General case of contact problems for a regular polygon weakened with full-strength hole

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The paper addresses a problem of plane elasticity theory for a doubly connected body whose external boundary is a regular polygon and the internal boundary is the required full-strength hole including the origin of coordinates. The full-strength hole is cycle symmetric. It is assumed that to every link of the broken line conforming the outer boundary of the given body are applied absolutely smooth rigid punches with rectilinear bases which are under the action of the force P, that applies to their middle points . There is no friction between the surface of the given elastic body and the punches. The edge of the unknown full-strength contour is subject to uniform pressure Q. Using the methods of complex analysis, the analytical image of Kolosov-Muskhelishvili's complex potentials (characterizing an elastic equilibrium of the body), and unknown parts of its boundary are determined under the condition that the tangential normal stress arising at it takes a constant value. A similar problem is considered for a square and an equilateral triangle which are weakened with full-strength holes. Using the method developed here the partially unknown boundary value problems under consideration is reduced to known boundary value problems of the theory of analytic functions. The solutions are presented in quadratures and full-strength contours are constructed.

1 Introduction

Applications of elastic plates weakened by holes are of great interest in several mechanical constructions (buildings, ships, aircrafts, etc.).

It is known that, in some cases, external action may cause stress concentration, especially along the borders of holes, leading to plastic deformations or even structural fractures.

In general, tangential normal stresses and tangential normal moments whose values depend on external loads and hole shapes play an important role in the plasticity zone formation in the plates with holes and also in the neighborhood of the plate's hole boundary. Proceeding from the above-mentioned, the following tasks were assigned: in conditions of provided external loads the shapes of the holes in plates should be chosen so that the maximal value of tangential normal stresses (tangential normal moments) on the boundaries will be the same and minimal in the same body in all other possible holes. It's proven that the minimum of maximal values of tangential normal stresses (tangential normal moments) will be obtained on such contours, where this value maintains the constant value; this contours are named full-strength contours.

Boundary-value problems of the plane theory of elasticity and plate bending for infinite plates weakened by unknown full-strength holes with normal stresses acting on their boundaries and forces applied at infinity were analyzed in [1, 2, 8, 12, 20].

Boundary-value problems for a finite doubly-connected domain with a part of its boundary being unknown full-strength and the other part being a polygonal line are solved in [6, 7].

The axis-symmetric and cycle symmetric problems of the plane theory of elasticity and plate bending with partially unknown boundaries are studied in [3–5, 13–19].

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In the present work we investigate the contact problem for a regular polygon which is weakened by a full-strength hole, when tangential stresses on the boundary are equal to zero and normal displacements are piecewise constant. The linear segments are endowed with the boundary conditions of the third problem. As specific cases, the similar problems are considered for a square and an equilateral triangle which are weakened with full-strength holes.

One of the most effective methods to solve such kind of problems turned out to be the methods of the theory of analytic functions. The solution is written in quadratures and full-strength holes are constructed.

2 Statement of the problem and solution technique

Let an isotropic elastic body on the plane $z = x + iy$ occupy a double connected domain S, whose external boundary is a regular polygon with m verticies and the internal boundary is the required full-strength hole.

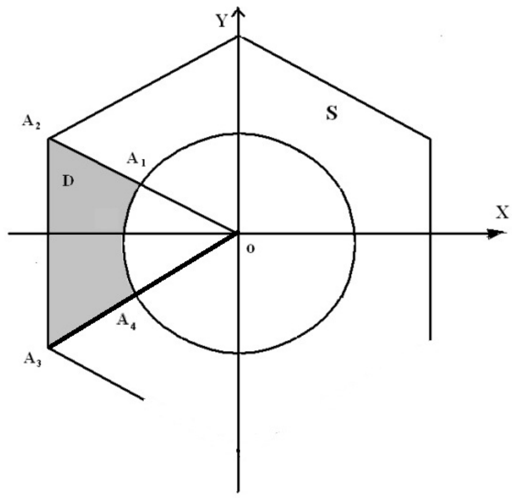

Fig. 1 Statement of the problem

The required full-strength hole includes the origin of coordinates. Let to every link of the broken line (outer boundary of the given body) be applied absolutely smooth rigid punches with rectilinear bases which displace to the normal under the action of concentrated normally compressive forces P applied to the punch midpoints of polygon's sides. There is no friction between the given elastic body and the punches. Let the edge of unknown full-strength contour be subject to uniform pressure Q. Under these assumptions, normal displacement of every link of broken line of the outer boundary $v_n = v = \text{const}$. Tangential stresses $\tau_{ns} = 0$ along the entire boundary of the domain S.

We formulate the following problem (A): Find the stress state of the body and the form of an *unknown full-strength contour* under the condition that the tangential normal stress σ_s on it takes constant value $\sigma_s = K = \text{const}$.

Since the problem is cyclic-symmetric it is sufficient to consider the curvilinear quadrangle $A_1A_2A_3A_4$ which be denoted by D. The normal displacements and the tangential stresses are equal to zero $v_n = \tau_{ns} = 0$ at each segment $[A_1A_2]$, $[A_3A_4]$.

Let introduce the following notations $\Gamma_1 = A_1A_2, \Gamma_2 = A_2A_3, \Gamma_3 = A_3A_4, \gamma = A_4A_1, \Gamma = \cup_{j=1}^{3}\Gamma_j$, $P_1 = \int_{\Gamma_1} \sigma_n \, ds$, $P_2 = \int_{\Gamma_2} \sigma_n \, ds$, $P_3 = \int_{\Gamma_3} \sigma_n \, ds$, σ_n is the normal stress. $P_2 = \int_{\Gamma_2} \sigma_n \, ds = -P$. Since D is in the equilibrium state, then we have:

$$P_2 = P_1 \cos\left(\frac{\pi}{2} - \frac{\pi}{m}\right) + P_3 \cos\left(\frac{\pi}{2} - \frac{\pi}{m}\right) + Q; \qquad P_3 = P_1.$$

Hence one obtains

$$P_1 = \frac{P_2 - Q}{2\sin\frac{\pi}{m}} = -\frac{P + Q}{2\sin\frac{\pi}{m}}$$

The boundary conditions are

$$v_n = \begin{cases} 0, & t \in \Gamma_3 \cup \Gamma_1 \\ v, & t \in \Gamma_2 \end{cases} \tag{2.1}$$

$$\tau_{ns} = 0, \qquad t \in \Gamma \cup \gamma \tag{2.2}$$

$$\sigma_n = -Q, \qquad \sigma_s = K, \qquad t \in \gamma \tag{2.3}$$

$$P_2 = -P, \qquad P_1 = P_3 = -\frac{P+Q}{2\sin\frac{\pi}{m}} \tag{2.4}$$

On the basis of the well-known Kolosov-Muskelishvili's formulas [10], the problem reduces to finding the functions ψ, φ which are holomophic in the domain D with the following conditions

$$\operatorname{Re} e^{-i\alpha(t)} \left(\chi\varphi(t) - t\,\overline{\varphi'(t)} - \overline{\psi(t)}\right) = 2\mu v_n(t), \qquad t \in \Gamma, \tag{2.5}$$

$$\operatorname{Re} e^{-i\alpha(t)} \left(\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)}\right) = C(t), \qquad t \in \Gamma, \tag{2.6}$$

$$\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)} = -Qt, \qquad t \in \gamma, \tag{2.7}$$

$$\operatorname{Re}\varphi'(t) = \frac{\sigma_n + \sigma_s}{4} = \frac{-Q+K}{4}, \qquad t \in \gamma, \tag{2.8}$$

where χ, μ are elasticity constants, $C(t)$ is a piecewise constant function, $\alpha(t)$ is the angle formed between the external normal n to contour and the abscissa axis Ox.

$$\alpha(t) = \alpha_k, \qquad t \in \Gamma_k, \qquad k = 1,2,3, \qquad \alpha_1 = \frac{\pi}{2} - \frac{\pi}{m}, \qquad \alpha_2 = \pi, \qquad \alpha_3 = \frac{3\pi}{2} + \frac{\pi}{m} \tag{2.9}$$

$$C(t) = \operatorname{Re}\left(e^{-i\alpha(t)} i \left(\int_{A_1}^{t} \sigma_n(s_0) e^{i\alpha(s_0)}\, ds_0\right)\right) \tag{2.10}$$

Taking into account (2.4) and (2.9), the (2.10) has following form:

$$C(t) = \begin{cases} 0, & t \in \Gamma_1 \\ -\frac{P+Q}{2}\operatorname{cotan}\frac{\pi}{m}, & t \in \Gamma_2 \\ Q\cos\frac{\pi}{m}, & t \in \Gamma_3 \end{cases} \tag{2.11}$$

Let $t \in A_k A_{k+1}$, $k = 1,2,3$, then $t - A_k = i|t - A_k| e^{i\alpha_k}$. Hence we obtain

$$\operatorname{Re} t e^{-i\alpha(t)} = \operatorname{Re} e^{-i\alpha(t)} A(t) \tag{2.12}$$

where $A(t)$ is a piecewise-constant function, $A(t) = A_k$, $k = 1,2,3$.
Taking into account (2.9) one obtains:

$$\operatorname{Re} e^{-i\alpha(t)} A(t) = \begin{cases} \frac{a}{2}\operatorname{cotan}\frac{\pi}{m}, & t \in \Gamma_2 \\ 0, & t \in \Gamma_1 \cup \Gamma_3 \end{cases} \tag{2.13}$$

where a is the length of the regular polygon side.
The functions $\varphi(z)$, $\bar{z}\varphi'(z) + \psi(z)$ will be assumed to be continuous everywhere on the boundary of the domain D, while functions $\varphi'(z)$, $\psi(z)$ are continuously extendable on the boundary D, with the exclusion of the points A_2, A_3 in the neighborhood of which is admitted the following estimate:

$$|\varphi'(z)| < M|z - A_i|^{-\delta}, \qquad |\psi(z)| < M|z - A_i|^{-\delta}, \qquad i = 2,3, \qquad 0 \le \delta < 1. \tag{2.14}$$

Summarizing (2.5) and (2.6) and differentiating with respect to arc abscissa s and taking into account that $\alpha(t)$, $C(t)$, $v_n(t)$ are the piecewise-constant functions on Γ, we obtain

$$\operatorname{Im}\varphi'(t) = 0, \qquad t \in \Gamma. \tag{2.15}$$

Equalities (2.8) and (2.15) are the Keldysh-Sedov problem for domain S:

$$\begin{aligned} &\operatorname{Re}\left(\varphi'(t) - \frac{K-Q}{4}\right) = 0, \quad t \in A_4A_1, \\ &\operatorname{Im}\left(\varphi'(t) - \frac{K-Q}{4}\right) = 0, \qquad t \in \Gamma. \end{aligned} \tag{2.16}$$

Problem (2.16) has a unique solution [9]

$$\varphi'(z) = \frac{K-Q}{4} \tag{2.17}$$

Hence we obtain

$$\varphi(z) = \frac{K-Q}{4} z. \tag{2.18}$$

Here we neglect the constant summands.

Substituting the values $\varphi(t)$, $C(t)$ into the boundary conditions (2.6)–(2.7) and taking into account (2.12) one gets the following problem

$$\operatorname{Re}\left[e^{-i\alpha(t)}\left(\frac{K-Q}{2}t + \overline{\psi(t)}\right)\right] = C(t) = \begin{cases} 0, & t \in \Gamma_1 \\ -\frac{P+Q}{2}\operatorname{cotan}\frac{\pi}{m}, & t \in \Gamma_2, \\ Q\cos\frac{\pi}{m}, & t \in \Gamma_3 \end{cases} \tag{2.19}$$

$$\operatorname{Re} te^{-i\alpha(t)} = \operatorname{Re} e^{-i\alpha(t)}A(t), \qquad t \in \Gamma, \tag{2.20}$$

$$\frac{K+Q}{2}t + \overline{\psi(t)} = 0, \qquad t \in \gamma. \tag{2.21}$$

Let the function $z = \omega(\zeta)$, $\zeta = \xi + i\eta$ maps the semicircle $|\zeta| < 1$, $\operatorname{Im}\zeta > 0$ conformally onto the domain D. It is assumed that the vertices A_k of polygonal line correspond to the points a_k of the semicircle $|\zeta| = 1$, $\operatorname{Im}\zeta > 0$, $a_k = \omega^{-1}(A_k)$, $k = 1,2,3,4$. It is assumed that $a_1 = 1$, $a_4 = -1$, $a_3 = i$. Here we can fix three points and the remaining ones are to be defined. Then the diameter $-1 \le \xi \le 1$ is mapped onto arc A_4A_1 and the semi-circumference γ_0: $|\gamma_0| = 1$, $\operatorname{Im}\zeta > 0$ is mapped onto the broken line Γ. The point $a_2 = e^{i\theta_2}$, $0 < \theta_2 < \frac{\pi}{2}$ is to be determined.

As a result of the mapping the boundary conditions (2.19)-(2.21) take the form:

$$\operatorname{Re} e^{-i\alpha(\sigma)}\overline{\psi_0(\sigma)} = -\frac{(K-Q)}{2}\operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma) + C(\sigma), \qquad \sigma \in \gamma_0 \tag{2.22}$$

$$\operatorname{Re} e^{-i\alpha(\sigma)}\omega(\sigma) = \operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma), \qquad \sigma \in \gamma_0 \tag{2.23}$$

$$\frac{K+Q}{2}\omega(\sigma) + \overline{\psi_0(\sigma)} = 0, \qquad \sigma \in (-1,1) \tag{2.24}$$

$$\psi_0(\zeta) = \psi(\omega(\zeta)). \tag{2.25}$$

Since $\alpha(t)$, $A(t)$, $C(t)$ are the piecewise-constant functions, $\alpha(\omega(\sigma))$, $A(\omega(\sigma))$, $C(\omega(\sigma))$ are denoted by $\alpha(\sigma)$, $A(\sigma)$, $C(\sigma)$.

Let us introduce a piecewise-holomorphic function $W(\zeta)$ in the circle $|\zeta| < 1$.

$$W(\zeta) = \begin{cases} \frac{K+Q}{2}\omega(\zeta), & |\zeta| < 1, \operatorname{Im}\zeta > 0 \\ -\overline{\psi_0(\bar{\zeta})}, & |\zeta| < 1, \operatorname{Im}\zeta < 0 \end{cases} \tag{2.26}$$

Taking into account (2.26)

$$W^+(\xi) = \frac{K+Q}{2}\omega(\xi), \qquad W^-(\xi) = -\overline{\psi_0(\xi)}, \qquad -1 < \xi < 1, \tag{2.27}$$

$$W^+(\sigma) = \frac{K+Q}{2}\omega(\sigma), \quad \sigma \in \gamma_0, \qquad W^-(\sigma) = -\overline{\psi_0(\bar{\sigma})}, \quad \sigma \in \gamma_0^*. \tag{2.28}$$

where the signs $(+)$ and $(-)$ denote boundary values of W taken, respectively, on the upper and lower edges of the diameter and γ_0^* is mirror image of γ_0 about the Ox-axis. By virtue of (2.24) and (2.27), we obtain $W^+(\xi) - W^-(\xi) = 0$, $-1 < \xi < 1$.

This means that $W(\zeta)$ is a holomorphic function on the circle $|\zeta| < 1$.
Considering (2.26), we get

$$\operatorname{Re} e^{-i\alpha(\sigma)} W(\sigma) = \frac{K+Q}{2} \operatorname{Re} e^{-i\alpha(\sigma)} A(\sigma), \qquad \sigma \in \gamma_0, \tag{2.29}$$

$$\operatorname{Re} e^{-i\alpha(\sigma)} W(\sigma) = -C(\sigma) + \frac{K-Q}{2} \operatorname{Re} e^{-i\alpha(\sigma)} A(\sigma), \qquad \sigma \in \gamma_0^*. \tag{2.30}$$

Conditions (2.27) and (2.28) can be rewritten in the following form

$$\operatorname{Re} e^{-i\alpha(\sigma)} W(\sigma) = f(\sigma), \qquad \sigma \in \gamma', \tag{2.31}$$

where

$$\gamma' = \gamma_0 \cup \gamma_0^*, \quad \alpha(\sigma) = \alpha(\bar\sigma), \quad \sigma \in \gamma_0^*, \qquad \alpha_1 = \frac{\pi}{2} - \frac{\pi}{m}, \quad \alpha_2 = \pi, \quad \alpha_3 = \frac{3\pi}{2} + \frac{\pi}{m}.$$

If $\sigma \in \gamma_0$, then

$$f(\sigma) = \frac{K+Q}{2} \operatorname{Re} e^{-i\alpha(\sigma)} A(\sigma) = \begin{cases} \frac{(K+Q)a}{4} \operatorname{cotan} \frac{\pi}{m}, & \sigma \in (a_2, a_3) \\ 0, & \sigma \in (a_1, a_2) \cup (a_3, a_4) \end{cases} \tag{2.32}$$

If $\sigma \in \gamma_0^*$, then

$$f(\sigma) = \frac{K-Q}{2} \operatorname{Re} e^{-i\alpha(\sigma)} A(\sigma) - C(\sigma) = \begin{cases} 0, & \sigma \in (\bar a_2, a_1^*) \\ \frac{(K-Q)a}{4} \operatorname{cotan} \frac{\pi}{m} + \frac{P+Q}{2} \operatorname{cotan} \frac{\pi}{m}, & \sigma \in (\bar a_3, \bar a_2) \\ -Q \cos \frac{\pi}{m}, & \sigma \in (a_4, \bar a_3) \end{cases} \tag{2.33}$$

where $a_1^* = e^{2\pi i}$.

Thus, the problem in question has been reduced to the Riemann-Hilbert problem with piecewise-constant coefficients. The solution of this problem was obtained in [11] (by reducing it to a linear-conjugation problem). Here we reduce the problem to the Dirichlet problem for a circle and represent its solution in terms of the Schwarz integral, which is computationally convenient.

Function $e^{2i\alpha(\sigma)}$ is given by

$$e^{2i\alpha(\sigma)} = \frac{X(\sigma)}{\overline{X(\sigma)}}, \qquad |\sigma| = 1, \tag{2.34}$$

where $X(\zeta)$ is given as

$$X(\zeta) = \frac{(\zeta - \bar a_2)^{\left(\frac{1}{2}+\frac{1}{n}\right)} (\zeta - \bar a_3)^{\left(\frac{1}{2}+\frac{1}{n}\right)} (\zeta - a_2)^{\left(\frac{1}{2}-\frac{1}{n}\right)} (\zeta - a_3)^{\left(\frac{1}{2}-\frac{1}{n}\right)} \sqrt{\overline{a_2 a_3}}}{\zeta} e^{-i\left(3\pi\left(\frac{1}{n}+\frac{1}{2}\right) - \left(1+\frac{2}{n}\right)\theta_2\right)}, \qquad |\zeta| < 1, \tag{2.35}$$

By virtue of (2.34) condition (2.31) becomes

$$\frac{W(\sigma)}{X(\sigma)} + \frac{\overline{W(\sigma)}}{\overline{X(\sigma)}} = \frac{2f e^{i\alpha(\sigma)}}{X(\sigma)}. \tag{2.36}$$

Condition (2.36) represents the boundary condition of the Dirichlet problem, whose solution is presented by Schwarz formula

$$\frac{W(\zeta)}{X(\zeta)} = \frac{1}{2\pi i} \int_{\gamma'} \frac{f(\sigma) e^{i\alpha(\sigma)} (\sigma + \zeta)\, d\sigma}{X(\sigma)\sigma(\sigma - \zeta)} \tag{2.37}$$

Since function $X(\zeta)$ has a simple pole at the point $\zeta = 0$, the function $W(\zeta)/X(\zeta)$ will have the first-order zero at this point $\zeta = 0$. Hence from (2.37) one gets

$$\int_{\gamma'} \frac{f(\sigma) e^{i\alpha(\sigma)}\, d\sigma}{X(\sigma)\sigma} = 0 \tag{2.38}$$

Taking into account (2.32)-(2.33) and (2.9), the (2.38) has the form:

$$\frac{(K+Q)a}{4}\,\mathrm{cotan}\,\frac{\pi}{m}e^{\pi i}\int_{a_2}^{a_3}\frac{d\sigma}{X(\sigma)\sigma}+\left(\frac{(K-Q)a}{4}\,\mathrm{cotan}\,\frac{\pi}{m}+\frac{(P+Q)}{2}\,\mathrm{cotan}\,\frac{\pi}{m}\right)e^{\pi i}\int_{\bar a_3}^{\bar a_2}\frac{d\sigma}{X(\sigma)\sigma}$$
$$-\,Q\cos\frac{\pi}{m}e^{\left(\frac{3\pi}{2}+\frac{\pi}{m}\right)i}\int_{a_4}^{\bar a_3}\frac{d\sigma}{X(\sigma)\sigma}=0 \quad (2.39)$$

Thus we obtain the equation with respect to two unknown parameters a_2, K.

We could choose some value of K and then determine the parameter a_2. In this case, the problem becomes more complicated. For the purpose of computations, it is more convenient to calculate K for each fixed point $a_2 = e^{i\theta_2}$, $0 < \theta_2 < \frac{\pi}{2}$ and for the given P and Q. From (2.39) one obtains:

$$K = -\frac{(A-aB+2B)Q+2PB+C}{A+aB} \quad (2.40)$$

Where

$$A=\frac{a\mathrm{cotan}\,\frac{\pi}{m}e^{\pi i}}{4}\int_{a_2}^{a_3}\frac{d\sigma}{X(\sigma)\sigma},\qquad B=\frac{\mathrm{cotan}\,\frac{\pi}{m}e^{\pi i}}{4}\int_{\bar a_3}^{\bar a_2}\frac{d\sigma}{X(\sigma)\sigma},\qquad C=-Q\cos\frac{\pi}{m}e^{\left(\frac{\pi}{2}-\frac{\pi}{m}\right)i}\int_{a_4}^{\bar a_3}\frac{d\sigma}{X(\sigma)\sigma}.$$

Remark 2.1 The solution of the problem exists if

$$A + aB \neq 0$$

By virtue of (2.38), the formula (2.37) has the form:

$$W(\zeta)=\frac{\zeta X(\zeta)}{\pi i}\int_{\gamma'}\frac{f(\sigma)e^{i\alpha(\sigma)}\,d\sigma}{X(\sigma)\sigma(\sigma-\zeta)} \quad (2.41)$$

i.e.

$$W(\zeta)=\frac{\zeta X(\zeta)}{\pi i}\left(\frac{(K+Q)a}{4}\,\mathrm{cotan}\,\frac{\pi}{m}e^{\pi i}\int_{a_2}^{a_3}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\zeta)}\right.$$
$$+\left(\frac{(K-Q)a}{4}\,\mathrm{cotan}\,\frac{\pi}{m}+\frac{(P+Q)}{2}\,\mathrm{cotan}\,\frac{\pi}{m}\right)e^{\pi i}\int_{\bar a_3}^{\bar a_2}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\zeta)}$$
$$\left.-Q\cos\frac{\pi}{m}e^{\left(\frac{3\pi}{2}+\frac{\pi}{m}\right)i}\int_{a_4}^{\bar a_3}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\zeta)}\right)$$

By virtue of (2.26), equation of the contour $z = \omega(\xi)$ is presented by

$$\omega(\xi)=\frac{2W(\xi)}{K+Q},\qquad -1<\xi<1. \quad (2.42)$$

Thus, the equation of the body's unknown full-strength contour and the stress state is defined.

2.1 Construction of the polygon's full-strength hole

To construct the required *polygon*'s full-strength hole, at first, the arc $\gamma = A_4A_1$ of the required full-strength contour is constructed.

Having defined K by formula (2.40) for every fixed point $a_2 = e^{i\theta_2}$, $0 < \theta_2 < \pi/2$ and given P, Q, we define the function $W(\xi)$ by formula (2.41), and then using formula (2.42) we define the function $\omega(\xi)$. As a result of conformal mapping $z = \omega(\xi)$, $-1 < \xi < 1$, we obtain $\gamma = A_4A_1$ the one m-th part of the required full-strength hole. Since the problem is cyclically symmetric, the other parts of required full-strength hole can be obtained by rotating the graphs z through an angle of π/m.

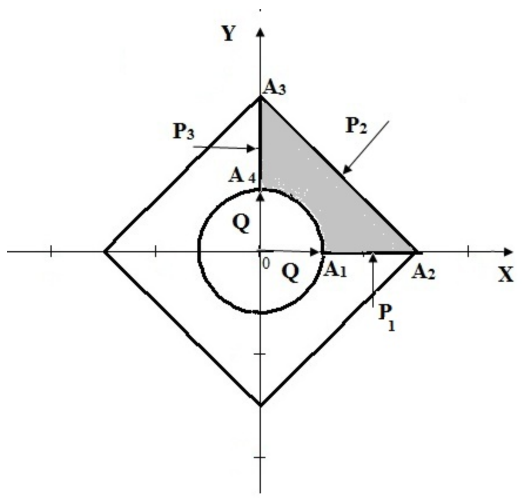

Fig. 2 Construction of full-strength hole for a square

3 Some Constructions of full-strength holes for a square (m is an even number)

Let us consider the concrete case when the regular polygon (m is an even number) is a square, which is weakened by full-strength hole. Diagonals of the square coincide with Ox and Oy coordinate axes and its center coincides with the origin of coordinate axes.

Let us consider the same above mentioned posed problem for a square. Let to every link of the outer boundary of the square be applied absolutely smooth rigid punches with rectilinear bases which displace to the normal under the action of concentrated normally compressive forces P applied to the punch midpoints of square's sides. There is no friction between the given elastic body and the punches. Let the edge of unknown full-strength contour be subject to uniform pressure Q. Under these assumptions, normal displacement of every link of the broken line of the outer boundary is $v_n = v =$ const. Tangential stresses $\tau_{ns} = 0$ along the entire boundary of the domain S.

Similarly as above let us consider the problem (A).

Problem (A): Find the stress state of the body and the form of an *unknown full-strength contour* under the condition that the tangential normal stress σ_s on it takes constant value $\sigma_s = K =$ const.

Since the problem is cyclic-symmetric it is sufficient to consider the curvilinear quadrangle $A_1A_2A_3A_4$ denoted by D (Figure 2).

The normal displacements and the tangential stresses are equal to zero $v_n = \tau_{ns} = 0$ at each segment $[A_1A_2]$, $[A_3A_4]$.

Let introduce the following notations $\Gamma_1 = A_1A_2$, $\Gamma_2 = A_2A_3$, $\Gamma_3 = A_3A_4$, $\gamma = A_4A_1$, $\Gamma = \cup_{j=1}^{3}\Gamma_j$. $P_1 = \int_{\Gamma_1} \sigma_n\, ds$, $P_2 = \int_{\Gamma_2} \sigma_n\, ds$, $P_3 = \int_{\Gamma_3} \sigma_n\, ds$, σ_n is the normal stress. According to the statement of the problem one gets: $P_2 = \int_{\Gamma_2} \sigma_n\, ds = -P$.

$$\alpha(t) = \alpha_k, \quad t \in \Gamma_k, \quad k = 1, 2, 3, \quad \alpha_1 = \frac{\pi}{2}, \quad \alpha_2 = \frac{\pi}{4}, \quad \alpha_3 = \pi. \tag{3.1}$$

Since D is in the equilibrium state, then we have:

$$\begin{cases} P_3 + Q = P_2 \cos\alpha_2, \\ P_1 + Q = P_2 \sin\alpha_2. \end{cases}$$

Hence one obtains $P_1 = P_3 = -Q - \frac{\sqrt{2}}{2}P$.

The boundary conditions are

$$v_n = \begin{cases} 0, & t \in \Gamma_3 \cup \Gamma_1 \\ v, & t \in \Gamma_2, \end{cases} \tag{3.2}$$

$$\tau_{ns} = 0, \qquad t \in \Gamma \cup \gamma, \tag{3.3}$$

$$\sigma_n = -Q, \quad \sigma_s = K, \qquad t \in \gamma, \tag{3.4}$$

$$P_2 = -P, \quad P_1 = P_3 = -Q - \frac{\sqrt{2}}{2} P. \tag{3.5}$$

On the basis of the well-known Kolosov-Muskelishvili's formulas [10], the problem reduces to finding the functions ψ, φ which are holomorphic in the domain D with the following conditions

$$\operatorname{Re} e^{-i\alpha(t)} \left(\chi\varphi(t) - t\,\overline{\varphi'(t)} - \overline{\psi(t)} \right) = 2\mu v_n(t), \qquad t \in \Gamma, \tag{3.6}$$

$$\operatorname{Re} e^{-i\alpha(t)} \left(\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)} \right) = C(t), \qquad t \in \Gamma, \tag{3.7}$$

$$\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)} = -Qt, \qquad t \in \gamma, \tag{3.8}$$

$$\operatorname{Re} \varphi'(t) = \frac{\sigma_n + \sigma_s}{4} = \frac{-Q + K}{4}, \qquad t \in \gamma \tag{3.9}$$

$$C(t) = \operatorname{Re} \left(e^{-i\alpha(t)} i \left(\int_{A_1}^{t} \sigma_n(s_0) e^{i\alpha(s_0)}\, ds_0 \right) \right) \tag{3.10}$$

Taking into account (3.1) and (3.5), $C(t)$ has the following form:

$$C(t) = \begin{cases} 0, & t \in \Gamma_1 \\ -\frac{\sqrt{2}}{2} \left(Q + \frac{P\sqrt{2}}{2} \right), & t \in \Gamma_2 \\ Q, & t \in \Gamma_3 \end{cases} \tag{3.11}$$

Taking into account (3.1) one obtains:

$$\operatorname{Re} e^{-i\alpha(t)} A(t) = \begin{cases} \frac{a}{2}, & t \in \Gamma_2 \\ 0, & t \in \Gamma_1 \cup \Gamma_3 \end{cases} \tag{3.12}$$

where a is the length of the square side.

Analogously as above, we obtain that

$$\varphi(z) = \frac{K - Q}{4} z \tag{3.13}$$

Here we neglect the constant summands.

Substituting the values, $\varphi(t)$, $C(t)$ into the boundary conditions (3.7) - (3.8) and taking into account (3.11), (3.12) one gets the following problem

$$\operatorname{Re} \left[e^{-i\alpha(t)} \left(\frac{K - Q}{2} t + \overline{\psi(t)} \right) \right] = C(t) = \begin{cases} 0, & t \in \Gamma_1 \\ -\frac{\sqrt{2}}{2} \left(Q + \frac{\sqrt{2}}{2} P \right), & t \in \Gamma_2, \\ Q, & t \in \Gamma_3 \end{cases} \tag{3.14}$$

$$\operatorname{Re} t e^{-i\alpha(t)} = \operatorname{Re} e^{-i\alpha(t)} A(t), \qquad t \in \Gamma, \tag{3.15}$$

$$\frac{K + Q}{2} t + \overline{\psi(t)} = 0, \qquad t \in \gamma \tag{3.16}$$

Let the function $z = \omega(\zeta)$, $\zeta = \xi + i\eta$ map the semicircle $|\zeta| < 1$, $\operatorname{Im} \zeta > 0$ conformally onto the domain D. It is assumed that the vertices A_k of square line correspond to the point a_k of the semicircle $|\zeta| = 1$, $\operatorname{Im} \zeta > 0$, $a_k = \omega^{-1}(A_k)$, $k = 1, 2, 3, 4$. It is assumed, that $a_1 = 1$, $a_4 = -1$, $a_3 = i$. The diameter $-1 \le \xi \le 1$ is mapped onto arc $A_4 A_1$ and the semi-circumference γ_0: $|\gamma_0| = 1$, $\operatorname{Im} \zeta > 0$ is mapped onto the broken line Γ. The point $a_2 = e^{i\theta_2}$, $0 < \theta_2 < \pi/2$ is to be determined.

As a result of the mapping the boundary conditions (3.14)- (3.16) take the form:

$$\operatorname{Re} e^{-i\alpha(\sigma)}\overline{\psi_0(\sigma)} = -\frac{(K-Q)}{2}\operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma) + C(\sigma), \qquad \sigma \in \gamma_0, \tag{3.17}$$

$$\operatorname{Re} e^{-i\alpha(\sigma)}\omega(\sigma) = \operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma), \qquad \sigma \in \gamma_0, \tag{3.18}$$

$$\frac{K+Q}{2}\omega(\sigma) + \overline{\psi_0(\sigma)} = 0, \qquad \sigma \in (-1,1), \tag{3.19}$$

$$\psi_0(\zeta) = \psi(\omega(\zeta)). \tag{3.20}$$

Let us introduce a piecewise-holomorphic function $W(\zeta)$ in the circle $|\zeta| < 1$.

$$W(\zeta) = \begin{cases} \frac{K+Q}{2}\omega(\zeta), & |\zeta| < 1, \quad \operatorname{Im}\zeta > 0, \\ -\overline{\psi_0(\bar{\zeta})}, & |\zeta| < 1, \quad \operatorname{Im}\zeta < 0. \end{cases} \tag{3.21}$$

Similarly as above it is proved that $W(\zeta)$ is a holomorphic function on the circle $|\zeta| < 1$.
By virtue of (3.17), (3.18) and (3.21) one gets

$$\operatorname{Re} e^{-i\alpha(\sigma)}W(\sigma) = \frac{K+Q}{2}\operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma), \qquad \sigma \in \gamma_0, \tag{3.22}$$

$$\operatorname{Re} e^{-i\alpha(\sigma)}W(\sigma) = -C(\sigma) + \frac{K-Q}{2}\operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma), \qquad \sigma \in \gamma_0^*. \tag{3.23}$$

Conditions (3.22) and (3.23) can be rewritten in the following form

$$\operatorname{Re} e^{-i\alpha(\sigma)}W(\sigma) = f(\sigma), \qquad \sigma \in \gamma', \tag{3.24}$$

where

$$\gamma' = \gamma_0 \cup \gamma_0^*, \quad \alpha(\sigma) = \alpha(\bar{\sigma}), \quad \sigma \in \gamma_0^*, \quad \alpha_1 = -\frac{\pi}{2}, \quad \alpha_2 = \frac{\pi}{4}, \quad \alpha_3 = \pi.$$

If $\sigma \in \gamma_0$, then

$$f(\sigma) = \frac{K+Q}{2}\operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma) = \begin{cases} \frac{(K+Q)a}{4}, & \sigma \in (a_2, a_3), \\ 0, & \sigma \in (a_1, a_2) \cup (a_3, a_4). \end{cases} \tag{3.25}$$

If $\sigma \in \gamma_0^*$, then

$$f(\sigma) = \frac{K-Q}{2}\operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma) - C(\sigma) = \begin{cases} 0, & \sigma \in (\bar{a}_2, a_1^*), \\ \frac{(K-Q)a}{4} + \frac{\sqrt{2}}{2}\left(Q + \frac{\sqrt{2}}{2}P\right), & \sigma \in (\bar{a}_3, \bar{a}_2), \\ -Q, & \sigma \in (a_4, \bar{a}_3) \end{cases} \tag{3.26}$$

where $a_1^* = e^{2\pi i}$.

Thus, the problem in question has been reduced to the Riemann–Hilbert problem with piecewise-constant coefficients. Solution of the problem is reduced to the Dirichlet problem for a circle and its solution is presented in terms of the Schwarz integral.

Function $e^{2i\alpha(\sigma)}$ is given by

$$e^{2i\alpha(\sigma)} = \frac{X(\sigma)}{\overline{X(\sigma)}}, \qquad |\sigma| = 1, \tag{3.27}$$

where $X(\zeta)$ is given as

$$X(\zeta) = \left(\frac{\zeta - a_2}{\zeta - a_1}\right)^{-\frac{1}{2}} \left(\frac{\zeta - a_3}{\zeta - a_2}\right)^{\frac{1}{4}} \left(\frac{\zeta - a_4}{\zeta - a_3}\right) \left(\frac{\zeta - \bar{a}_3}{\zeta - a_4}\right) \left(\frac{\zeta - \bar{a}_2}{\zeta - \bar{a}_3}\right)^{\frac{1}{4}} \left(\frac{\zeta - a_1^*}{\zeta - \bar{a}_2}\right)^{-\frac{1}{2}} \frac{(\zeta - a_2)(\zeta - a_3)}{\zeta}\sqrt{\bar{a}_2\bar{a}_3}, \quad |\zeta| < 1. \tag{3.28}$$

By virtue of (3.27) condition (3.24) becomes

$$\frac{W(\sigma)}{X(\sigma)} + \frac{\overline{W(\sigma)}}{\overline{X(\sigma)}} = \frac{2fe^{i\alpha(\sigma)}}{X(\sigma)} \tag{3.29}$$

Condition (3.29) represents the boundary condition of the Dirichlet problem, whose solution is presented by the Schwarz formula

$$\frac{W(\zeta)}{X(\zeta)} = \frac{1}{2\pi i}\int_{\gamma'} \frac{f(\sigma)e^{i\alpha(\sigma)}(\sigma+\zeta)\,d\sigma}{X(\sigma)\sigma(\sigma-\zeta)} \tag{3.30}$$

Since function $X(\zeta)$ has a simple pole at the point $\zeta = 0$, the function $W(\zeta)/X(\zeta)$ will have the first-order zero at this point $\zeta = 0$. Hence from (3.30) we get

$$\int_{\gamma'} \frac{f(\sigma)e^{i\alpha(\sigma)}\,d\sigma}{X(\sigma)\sigma} = 0 \tag{3.31}$$

Taking into account (3.25)-(3.26) and (3.1), the (3.31) has the form:

$$\frac{(K+Q)a}{4}e^{\frac{\pi}{4}i}\int_{a_2}^{a_3}\frac{d\sigma}{X(\sigma)\sigma} + \left(\frac{(K-Q)a}{4} + \frac{\sqrt{2}}{2}\left(Q + \frac{\sqrt{2}}{2}P\right)\right)e^{\frac{\pi}{4}i}\int_{\bar{a}_3}^{\bar{a}_2}\frac{d\sigma}{X(\sigma)\sigma} - Qe^{\pi i}\int_{a_4}^{\bar{a}_3}\frac{d\sigma}{X(\sigma)\sigma} = 0 \tag{3.32}$$

Thus we obtain the equation with respect to two unknown parameters a_2, K.

Analogously, for the purpose of computations, it is more convenient to calculate K for each fixed point $a_2 = e^{i\theta_2}$, $0 < \theta_2 < \pi/2$ and for the given P and Q. From (3.32) one obtains:

$$K = -\frac{(A - aB + 2\sqrt{2}B)Q + 2PB + C}{A + aB} \tag{3.33}$$

Where

$$A = \frac{ae^{\frac{\pi}{4}i}}{4}\int_{a_2}^{a_3}\frac{d\sigma}{X(\sigma)\sigma}, \qquad B = \frac{e^{\frac{\pi}{4}i}}{4}\int_{\bar{a}_3}^{\bar{a}_2}\frac{d\sigma}{X(\sigma)\sigma}, \qquad C = -Qe^{\pi i}4\int_{a_4}^{\bar{a}_3}\frac{d\sigma}{X(\sigma)\sigma}.$$

Remark 3.1 A solution of the problem exists if

$$A + aB \neq 0$$

By virtue of (3.31) the formula (3.30) has the form:

$$W(\zeta) = \frac{\zeta W(\zeta)}{\pi i}\int_{\gamma}\frac{f(\sigma)e^{i\alpha(\sigma)}\,d\sigma}{X(\sigma)\sigma(\sigma-\zeta)} \tag{3.34}$$

i.e.:

$$W(\zeta) = \frac{\zeta X(\zeta)}{\pi i}\left(\frac{(K+Q)a}{4}e^{\frac{\pi}{4}i}\int_{a_2}^{a_3}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\zeta)} + \left(\frac{(K-Q)a}{4} + \frac{\sqrt{2}}{2}\left(Q+\frac{\sqrt{2}}{2}P\right)\right)e^{\frac{\pi}{4}i}\int_{\bar{a}_3}^{\bar{a}_2}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\zeta)} - Qe^{\pi i}\int_{a_4}^{\bar{a}_3}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\zeta)}\right)$$

By virtue of (3.21), the equation of the contour $z = \omega(\xi)$ is presented by

$$\omega(\xi) = \frac{2W(\xi)}{K+Q}, \qquad -1 < \xi < 1. \tag{3.35}$$

Thus equation of the unknown full-strength contour and the stress state is defined.

3.1 Construction of the square's full-strength hole

Let us consider a concrete case: let the length of square side $a = 3/\sqrt{2}$. To construct the required *square*'s full-strength hole for given P, Q and fixed point $a_2 = e^{i\theta_2}$, $0 < \theta_2 < \pi/2$, at first we construct the arc $\gamma = A_4A_1$ of the required full-strength contour by (3.35) $\omega(\xi) = \frac{2W(\xi)}{K+Q}$, $-1 < \xi < 1$, where

$$W(\xi) = \frac{\xi X(\xi)}{\pi i}\left(\frac{(K+Q)a}{4}e^{\frac{\pi}{4}i}\int_{a_2}^{a_3}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\xi)}\right.$$
$$\left.+\left(\frac{(K-Q)a}{4}+\frac{\sqrt{2}}{2}\left(Q+\frac{\sqrt{2}}{2}P\right)\right)e^{\frac{\pi}{4}i}\int_{\bar{a}_3}^{\bar{a}_2}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\xi)} - Qe^{\pi i}\int_{a_4}^{\bar{a}_3}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\xi)}\right),$$
$$-1 < \xi < 1.$$

and K defined by formula (3.33)

Since the problem is cyclically symmetric, the other parts of the required full-strength hole can be obtained by rotating the graph $\omega(\xi)$, $-1 < \xi < 1$ through an angle of $\pi/2$

$$\omega_1(\xi) = \omega(\xi), \quad u(\xi) = \omega_1(\xi)e^{\frac{\pi}{2}i}, \quad h(\xi) = u(\xi)e^{\frac{\pi}{2}i}, \quad r(\xi) = h(\xi)e^{\frac{\pi}{2}i}.$$

As an illustration, some graphics of full-strength contours are presented for different parameters in Figure 3

4 Some constructions of full-strength holes for an equilateral triangle (m is an odd number)

Now let us consider the case when the regular polygon (m is an odd number) is an equilateral triangle S, which is weakened with full-strength hole. Let one of the sides of the equilateral triangle be parallel to axis Ox and its center coincide with the origin of coordinates.

Let us consider the same above mentioned posed problem for an equilateral triangle. Let to every link of the outer boundary of the equilateral triangle be applied absolutely smooth rigid punches with rectilinear bases which displace to the normal under the action of concentrated normally compressive forces P applied to the punch midpoints of the equilateral triangle's sides. There is no friction between the given elastic body and the punches. Let the edge of unknown full-strength contour be subject to uniform pressure Q. Under these assumptions, normal displacement of every link of the broken line of the outer boundary is $v_n = v = \text{const}$. Tangential stresses $\tau_{ns} = 0$ along the entire boundary of the domain S.

Similarly as above let us consider the problem (A):

Problem (A): Find the stress state of the body and the form of an unknown full-strength contour under the condition that the tangential normal stress σ_s on it takes constant value $\sigma_s = K = \text{const}$.

Since the problem is cyclic-symmetric it is sufficient to consider the curvilinear quadrangle $A_1A_2A_3A_4$ which will be denoted by D (Figure 4).

The normal displacements and the tangential stresses are equal to zero $v_n = \tau_{ns} = 0$ at each segment $[A_1A_2]$, $[A_3A_4]$.

Let $\Gamma_i = A_iA_{i+1}$, $\Gamma = \cup_{i=1}^{3}\Gamma_i$, $P_i = \int_{\Gamma_i}\sigma_n\,ds$, $i = 1,2,3$, $\gamma = A_4A_1$. According to the statement of the problem one gets: $P_2 = \int_{\Gamma_2}\sigma_n\,ds = -P$.

$$\alpha(t) = \alpha_k, \quad t \in A_kA_{k+1}, \qquad k = 1,2,3. \qquad \alpha_1 = -\frac{\pi}{3}, \quad \alpha_2 = \frac{\pi}{2}, \quad \alpha_3 = \frac{4\pi}{3}. \tag{4.1}$$

Proceeding from the equilibrium state of D we have the following expressions:

$$P_1 = P_3 = \frac{P_2 - Q}{\sqrt{3}} = -\frac{P+Q}{\sqrt{3}}.$$

From the statement of the problem it follows that the boundary conditions have the form

$$v_n = \begin{cases} 0, & t \in \Gamma_1 \cup \Gamma_3, \\ v, & t \in \Gamma_2 \end{cases} \tag{4.2}$$

$$\tau_{ns} = 0, \qquad t \in \Gamma \cup \gamma, \tag{4.3}$$

$$\sigma_n = -Q, \quad \sigma_s = K, \qquad t \in \gamma, \tag{4.4}$$

$$P_2 = -P, \quad P_1 = P_3 = -\frac{P+Q}{\sqrt{3}}. \tag{4.5}$$

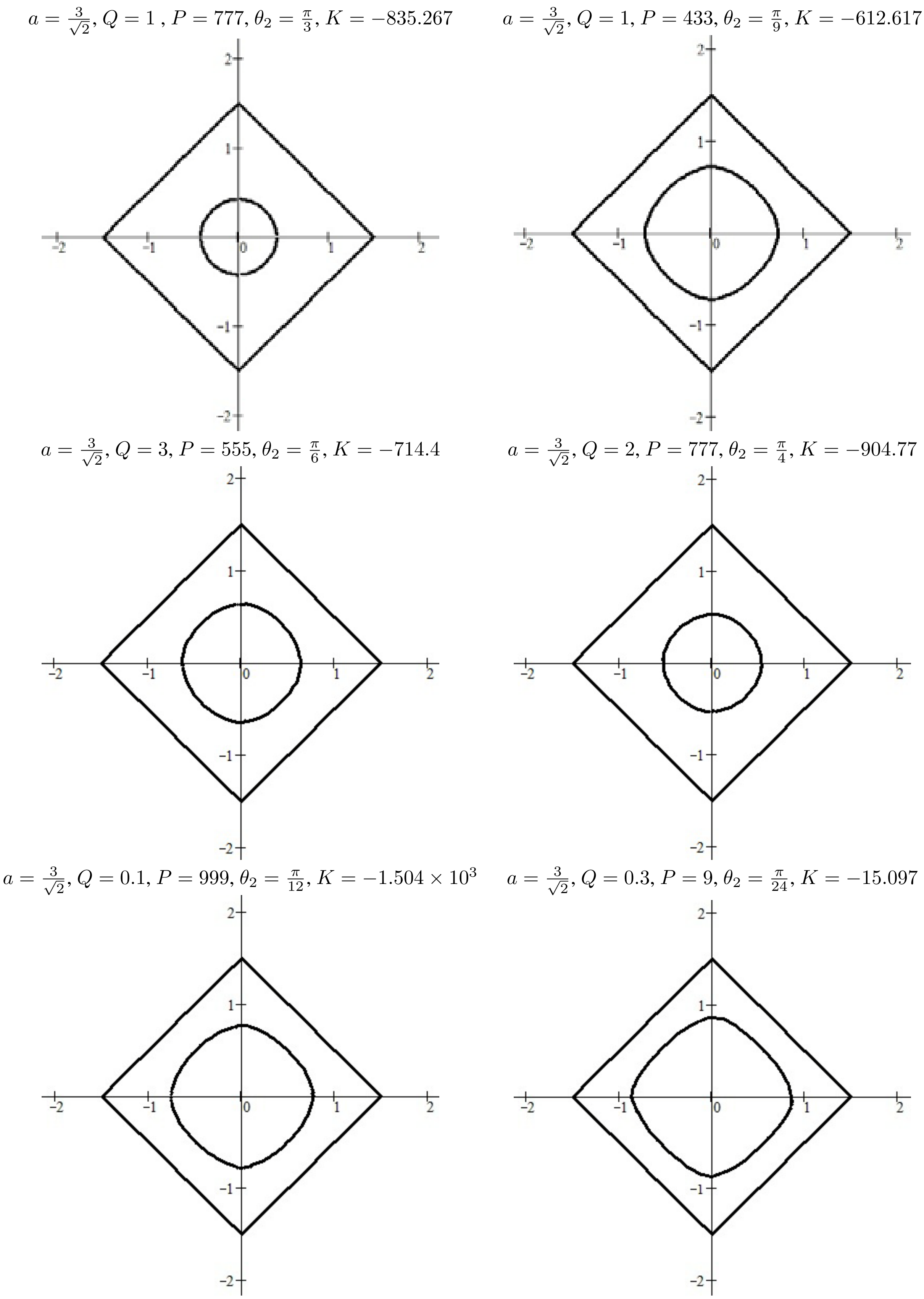

Fig. 3 Graphics of full-strength contours for different parameters

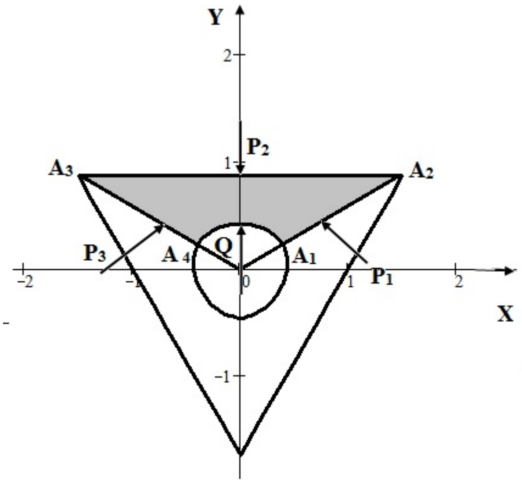

Fig. 4 Construction of full-strength hole for a triangle

Analogously as above, the solution of problem (4.2)- (4.5) reduces to the problem of finding the functions ψ, φ which are holomorphic in the domain D with the following conditions

$$\operatorname{Re} e^{-i\alpha(t)}\left(\chi\varphi(t) - t\,\overline{\varphi'(t)} - \overline{\psi(t)}\right) = 2\mu v_n(t), \qquad t \in \Gamma, \tag{4.6}$$

$$\operatorname{Re} e^{-i\alpha(t)}\left(\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)}\right) = C(t), \qquad t \in \Gamma, \tag{4.7}$$

$$\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)} = -Qt, \qquad t \in \gamma, \tag{4.8}$$

$$\operatorname{Re}\varphi'(t) = \frac{\sigma_n + \sigma_s}{4} = \frac{-Q + K}{4}, \qquad t \in \gamma. \tag{4.9}$$

By virtue of (4.1) and (4.5) one gets:

$$C(t) = \begin{cases} 0, & t \in \Gamma_1, \\ -\frac{P+Q}{2\sqrt{3}}, & t \in \Gamma_2 \\ \frac{Q}{2}, & t \in \Gamma_3 \end{cases} \tag{4.10}$$

If $t \in A_kA_{k+1}$, $k = 1, 2, 3$ then $t - A_k = i\rho e^{i\alpha_k}$, $\rho = |t - A_k|$. Hence one obtains

$$\operatorname{Re} te^{-i\alpha(t)} = \operatorname{Re} A(t)e^{-i\alpha(t)}, \tag{4.11}$$

where $A(t)$ is a piecewise constant function, $A(t) = A_k$, $t \in A_kA_{k+1}$, $k = 1, 2, 3$.

By virtue of (4.1), (4.11)

$$\operatorname{Re} A(t)e^{-i\alpha(t)} = \begin{cases} \frac{a}{2\sqrt{3}}, & t \in \Gamma_2, \\ 0, & t \in \Gamma_1 \cup \Gamma_3, \end{cases} \tag{4.12}$$

where a is the length of the equilateral triangle side.

Analogously one gets

$$\varphi(z) = \frac{K - Q}{4} z \tag{4.13}$$

where the constant was neglect.

Substituting the value $\varphi(t)$ defined by formula (4.13) into the boundary conditions (4.7)-(4.8) and taking into account (4.10)-(4.11), one gets the following problem

$$\operatorname{Re} e^{-i\alpha(t)}\left(\frac{K-Q}{2}t+\overline{\psi(t)}\right)=C(t)=\begin{cases}0, & t\in\Gamma_1,\\ -\frac{P+Q}{2\sqrt{3}}, & t\in\Gamma_2,\\ \frac{Q}{2}, & t\in\Gamma_3,\end{cases} \tag{4.14}$$

$$\operatorname{Re} te^{-i\alpha(t)}=\operatorname{Re} A(t)e^{-i\alpha(t)}, \qquad t\in\Gamma, \tag{4.15}$$

$$\frac{K+Q}{2}t+\overline{\psi(t)}=0, \qquad t\in\gamma. \tag{4.16}$$

Let the function $z=\omega(\zeta)$, $\zeta=\xi+i\eta$ map the semicircle $|\zeta|<1$, $\operatorname{Im}\zeta>0$ conformally onto the domain D. It is assumed that vertices A_k of broken line Γ correspond the points a_k of semicircle $|\zeta|=1$, $\operatorname{Im}\zeta>0$, $a_k=\omega^{-1}(A_k)$, $k=1,2,3,4$. It is assumed, that $a_1=1$, $a_4=-1$, $a_2=e^{i\theta_2}$, $a_3=e^{i(\pi-\theta_2)}$, $0<\theta_2<\pi/2$. The $-1\le\xi\le1$ diameter is mapped onto the arc A_4A_1 and semi-circumference γ_0: $|\gamma_0|=1$, $\operatorname{Im}\zeta>0$ is mapped onto the broken line Γ. The points $a_2=e^{i\theta_2}$, $a_3=e^{i(\pi-\theta_2)}$, $0<\theta_2<\pi/2$ are to be defined. Hence, by virtue of (4.14)-(4.16) for functions $\psi_0(\zeta)=\psi(\omega(\zeta))$ and $\omega(\zeta)$ one obtains:

$$\operatorname{Re} e^{-i\alpha(\sigma)}\overline{\psi_0(\sigma)}=-\frac{K-Q}{2}\operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma)+C(\sigma), \qquad \sigma\in\gamma_0 \tag{4.17}$$

$$\operatorname{Re} e^{-i\alpha(\sigma)}\omega(\sigma)=\operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma), \qquad \sigma\in\gamma_0, \tag{4.18}$$

$$\frac{K+Q}{2}\omega(\sigma)+\overline{\psi_0(\sigma)}=0, \qquad \sigma\in(-1,1). \tag{4.19}$$

For simplicity the piecewise constant functions $\alpha(\omega(\sigma))$, $A(\omega(\sigma))$ will again be denoted by $\alpha(\sigma)$, $A(\sigma)$.

Let us introduce a piecewise-holomorphic function $W(\zeta)$ in the circle $|\zeta|<1$.

$$W(\zeta)=\begin{cases}\frac{K+Q}{2}\omega(\zeta), & |\zeta|<1, \quad \operatorname{Im}\zeta>0,\\ -\overline{\psi_0(\bar\zeta)}, & |\zeta|<1, \quad \operatorname{Im}\zeta<0.\end{cases} \tag{4.20}$$

Similarly, as above, it easy to verify that $W(\zeta)$ is a holomorphic function into the circle $|\zeta|<1$.

By virtue of (4.17)-(4.19) the function $W(\zeta)$ defined by (4.20) satisfies the boundary conditions:

$$\operatorname{Re} e^{-i\alpha(\sigma)}W(\sigma)=\frac{K+Q}{2}\operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma), \qquad \sigma\in\gamma_0, \tag{4.21}$$

$$\operatorname{Re} e^{-i\alpha(\bar\sigma)}W(\sigma)=\frac{K-Q}{2}\operatorname{Re} e^{-i\alpha(\bar\sigma)}A(\sigma)-C(\sigma), \qquad \sigma\in\gamma_0^*. \tag{4.22}$$

where γ_0^* is mirror image of γ_0 about the Ox axis.

The conditions (4.21), (4.22) may be rewritten in the following form

$$\operatorname{Re} e^{-i\alpha(\sigma)}W(\sigma)=f(\sigma), \qquad \sigma\in\gamma' \tag{4.23}$$

where $\gamma'=\gamma_0\cup\gamma_0^*$, $\alpha(\sigma)=\alpha(\bar\sigma)$, $\sigma\in\gamma_0^*$, $\alpha_0=-\pi/3$, $\alpha_1=\pi/2$, $\alpha_2=4\pi/3$.

If $\sigma\in\gamma_0$, then

$$f(\sigma)=\frac{K+Q}{2}\operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma)=\begin{cases}0, & \sigma\in(a_1,a_2),\\ \frac{K+Q}{4\sqrt{3}}a, & \sigma\in(a_2,a_3),\\ 0, & \sigma\in(a_3,a_4),\end{cases} \tag{4.24}$$

where a is the length of the equilateral triangle side.

If $\sigma\in\gamma_0^*$, then

$$f(\sigma)=\frac{K-Q}{2}\operatorname{Re} e^{-i\alpha(\sigma)}A(\sigma)-C(\sigma)=\begin{cases}-\frac{Q}{2}, & \sigma\in(a_4,\bar a_3),\\ \frac{K-Q}{4\sqrt{3}}a+\frac{P+Q}{2\sqrt{3}}, & \sigma\in(\bar a_3,\bar a_2),\\ 0, & \sigma\in(\bar a_2,a_1^*).\end{cases} \tag{4.25}$$

where $a_1^* = e^{2\pi i}$.

Thus, our problem is reduced to Riemann-Hilbert problem with piecewise constant coefficients. Similarly the solution of the problem is reduced to the Dirichlet problem for circle and its solution is presented by Schwartz formula.

Let function $e^{2\pi i}$ be presented in the following form

$$e^{2i\alpha(\sigma)} = \frac{X(\sigma)}{\overline{X(\sigma)}}, \qquad |\sigma| = 1, \tag{4.26}$$

where $X(\zeta)$ is presented as

$$X(\zeta) = \left(\frac{\zeta - a_2}{\zeta - a_1}\right)^{-\frac{1}{3}} \left(\frac{\zeta - a_3}{\zeta - a_2}\right)^{\frac{1}{2}} \left(\frac{\zeta - a_4}{\zeta - a_3}\right)^{\frac{4}{3}} \left(\frac{\zeta - \bar{a}_3}{\zeta - a_4}\right)^{\frac{4}{3}} \left(\frac{\zeta - \bar{a}_2}{\zeta - \bar{a}_3}\right)^{\frac{1}{2}} \left(\frac{\zeta - a_1^*}{\zeta - \bar{a}_2}\right)^{\frac{1}{3}} \cdot \frac{(\zeta - a_2)(\zeta - a_3)}{\zeta} \sqrt{\bar{a}_2 \bar{a}_3} \tag{4.27}$$

By virtue of conditions (4.26), the condition (4.23) will have the form

$$\frac{W(\sigma)}{X(\sigma)} + \frac{\overline{W(\sigma)}}{\overline{X(\sigma)}} = \frac{2fe^{i\alpha(\sigma)}}{X(\sigma)}. \tag{4.28}$$

The condition (4.28) presents the boundary condition of Dirihlet problem, whose solution is presented by Schwartz formula

$$\frac{W(\zeta)}{X(\zeta)} = \frac{1}{2\pi i} \int_\gamma \frac{f(\sigma)e^{i\alpha(\sigma)}(\sigma + \zeta)\, d\sigma}{X(\sigma)\sigma(\sigma - \zeta)}. \tag{4.29}$$

Since the function $X(\zeta)$ has pole of the first order at the point $\zeta = 0$, then the function $W(\zeta)/X(\zeta)$ has the first order zero at this point. Hence from (4.29) one holds:

$$\int_\gamma \frac{f(\sigma)e^{i\alpha(\sigma)}\, d\sigma}{X(\sigma)\sigma} = 0. \tag{4.30}$$

By virtue of (4.24), (4.25) and (4.1) one gets:

$$e^{\frac{\pi}{2}i} \frac{K+Q}{4\sqrt{3}} a \int_{a_2}^{a_3} \frac{d\sigma}{X(\sigma)\sigma} - e^{\frac{4\pi}{3}i} \frac{Q}{2} \int_{a_4}^{\bar{a}_3} \frac{d\sigma}{X(\sigma)\sigma} + e^{\frac{\pi}{2}i} \left(\frac{(K-Q)a}{4\sqrt{3}} + \frac{P+Q}{2\sqrt{3}}\right) \int_{\bar{a}_3}^{\bar{a}_2} \frac{d\sigma}{X(\sigma)\sigma} = 0 \tag{4.31}$$

Thus we obtain equation with respect to two unknown parameters θ_2, K ($a_2 = e^{i\theta_2}$, $a_3 = e^{i(\pi - \theta_2)}$).

Taking into account (4.30), the formula (4.29) will have the form

$$W(\zeta) = \frac{\zeta X(\zeta)}{\pi i} \int_\gamma \frac{f(\sigma)e^{i\alpha(\sigma)}\, d\sigma}{X(\sigma)\sigma(\sigma - \zeta)} \tag{4.32}$$

That means

$$W(\zeta) = \frac{\zeta X(\zeta)}{\pi i} \left(e^{\frac{\pi}{2}i} \frac{K+Q}{4\sqrt{3}} a \int_{a_2}^{a_3} \frac{d\sigma}{X(\sigma)\sigma(\sigma - \zeta)} - e^{\frac{4\pi}{3}i} \frac{Q}{2} \int_{a_4}^{\bar{a}_3} \frac{d\sigma}{X(\sigma)\sigma(\sigma - \zeta)} + e^{\frac{\pi}{2}i} \left(\frac{(K-Q)a}{4\sqrt{3}} + \frac{P+Q}{2\sqrt{3}}\right) \int_{\bar{a}_3}^{\bar{a}_2} \frac{d\sigma}{X(\sigma)\sigma(\sigma - \zeta)} \right) \tag{4.33}$$

From relationship (4.20) functions ω, ψ are defined. Thus the equation of the required contour and the stress state of body are obtained.

Remark 4.1 Let us introduce the following notations

$$A = \frac{e^{\frac{\pi}{2}i}}{4\sqrt{3}} a \int_{a_2}^{a_3} \frac{d\sigma}{X(\sigma)\sigma}, \quad B = \frac{e^{\frac{\pi}{2}i}}{4\sqrt{3}} \int_{\bar{a}_3}^{\bar{a}_2} \frac{d\sigma}{X(\sigma)\sigma}, \quad C = -e^{\frac{4\pi}{3}i} \frac{Q}{2} \int_{a_4}^{\bar{a}_3} \frac{d\sigma}{X(\sigma)\sigma} \tag{4.34}$$

From equality (4.31) one obtains

$$K = -\frac{Q(A - aB + 2B) + C + 2PB}{A + aB}. \tag{4.35}$$

The solution of the problem exists if

$$A + aB \neq 0 \tag{4.36}$$

4.1 Construction of full-strength hole for an equilateral triangle

Let us consider a concrete case: let the length of an equilateral triangle side be $a = 3$. To construct the required *full-strength hole for an equilateral triangle* for given P, Q and fixed θ_2, $0 < \theta_2 < \pi/2$, at first we construct as above the arc A_4A_1 by $\omega(\xi) = \frac{2W(\xi)}{K+Q}$, $-1 < \xi < 1$, where $W(\xi)$ is defined by formula (4.33):

$$W(\xi) = \frac{\xi W(\xi)}{\pi i}\left(e^{\frac{\pi}{2}i}\frac{K+Q}{4\sqrt{3}}a\int_{a_2}^{a_3}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\zeta)} - e^{\frac{4\pi}{3}i}\frac{Q}{2}\int_{a_4}^{\bar{a}_3}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\zeta)}\right.$$
$$\left. + e^{\frac{\pi}{2}i}\left(\frac{(K-Q)a}{4\sqrt{3}} + \frac{P+Q}{2\sqrt{3}}\int_{\bar{a}_3}^{\bar{a}_2}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\zeta)}\right)\right), \qquad -1 < \xi < 1$$

and K is defined by (4.35).

Piecewise constant functions $\alpha(t) = a_k$, $t \in A_kA_{k+1}$, $k = 1, 2, 3$, $\alpha_1 = -\pi/3$.

Since the problem is cycle symmetric, the others parts of this hole are constructed by its rotation with angle $2\pi/3$. For simplicity the coordinate origin is made to coincide with the center O of an equilateral triangle.

$$\omega_1(\xi) = \omega(\xi), \quad u(\xi) = \omega_1(\xi)e^{\frac{2\pi}{3}i}, \quad h(\xi) = u(\xi)e^{\frac{2\pi}{3}i}.$$

As an illustration, some graphics of full-strength contours are presented for the different parameters in Figure 5.

5 Conclusion

The boundary value problems of the plane theory of elasticity and plate bending with partially unknown boundaries (the problems of finding a *full-strength contour*) are closely related to very important problems of optimal projecting. Their solution ensures optimal distribution of stresses on the hole boundary by means of appropriate inspection of its form and location. The solution of these problems provides the rise of the construction strength and minimization of its weight.

The most effective methods for studying this problem are those of the theory of analytical functions of complex variable. This problem is both of mechanics and of geometry since the shape of a hole of a plate is required and the conformal mapping function is used to define it. Thus, formulas of Kolosov-Muskhelishvili are used for investigating this problem. Using the conformal mapping, the investigation of the problem is reduced to a Riemann-Hilbert problem. The solution is written in quadratures and the graphs of full-strength contours are constructed.

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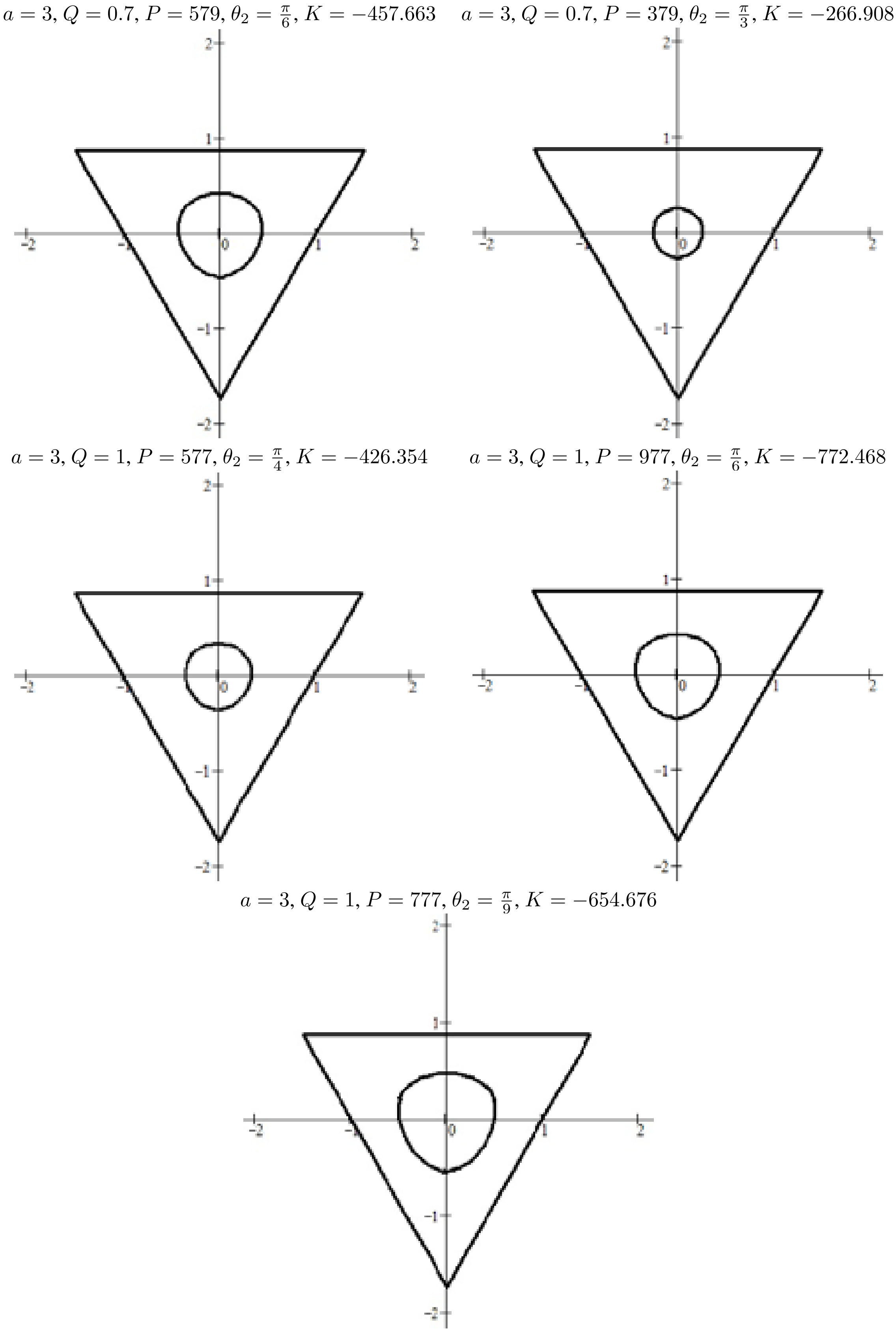

Fig. 5 Graphics of full-strength contours for different parameters

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