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On the Use of the L_p Distance in Reference Point-based Approaches for Multiobjective Optimization

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Abstract Reference point-based methods are very useful techniques for solving multiobjective optimization problems. In these methods, the most commonly used achievement scalarizing functions are based on the Tchebychev distance (minmax approach), which generates every Pareto optimal solution in any multiobjective optimization problem, but does not allow compensation among the deviations to the reference values given that it minimizes the value of the highest deviation. At the same time, for any $1 \leq p \leq \infty$, compromise programming minimizes the L_p distance to the ideal objective vector from the feasible objective region. Although the ideal objective vector can be replaced by a reference point, achievable reference points are not supported by this approach, and special care must be taken in the unachievable case. In this paper, for $1 \leq p < \infty$, we propose a new scheme based on the L_p distance, in which different single-objective optimization problems are designed and solved depending on the achievability of the reference point. The formulation proposed allows different compensation degrees among the deviations to the reference values. It is proven that, in the achievable case, any optimal solution obtained is efficient, and, in the unachievable one, it is at least weakly efficient, although it is assured to be efficient if an augmentation term is added to the new formulation. Besides, we suggest an interactive algorithm where the new formulation is embedded. Finally, we show the empirical advantages of the new formulation by its application to both numerical problems and a real multiobjective optimization problem, for achievable and unachievable reference points.

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1 Introduction

In general, *optimization* consists in finding a solution that best fits one or several criteria at the same time, within a set of feasible alternatives. When both the problem's criteria and the constraints that determine the feasible set of alternatives can be mathematically expressed by functions, we refer to these problems as *multiobjective optimization problems*. Unfortunately, finding a single solution optimizing all the criteria at the same time is impossible in most cases, because of the degree of conflict among the criteria. In this regard, the so-called *efficient* or *Pareto optimal* solutions are very useful.

There are many methods reported in the literature for solving multiobjective optimization problems [5,6,9,14]. Usually, they are classified into a priori, a posteriori and interactive methods, depending on the role of the decision maker (DM) in the solution process. In *interactive methods*, the idea is the gradual incorporation of the DM's preferences into the solution process, in an interactive and iterative way (see [9,14] for further details).

The preference information can be expressed in several ways [9,14] of which one of the most intuitive is given by the so-called *reference point*, which consists of desirable reference or aspiration values that the DM wants to achieve on each objective function. The reference point preferential scheme is used in many multiobjective optimization methods, and most of these methods minimize an achievement scalarizing function over the feasible set in order to find a Pareto optimal solution. The majority of the achievement scalarizing functions used are extensions of the Tchebychev distance [15,21–23], in which the maximum deviation to the reference point is minimized. These functions are able to support every Pareto optimal solution for any kind of multiobjective optimization problem. However, there are other distances that can be considered for formulating achievement scalarizing functions. For example, an additive achievement scalarizing function based on the L_1 distance (linear compensation among the deviations) was proposed in [18], which also enables the generation of any Pareto optimal solution, including unsupported solutions (see [20] for a definition of unsupported solutions). This function only aggregates the positive deviations to the reference point, in other words the deviations to the reference values which can be improved.

On the one side, minimizing an achievement scalarizing function based on the L_∞ distance, as the one proposed by Wierzbicki in [21], means minimizing the maximum deviation, which can be positive or negative, to the reference point. On the other side, by minimizing the achievement scalarizing function suggested in [18], which extends the idea of the L_1 distance, we minimize the sum of the positive deviations to the reference values. That way, the former does not allow balancing among the individual deviations to the reference values, and the latter compensates for the positive deviations linearly. But between the global compensation for the deviations (L_1 distance) and no compensation (L_∞ distance), there are different compensation degrees[1] given by the other L_p distances, for $1 < p < \infty$, which can be used for formulating new achievement scalarizing functions.

The L_p distances came into use in the early seventies through *compromise programming* [24–27]. Assuming that the DM seeks a solution as close as possible to the ideal objective vector, compromise programming determines the Pareto optimal solution that minimizes the L_p distance (for $1 \leq p \leq \infty$) between the feasible objective region and the ideal objective vector. In compromise programming, a reference point can be used instead of the ideal objective vector. However, the reference point must be carefully selected. On the one hand, achievable reference points must be avoided given that, in this case, the objective vector which minimizes the L_p distance to the reference point is either feasible or the reference point itself, so it is a dominated objective vector. On the other hand, when the reference point is unachievable, the minimization of the L_p distance can fail in finding the closest nondominated objective vector to the reference point in some situations. For unachievable reference points, there is no objective vector that improves all the reference values at the same time, but there may exist an objective vector which achieves or improves some of the reference values (but not all of them). In this situation, the minimization of the L_p distance may produce undesirable effects because these distances are defined using the absolute value of the deviations to the reference values, and allow compensation between positive and negative deviations.

[1] By *compensation degrees between the deviations to the reference values* we refer to the effect that is obtained by aggregating the power p of all the individual deviations to the reference values. In the L_p distance, the higher the value of p, the more importance is given to the higher deviation (in absolute value).

In this paper, we propose a new formulation, called *extended formulation*, for generating Pareto optimal solutions in reference point-based methods, including those that are unsupported, which allows certain compensation degrees between the positive deviations to the reference values, for $1 \leq p < \infty$. The new formulation combines the strengths of the L_p distances and the advantages of achievement scalarizing functions. To achieve our purpose, the formulation of compromise programming has been adapted in order, firstly, to enable the use of achievable reference points, and, secondly, to overcome the weaknesses of the L_p distance when the reference point is unachievable. The new scheme proposed can be considered as an extension of compromise programming in the sense that it allows the use of any reference point instead of the ideal objective vector. But, it can be also understood as an extension of the achievement scalarizing function proposed in [18] which fills the gap existing between this function (based on the L_1 distance) and the achievement scalarizing functions based on the L_∞ distance.

The remainder of this paper is organized as follows. In Section 2, the main concepts and notations used in multiobjective optimization are described. The extended formulation proposed for reference point-based approaches is defined and analysed in Section 3. Subsequently, the performance of the new formulation is illustrated with numerical examples for two and three objective problems in Section 4, followed by its application to an empirical optimization problem -with seven objectives-, based on real data, in Section 5. Finally, the main contributions of this paper are described in Section 6.

2 Background Concepts of Multiobjective Optimization

Let us consider a *multiobjective optimization problem* which maximizes k (≥ 2) conflicting *objective functions* $f_i : S \rightarrow \mathbf{R}$ simultaneously, defined as follows:

$$\begin{array}{ll} \text{maximize} & \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})) \\ \text{subject to} & \mathbf{x} \in S. \end{array} \tag{1}$$

Let us suppose that the *feasible set* $S \subset \mathbf{R}^n$ of *decision variables* $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is nonempty and compact. The image of the feasible set in the *objective space* $\mathbf{R}^k$ is called the *feasible objective region* $Z = \mathbf{f}(S)$ and it consists of *objective vectors* $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x}))^T$, formed by the objective values $f_i(\mathbf{x})$, for $i = 1, \ldots, k$ and $\mathbf{x} \in S$.

In multiobjective optimization, a solution $\mathbf{x}' \in S$ is said to be *efficient* or *Pareto optimal* if there does not exist another $\mathbf{x} \in S$ such that $f_i(\mathbf{x}) \geq f_i(\mathbf{x}')$ for all $i = 1, \ldots, k$, and $f_j(\mathbf{x}) > f_j(\mathbf{x}')$ for at least one index j. Alternatively, given $\mathbf{z}, \mathbf{z}' \in Z$, we say that $\mathbf{z}$ *dominates* $\mathbf{z}'$ if and only if $z_i \geq z'_i$ for all $i = 1, \ldots, k$ and $z_j > z'_j$ for at least one index j. Then, we can also define Pareto optimal solutions as those $\mathbf{x}' \in S$ for which there is no $\mathbf{x} \in S$ such that $\mathbf{f}(\mathbf{x})$ dominates $\mathbf{f}(\mathbf{x}')$. On the other hand, a solution $\mathbf{x}' \in S$ is said to be *weakly efficient* or *weakly Pareto optimal* if there does not exist another $\mathbf{x} \in S$ such that $f_i(\mathbf{x}) > f_i(\mathbf{x}')$ for all $i = 1, \ldots, k$. The corresponding objective vector $\mathbf{f}(\mathbf{x})$ is called *(weakly) nondominated objective vector*. The set of all (weakly) Pareto optimal solutions is called the *(weakly) efficient* or *(weakly) Pareto optimal set*, and all the (weakly) nondominated objective vectors form the *(weakly) nondominated objective set*. The efficient set is denoted by E and the nondominated objective set by $\mathbf{f}(E)$. It is noteworthy that the Pareto optimal set is a subset of the set of weakly Pareto optimal solutions.

In this paper, we assume that the nondominated objective set of problem (1) contains more than one vector. Usually, it is useful to know the so-called ideal and nadir objective vectors, because they represent the best and the worst values that each objective function can reach in the Pareto optimal set, respectively. The *ideal objective vector* $\mathbf{z}^\star = (z_1^\star, \ldots, z_k^\star)^T \in \mathbf{R}^k$ gives upper bounds for the objectives and is obtained by maximizing each objective function over the feasible set, that is, $z_i^\star = \max_{\mathbf{x} \in S} f_i(\mathbf{x}) = \max_{\mathbf{x} \in E} f_i(\mathbf{x})$ for all $i = 1, \ldots, k$. Sometimes, a vector which is strictly better than the ideal objective vector is needed. This vector is called an *utopian objective vector*, denoted by $\mathbf{z}^{\star\star} = (z_1^{\star\star}, \ldots, z_k^{\star\star})^T$, and it can be obtained as $z_i^{\star\star} = z_i^\star + \varepsilon_i$, where $\varepsilon_i > 0$ is a small real number, for every $i = 1, \ldots, k$. The *nadir objective vector* $\mathbf{z}^{\text{nad}} = (z_1^{\text{nad}}, \ldots, z_k^{\text{nad}})^T$ contains lower objective function bounds and it is defined as $z_i^{\text{nad}} = \min_{\mathbf{x} \in E} f_i(\mathbf{x})$ for all $i = 1, \ldots, k$. In general, the nadir objective vector is difficult to calculate and, it is commonly estimated [1–3,7]. One possibility is that the DM specifies the worst possible objective function values that one could consider, and use them as components of the nadir objective vector. Another alternative may be to approximate it using a pay-off table although, with this, the nadir objective vector may be overestimated (see [14] and references therein).

From the practical point of view, we need a DM to identify the most preferred among all Pareto optimal solutions. We assume that the DM can express preference information related to the conflicting objectives and that more is preferred to less on each objective. Different types of preference information asked from the DM can be found in [9,12,14] and references therein. Here, we suppose that the DM specifies her/his preferences in the form of a reference point. A *reference point* is an objective vector $\mathbf{q} = (q_1, \ldots, q_k)^T$, where q_i is a desirable *reference value* or *reference level* for the objective function f_i provided by the DM, for every $i = 1, \ldots, k$. Usually, a reference point is said to be *achievable* if there exists a feasible solution $\mathbf{x} \in S$ such that $f_i(\mathbf{x}) \geq q_i$ for every $i = 1, \ldots, k$; otherwise, the reference point is said to be *unachievable*.

When solving multiobjective optimization problems, the preferential scheme based on reference points is often associated with the use of achievement scalarizing functions. An *achievement scalarizing function* is a real-valued function which combines the original objective functions and the DM's preferences. It is specially formulated to generate (weakly) efficient solutions for the original problem (1) when it is minimized over the feasible set [23]. Further details about achievement scalarizing functions can be found in [15]. Most of the achievement scalarizing functions used are based on the L_∞ distance [11,15]. One of them was proposed by Wierzbicki in 1980 [21] which, for a given reference point $\mathbf{q} = (q_1, \ldots, q_k)^T$ and a vector of positive weights $\mathbf{w} = (w_1, \ldots, w_k)^T$, is given by:

$$s(\mathbf{q}, \mathbf{f}(\mathbf{x}), \mathbf{w}) = \max_{i=1,\ldots,k} \{\, w_i(q_i - f_i(\mathbf{x}))\}, \tag{2}$$

which must be minimized over S:

$$\begin{array}{ll} \text{minimize} & s(\mathbf{q}, \mathbf{f}(\mathbf{x}), \mathbf{w}) \\ \text{subject to} & \mathbf{x} \in S. \end{array} \tag{3}$$

The weights $w_i > 0$ assigned to the objective functions can have a purely normalizing meaning or they can be set as preferential parameters [10,17]. Apart from the weights, (2) differs from the formulation of the L_∞ distance in that the absolute value operator is removed so that the negative values of $q_i - f_i(\mathbf{x})$ are not penalized (that is, values in which $f_i(\mathbf{x})$ improves q_i). In practice, if the reference point used is unachievable, problem (3) is equivalent to the minmax distance, where the maximum deviation to the reference point is minimized. If the reference point is achievable, the objective values at the solution of problem (3) improve the given reference levels "as much as possible". It is assured that the solution of problem (3) is a weakly Pareto optimal solution of the original problem (1), and it is Pareto optimal if it is unique [14]. In order to secure that only Pareto optimal solutions with bounded trade-offs between objectives (the so-called *properly efficient solutions*) are generated by solving (3), we can add a so-called *augmentation term* to the function given in (2). This is shown in (4) for a small positive value ρ [14]:

$$\bar{s}(\mathbf{q}, \mathbf{f}(\mathbf{x}), \mathbf{w}) = \max_{i=1,\ldots,k} \{\, w_i(q_i - f_i(\mathbf{x}))\} + \rho \sum_{i=1}^{k} w_i \left(q_i - f_i(\mathbf{x})\right), \tag{4}$$

which must also be minimized over S:

$$\begin{array}{ll} \text{minimize} & \bar{s}(\mathbf{q}, \mathbf{f}(\mathbf{x}), \mathbf{w}) \\ \text{subject to} & \mathbf{x} \in S. \end{array} \tag{5}$$

In [18], an additive achievement scalarizing function based on the L_1 distance is suggested. For a reference point $\mathbf{q} = (q_1, \ldots, q_k)^T$ which is not dominated by the objective vector of any feasible solution (unachievable case), and a vector of positive weights $\mathbf{w} = (w_1, \ldots, w_k)^T$, the formulation of the proposed function is given by:

$$\tilde{s}_1(\mathbf{q}, \mathbf{f}(\mathbf{x}), \mathbf{w}) = \sum_{\{i: q_i \geq f_i(x)\}} w_i(q_i - f_i(\mathbf{x})) = \sum_{i=1}^{k} \max\{\, w_i(q_i - f_i(\mathbf{x})), 0\}, \tag{6}$$

which must also be minimized over S:

$$\begin{array}{ll} \text{minimize} & \tilde{s}_1(\mathbf{q}, \mathbf{f}(\mathbf{x}), \mathbf{w}) \\ \text{subject to} & \mathbf{x} \in S. \end{array} \tag{7}$$

It is clear that (6) only considers positive deviations to the reference values in order to avoid penalizing both positive and negative ones, which has no meaning unless the DM wants to reach the reference point exactly. When $\mathbf{q}$ is dominated by the objective vector of a feasible solution (achievable case), [18] proposes to solve the following problem:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{k} w_i \cdot f_i(\mathbf{x}) \\ \text{subject to} & f_i(\mathbf{x}) \geq q_i \quad \text{for every } i = 1, \dots, k, \\ & \mathbf{x} \in S. \end{array} \tag{8}$$

Assuming that the DM searches for a Pareto optimal solution as close as possible to the ideal objective vector, *compromise programming* is the approach which minimizes the L_p distance between the ideal objective vector and the objective space, for $1 \leq p \leq \infty$. Accordingly, for a vector of weights $\mathbf{w} = (w_1, \dots, w_k)^T$, with $w_i \geq 0$ for all $i = 1, \dots, k$, the *(weighted) compromise programming problem* (see [24, 27]) is formulated as follows for $1 \leq p < \infty$:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{k} w_i \left| \frac{z_i^\star - f_i(\mathbf{x})}{z_i^\star - z_i^{\text{nad}}} \right|^p \\ \text{subject to} & \mathbf{x} \in S. \end{array} \tag{9}$$

and for $p = \infty$:

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\dots,k} \left\{ w_i \left| \frac{z_i^\star - f_i(\mathbf{x})}{z_i^\star - z_i^{\text{nad}}} \right| \right\} \\ \text{subject to} & \mathbf{x} \in S. \end{array} \tag{10}$$

It can be observed that problems (9) and (10) have been defined using relative deviations from the ideal objective vector, instead of the deviations themselves. This normalization is done in order to avoid problems derived from adding objective values on very different scales. In this paper, we assume that $z_i^\star - z_i^{\text{nad}} > 0$ for all $i = 1, \dots, k$. If this is not true, a utopian objective vector can be used instead of the ideal objective vector.

Besides, for $1 \leq p < \infty$, the exponent $1/p$ has been dropped from the L_p distance in (9), since problems with and without it have the same optimal solution or solutions. Likewise, problems (9) and (10) introduce different weights to the objective functions, but the weights can be removed if so desired, as the original formulation of compromise programming did not incorporate them [24, 26]. It is proven in [24, 26] that the solution of problem (9) is a Pareto optimal solution of problem (1) if $w_i > 0$ for all $i = 1, \dots, k$, or if it is unique. Also, the solution of problem (10) is a weakly Pareto optimal solution of problem (1) if $w_i > 0$ for all $i = 1, \dots, k$, and it is Pareto optimal if it is unique. More information about compromise programming can be found in [24–27].

3 An Extended Formulation for Reference Point-based Approaches

Between the achievement scalarizing function based on the L_1 distance (linear compensation among the deviations) and the functions based on the L_∞ distance (no compensation among the deviations), there are different compensation degrees that can be considered for formulating new schemes which extend the idea of the L_p distances, for $1 < p < \infty$. As explained in [18], using the achievement scalarizing function based on the L_1 distance may be convenient when the DM wants to achieve certain reference values for the objectives (e.g., when dealing with monetary objectives) and if (s)he is interested in minimizing the aggregated deviation to only these reference values which are not achieved (e.g., if the reference point represents a budget limit, with (6), we minimize the aggregation of only the positive budgetary deviations to the desired values, which means in practice improving only the objectives which exceed the budget limits). Similarly, new formulations based on the L_p distances may be useful in multiobjective optimization problems when, e.g., the DM wants, or if it is necessary, to minimize the Euclidean distance (L_2 distance) to a reference point, such as in transport network design problems, locations problems, etc.

In compromise programming, replacing the ideal objective vector by a reference point can be useful whenever the DM wants to introduce additional preferential information into the solution process. But one must be extremely careful in selecting this reference point because of the meaning of L_p distance in practice. If the reference point considered is achievable, the solutions of problems (9) and (10) are the reference point itself, or the closest feasible solution to it, regarding the L_p distance used. Consequently, in the achievable case, the minimization of the L_p distance from the reference point does not find an efficient solution of problem (1). Besides, in the unachievable case, there are some situations in which the minimization of the L_p distance from the reference point may emphasize dominated objective vectors over nondominated ones. For example, let $\mathbf{q} = (0.9, 0.8)^T$ be an unachievable reference point of a biobjective optimization problem, the reference value $q_2 = 0.8$ of which is improved by the objective vectors $\mathbf{z}^1 = (0.2, 0.81)^T$ and $\mathbf{z}^2 = (0.3, 0.95)^T$. For $p = 1$, $L_1(\mathbf{q}, \mathbf{z}^1) = |0.7| + |-0.01| = 0.71$ and $L_1(\mathbf{q}, \mathbf{z}^2) = |0.6| + |-0.15| = 0.75$, so $L_1(\mathbf{q}, \mathbf{z}^1) < L_1(\mathbf{q}, \mathbf{z}^2)$ and then, $\mathbf{z}^1$ would be closer to the reference point than $\mathbf{z}^2$ regarding the L_1 distance. However, it is clear that $\mathbf{z}^2$ dominates $\mathbf{z}^1$ and, then, it should be preferred.

For $1 \leq p < \infty$, let us introduce the new scheme, called *extended formulation*, which can be used in any multiobjective optimization method based on reference points. For a reference point $\mathbf{q} = (q_1, \ldots, q_k)^T$ and a vector of weights $\mathbf{w} = (w_1, \ldots, w_k)^T$, with $w_i > 0$ for all $i = 1, \ldots, k$, the proposed *extended formulation* first considers the following scalarized problem:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{k} \left(w_i \frac{f_i(\mathbf{x}) - q_i}{z_i^\star - z_i^{\text{nad}}} \right)^{1/p} \\ \text{subject to} & f_i(\mathbf{x}) \geq q_i \quad \text{for every } i = 1, \ldots, k, \\ & \mathbf{x} \in S. \end{array} \tag{11}$$

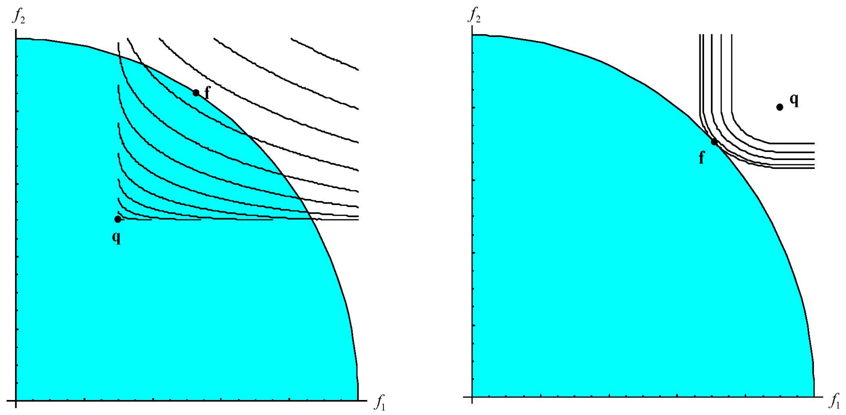
If the feasible set of problem (11) is not empty, then the reference point $\mathbf{q}$ is achievable. In this case, this problem is solved and the process ends. Otherwise, if problem (11) is infeasible, the reference point $\mathbf{q}$ is unachievable and the following scalarized problem is subsequently formulated and solved:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{k} \left(w_i \cdot \max \left\{ \frac{q_i - f_i(\mathbf{x})}{z_i^\star - z_i^{\text{nad}}}, 0 \right\} \right)^{p} \\ \text{subject to} & \mathbf{x} \in S. \end{array} \tag{12}$$

In problems (11) and (12), we have normalized the deviations to the reference values in order to manage values on the same scales. If no normalization is needed, or if another system is used for normalizing the objective values, problems (11) and (12) can be adjusted accordingly.

Let $\mathbf{x}^p$ be an optimal solution of the proposed formulation, that is, $\mathbf{x}^p$ denotes an optimal solution of problem (11) if $\mathbf{q}$ is achievable, and $\mathbf{x}^p$ is an optimal solution of problem (12) otherwise. Its corresponding objective vector is $\mathbf{f}(\mathbf{x}^p)$.

When the reference point is achievable, the function to be maximized in problem (11) is somehow similar to a $L_{1/p}$ distance from the reference point. In this case, the feasible set is restricted to the feasible solutions whose objective vectors dominate or match the reference point. Because of this, $f_i(\mathbf{x}) - q_i \geq 0$ for every $i = 1, \ldots, k$ and we do not need to consider the absolute value of the relative deviations. By optimizing this function, we maximize the sum of the relative deviation to the reference point elevated to the power $1/p$. Then, $\mathbf{x}^p$ is the efficient solution of problem (1) which is "further" from the reference point. It should be explained that we have elevated the terms in the function minimized in (11) to the power $1/p$ for the following reasons. Firstly, if all the reference values are achievable, all the objective functions must be improved at the same time, although we also need to balance between the improvements achieved in all of them. This is somehow controlled by minimizing the aggregation of the deviations elevated to the power $1/p$. Secondly, by using the power $1/p$, the level curves of the function minimized in (11) are convex, as it will be seen hereafter in Figure 1. Instead, if we elevate the terms of this function to the power p ($p \geq 1$), the level curves of the function minimized in (11) would be concave. In this situation, for multiobjective maximization problems with convex nondominated objective sets, the new formulation would only generate



(a) Achievable reference point (b) Unachievable reference point

Fig. 1 Level curves of the extended formulation for $p = 2$.

the efficient solutions which individually optimize each objective (extreme solutions), being unable to generate unsupported solutions.

If the reference point is unachievable, the problem to be solved is (12), the function of which is similar to the L_p distance from the reference point to the feasible set, but only the positive relative deviations are taken into account. This way, the function to be minimized only considers the positive values of $q_i - f_i(\mathbf{x})$, which correspond to the objective values $f_i(\mathbf{x})$ which do not achieve (or are worse than) their corresponding reference values q_i. With this, we avoid penalizing positive and negative deviations to the reference point, as in the L_p distance, which makes no sense unless the reference point is to be reached exactly (which is impossible in this case). By using power p in problem (12), we allow different compensation degrees just between the positive deviations to the reference values. This way, the higher the value of p, the more important is a positive deviation if the aggregation of the positive deviations is minimized.

Figure 1 gives a graphical idea of the level curves in the objective space of the proposed formulation for $p = 2$ in a biobjective optimization problem, in the achievable and unachievable cases. Here, $\mathbf{f}$ denotes the objective vector associated to the optimal solution obtained in each case. The level curves in the achievable case (figure (a)) are those of problem (11), where a function similar to a $L_{1/2}$ distance (sum of square roots) to the reference point is maximized. In this case, we can see that the feasible set is the subset of solutions whose objective vectors dominate the reference point. In the unachievable case, problem (11) is infeasible and the level curves of figure (b) are those of problem (12), which is the problem solved in this case; the negative deviations are not considered and that is the reason for the straight parts of the level curves that can be seen in figure (b) (i.e. these parts correspond to the objective vectors which reach or improve some of the reference values).

As explained above, different single-objective optimization problems are formulated and solved in the extended formulation for achievable and unachievable reference points. In both cases, we optimize functions which are extensions of the L_p distances and which allow different compensation degrees among the positive deviations to the reference values. With this, our purpose is twofold. Firstly, the formulation proposed permits the use of any type of reference point under the philosophy of compromise programming, overcoming the previously mentioned difficulties of the L_p distance. And secondly, the new formulation fills the gap existing between achievement scalarizing functions based on the L_∞ distance and the one based on the L_1 distance for reference point-based approaches.

The extended formulation has been suggested for $1 \leq p < \infty$. It can be noted that for $p = 1$, problems (11) and (12) are equivalent to problems (8) and (7), respectively. It is also noteworthy that the new formulation proposed is not intended to replace or to outperform the minimization of Wierzibicki's achievement scalarizing function given in (2). It just represents a new scheme that enables us to generate efficient solutions in reference point-based approaches in a different way. We would like to clarify that our formulation differs from the scalarizing function proposed in [16] in that we just consider the positive deviations to the reference values. Besides, our scheme does not try to bound the trade-offs among the objectives, but focuses on improving all the objective values which do not reach their reference levels at the same time, using a compensation degree between them.

In what follows, we show that, for any achievable reference point, any optimal solution $\mathbf{x}^p$ found by the extended formulation is an efficient solution of problem (1) and, if the reference point happens to be unachievable, $\mathbf{x}^p$ is a weakly efficient solution of problem (1) and, if it is unique, it is efficient.

If reference point $\mathbf{q}$ is achievable, the extended formulation solves problem (11) since its feasible set is not empty. In this case, it is obvious that $\mathbf{f}(\mathbf{x}^p)$ dominates or matches $\mathbf{q}$. Let us prove this result.

Theorem 1 *Let* $\mathbf{q}$ *be an achievable reference point for problem* (1)*. Given a value* p *with* $1 \leq p < \infty$*, an optimal solution* $\mathbf{x}^p$ *of problem* (11) *is an efficient solution of problem* (1)*.*

Proof Let us prove the theorem by *reductio ad absurdum*. Let us suppose that $\mathbf{x}^p$ is not efficient for problem (1). This means that there exists a feasible solution $\mathbf{x}^* \in S$ with $\mathbf{x}^* \neq \mathbf{x}^p$ such that:

$$f_i(\mathbf{x}^*) \geq f_i(\mathbf{x}^p) \text{ for all } i = 1, \ldots, k, \text{ and } f_j(\mathbf{x}^*) > f_j(\mathbf{x}^p) \text{ for some } j. \tag{13}$$

This implies that $\mathbf{x}^*$ is also a feasible solution of problem (11), since:

$$f_i(\mathbf{x}^*) \geq f_i(\mathbf{x}^p) \geq q_i, \quad \text{for all } i = 1, \ldots, k.$$

Then, we have:

$$f_i(\mathbf{x}^*) - q_i \geq f_i(\mathbf{x}^p) - q_i \geq 0, \quad \text{for all } i = 1, \ldots, k,$$

and, because of (13), $f_j(\mathbf{x}^*) - q_j > f_j(\mathbf{x}^p) - q_j$ for the index j. Thus, since $w_i > 0$ and $z_i^\star - z_i^{\text{nad}} > 0$, for all $i = 1, \ldots, k$, it is verified that:

$$\sum_{i=1}^{k} \left(w_i \frac{f_i(\mathbf{x}^*) - q_i}{z_i^\star - z_i^{\text{nad}}} \right)^{1/p} > \sum_{i=1}^{k} \left(w_i \frac{f_i(\mathbf{x}^p) - q_i}{z_i^\star - z_i^{\text{nad}}} \right)^{1/p},$$

which contradicts the fact that $\mathbf{x}^p$ is an optimal solution of problem (11). With this, the theorem is proven.

When the reference point $\mathbf{q}$ is unachievable, problem (11) is infeasible and, then, the extended formulation solves problem (12). In this case, the following result holds.

Theorem 2 *Let* $\mathbf{q}$ *be an unachievable reference point for problem* (1)*. Given a value* p *with* $1 \leq p < \infty$*, an optimal solution* $\mathbf{x}^p$ *of problem* (12) *is a weakly efficient solution of problem* (1)*. In addition, it is an efficient solution of problem* (1) *if it is unique.*

Proof We will prove the theorem by *reductio ad absurdum*. Let us suppose that $\mathbf{x}^p$ is not weakly efficient for problem (1). This means that there exists a feasible solution $\mathbf{x}^* \in S$ with $\mathbf{x}^* \neq \mathbf{x}^p$ such that:

$$f_i(\mathbf{x}^*) > f_i(\mathbf{x}^p), \quad \text{for all } i = 1, \ldots, k. \tag{14}$$

Taking into account that $w_i > 0$ and $z_i^\star - z_i^{\text{nad}} > 0$, for all $i = 1, \ldots, k$, (14) implies that:

$$0 \leq \max\left\{ \frac{q_i - f_i(\mathbf{x}^*)}{z_i^\star - z_i^{\text{nad}}}, 0 \right\} \leq \max\left\{ \frac{q_i - f_i(\mathbf{x}^p)}{z_i^\star - z_i^{\text{nad}}}, 0 \right\}, \quad \text{for all } i = 1, \ldots, k \quad \Rightarrow$$
$$\left(w_i \cdot \max\left\{ \frac{q_i - f_i(\mathbf{x}^*)}{z_i^\star - z_i^{\text{nad}}}, 0 \right\} \right)^p \leq \left(w_i \cdot \max\left\{ \frac{q_i - f_i(\mathbf{x}^p)}{z_i^\star - z_i^{\text{nad}}}, 0 \right\} \right)^p, \quad \text{for all } i = 1, \ldots, k.$$

On the other hand, since $\mathbf{q}$ is unachievable, there exists some j such that $q_j > f_j(\mathbf{x}^p)$. In addition, from (14), it holds that $f_j(\mathbf{x}^*) > f_j(\mathbf{x}^p)$. Then, for this index j, we have:

$$q_j - f_j(\mathbf{x}^*) < q_j - f_j(\mathbf{x}^p) \quad \text{and} \quad q_j - f_j(\mathbf{x}^p) > 0 \quad \Rightarrow$$
$$\max\left\{\frac{q_j - f_j(\mathbf{x}^*)}{z_j^\star - z_j^{\text{nad}}}, 0\right\} < \max\left\{\frac{q_j - f_j(\mathbf{x}^p)}{z_j^\star - z_j^{\text{nad}}}, 0\right\} = \frac{q_j - f_j(\mathbf{x}^p)}{z_j^\star - z_j^{\text{nad}}} \quad \Rightarrow$$
$$\left(w_j \cdot \max\left\{\frac{q_j - f_j(\mathbf{x}^*)}{z_j^\star - z_j^{\text{nad}}}, 0\right\}\right)^p < \left(w_j \cdot \max\left\{\frac{q_j - f_j(\mathbf{x}^p)}{z_j^\star - z_j^{\text{nad}}}, 0\right\}\right)^p.$$

This means that

$$\sum_{i=1}^k \left(w_i \cdot \max\left\{\frac{q_i - f_i(\mathbf{x}^*)}{z_i^\star - z_i^{\text{nad}}}, 0\right\}\right)^p < \sum_{i=1}^k \left(w_i \cdot \max\left\{\frac{q_i - f_i(\mathbf{x}^p)}{z_i^\star - z_i^{\text{nad}}}, 0\right\}\right)^p,$$

which contradicts the fact that $\mathbf{x}^p$ is an optimal solution of problem (12). With this, it is proven that $\mathbf{x}^p$ is a weakly efficient solution of (1).

Next, let us assume that $\mathbf{x}^p$ is the unique solution of problem (12), and let us see that it is an efficient solution of problem (1). Again, we will use *reductio ad absurdum*, so let us consider that $\mathbf{x}^p$ is not an efficient solution of (12). Then, there exists a feasible solution $\mathbf{x}^* \in S$ with $\mathbf{x}^* \neq \mathbf{x}^p$ such that:

$$f_i(\mathbf{x}^*) \geq f_i(\mathbf{x}^p), \text{ for all } i = 1, \ldots, k, \text{ and } f_j(\mathbf{x}^*) > f_j(\mathbf{x}^p) \text{ for some } j. \tag{15}$$

This implies that:

$$0 \leq \max\left\{\frac{q_i - f_i(\mathbf{x}^*)}{z_i^\star - z_i^{\text{nad}}}, 0\right\} \leq \max\left\{\frac{q_i - f_i(\mathbf{x}^p)}{z_i^\star - z_i^{\text{nad}}}, 0\right\}, \quad \text{for all } i = 1, \ldots, k \quad \Rightarrow$$
$$\sum_{i=1}^k \left(w_i \cdot \max\left\{\frac{q_i - f_i(\mathbf{x}^*)}{z_i^\star - z_i^{\text{nad}}}, 0\right\}\right)^p \leq \sum_{i=1}^k \left(w_i \cdot \max\left\{\frac{q_i - f_i(\mathbf{x}^p)}{z_i^\star - z_i^{\text{nad}}}, 0\right\}\right)^p.$$

The fact that this inequality includes the equality contradicts that $\mathbf{x}^p$ is the unique solution of (12). Therefore, the proof is complete.

Following the same idea used to formulate (4), we can avoid generating weakly efficient solutions in the unachievable case if we add an augmentation term to the function of problem (12):

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^k \left(w_i \cdot \max\left\{\frac{q_i - f_i(\mathbf{x})}{z_i^\star - z_i^{\text{nad}}}, 0\right\}\right)^p + \rho \sum_{i=1}^k w_i \cdot \left(\frac{q_i - f_i(\mathbf{x})}{z_i^\star - z_i^{\text{nad}}}\right) \\ \text{subject to} & \mathbf{x} \in S, \end{array} \tag{16}$$

for a value $\rho > 0$. In practice, as in problem (4), ρ should take a small positive value (e.g. $\rho = 0.001$ or $\rho = 0.0001$). Next, we prove that, for any value $\rho > 0$, an optimal solution of problem (16) is always a Pareto optimal solution of the original problem (1):

Theorem 3 *Let* $\mathbf{q}$ *be an unachievable reference point for problem* (1). *Given a value* p *with* $1 \leq p < \infty$, *for any* $\rho > 0$, *an optimal solution* $\mathbf{x}^p$ *of problem* (16) *is an efficient solution of problem* (1).

Proof By *reductio ad absurdum*, let us suppose that $\mathbf{x}^p$ is not an efficient solution of problem (1), that is, let us assume that there exists a feasible solution $\mathbf{x}^* \in S$ with $\mathbf{x}^* \neq \mathbf{x}^p$ such that:

$$f_i(\mathbf{x}^*) \geq f_i(\mathbf{x}^p), \text{ for all } i = 1, \ldots, k, \text{ and } f_j(\mathbf{x}^*) > f_j(\mathbf{x}^p) \text{ for some } j. \tag{17}$$

Since $w_i > 0$ and $z_i^\star - z_i^{\text{nad}} > 0$, for all $i = 1, \ldots, k$, from (17), we have:

$$0 \leq \max\left\{\frac{q_i - f_i(\mathbf{x}^*)}{z_i^\star - z_i^{\text{nad}}}, 0\right\} \leq \max\left\{\frac{q_i - f_i(\mathbf{x}^p)}{z_i^\star - z_i^{\text{nad}}}, 0\right\}, \quad \text{for all } i = 1, \ldots, k \quad \Rightarrow$$

$$\sum_{i=1}^{k}\left(w_i \cdot \max\left\{\frac{q_i - f_i(\mathbf{x}^*)}{z_i^\star - z_i^{\rm nad}}, 0\right\}\right)^p \leq \sum_{i=1}^{k}\left(w_i \cdot \max\left\{\frac{q_i - f_i(\mathbf{x}^p)}{z_i^\star - z_i^{\rm nad}}, 0\right\}\right)^p. \qquad (18)$$

On the other hand, taking into account (17):

$$\rho\sum_{i=1}^{k} w_i \cdot \left(\frac{q_i - f_j(\mathbf{x}^*)}{z_i^\star - z_i^{\rm nad}}\right) < \rho\sum_{i=1}^{k} w_i \cdot \left(\frac{q_i - f_i(\mathbf{x}^p)}{z_i^\star - z_i^{\rm nad}}\right). \qquad (19)$$

Therefore, inequalities (18) and (19) imply that:

$$\sum_{i=1}^{k}\left(w_i \cdot \max\left\{\frac{q_i - f_i(\mathbf{x}^*)}{z_i^\star - z_i^{\rm nad}}, 0\right\}\right)^p + \rho\sum_{i=1}^{k} w_i \cdot \left(\frac{q_i - f_j(\mathbf{x}^*)}{z_i^\star - z_i^{\rm nad}}\right) <$$
$$< \sum_{i=1}^{k}\left(w_i \cdot \max\left\{\frac{q_i - f_i(\mathbf{x}^p)}{z_i^\star - z_i^{\rm nad}}, 0\right\}\right)^p + \rho\sum_{i=1}^{k} w_i \cdot \left(\frac{q_i - f_i(\mathbf{x}^p)}{z_i^\star - z_i^{\rm nad}}\right),$$

which cannot be true given that $\mathbf{x}^p$ is an optimal solution of problem (16). With this, we complete the proof.

Taking into account some results pointed out in [19] about compromise programming using L_p norms, we think that, in order to generate the whole Pareto optimal set with our formulation, we may need to use a large value of p. In the case of $p = 1$, it is only possible to generate the whole efficient set if the nondominated objective set is $\mathbf{R}^k_-$−convex (for maximization problems), that is, if the set defined by $A = \{\mathbf{a} = \mathbf{z} + \mathbf{v} \text{ with } \mathbf{z} \in \mathbf{f}(E), \mathbf{v} \in \mathbf{R}^k_-\}$ is convex, where $\mathbf{R}^k_- = \{\mathbf{v} \in \mathbf{R}^k | v_i \leq 0 \text{ for } i = 1, \ldots, k\}$. Probably, the magnitude of p needed to generate the whole efficient set is related to the nonconvexity of A. Note that the larger the value of p, the more similar are the level curves of the function defined in the extended formulation in both cases to the level curves of the Tchebychev metric.

3.1 An Interactive Algorithm

The extended formulation can be embedded in an interactive algorithm which, using a set of values for p, generates a set of solutions at each iteration for the reference point given by the DM. Let us describe the step-by-step procedure of the interactive algorithm proposed. Let us consider the solutions for $p \in \{1, 2, 3, 5, \infty\}$ (these values can be changed if so desired). In what follows, we denote by h the current iteration, by $\mathbf{q}^h$ the current reference point, by $\mathbf{w}^h$ the vector of positive weights, by $\mathbf{x}^{h,p}$ the current solution found by the extended formulation for each value of p, and by $\mathbf{f}^{h,p}$ its corresponding objective vector:

Step 1. Initialization. Calculate or estimate the ideal and the nadir objective vectors, $\mathbf{z}^\star$ and $\mathbf{z}^{\rm nad}$. Set $h = 0$, $\mathbf{w}^0 = (1, \ldots, 1)^T$ and $\mathbf{q}^0 = \mathbf{z}^\star$.

Step 2. Preferential information. Ask the DM if (s)he desires to change the reference point and/or the vector of positive weights. If (s)he gives a new reference point $\mathbf{q}$, set $\mathbf{q}^h = \mathbf{q}$. If (s)he provides new positive weights for the objective functions, update $w_i^h > 0$ for $i = 1, \ldots, k$.

Step 3. Extended formulation (I). For each value of p considered, formulate problem (11) using $\mathbf{w}^h$ and $\mathbf{q}^h$. If the feasible set of the first problem solved is empty, go to Step 4. Otherwise, let $\mathbf{x}^{h,p}$ be an optimal solution of (11) for each p, $\mathbf{f}^{h,p} = \mathbf{f}(\mathbf{x}^{h,p})$ and go to Step 5.

Step 4. Extended formulation (II). For each value of p considered, formulate and solve problem (16) using $\mathbf{w}^h$ and $\mathbf{q}^h$. Let $\mathbf{x}^{h,p}$ be the optimal solution of (16) for each p, $\mathbf{f}^{h,p} = \mathbf{f}(\mathbf{x}^{h,p})$ and go to Step 5.

Step 5. Termination criterion. Show the DM the solutions obtained $\mathbf{x}^{h,p}$ and their corresponding objective vectors $\mathbf{f}^{h,p}$ for each p considered. If (s)he is satisfied with some of these solutions, this is the final solution of the original problem and *Stop*. Otherwise, set $h = h + 1$ and go to Step 2.

In this algorithm, several solutions are generated at each iteration so that the DM can compare different solutions that can be obtained for the reference point given. In our opinion, this is better than generating just one solution at each iteration because, in this way, the DM can have a good understanding of the trade-offs that can be obtained among the objective functions, being in a better position to make a decision. Different values of p can be used if so desired. If the DM is particularly interested in the solutions generated for one or several specific values of p, the algorithm can be adjusted correspondingly.

The weights assigned to the objective functions can be set as preferential parameters, as explained in [10,17], giving the DM the opportunity of changing the weights if needed at Step 2 of the algorithm. However, it is important to keep in mind that the preferences are mainly expressed by means of the reference point, and the vector of weights can be kept unaltered during the entire solution process if so desired.

According to our formulation, the solutions found depend on the nadir objective vector. It should be said that the nadir objective vector is only used to normalize the objective functions, if needed, but any other kind of normalization can be used if desired. Besides, other available lower bounds can be used for the normalization.

4 Illustrative Examples

In this section, we illustrate the behaviour of the extended formulation in a biobjective optimization problem and in an optimization problem with three objectives, taking into account achievable and unachievable reference points and several values of p. Besides, we compare the solutions obtained by the extended formulation and by the original compromise programming in a linear biobjective problem.

In these examples, for $1 \leq p < \infty$, we consider problem (11) for the achievable case and problem (16) for the unachievable case so that only Pareto optimal solutions are generated. For $p = \infty$, we solve problem (5) for any type of reference point. In each example, we refer to the achievable reference point as $\mathbf{q}^a$ and to the unachievable one as $\mathbf{q}^u$. For each value of p, we denote the Pareto optimal solution obtained in the extended formulation for p and $\mathbf{q}^a$ (respectively, for $\mathbf{q}^u$) by $\mathbf{x}^{a,p}$ (respectively, by $\mathbf{x}^{u,p}$), and its objective vector by $\mathbf{f}^{a,p} = \mathbf{f}(\mathbf{x}^{a,p})$ (respectively, by $\mathbf{f}^{u,p} = \mathbf{f}(\mathbf{x}^{u,p})$).

4.1 Biobjective Optimization Problem

Let us consider the following non-linear biobjective optimization problem:

$$\begin{array}{rl} \text{maximize} & 4x_1 + x_2 \\ \text{maximize} & -x_1 + 2x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 6 \\ & x_1^2 + x_2^2 \leq 9 \\ & x_1, x_2 \geq 0. \end{array} \tag{20}$$

The feasible set is $S = \{(x_1, x_2)^T \in \mathbf{R}^2 : 2x_1 + x_2 \leq 6, x_1^2 + x_2^2 \leq 9, x_1, x_2 \geq 0\}$, and the ideal and nadir objective vectors are $\mathbf{z}^\star = (12, 6)^T$ and $\mathbf{z}^{\text{nad}} = (3, -3)^T$, respectively. The objective region $Z = \mathbf{f}(S)$ is shown in Figure 2, where it can be seen that the nondominated objective set is the curve ABC. We have obtained Z by solving the following system of equations for the variables x_1 and x_2:

$$z_1 = 4x_1 + x_2, \quad z_2 = -x_1 + 2x_2,$$

and substituting the expressions obtained for x_1 and x_2 in the constraints which define S. With that, we obtain that Z is:

$$Z = \{(z_1, z_2)^T \in \mathbf{R}^2 : \tfrac{1}{81}(5z_1^2 + 17z_2^2 + 4z_1z_2) \leq 9, \tfrac{1}{9}(5z_1 + 2z_2) \leq 6, \\ \tfrac{1}{9}(2z_1 - z_2) \geq 0, \tfrac{1}{9}(z_1 + 4z_2) \geq 0\}.$$

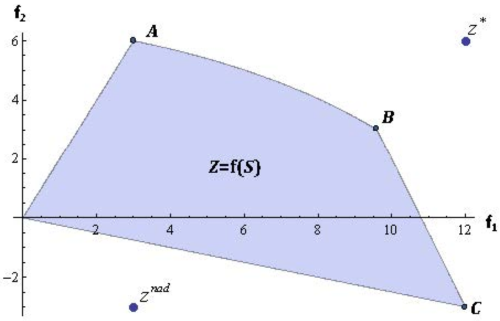


Fig. 2 Objective space of problem (20).

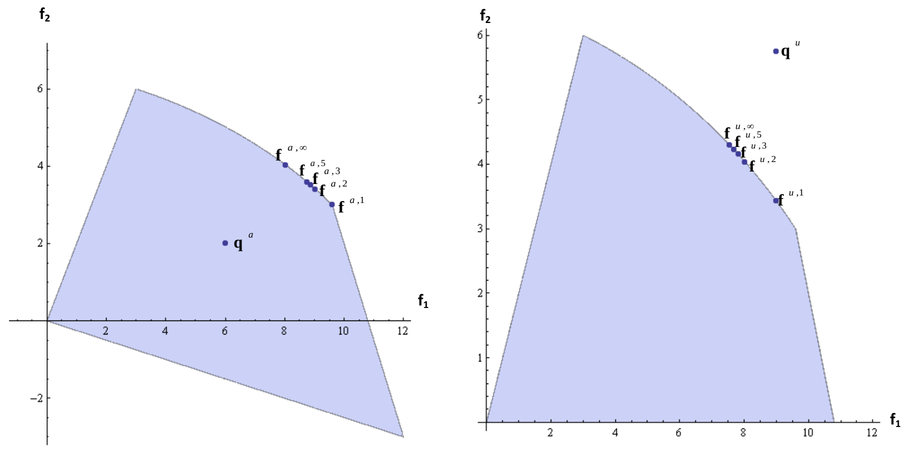
Let us consider the vector of weights $\mathbf{w} = (1,1)^T$ and the values of p in the set $\{1, 2, 3, 5, \infty\}$. The achievable reference point considered is $\mathbf{q}^a = (6,2)^T$. In the extended formulation, problem (11) is initially formulated. In this case, the feasible set of this problem is not empty, so it is solved for the different values of p and the process ends. The solutions obtained for $p \in \{1, 2, 3, 5, \infty\}$ are given in Table 1, and they have been represented in Figure 3(a). It can be seen that all these solutions are Pareto optimal for the original problem (20) and they improve all the reference values given in $\mathbf{q}^a$. For the unachievable case, we have selected the reference point $\mathbf{q}^u = (9, 5.75)^T$. In this case, problem (11) has no solution (since its feasible region is empty) and, then, the extended formulation solves problem (16). As previously explained, only the deviations from the unachievable values of $\mathbf{q}^u$ (that is, the positive deviations) are considered in (16). Table 2 contains the solutions obtained for $p \in \{1, 2, 3, 5, \infty\}$, and they have been plotted in Figure 3(b) (in this case, we have shown a closer view of the objective space in order to better distinguish the solutions). It can be seen that in both the achievable and the unachievable cases the solutions obtained for the different values of p differ from each other. This demonstrates that the extended formulation provides different schemes (one for each p) for the generation of different Pareto optimal solutions in reference point-based methods.

Table 1 Solutions obtained by the extended formulation for problem (20) and $\mathbf{q}^a$.

Values of p	**Solutions**
1	$\mathbf{f}^{a,1} = \mathbf{f}(\mathbf{x}^{a,1}) = (9.6, 3)^T$
2	$\mathbf{f}^{a,2} = \mathbf{f}(\mathbf{x}^{a,2}) = (9.03, 3.41)^T$
3	$\mathbf{f}^{a,3} = \mathbf{f}(\mathbf{x}^{a,3}) = (8.85, 3.53)^T$
5	$\mathbf{f}^{a,5} = \mathbf{f}(\mathbf{x}^{a,5}) = (8.75, 3.6)^T$
∞	$\mathbf{f}^{a,\infty} = \mathbf{f}(\mathbf{x}^{a,\infty}) = (8.03, 4.03)^T$

Table 2 Solutions obtained by the extended formulation for problem (20) and $\mathbf{q}^u$.

Values of p	**Solutions**
1	$\mathbf{f}^{u,1} = \mathbf{f}(\mathbf{x}^{u,1}) = (9, 3.43)^T$
2	$\mathbf{f}^{u,2} = \mathbf{f}(\mathbf{x}^{u,2}) = (8.03, 4.04)^T$
3	$\mathbf{f}^{u,3} = \mathbf{f}(\mathbf{x}^{u,3}) = (7.82, 4.15)^T$
5	$\mathbf{f}^{u,5} = \mathbf{f}(\mathbf{x}^{u,5}) = (7.69, 4.22)^T$
∞	$\mathbf{f}^{u,\infty} = \mathbf{f}(\mathbf{x}^{u,\infty}) = (7.55, 4.30)^T$



(a) Achievable reference point $\mathbf{q}^a$ (b) Unachievable reference point $\mathbf{q}^u$

Fig. 3 Solutions obtained by the extended formulation for problem (20) and $p \in \{1, 2, 3, 5, \infty\}$.

4.2 Three-objective Optimization Problem

Let us consider the three-objective minimization problem DTLZ2 with 12 variables proposed in [4]. This is a nonconvex problem, the formulation of which is given by:

$$\begin{array}{ll} \text{minimize} & f_1(x_1,\ldots,x_{12}) = (1+g)\cdot cos(\frac{\Pi}{2}x_1)\cdot cos(\frac{\Pi}{2}x_2) \\ \text{minimize} & f_2(x_1,\ldots,x_{12}) = (1+g)\cdot cos(\frac{\Pi}{2}x_1)\cdot sin(\frac{\Pi}{2}x_2) \\ \text{minimize} & f_3(x_1,\ldots,x_{12}) = (1+g)\cdot sin(\frac{\Pi}{2}x_2) \\ \text{subject to} & g = \sum_{i=3}^{12}(x_i-0.5)^2 \\ & 0 \le x_i \le 1, \quad i = 1,\ldots,12. \end{array} \tag{21}$$

This problem has a spherical nondominated objective set which is given by the equation $(z_1)^2 + (z_2)^2 + (z_3)^2 = 1$, where $z_i = f_i(x_1, \ldots, x_{12})$ for $i = 1, 2, 3$. The ideal and nadir objective vectors are $\mathbf{z}^\star = (0, 0, 0)^T$ and $\mathbf{z}^{\text{nad}} = (1, 1, 1)^T$, respectively (see [4] for more details).

Let us consider $\mathbf{w} = (1, 1, 1)^T$ and $p \in \{1, 2, 3, 5, \infty\}$. Let us remember that problems (11) and (16) have been formulated for a multiobjective maximization problem such as (1). It is important to clarify that since all the objectives of the DTLZ2 problem must be minimized, we have rewritten these problems to adapt them to a multiobjective minimization problem.

Both achievable and unachievable reference points, have been uniformly generated between 0 and 1. In this case, the achievable reference point considered is $\mathbf{q}^a = (0.52, 0.17, 0.98)^T$. In the extended formulation, problem (11) is formulated first and, since its feasible set is not empty, it is solved for each value of p. The solutions obtained for $p \in \{1, 2, 3, 5, \infty\}$ are given in Table 3, and can be seen in Figure 4(a). In the unachievable case, the reference point considered is $\mathbf{q}^u = (0.23, 0.0, 0.48)^T$. Since the feasible set of problem (11) is empty for this reference point, problem (16) is solved. Table 4 contains the solutions obtained for $p \in \{1, 2, 3, 5, \infty\}$, and Figure 4(b) shows them in the objective space. As in the previous example, for every p, the extended formulation provides a Pareto optimal solution which is different from the ones generated for the other values of p.

5 Example with real data

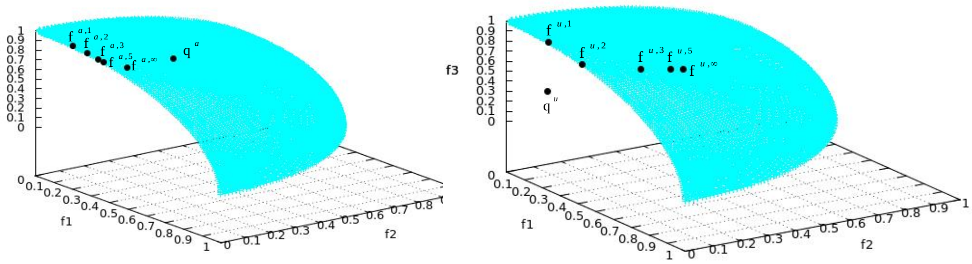
In this section, we solve a real multiobjective optimization model by using the new formulation. This model was proposed in [13] and the main idea behind it is to determine the profile(s)

Table 3 Solutions obtained by the extended formulation for problem (21) and $\mathbf{q}^a$.

Values of p	**Solutions**
1	$\mathbf{f}^{a,1} = \mathbf{f}(\mathbf{x}^{a,1}) = (0.20, 0.0, 0.98)^T$
2	$\mathbf{f}^{a,2} = \mathbf{f}(\mathbf{x}^{a,2}) = (0.28, 0.0, 0.96)^T$
3	$\mathbf{f}^{a,3} = \mathbf{f}(\mathbf{x}^{a,3}) = (0.34, 0.0, 0.94)^T$
5	$\mathbf{f}^{a,5} = \mathbf{f}(\mathbf{x}^{a,5}) = (0.37, 0.0, 0.93)^T$
∞	$\mathbf{f}^{a,\infty} = \mathbf{f}(\mathbf{x}^{a,\infty}) = (0.44, 0.04, 0.90)^T$

Table 4 Solutions obtained by the extended formulation for problem (21) and $\mathbf{q}^u$.

Values of p	**Solutions**
1	$\mathbf{f}^{u,1} = \mathbf{f}(\mathbf{x}^{u,1}) = (0.23, 0.0, 0.97)^T$
2	$\mathbf{f}^{u,2} = \mathbf{f}(\mathbf{x}^{u,2}) = (0.43, 0.0, 0.90)^T$
3	$\mathbf{f}^{u,3} = \mathbf{f}(\mathbf{x}^{u,3}) = (0.51, 0.16, 0.84)^T$
5	$\mathbf{f}^{u,5} = \mathbf{f}(\mathbf{x}^{u,5}) = (0.53, 0.25, 0.81)^T$
∞	$\mathbf{f}^{u,\infty} = \mathbf{f}(\mathbf{x}^{u,\infty}) = (0.54, 0.30, 0.79)^T$



(a) Achievable reference point $\mathbf{q}^a$

(b) Unachievable reference point $\mathbf{q}^u$

Fig. 4 Solutions obtained by the extended formulation for problem (21) and $p \in \{1, 2, 3, 5, \infty\}$.

of the most satisfied worker(s) in Spain, as well as the policies that may be carried out to improve employees' satisfaction. The model solved here is based on a new elaboration of the estimates included in the said work, in which we just consider the data concerning Spanish female workers.

The idea behind this implementation is that measures of job satisfaction, as a proxy for job quality, appear to be useful predictors of future labour market behaviour. Workers' decisions about whether to work or not, what kind of job to accept or stay in, and how hard to work, are all likely to depend in part on the workers' subjective evaluation of their work, in other words, on their job satisfaction. Consequently, job satisfaction acts as a summary measure of the different aspects of job quality, a number of which are difficult to observe or measure; e.g., using American panel data, [8] showed that job satisfaction is a strongly significant predictor of quits, even more than wages in some cases. However, it seems clear that job satisfaction is not a single dimensional measure as there can be different (and conflicting) aspects of job satisfaction. Thus, we have concentrated on different aspects of job satisfaction as a proxy for job quality in an attempt to quantify a worker's individual preferences. Here, the decision makers are governments which can affect factors influencing different aspects of workers' satisfaction balance (such as salaries, working hours, etc.) in order to boost job quality. In this sense, we have simulated governments' potential aim of reaching women with the highest satisfaction levels in Europe -as desirable values for each objective-, i.e. we have used Danish female workers' satisfaction as an "ideal" target, as explained below.

The data set used comes from the European Community Household Panel (ECHP) for the period 1995-2001, which is the only European wide survey containing microdata on worker satisfaction with different work aspects, as well as on several individual personal and family characteristics[2].

Considering the degree of conflict between the different satisfaction aspects, the multiobjective methodology is especially suitable for finding the individual profiles which achieve the highest values for the satisfaction aspects (objective functions) simultaneously. Based on an econometric analysis of the data, as in [13], a seven-objective optimization model is proposed. In [13], a reference point approach is used to solve the problem at the first stage and a goal programming approach is considered at the second stage. The latter allowed the authors to carry out a sensitivity analysis of the solution obtained by relaxing several constraints.

By contrast, in what follows, we carry out a sensitivity analysis without relaxing any constraint. For this purpose, several solutions are generated for the same reference point by solving the extended formulation proposed. We use the Danish female workers' satisfaction levels as reference values for the satisfaction objective functions, as they reported the highest average work satisfaction level within the 15 countries participating in the survey. More precisely, regarding the information on satisfaction, workers in the ECHP were asked to evaluate seven different satisfaction aspects of their jobs, on a scale from 1 to 6, where 1 is "not satisfied at all" and 6 is "fully satisfied". As mentioned previously, we restricted the sample to female workers working in the private sector aged between 25 and 65 (retirement age). The job satisfaction aspects evaluated were: (1) earnings, (2) job security, (3) type of work, (4) number of working hours, (5) working times, (6) working conditions/environment and (7) distance from work/commuting[3].

On the ground of a sample of data contained in the aforementioned survey, an econometric analysis was carried out in order to find dependence relations of the satisfaction aspects with respect to a set of personal and family characteristics of the individuals, as well as possible correlations among some data. Based on these results, in order to model the multiobjective optimization problem, the statistically significant decision variables of the problem were identified, and the objective functions (the seven job satisfaction aspects) and constraints were built. To achieve this, the level for each satisfaction aspect was estimated from the combination of a set of 35 explanatory variables related to individual and contextual features, unobservable factors and a random disturbance (ε). These variables, which are the problem's decision variables, are shown in Table 1 of [13]. If we assume that, for a sample of size N, the individuals of the sample are indexed by r ($r = 1, \ldots, N$), the econometric model for the seven job satisfaction aspects can be represented by the following set of equations:

$$S_j(r) = \alpha^j + \beta_1^j \cdot ghw(r) + \beta_2^j \cdot edh(r) + \cdots + \beta_{35}^j \cdot y_7(r) + \varepsilon_j(r), \tag{22}$$

where $S_j(r)$ is a measure of the satisfaction aspect j of individual r, for each $j = 1, \ldots, 7$; $ghw(r), edh(r), \ldots, y_7(r)$ are the group of 35 explanatory variables; the vector of slope coefficients is represented by $\beta^j = (\beta_1^j, \beta_2^j, \ldots, \beta_{35}^j)^T$; α^j is a fixed but unknown population intercept; and $\varepsilon_j(r)$ is a random disturbance. Therefore, we are assuming that each individual's job satisfaction aspect is affected by random factors which are inherently unobservable and distributed normally.

From the seven equations represented by (22)[4], using econometric techniques, we obtained the estimated regression coefficients $\hat{\beta}^j = (\hat{\beta}_1^j, \hat{\beta}_2^j, \ldots, \hat{\beta}_{35}^j)^T$ and $\hat{\alpha}^j$, for every $j = 1, \ldots, 7$, on the key variables of interest, which were quite consistent with those found in the previous literature (the estimations obtained are not presented here to conserve space). Based on these estimates, we go a step forward presenting and solving the multiobjective optimization problem using the extended formulation. If, for simplicity, we rename the 35 variables $ghw, edh, \ldots, y_7$ as x_i, for $i = 1, \ldots, 35$, the estimated satisfaction aspects are defined by $ES_j(\mathbf{x}) = \sum_{i=1}^{35} \hat{\beta}_i^j \cdot x_i + \hat{\alpha}^j$ for every $j = 1, \ldots, 7$, and the multiobjective problem which optimizes the job satisfaction

[2] There is more up to date data from this survey but it does not contain data on satisfaction related variables

[3] The precise wording of the questions was: How satisfied are you with your present job in terms of ...?

[4] For each $j = 1, \ldots, 7$, the index r in equation (22) refers to the number of individuals (observations) in the sample of size N considered for the survey but, once the econometric techniques are applied to estimate the coefficients β_i^j ($i = 1, \ldots, 35$) and α^j, r is eliminated. Thus, (22) leads to seven equations.

aspects is the following one:

$$\begin{array}{ll} \text{maximize} & \mathbf{ES}(\mathbf{x}) = (ES_1(\mathbf{x}), ES_2(\mathbf{x}), \ldots, ES_7(\mathbf{x})) = \\ & = \left(\sum_{i=1}^{35} \hat{\beta}_i^1 \cdot x_i + \hat{\alpha}^1, \sum_{i=1}^{35} \hat{\beta}_i^2 \cdot x_i + \hat{\alpha}^2, \ldots, \sum_{i=1}^{35} \hat{\beta}_i^7 \cdot x_i + \hat{\alpha}^7 \right) \\ \text{subject to} & \text{two sets of constraints}^5. \end{array} \quad (23)$$

In our formulation, it is implicitly assumed that the achievements of all the reference levels are equally important for the decision maker ($w_i = 1$ for all $i = 1, \ldots, k$). Besides, note that, given that all the satisfaction aspects are specified on a $1-6$ scale, no normalization is needed.

To carry out the sensitivity analysis with our extended formulation, different solutions are generated for different p values using the extended formulation for the reference point $\mathbf{q} = (4.36, 4.74, 4.78, 4.76, 4.92, 4.80, 4.89)$, which are the satisfaction levels reached by the Danish female workers. The solution obtained for $p = \infty$ by solving (5), denoted by $\mathbf{x}^\infty$, reached the following satisfaction function values:

$$\mathbf{ES}(\mathbf{x}^\infty) = (3.72, 4.94, 5.16, 4.48, 4.28, 5.04, 4.86),$$

which indicates that the reference point is unachievable since some reference values are reached (the corresponding ones for satisfaction functions ES_2, ES_3 and ES_6) but not others (the ones for satisfaction functions ES_1, ES_4, ES_5 and ES_7). As it was pointed out in previous sections, using the L_p distance directly may produce undesirable effects in this case. In the extended formulation, problem (16) is solved for different values of p indicated in Table 5, which also reports the satisfaction function objective values of the solutions[6].

Table 5 Solutions obtained by the extended formulation for problem (23) and **q**.

Satisfaction Function	**Reference Point**	**Solutions**				
		$p=1$	$p=2$	$p=3$	$p=5$	$p=\infty$
ES_1	4.36	3.67	3.82	3.82	3.69	3.72
ES_2	4.74	5.04	4.98	4.96	4.64	4.94
ES_3	4.78	5.12	5.20	5.20	4.79	5.16
ES_4	4.76	4.40	4.40	4.41	4.24	4.48
ES_5	4.92	4.22	4.21	4.22	4.51	4.28
ES_6	4.80	5.00	5.00	4.99	4.75	5.04
ES_7	4.89	4.91	4.93	4.91	4.67	4.86

With respect to the satisfaction values of the solutions, as reported in Table 5, for $p = \infty$ women achieve three out of their seven reference levels (job security -ES_2-, type of work -ES_3- and working conditions -ES_6-); similarly, when assuming $p = 1$, the same reference levels are achieved in addition to the satisfaction level regarding the distance to work (ES_7) which, nevertheless, showed a small distance to the reference point when $p = \infty$. Interestingly enough, despite the tiny differences in results for $p = 2$ and $p = 3$, the results are sensitive to the choice of values for p, which support the importance of the extended formulation for the multiobjective optimization problem.

What is more, when pairwise correlations between deviations from the reference point for each solution were computed, we found statistically significant coefficients which highlight the existence of important trade-offs between different objective functions (e.g. the correlation between deviation to the reference level for satisfaction with working time (ES_5) and for

[5] The two sets of constraints (not reported to conserve space -available upon request-) are: a) technical constraints, which ensure that certain binary variables do not take the value 1 simultaneously, and b) some constraints to ensure that the profile of the worker that we are looking for is sufficiently realistic (e.g. defining bounds for the salary depending on the individuals' highest education level).

[6] We have not included the values of the variables due to space reasons.

working conditions (ES_6) is -0.928)[7]. This may indicate that the shape of the nondominated objective set may have a linear shape in the considered area.

Regarding the results obtained, it can be concluded that the profile of the most satisfied female workers in Spain is that of a middle-aged person with a high income (situated in the top quartile), and with high education level. In general, higher income (salary and family income) are needed to achieve higher satisfaction levels, but there is another factor: part of the current satisfaction levels is derived from high unemployment rates. That is, workers expectations concerning the labour market may affect workers' relative satisfaction depending on the unemployment rate of the region where they live. Obviously, this means by no way that higher unemployment rates should be a target. However, the implementation of equity policies, aimed at reducing the concentration of incomes in some socioeconomic groups, such as for example higher tax progressivity and incentives to achieve high education levels, would contribute significantly to shorten the distance between "in"-satisfied Spanish female workers and some "top" satisfied workers defined.

6 Conclusion

Usually, reference point-based approaches minimize an achievement scalarizing function over the feasible set in order to find the Pareto optimal solution which best fits the reference point. The majority of these functions are based on the L_∞ distance. There is also an additive achievement scalarizing function [18] based on the L_1 distance and, when it is minimized over the feasible set, we minimize the sum of the deviations to the reference values which are not achieved (positive deviations). In this paper, we generalize this approach and, for $1 \leq p < \infty$, we formulate a new scheme based on the L_p distance which is called *extended formulation*. Our main motivation is that, between the total compensation of the deviations to the reference values provided by the L_1 distance and no compensation that is implied by the L_∞ distance, there are different compensation degrees given by the rest of L_p distances which can be used for generating efficient solutions in reference point-based approaches. Additionally, the new formulation proposed is also intended at enabling the use of any type of reference point under the compromise programming philosophy, including achievable ones.

Depending on the achievability of the reference point, different single-objective optimization problems are formulated and solved in the extended formulation. It has been shown that any optimal solution obtained by the extended formulation is Pareto optimal if the reference point is achievable. When the reference point is unachievable, any optimal solution generated is, at least, weakly Pareto optimal, and it is Pareto optimal if it is unique. Furthermore, in this case, an augmentation term can be added to generate only efficient solutions. Besides, we have proposed an interactive algorithm based on the extended formulation, in which several efficient solutions are shown to the DM at each iteration according to the reference point used.

Two numerical examples of multiobjective optimization problems have illustrated the performance of the extended formulation for achievable and unachievable reference points. It has been shown that, for the values of p considered, the Pareto optimal solutions generated differ from each other. This means that the new formulation provides different schemes (one for each p) for obtaining different efficient solutions in reference point-based methods. Moreover, the application of the extended formulation to a multiobjective optimization problem, based on real data, provides further evidence of the sensitivity of the results to different values of p, when substantial trade-offs are into play.

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[7] The deviation of satisfaction with working time from the reference point also holds negative correlation with earnings (-0.490), job security (-0.978), type of work (-0.968) and number of working hours (-0.821).

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