n = O\left(\frac{1}{n \log(n+1)}\right). \quad (18)$$

Take $f \in X$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$). Since $X \subset \mathcal{B}$, we have that $f \in \mathcal{B}$. Then, using a result of Kayumov and Wirths (see [27, Corollary 4] or [28, Corollary D]), we have

$$\left| \sum_{k=0}^n a_k \right| \lesssim \|f\|_{\mathcal{B}} \log(n+1). \quad (19)$$

The estimates (18) and (19) yield

$$M_2^2(r, \mathcal{C}_\mu(f)') = \sum_{n=1}^{\infty} n^2 \mu_n^2 \left| \sum_{k=0}^n a_k \right|^2 r^{2n-2} \lesssim \sum_{n=1}^{\infty} r^{2n-2} \lesssim \frac{1}{1-r}.$$

Hence $\mathcal{C}_\mu(f) \in \Lambda_{1/2}^2 \subset Y$.

Suppose now that $\mathcal{C}_\mu(X) \subset Y$. As above, set $\lambda(z) = \log \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{z^n}{n}$ ($z \in \mathbb{D}$).

We have that $\lambda \in X$ and then $\mathcal{C}_\mu(\lambda) \in Y \subset \mathcal{B}$. Now, $\mathcal{C}_\mu(\lambda)(z) = \sum_{n=1}^{\infty} \mu_n \left(\sum_{k=1}^n \frac{1}{k} \right) z^n$ and then it follows that

$$\sum_{n=1}^{\infty} n \mu_n \left(\sum_{k=1}^n \frac{1}{k} \right) r^n \lesssim \frac{1}{1-r}, \quad r \in (0, 1).$$

For $N \geq 2$ take $r_N = 1 - \frac{1}{N}$. Bearing in mind that the sequence $\{\mu_n\}$ is decreasing, simple estimations lead us to the following

$$\begin{aligned}
 N^2 \mu_N \log N &\asymp \mu_N \sum_{n=1}^N n \log n \\
 &\lesssim \sum_{n=1}^N n \mu_n (\log n) r_N^n \\
 &\lesssim \sum_{n=1}^N n \mu_n \left(\sum_{k=1}^n \frac{1}{k} \right) r_N^n \\
 &\lesssim \sum_{n=1}^{\infty} n \mu_n \left(\sum_{k=1}^n \frac{1}{k} \right) r_N^n \\
 &\lesssim N.
 \end{aligned}$$

Hence $\mu_N \lesssim \frac{1}{N \log N}$ which implies that μ is a 1-logarithmic 1-Carleson measure. $\square$

We have the following result concerning compactness.

Theorem 8 *Let μ be a positive finite Borel measure on $[0, 1)$ and let X and Y be two Banach subspaces of $\text{Hol}(\mathbb{D})$ with $\Lambda_{1/2}^2 \subset X \subset \mathcal{B}$ and $\Lambda_{1/2}^2 \subset Y \subset \mathcal{B}$. Then the following four conditions are equivalent.*

- (i) μ is a vanishing 1-logarithmic 1-Carleson measure.
- (ii) The operator C_μ is a compact operator from X into Y .
- (iii) The operator C_μ is a compact operator from the Bloch space $\mathcal{B}$ into itself.
- (iv) The operator C_μ is a compact operator from the $BMOA$ into itself.

Proof Clearly, it suffices to prove that (i) and (ii) are equivalent. Let us prove first that (i) implies (ii). So, assume that μ is a vanishing 1-logarithmic 1-Carleson measure and $\Lambda_{1/2}^2 \subset X, Y \subset \mathcal{B}$.

Take $\{f_j\} \subset X$ with $\|f_j\|_X \leq 1$, for all j , and $f_j \rightarrow 0$, as $j \rightarrow \infty$, uniformly on compact subsets of $\mathbb{D}$. Since X is continuously embedded in $\mathcal{B}$, $\{f_j\} \subset \mathcal{B}$ and there exists $K_1 > 0$ such that $\|f\|_{\mathcal{B}} \leq K_1$, for all j .

Say $f_j(z) = \sum_{k=0}^{\infty} a_k^{(j)} z^k$ ($z \in \mathbb{D}$). Using the result of Kayumov and Wirhth that we have mentioned above, we see that there exists $K_2 > 0$ such that

$$\left| \sum_{k=0}^n a_k^{(j)} \right| \leq K_2 \|f_j\|_{\mathcal{B}} \log(n+1) \leq K_1 K_2 \log(n+1), \quad \text{for all } n \text{ and } j.$$

Set $K = K_1 K_2$.

Since μ is a vanishing 1-logarithmic 1-Carleson measure, $\mu_n = o\left(\frac{1}{n \log(n+1)}\right)$. Say $\mu_n = \frac{\varepsilon_n}{n \log(n+1)}$, with $\{\varepsilon_n\} \rightarrow 0$. Take $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $\varepsilon_n^2 K^2 < \frac{\varepsilon}{2}$ if

$n \geq N$. We have, for all $j \in \mathbb{N}$ and $0 < r < 1$,

$$\begin{aligned} M_2^2(r, C_\mu(f_j)') &= \sum_{n=1}^{\infty} n^2 \mu_n^2 \left| \sum_{k=0}^n a_k^{(j)} \right|^2 r^{2n-2} \\ &\leq \sum_{n=1}^N n^2 \mu_n^2 \left| \sum_{k=0}^n a_k^{(j)} \right|^2 + \sum_{n=N+1}^{\infty} n^2 \mu_n^2 K^2 [\log(n+1)]^2 r^{2n-2} \\ &\leq \sum_{n=1}^N n^2 \mu_n^2 \left| \sum_{k=0}^n a_k^{(j)} \right|^2 + \frac{\varepsilon/2}{1-r}. \end{aligned}$$

Thus,

$$\sup_{0 \leq r < 1} (1-r) M_2^2(r, C_\mu(f_j)') \leq \frac{\varepsilon}{2} + \sum_{n=1}^N n^2 \mu_n^2 \left| \sum_{k=0}^n a_k^{(j)} \right|^2, \quad j \in \mathbb{N}.$$

Now, since $\sum_{n=1}^N n^2 \mu_n^2 \left| \sum_{k=0}^n a_k^{(j)} \right|^2 \rightarrow 0$ and $f_j(0) \rightarrow 0$, as $j \rightarrow \infty$, it follows that there exists $j_0 \in \mathbb{N}$ such that

$$|f_j(0)| + \sum_{n=1}^N n^2 \mu_n^2 \left| \sum_{k=0}^n a_k^{(j)} \right|^2 < \frac{\varepsilon}{2}$$

for all $j \geq j_0$. With this we have proved that $C_\mu(f_j) \rightarrow 0$ in $\Lambda_{1/2}^2$. Since $\Lambda_{1/2}^2$ is continuously embedded in Y , it follows that $C_\mu(f_j) \rightarrow 0$ in Y .

Let us prove now that (ii) implies (i). Assume that $\Lambda_{1/2}^2 \subset X$, $Y \subset \mathcal{B}$ and that C_μ is compact from X into Y . For $0 < a < 1$, set

$$f_a(z) = \left(\log \frac{2}{1-a} \right)^{-1} \left(\log \frac{2}{1-az} \right)^2, \quad z \in \mathbb{D}.$$

We have that

$$f'_a(z) = \left(\log \frac{2}{1-a} \right)^{-1} \left(\log \frac{2}{1-az} \right) \frac{2a}{1-az}, \quad z \in \mathbb{D}, \quad 0 < a < 1.$$

Then it is clear that $f_a \in \Lambda_{1/2}^2$ for all $a \in [0, 1)$ and that there exists a constant $M_1 > 0$ such that $\|f_a\|_{2,1/2} \leq M_1$, for all $a \in (0, 1)$. Since $\Lambda_{1/2}^2$ is continuously embedded in X , it follows that $f_a \in X$ for all $a \in [0, 1)$ and that there exists $M > 0$ such that $\|f_a\|_X \leq M$, for all $a \in (0, 1)$. Also, $f_a \rightarrow 0$, as $a \rightarrow 1$, uniformly on compact subsets of $\mathbb{D}$. Since C_μ is compact from X into Y , we have that $\|C_\mu(f_a)\|_Y \rightarrow 0$, as $a \rightarrow 1$. This, together with the fact that Y is continuously embedded in $\mathcal{B}$, implies

that

$$\|\mathcal{C}_\mu(f_a)\|_{\mathcal{B}} \rightarrow 0, \quad \text{as } a \rightarrow 1. \quad (20)$$

A simple calculation gives that for $0 < a < 1$ and $z \in \mathbb{D}$,

$$\mathcal{C}_\mu(f_a)'(z) = \int_{[0,1)} \left[\frac{tf_a'(tz)}{1-tz} + \frac{tf_a(tz)}{(1-tz)^2} \right] d\mu(t).$$

Then it follows that, for $0 < a < 1$,

$$\begin{aligned} |\mathcal{C}_\mu(f_a)'(a)| &= \mathcal{C}_\mu(f_a)'(a) \\ &\geq \int_{[0,1)} \frac{tf_a(ta)}{(1-ta)^2} d\mu(t) \\ &= \left(\log \frac{2}{1-a} \right)^{-1} \int_{[0,1)} \frac{t \left(\log \frac{2}{1-ta} \right)^2}{(1-ta)^2} d\mu(t) \\ &\geq \left(\log \frac{2}{1-a} \right)^{-1} \int_{[a,1)} \frac{t \left(\log \frac{2}{1-ta} \right)^2}{(1-ta)^2} d\mu(t) \\ &\geq \left(\log \frac{2}{1-a} \right)^{-1} \mu([a, 1)) \frac{a \left(\log \frac{2}{1-a^2} \right)^2}{(1-a^2)^2}. \end{aligned}$$

This gives that

$$\mu([a, 1)) \lesssim (1-a) \left(\log \frac{2}{1-a} \right)^{-1} \|\mathcal{C}_\mu(f_a)\|_{\mathcal{B}}.$$

This and (20) imply that μ is a vanishing 1-logarithmic 1-Carleson measure. $\square$

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