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Necessary and sufficient conditions of optimality for an optimal control problem with non-local boundary conditions and quadratic functional

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SUMMARY

In this paper the optimal control problem with a quadratic functional for Helmholtz equation with non-local boundary conditions is considered. Necessary and sufficient conditions of optimality are obtained on the basis of which the existence and uniqueness of a solution to the optimal problem is proved.

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KEY WORDS: Helmholtz equation, boundary values problems, optimality for control problems.

1. INTRODUCTION

Non-classical boundary and initial boundary value problems often arise when constructing mathematical models of various phenomena of physics, biology and ecology. Non-local boundary and initial value problems formulated for equations of mathematical physics are non-classical problems, where instead of the boundary or initial conditions a certain dependence of the values of the unknown function on the boundary on its values in internal points of the considered domain is given. The non-classical problems with non-local boundary conditions are used for mathematical modelling of pollution processes in rivers, seas, which are caused by sewage. Non-local boundary conditions simulate decreasing of pollution under influence of natural factors of filtration and settling that cause self-purification of the environment. In non-classical problems with non-local initial conditions instead of classical initial conditions combinations of the initial values of the unknown function and its values at later times are given. Non-local in time problems are obtained when modelling the processes of radionuclides propagation in Stokes fluid, diffusion and flow in porous media.

The first investigation of non-classical boundary value problem for the homogeneous heat equation with integral non-local boundary condition was carried out by J. Cannon [1]. Further, similar problems were studied by L. Kamynin [2], N. Ionkin [3], A. Samarskii [4], A. Friedman [5], A. Bouziani [6], etc.

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The first basic study of non-local boundary problem arises in connection with the mathematical modelling of plasma physics processes, where non-local problems where posed and generalised for a certain class of elliptic functions. The authors proved the existence and uniqueness of solution in the case of rectangular domain and Laplace operator applying methods of the theory of integral equations [7].

In [8, 9] several results concerning the investigation of non-local boundary problems of the Bitsadze-Samarski type and its generalisation for some equations of mathematical physics, mainly elasticity and shell theory are presented. Iterative methods were also suggested for solving such problems in the case of quite general elliptic equations. Later on, non-local problems for elliptic, parabolic and hyperbolic equations subject to discrete and integral non-local boundary conditions were considered and investigated by V. Il'in, E. I.Moiseev [10], E. Moiseev [11], B. Paneyakh [12], A. Skubachevskii [13], etc.

Non-local in time problems for evolution equations of mathematical physics were investigated by V. V. Shelukhin [14], C. V. Pao [15], R. Lazarov et al. [16], etc.

This paper scrutinises an unconstrained (i.e. without control constraints) optimal control problem governed by a two-dimensional linear PDEs with a quadratic functional. The results of this paper may be of a great interest for physical applications since control constraints are usually a reflection of physical limitations as well as to impose severe complications on the theoretical derivation of necessary and sufficient conditions.

As to analytical framework of this paper, it is assumed that each control function is a distribution defined on an open set D and ranging in an open interval V contained in $I\!R$. Hence, the set of control function V_{ad} is not complete in any reasonable topology. This is not compatible with conventional existence theories of the optimal control. Other peculiarities of this paper are in the non-local boundary conditions for the PDEs involved, since by non-local boundary condition some general integral operator involved on the boundary is usually considered and the adjoint equation is an elliptic equation with a non-local transmission. This implies that solving the adjoin system does not require the boundary condition. Therefore, these conditions make the topic slightly distinct to the existing literature on the control of elliptic PDEs.

2. STATEMENT OF THE PROBLEM AND BASIC PROPERTIES

Let $\bar{D} = [0, l_1] \times [0, l_2]$ be a rectangle, ∂D the boundary of the rectangular domain, $\gamma = \{(l_1, y) : 0 \le y \le l_2\}$, $\gamma_0 = \{(x_0, y) : 0 \le y \le l_2\}$, x_0 the fixed point of the interval $]0, l_1[$, $V_{ad} = \{v : D \to V, v \in L_2(D)\}$ the set of control functions, V being an open interval contained in $I\!R$.

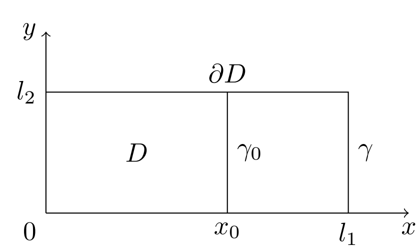

Figure 1. The rectangular domain

Let us consider the non-local boundary problem for Helmholtz's equation for each fixed $v \in V_{ad}$ in the domain D [7, 17].

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - q(x,y)u = a(x,y)v + b(x,y), \qquad (x,y) \in D \tag{1}$$

$$\begin{aligned} u(x,y) &= 0, & (x,y) &\in \partial D \backslash \gamma \\ u(l_1,y) &= \sigma u(x_0,y), & 0 &\le y \le l_2 \end{aligned} \tag{2}$$

where: $a \in L_\infty(D)$, $b \in L_2(D)$, $0 \le q \in L_\infty(D)$, $0 < \sigma < 1$, $\sigma = \text{const}$.

We denote by $L_2(D)$ the class of all real-valued measurable functions v defined on D for which the norm in $L_2(D)$ is as follows

$$\|v\|_{L_2(D)} = \left(\iint_D |v(x,y)|^2 \, dx \, dy \right)^{1/2} < +\infty \tag{3}$$

We identify in $L_2(D)$ functions that are equal almost everywhere in D; the elements of $L_2(D)$ are thus equivalence classes of measurable functions satisfying (3), two functions being equivalent if they are equal a. e. in D. For convenience, we ignore this distinction, and write $v \in L_2(D)$ if v satisfies (3), and $v = 0$ in $L_2(D)$ if $v(x,y) = 0$ a. e. in D.

By $V_{ad} = \{L_2(D), V\}$ we identify the class of all real-valued measurable functions satisfying (3) and ranging in V.

The Sobolev linear space $H^m(D)$, m being a non-negative integer, is defined to be the set of all functions u such that for $0 \le |\alpha| \le m$ all the weak derivatives $D^\alpha u$ exist and are in $L_2(D)$, and equip H^m with a norm as follows

$$\|u\|_{H^m(D)} = \left(\sum_{|\alpha| \le m} \iint_D |D^\alpha u|^2 \, dx \, dy \right)^{1/2} < +\infty \tag{4}$$

In this paper are considered spaces with $m = 0, 1, 2$. It is evident that $H^0(D) = L_2(D)$.

As a direct consequence

$$H^2(D) \subset H^1(D) \subset H^0(D) = L_2(D)$$
$$\|u\|_{H^2(D)} \ge \|u\|_{H^1(D)} \ge \|u\|_{L_2(D)}$$

Since a function $u \in L_2(D)$ need not be differentiable, the meaning of the condition $D^\alpha u \in L_2(D)$ requires an explanation. If $u \in L_2(D)$, then u can be identified with a regular distribution, that is to say, a continuous linear functional defined on the space of all test functions, and by a test function, we mean an infinitely differentiable function on D vanishing outside of some bounded set.

If $D^\alpha F u$ is a regular distribution corresponding to a function in $L_2(D)$, then we simply write $D^\alpha u \in L_2(D)$

The space $L_\infty(D)$ A function $a(x,y)$ that is measurable on D is said to be essentially bounded on D if there is a constant k such that $|a(x,y)| \le k$ a. e. on D. The greatest lower bound of such constants k is called the essential supremum of $a(x,y)$ on D and is denoted by $\operatorname{ess\,sup}_{(x,y)\in D} |a(x,y)|$.

We denote by $L_\infty(D)$ the vector space of all functions $a(x,y)$ that are essentially bounded on D, functions once again identified if they are equal a. e. on D. The functional $\|\cdot\|_\infty$ defined by

$$\|a\|_\infty = \operatorname*{ess\,sup}_{(x,y)\in D} a(x,y)$$

is a norm in $L_\infty(D)$.

Returning to problem (1)-(2), let us define the subspace of $H^1(D)$, obtained by closing the set

$$\overset{*}{C}{}^\infty(\bar{D}) = \begin{cases} u \in C^\infty(\bar{D}) & \operatorname{supp} u \cap \partial D \backslash \gamma = \emptyset \\ u(l_1,y) = \sigma u(x_0,y) & l_1 < y < l_2 \end{cases}$$

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with respect to norm $H^1(D)$, which is denoted by $\overset{*}{H}{}^1(D)$, being $\overset{*}{H}{}^2 = \overset{*}{H}{}^1 \cap H^2(D)$ the space of test functions.

Similarly as in [10, 18], it can be shown that the solution to problem (1)-(2) exists in weak sense†, is unique and belongs to Sobolev's space $\overset{*}{H}{}^2(D)$

Main results Let us consider the functional

$$I(v) = \iint\limits_D [c(x,y)u - z_d(x,y)]^2\, dx\, dy + \iint\limits_D N(x,y)v^2\, dx\, dy \tag{5}$$

where: $c \in L_\infty(D)$, $z_d \in L_2(D)$, $N \in L_\infty(D)$, $N > 0$.

Let the following control problem be formulated: finding function $v_0 \in V_{ad}$, whose corresponding solution u_0 of boundary value problem (1)-(2), together with v_0, results in a minimal functional value.

Conditions of optimality are obtained following the scheme developed in [19].

Let us consider arbitrary permissible control $v_\epsilon \in V_{ad}$ and let u_ϵ be the corresponding solution to problem (1)-(2).

Let us introduce the notation

$$\widetilde{v} = v_\epsilon - v_0, \qquad \widetilde{u} = u_\epsilon - u_0 \tag{6}$$

Then one obtains the following problem:

$$\frac{\partial^2 \widetilde{u}}{\partial x^2} + \frac{\partial^2 \widetilde{u}}{\partial y^2} - q(x,y)\widetilde{u} = a(x,y)\widetilde{v}, \qquad (x,y) \in D, \tag{7}$$

$$\begin{aligned} &\widetilde{u}(x,y) = 0, \quad (x,y) \in \partial D \backslash \gamma, \\ &\widetilde{u}(l_1,y) = \sigma \widetilde{u}(x_0,y), \quad 0 \le y \le l_2. \end{aligned} \tag{8}$$

Let us consider function $\Psi \neq 0$, $\Psi \in H^2(D\backslash\gamma_0) \cap H^1(D)$. Multiplying equation (7) by Ψ and integrating over domain D, the following equality is obtained:

$$\iint\limits_D \Psi(x,y) \left[\frac{\partial^2 \widetilde{u}}{\partial x^2} + \frac{\partial^2 \widetilde{u}}{\partial y^2} - q(x,y)\widetilde{u} \right] dx\, dy = \iint\limits_D a(x,y)\Psi(x,y)\widetilde{v}\, dx\, dy \tag{9}$$

The increment of functional (5) with fixed v_0, v_ϵ is

$$\begin{aligned} \widetilde{I} = I(v_\epsilon) - I(v_0) = &\iint\limits_D \left[(c(x,y)u_\epsilon - z_d)^2 + N(x,y){v_\epsilon}^2 \right] dx\, dy - \\ &- \iint\limits_D \left[(c(x,y)u_0 - z_d)^2 + N(x,y){v_0}^2 \right] dx\, dy = \\ = \iint\limits_D &\Big[2c(x,y)\widetilde{u}\big(c(x,y)u_0 - z_d\big) + 2N(x,y)v_0\widetilde{v} + c^2(x,y)\widetilde{u}^2 + N(x,y)\widetilde{v}^2 \Big] dx\, dy \end{aligned} \tag{10}$$

†By a weak solution of $Lu = f$, we mean the following: Given that $f \in L_2(D)$, $Lu = f$ makes sense if $D^\alpha u \in L_2(D)$. As the derivatives are considered in the generalised sense, then it follows from the definition of the Sobolev's space that

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2} \in L_2(D).$$

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From relationships (9)-(10), one obtains for the increment the following expression:

$$
\begin{aligned}
&\widetilde{I} = I(v_\epsilon) - I(v_0) = \\
&= \iint\limits_D \Psi(x,y)\left(\frac{\partial^2 \widetilde{u}}{\partial x^2} + \frac{\partial^2 \widetilde{u}}{\partial y^2} - q(x,y)\widetilde{u}\right) dx\,dy - \iint\limits_D a(x,y)\Psi(x,y)\widetilde{v}\,dx\,dy + \\
&\quad + \iint\limits_D \Big[2c(x,y)\widetilde{u}\big(c(x,y)u_0 - z_d\big) + N(x,y)\widetilde{v}^2 + 2N(x,y)v_0\widetilde{v} + c^2(x,y)\widetilde{u}\Big]\,dx\,dy = \\
&= \iint\limits_D \Psi(x,y)\left[\frac{\partial^2 \widetilde{u}}{\partial x^2} + \frac{\partial^2 \widetilde{u}}{\partial y^2}\right] dx\,dy + \\
&\qquad + \iint\limits_D \Big[\big(c(x,y)u_0 - z_d(x,y) - q(x,y)\Psi\big)2c(x,y)\widetilde{u}\Big]\,dx\,dy + \\
&\qquad\quad + \iint\limits_D \Big[\big(2N(x,y)v_0 - a(x,y)\Psi\big)\widetilde{v} + c^2(x,y)\widetilde{u}^2 + N(x,y)\widetilde{v}^2\Big]\,dx\,dy
\end{aligned}
\tag{11}
$$

To obtain the adjoint equation ‡, let us make the following transformations:

$$
\begin{aligned}
&\int_0^{l_1}\!\!\int_0^{l_2} \Psi(x,y)\frac{\partial^2 \widetilde{u}}{\partial x^2}\,dx\,dy = \int_0^{l_2}\left(\int_0^{x_0}\Psi(x,y)\frac{\partial^2 \widetilde{u}}{\partial x^2}\,dx + \int_{x_0}^{l_1}\Psi(x,y)\frac{\partial^2 \widetilde{u}}{\partial x^2}\,dx\right) dy = \\
&= \int_0^{l_2}\Bigg(\Psi(l_1,y)\widetilde{u}_x(l_1,y) - \Psi(0,y)\widetilde{u}_x(0,y) + \big(\Psi(x_0^-,y) - \Psi(x_0^+,y)\big)\,\widetilde{u}_x(x_0,y) + \\
&\qquad + \big(\Psi_x(x_0^-,y) - \Psi_x(x_0^+,y) - \sigma\Psi_x(l_1,y)\big)\,\widetilde{u}(x_0,y) + \int_0^{l_1}\frac{\partial^2 \Psi}{\partial x^2}\widetilde{u}(x,y)\,dx\Bigg)\,dy
\end{aligned}
$$

In a similar way the following is obtained:

$$
\int_0^{l_1}\!\!\int_0^{l_2} \Psi(x,y)\frac{\partial^2 \widetilde{u}}{\partial y^2}\,dx\,dy = \int_0^{l_1}\left(\int_0^{l_2}\widetilde{u}(x,y)\frac{\partial^2 \Psi}{\partial y^2}\,dy + \Psi(x,l_2)\widetilde{u}_y(x,l_2) - \Psi(x,0)\widetilde{u}_y(x,0)\right) dx
$$

Since $\Psi \in H^2(D\backslash\gamma_0) \cap H^1(D)$, according to the embedding theory§, one has $\Psi(x_0,y) \in \mathcal{C}(D)$¶.

‡Corresponding to the homogeneous boundary value problem $L\widetilde{u} = f$, $B_1(\widetilde{u}) = B_2(\widetilde{u}) = 0$, where B_1 and B_2 are functions of the unknown solution $\widetilde{u}$ there is an adjoint boundary value problem:

$$
\begin{aligned}
L^*\Psi &= f \\
B_1^*(\Psi) = B_2^*(\Psi) &= 0
\end{aligned}
$$

These are referred to as adjoint boundary conditions [20].

§We say the normed space H^p is embedded in the normed space H^q if the two following conditions hold:

1. $p > q$, that is to say, H^p is a vector subspace of H^q,
2. the identity operator I defined on H^p into H^q by $I(u) = u$ for all $u \in H^p$ is continuous.

Since I is linear, 2. is equivalent to the existence of a constant M such that

$$\|I(u)\|_{H^q} \leq M\|u\|_{H^p}, \qquad u \in H^p$$

¶$\mathcal{C}(D)$ denotes the set of continous bounded functions defined on D.

Therefore, by grouping terms in the above expressions it is possible to conclude that if Ψ is a solution to the following problem:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - q(x,y)\Psi = -2c(x,y)\left(c(x,y)u_0 - z_d(x,y)\right), \quad (x,y) \in D\backslash\gamma_0 \tag{12}$$

$$\begin{gathered}\Psi(x,y) = 0, \qquad (x,y) \in \partial D, \\ \Psi_x(x_0^+, y) - \Psi_x(x_0^-, y) = \sigma\Psi_x(l_1, y), \quad 0 \le y \le l_2\end{gathered} \tag{13}$$

then the increment of the functional will take the form

$$\widetilde{I} = \iint\limits_D \left(2N(x,y)v_0 - a(x,y)\Psi\right)\widetilde{v}\,dx\,dy + \iint\limits_D \left(c^2(x,y)\widetilde{u}^2 + N(x,y)\widetilde{v}^2\right)dx\,dy \tag{14}$$

Integrating partially equation (12) and using the properties of Dirac's distribution [20], one obtains the following problem equivalent to problem (12)-(13):

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - q(x,y)\Psi = -2c(x,y)\left(c(x,y)u_0 - z_d\right) - \delta(x_0 - x)\sigma\Psi_x(l_1, y), \quad (x,y) \in D, \tag{15}$$

$$\Psi(x,y) = 0, \qquad (x,y) \in \partial D, \tag{16}$$

where $\delta(x_0 - x)$ is the Dirac distribution.

Theorem 1
Let Ψ_0 be a solution in weak sense to the adjoint problem (12)-(13) and $N(x,y) > 0$, then for (u_0, v_0) to be optimal a necessary and sufficient condition is that the following relation be true almost everywhere on D:

$$2N(x,y)v_0 - a(x,y)\Psi_0 = 0. \tag{17}$$

Proof
Let (u_0, v_0) be an optimal pair and $N(x,y) > 0$. Let it be shown that condition (17) holds. Supposing the contrary, that is to say

$$2N(x,y)v_0 - a(x,y)\Psi_0 \neq 0$$

on a set of positive Lebesgue's measure.

Therefore

$$\begin{aligned}0 &< \mu\left[\{(x,y) \in D/2N(x,y)v_0 - a(x,y)\Psi_0 \neq 0\}\right] = \\ &= \mu\left[\{(x,y) \in D/2N(x,y)v_0 - a(x,y)\Psi_0 > 0\} \cup \{(x,y) \in D/2N(x,y)v_0 - a(x,y)\Psi_0 < 0\}\right] = \\ &= \mu\left[\{(x,y) \in D/2N(x,y)v_0 - a(x,y)\Psi_0 > 0\}\right] + \mu\left[\{(x,y) \in D/2N(x,y)v_0 - a(x,y)\Psi_0 < 0\}\right]\end{aligned}$$

where μ is the Lebesgue's measure on D.

Let us introduce notation

$$\begin{aligned}D_+ &= \left[\{(x,y) \in D/2N(x,y)v_0 - a(x,y)\Psi_0 > 0\}\right] \\ D_- &= \left[\{(x,y) \in D/2N(x,y)v_0 - a(x,y)\Psi_0 < 0\}\right]\end{aligned}$$

Consider two cases: $\mu(D_+) > 0$, $\mu(D_-) > 0$.

Suppose $\mu(D_+) > 0$. As V is open, there exists k_0 such that $v_\epsilon = v_0 + k\chi_{D_+} \in V_{ad}$ with $|k| \le k_0$ where χ_{D_+} denotes the characteristic function of set D_+.

Denote by

$$T = 2N(x,y)v_o - a(x,y)\Psi_0,$$

then there exists $\epsilon > 0$, such that $\iint\limits_{D_+} T\,dx\,dy > \epsilon$.

Let $k < 0$, $\widetilde{v} = k\chi_{D_+}$ and $\widetilde{u}(x, y, k\chi_{D_+})$ be the corresponding solution to problem (7)-(8). Then it follows that

$$\widetilde{u}(x, y, k\chi_{D_+}) = k\widetilde{u}(x, y, \chi_{D_+}) \tag{18}$$

Now, taking into account (18) and $\widetilde{v} = k\chi_{D_+}$, the following equalities are obtained from (14):

$$\iint\limits_D (2N(x,y)v_0 - a(x,y))\,\widetilde{v}\,dx\,dy = \iint\limits_D Tk\chi_{D_+}\,dx\,dy = k\iint\limits_{D_+} T\,dx\,dy,$$

$$\iint\limits_D N(x,y)\widetilde{v}^2\,dx\,dy = \iint\limits_D N(x,y)(k\chi_{D_+})^2\,dx\,dy = k^2\iint\limits_{D_+} N(x,y)\,dx\,dy.$$

$$\iint\limits_D c^2(x,y)\widetilde{u}^2\,dx\,dy = \iint\limits_D c^2(x,y)\widetilde{u}^2(x,y,k\chi_{D_+})\,dx\,dy = k^2\iint\limits_D c^2(x,y)\widetilde{u}^2(x,y,\chi_{D_+})\,dx\,dy.$$

Since v_0 is optimal, then for a sufficiently small increment $\widetilde{v}$: $I(\widetilde{v} + v_0) - I(v_0) = I(v_\epsilon) - I(v_0) \geq 0$.

Taking into account the above expressions, the increment of functional (14) can be written in the following form:

$$I(v_\epsilon) - I(v_0) = I(v_0 + \widetilde{v}) - I(v_0) = \iint\limits_D T\widetilde{v}\,dx\,dy + \iint\limits_D N(x,y)\widetilde{v}^2\,dx\,dy + \iint\limits_D c^2(x,y)\widetilde{u}\,dx\,dy =$$

$$= k\left(\iint\limits_{D_+} T\,dx\,dy + k\left(\iint\limits_{D_+} N(x,y)\,dx\,dy + \iint\limits_D c^2(x,y)\widetilde{u}^2(x,y,\chi_{D_+})\,dx\,dy\right)\right).$$

There exists $\delta > 0$ such that for $-\delta \leq k < 0$ one has

$$\left|k\left(\iint\limits_{D_+} N(x,y)\,dx\,dy + \iint\limits_D c^2(x,y)\widetilde{u}^2(x,y,\chi_{D_+}\,dx\,dy\right)\right| < \frac{\epsilon}{2}$$

Consequently, for $-\delta \leq k < 0$ and taking into account that $\iint_{D_+} T\,dx\,dy > \epsilon$, one has

$$0 < \frac{\epsilon}{2} < \iint\limits_{D_+} T\,dx\,dy + k\left(\iint\limits_{D_+} N(x,y)\,dx\,dy + \iint\limits_D c^2(x,y)\widetilde{u}^2(x,y,\chi_{d_+})\,dx\,dy\right)$$

and

$$k\left(\iint\limits_{D_+} T\,dx\,dy + k\left(\iint\limits_{D_+} N(x,y)\,dx\,dy + \iint\limits_D c^2(x,y)\widetilde{u}^2(x,y,\chi_{D_+})\,dx\,dy\right)\right) < 0$$

that is, for $-\delta \leq k < 0$: $I(v_0 + k\chi_{D_+}) - I(v_0) < 0$.

Taking into account that for $|k| \leq k_0$:

$$v_\epsilon = v_0 + k\chi_{D_+} \in V_{ad} \text{ and } \lim_{k\to 0}\|k\chi_{D_+}\|_{L_2(D)} = 0$$

this contradicts the optimality of v_0. Hence $\mu(D_+) = 0$. $\mu(D_-) = 0$ is proved in a similar way.

Thus, the necessary condition of optimality is proved.

To prove sufficiency: Let $2N(x,y)v_0 - a(x,y)\Psi_0 = 0$ almost everywhere on D and $N(x,y) > 0$. From these assumptions, it then follows that for each v_ϵ

$$I(v_\epsilon) - I(v_0) \geq 0$$

i.e., (u_0, v_0) is the optimal pair. □

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3. EXISTENCE AND UNIQUENESS

On the basis of the obtained conditions of optimality the proof of the existence and uniqueness of the solution to problem (1)-(2)-(5) is reduced to the proof of the existence and uniqueness of the solution to the following systems:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} - q(x,y)u = \dfrac{a^2(x,y)}{2N(x,y)}\Psi + b(x,y), & (x,y) \in D, \\ u(x,y) = 0, & (x,y) \in \partial D \backslash \gamma, \\ u(l_1,y) = \sigma u(x_0,y), & 0 \le y \le l_2. \end{cases} \tag{19}$$

$$\begin{cases} \dfrac{\partial^2 \Psi}{\partial x^2} + \dfrac{\partial^2 \Psi}{\partial y^2} - q(x,y)\Psi = -2c(x,y)\left(c(x,y)u - z_d(x,y)\right), & (x,y) \in D \backslash \gamma_0 \\ \Psi(x,y) = 0, & (x,y) \in \partial D \\ \Psi_x(x_0^-,y) - \Psi_x(x_0^+,y) = \sigma \Psi_x(l_1,y), & 0 \le y \le l_2, \end{cases} \tag{20}$$

where

A1: $a \in L_\infty(D)$, $b \in L_2(D)$, $q \in L_\infty(D)$, $c \in L_\infty(D)$, $z_d \in L_2(D)$,
A2: $\frac{a^2(x,y)}{2N(x,y)} \in L_\infty(D)$, $0 < \sigma < 1$, $\sigma, \mu_i =$ const, $i = 0,1,2$, where

$$\mu_0 = \inf\left\{\mu > 0 : q(x,y) \le \mu, \text{ for a.e. } (x,y) \in D\right\},$$
$$\mu_1 = \inf\left\{\mu > 0 : \frac{a^2(x,y)}{2N(x,y)} \le \mu, \text{ for a.e. } (x,y) \in D\right\},$$
$$\mu_2 = \inf\left\{\mu > 0 : 2c^2(x,y) \le \mu, \text{ for a.e. } (x,y) \in D\right\},$$

then $0 \le q(x,y) \le \mu_0$, $\frac{a^2(x,y)}{2N(x,y)} \le \mu_1$ and $2c^2(x,y) \ge \mu_2$ for a.e $(x,y) \in D$.

Let us consider the following iterative procedure, taking into account the equivalence of (12)-(13) and (15)-(16):

$$\begin{cases} \dfrac{\partial^2 u^{k+1}}{\partial x^2} + \dfrac{\partial^2 u^{k+1}}{\partial y^2} - q(x,y)u^{k+1} = \dfrac{a^2(x,y)}{2N(x,y)}\Psi^k + b(x,y), & (x,y) \in D, \\ u^{k+1}(x,y) = 0, & (x,y) \in \partial D \backslash \gamma \\ u^{k+1}(l_1,y) = \sigma u^k(x_0,y), & 0 \le y \le l_2. \end{cases} \tag{21}$$

$$\begin{cases} \dfrac{\partial^2 \Psi^{k+1}}{\partial x^2} + \dfrac{\partial^2 \Psi^{k+1}}{\partial y^2} - q(x,y)\Psi^{k+1} = & (x,y) \in D \\ \quad = -2c(x,y)\big(c(x,y)u^k - z_d(x,y)\big) - \delta(x_0 - x)\sigma \Psi_x^k(l_1,y), & \\ \Psi^{k+1}(x,y) = 0, \qquad (x,y) \in \partial D, & k = 0,1,2,\ldots \end{cases} \tag{22}$$

where (u^0, Ψ^0) is a initial approximation.

Let $\Psi^{k+1} = \Psi_1^{k+1} + \Psi_2^{k+1}$, where Ψ_1^{k+1} is the solution to the problem

$$\begin{cases} \dfrac{\partial^2 \Psi_1^{k+1}}{\partial x^2} + \dfrac{\partial^2 \Psi_1^{k+1}}{\partial y^2} - q(x,y)\Psi_1^{k+1} = -2c(x,y)\big(c(x,y)u^k - z_d(x,y)\big) & (x,y) \in D \\ \Psi_1^{k+1}(x,y) = 0, & (x,y) \in \partial D, \qquad k = 0,1,2,\ldots \end{cases} \tag{23}$$

and Ψ_2^{k+1} is the solution to the following problem:

$$\begin{cases} \dfrac{\partial^2 \Psi_2^{k+1}}{\partial x^2} + \dfrac{\partial^2 \Psi_2^{k+1}}{\partial y^2} - q(x,y)\Psi_2^{k+1} = -\delta(x_0 - x)\sigma \Psi_x^k(l_1,y), & (x,y) \in D \\ \Psi_2^{k+1}(x,y) = 0, & (x,y) \in \partial D, \end{cases} \tag{24}$$

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Let $z^k = \Psi^{k+1} - \Psi^k = (\Psi_1^{k+1} - \Psi_1^k) + (\Psi_2^{k+1} - \Psi_2^k) = z_1^k + z_2^k$, $t^k = u^{k+1} - u^k$, then from (21), (23), (24) one has

$$\begin{cases} \dfrac{\partial^2 t^k}{\partial x^2} + \dfrac{\partial^2 t^k}{\partial y^2} - q(x,y)t^k = \dfrac{a^2(x,y)}{2N(x,y)} t^{k-1}, & (x,y) \in D, \\ t^k(x,y) = 0, & (x,y) \in \partial D \backslash \gamma \\ t^k(l_1, y) = \sigma t^{k-1}(x_0, y), \qquad 0 \le y \le l_2. \end{cases} \tag{25}$$

$$\begin{cases} \dfrac{\partial^2 z_1^k}{\partial x^2} + \dfrac{\partial^2 z_1^k}{\partial y^2} - q(x,y)z_1^k = -2c^2(x,y)t^{k-1} & (x,y) \in D \\ z_1^k(x,y) = 0, & (x,y) \in \partial D, \end{cases} \tag{26}$$

$$\begin{cases} \dfrac{\partial^2 z_2^k}{\partial x^2} + \dfrac{\partial^2 z_2^k}{\partial y^2} - q(x,y)z_2^k = -\delta(x_0 - x)\sigma z_x^{k-1}(l_1, y), & (x,y) \in D \\ z_2^k(x,y) = 0, & (x,y) \in \partial D, \end{cases} \tag{27}$$

Let $f(x,y) = -\delta(x_0 - x)\sigma z_x^{k-1}(l_1, y)$.

Since $f \in H^{-1}(D)$ (H^{-1} is the dual space of the Sobolev's H^1 space), the generalised solution to problem (27) exists, is unique and belongs to Sobolev's space $\overset{\circ}{H}{}^1(D)$ [21]. Moreover, if $G(x,y,\xi,\eta)$ is Green's function, the solution can be presented in the following form[‖] [22, 23]:

$$z_2^k(x,y) = \int_0^{l_1}\!\!\int_0^{l_2} G(x,y,\xi,\eta)\delta(x - x_0)\sigma z_x^{k-1}(l_1,\eta)\, d\xi\, d\eta = \int_0^{l_2} G(x,y,x_0,\eta)\sigma z_x^{k-1}(l_1,\eta)\, d\eta \tag{28}$$

Furthermore, taking into account the properties of Green's function [23], it can be concluded that $z_2^k(x,y) \in \overset{\circ}{H}{}^2(D\backslash\gamma_0) \cap \overset{\circ}{H}{}^1(D)$. Therefore, the trace of function $z_x^k(l_1, y)$ is defined, which belongs to Sobolev's space $H^{1/2}(\partial D)$. Let $\|z_2^k\|_{H^1(D)}$ be estimated. To this end, it is first necessary to estimate $\|z_2^k\|_{L_2(D)}$. From relationship (28), using the Cauchy inequality one gets:

$$|z_2^k(x,y)| = \left| \int_0^{l_2} G(x,y,x_0,\eta)\sigma z_x^{k-1}(l_1,\eta)\, d\eta \right| \le \|G(x,y,x_0,\cdot)\|_{L_2[0,l_2]} \cdot \sigma \left\| z_x^{k-1}(l_1,\cdot) \right\|_{L_2[0,l_2]}$$

Hence

$$\|z_2^k\|_{L_2(D)} \le \left(\int_0^{l_1}\!\!\int_0^{l_2} \|G(x,y,x_0,\cdot)\|^2_{L_2[0,l_2]} \sigma^2 \left\| z_x^{k-1}(l_1,\cdot) \right\|^2_{L_2[0,l_2]} dx\, dy \right)^{\frac{1}{2}} =$$
$$= \sigma \|z_x^{k-1}(l_1,\cdot)\|_{L_2[0,l_2]} \|G(\cdot,\cdot,x_0,\cdot)\|_{L_2(D\times[0,l_2])}. \tag{29}$$

Taking into account $z = z_1 + z_2$ and according to the embedding theory, from (29) one gets

$$\|z_2^k\|_{L_2(D)} \le \left(\sigma \|z_{2x}^{k-1}(l_1,\cdot)\|_{L_2[0,l_2]} + \sigma M \|z_1^{k-1}\|_{H^2(D)}\right) \cdot \|G(\cdot,\cdot,x_0,\cdot)\|_{L_2(D\times[0,l_2])},$$

where $M = \text{const} > 0$ independent from z_1^{k-1}.

Denote: $T = z_{2x}(l_1,\cdot)$, one obtains:

$$\|z_2^k\|_{L_2(D)} \le \left(\sigma \|T^{k-1}\|_{L_2[0,l_2]} + \sigma M \|z_1^{k-1}\|_{H^2(D)}\right) \|G(\cdot,\cdot,x_0,\cdot)\|_{L_2(D\times[0,l_2])} \tag{30}$$

[‖] This formula is a consequence of the definition of Green's function: If we consider a problem $Lu = f$ for domain D and $u = 0$ on ∂D, then Green's function satisfies homogeneous equation and the solution can be expressed as (28). If operator L is Laplacian ($\Delta u = f$), then G is a harmonic function and satisfies homogeneous boundary conditions ($G|_{\partial D} = 0$).

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Now, let us estimate $\|T^k\|_{L_2[0,l_2]}$. Let us differentiate (28) with respect to x, then, substitute $x = l_1$ and use the Cauchy's inequality:

$$|z_{2x}^k(l_1, y)| \le \left| \int_0^{l_2} G_x(l_1, y, x_0, \eta)\sigma z_x^{k-1}(l_1, \eta)\, d\eta \right| \le \|G_x(l_1, y, x_0, \cdot)\|_{L_2[0,l_2]}\sigma\|z_x^{k-1}(l_1, \cdot)\|^2_{L_2[0,l_2]}.$$

Hence, one obtains

$$\|T^k\|_{L_2[0,l_2]} \le \left(\int_0^{l_2} \|G_x(l_1, y, x_0, \cdot)\|^2_{L_2[0,l_2]} \cdot \sigma^2\|z_x^{k-1}(l_1, \cdot)\|^2_{L_2[0,l_2]}\, dy \right)^{\frac{1}{2}} =$$
$$= \sigma\|z_x^{k-1}(l_1, \cdot)\|_{L_2[0,l_2]} \cdot \|G_x(l_1, \cdot, x_0, \cdot)\|_{L_2([0,l_2]\times[0,l_2])}.$$

Denote: $\widetilde{\Theta} = \|G_x(l_1, \cdot, x_0, \cdot)\|_{L_2([0,l_2]\times[0,l_2])}$.
As $z = z_1 + z_2$ one gets:

$$\|T^k\|_{L_2[0,l_2]} \le \widetilde{\Theta}\left(\sigma\|T^{k-1}\|_{L_2[0,l_2]} + M\sigma\|z_1^{k-1}\|_{H^2(D)}\right) \tag{31}$$

Analogously, for $\|z_{2x}^k\|_{L_2(D)}$ and $\|z_{2y}^k\|_{L_2(D)}$ one has

$$\|z_{2x}^k\|_{L_2(D)} \le \left(\sigma\|T^{k-1}\|_{L_2[0,l_2]} + M\sigma\|z_1^{k-1}\|_{H^2(D)}\right) \cdot \|G_x(\cdot, \cdot, x_0, \cdot)\|_{L_2(D\times[0,l_2])} \tag{32}$$
$$\|z_{2y}^k\|_{L_2(D)} \le \left(\sigma\|T^{k-1}\|_{L_2[0,l_2]} + M\sigma\|z_1^{k-1}\|_{H^2(D)}\right) \cdot \|G_y(\cdot, \cdot, x_0, \cdot)\|_{L_2(D\times[0,l_2])} \tag{33}$$

From (30), (32), (33) one obtains

$$\|z_2^k\|_{H^1(D)} \le \left(\sigma\|T^{k-1}\|_{L_2[0,l_2]} + M\sigma\|z_1^{k-1}\|_{H^2(D)}\right) \times$$
$$\times \Big(\|G_y(\cdot, \cdot, x_0, \cdot)\|^2_{L_2(D\times[0,l_2])} + \|G_x(\cdot, \cdot, x_0, \cdot)\|^2_{L_2(D\times[0,l_2])} +$$
$$+ \|G(\cdot, \cdot, x_0, \cdot)\|^2_{L_2(D\times[0,l_2])}\Big)^{\frac{1}{2}} \tag{34}$$

Let us denote

$$\Theta = \|G(\cdot, \cdot, x_0, \cdot)\|_{H^1(D\times[0,l_2])}$$

Then, from (34) one has

$$\|z_2^k\|_{H^1(D)} \le \Theta\left(\sigma\|T^{k-1}\|_{L_2[0,l_2]} + M\sigma\|z_1^{k-1}\|_{H^2(D)}\right) \tag{35}$$

For solving problem (25)-(26) the following estimations [21] are obtained:

$$\|t^k\|_{\overset{*}{H}^2(D)} \le M'\left(\|\sigma t^{k-1}\|_{\overset{*}{H}^2(D)} + \left\| \frac{a^2(x, y)}{2N(x, y)} z^{k-1} \right\|_{L_2(D)} \right) \le$$
$$\le M'\left(\sigma\|t^{k-1}\|_{\overset{*}{H}^2(D)} + \mu_1\left(\|z_2^{k-1}\|_{H^1(D)} + \|z_1^{k-1}\|_{H^2(D)}\right) \right) \tag{36}$$

$$\|z_1^k\|_{H^2(D)} \le M'\|2c^2(x, y)t^{k-1}\|_{L_2(D)} \le M'\mu_2\|t^{k-1}\|_{\overset{*}{H}^2(D)} \tag{37}$$

where $\mu_i = \text{const}, i = 1, 2$, $M' = \min(\mu_0/2; 1 - 1/(2\mu_0))$.
From relationships (31), (35), (36), (37) one gets

$$X^k \le AX^{k-1} \tag{38}$$

where vector $X^k = (\|t^k\|_{\overset{*}{H}^2(D)}, \|z_1^k\|_{H^2(D)}, \|z_2^k\|_{H^1(D)}, \|T^k\|_{L_2[0,l_2]})^T$ and the matrix

$$A = \begin{pmatrix} M'\sigma & M'\mu_1 & M'\mu_1 & 0 \\ M'\mu_2 & 0 & 0 & 0 \\ 0 & \Theta\sigma M & 0 & \Theta\sigma \\ 0 & \widetilde{\Theta}\sigma M & 0 & \widetilde{\Theta}\sigma \end{pmatrix}$$

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A3: Let σ, M, M', μ_1, μ_2 be such that a_{ij}-elements matrix A satisfy the condition $0 \leq a_{ij} < \bar{q}$, $0 < \bar{q} < 1/4$.

Since $a_{ij} \geq 0$, then from (38) one obtains:

$$X^k \leq AX^{k-1} \leq \ldots \leq A^k X^0 \tag{39}$$

By induction it is not difficult to prove that

$$0 \leq a_{ij}^{(k)} \leq 4^{k-1} q^{-k}, \tag{40}$$

where $a_{ij}^{(k)}$ are elements of matrix A^k.

Let it be shown that sequences $\{u^n\}$, $\{\Psi^n\}$ are fundamental. In fact

$$\begin{aligned}
&\left(\|u^{n+m} - u^n\|_{\overset{*}{H}^2(D)}, \|\Psi_1^{n+m} - \Psi_1^n\|_{H^2(D)}, \|\Psi_2^{n+m} - \Psi_2^n\|_{H^1(D)}, \|\Psi_{2x}^{n+m}(l_1, y) - \Psi_{2x}^n(l_1, y)\|_{L_2[0,l_2]}\right)^T \leq \\
&\leq \Big(\|u^{n+m} - u^{n+m-1}\|_{\overset{*}{H}^2(D)}, \|\Psi_1^{n+m} - \Psi_1^{n+m-1}\|_{H^2(D)}, \|\Psi_2^{n+m} - \Psi_2^{n+m-1}\|_{H^1(D)}, \\
&\qquad \|\Psi_{2x}^{n+m}(l_1, y) - \Psi_{2x}^{n+m-1}(l_1, y)\|_{L_2[0,l_2]}\Big)^T + \cdots + \\
&+ \left(\|u^{n+1} - u^n\|_{\overset{*}{H}^2(D)}, \|\Psi_1^{n+1} - \Psi_1^n\|_{H^2(D)}, \|\Psi_2^{n+1} - \Psi_2^n\|_{H^1(D)}, \|\Psi_{2x}^{n+1}(l_1, y) - \Psi_{2x}^n(l_1, y)\|_{L_2[0,l_2]}\right)^T = \\
&= X^{n+m-1} + X^{n+m-2} + \ldots + X^n
\end{aligned} \tag{41}$$

Taking into account (39) and (41), one gets

$$\begin{aligned}
&\left(\|u^{n+m} - u^n\|_{\overset{*}{H}^2(D)}, \|\Psi_1^{n+m} - \Psi_1^n\|_{H^2(D)}, \|\Psi_2^{n+m} - \Psi_2^n\|_{H^1(D)}, \|\Psi_{2x}^{n+m}(l_1, y) - \Psi_{2x}^n(l_1, y)\|_{L_2[0,l_2]}\right)^T \leq \\
&\leq \Big(A^{n+m-1} + A^{n+m-2} + \ldots + A^n\big) \cdot \big(\|u^1 - u^0\|_{\overset{*}{H}^2(D)}, \|\Psi_1^1 - \Psi_1^0\|_{H^2(D)}, \\
&\qquad \|\Psi_2^1 - \Psi_2^0\|_{H^1(D)}, \|\Psi_{2x}^1(l_1, y) - \Psi_{2x}^0(l_1, y)\|_{L_2[0,l_2]}\big)^T
\end{aligned} \tag{42}$$

Denote

$$B = A^{n+m-1} + A^{n+m-2} + \ldots + A^n \tag{43}$$

From (42) and (43) one has:

$$\begin{aligned}
\|u^{n+m} - u^n\|_{\overset{*}{H}^2(D)} &\leq b_{11}\|t^0\|_{\overset{*}{H}^2(D)} + b_{12}\|z_1^0\|_{H^2(D)} + b_{13}\|z_2^0\|_{H^1(D)} + b_{14}\|T^0\|_{L_2[0,l_2]} \\
\|\Psi_1^{n+m} - \Psi_1^n\|_{H^2(D)} &\leq b_{21}\|t^0\|_{\overset{*}{H}^2(D)} + b_{22}\|z_1^0\|_{H^2(D)} + b_{23}\|z_2^0\|_{H^1(D)} + b_{24}\|T^0\|_{L_2[0,l_2]} \\
\|\Psi_2^{n+m} - \Psi_2^n\|_{H^1(D)} &\leq b_{31}\|t^0\|_{\overset{*}{H}^2(D)} + b_{32}\|z_1^0\|_{H^2(D)} + b_{33}\|z_2^0\|_{H^1(D)} + b_{34}\|T^0\|_{L_2[0,l_2]} \\
\|\Psi_{2x}^{n+m}(l_1, y) - \Psi_{2x}^n(l_1, y)\|_{L_2[0,l_2]} &\leq b_{41}\|t^0\|_{\overset{*}{H}^2(D)} + b_{42}\|z_1^0\|_{H^2(D)} + b_{43}\|z_2^0\|_{H^1(D)} + b_{44}\|T^0\|_{L_2[0,l_2]}
\end{aligned} \tag{44}$$

For each element b_{ij}, $i, j = 1, 2, 3, 4$ of matrix B it can be obtained the following estimation:

$$0 \leq b_{ij} \leq \frac{1}{4}(4\bar{q})^n \frac{1 - (4\bar{q})^m}{1 - 4\bar{q}}, \qquad 0 < 4\bar{q} < 1, \qquad n, m \in \mathbb{N} \tag{45}$$

From (44) and (45) one gets:

$$\begin{aligned}
&\|u^{n+m} - u^n\|_{\overset{*}{H}^2(D)} \leq \\
&\qquad \leq (4\bar{q})^n \cdot \frac{1 - (4\bar{q})^m}{4(1 - 4\bar{q})} (\|t^0\|_{\overset{*}{H}^2(D)} + \|z_1^0\|_{H^2(D)} + \|z_2^0\|_{H^1(D)} + \|T^0\|_{L_2[0,l_2]]}),
\end{aligned}$$

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$$\|\Psi^{n+m} - \Psi^n\|_{H^1(D)} = \|(\Psi_1+\Psi_2)^{n+m} - (\Psi_1+\Psi_2)^n\|_{H^1(D)} \leq$$
$$\leq \|\Psi_1^{n+m} - \Psi_1^n\|_{H^2(D)} + \|\Psi_2^{n+m} - \Psi_2^n\|_{H^1(D)} \leq$$
$$\leq (4\bar{q})^n \frac{1-(4\bar{q})^m}{4(1-4\bar{q})} (\|t^0\|_{\overset{*}{H}^2(D)} + \|z_1^0\|_{H^2(D)} + \|z_2^0\|_{H^1(D)} + \|T^0\|_{L_2[0,l_2]}).$$

Since $0 < 4\bar{q} < 1$, this value can be as small as one wishes with n being sufficiently large. Since spaces $\overset{*}{H}^2$, H^1 are complete fundamental sequences $\{u^n\}$, $\{\Psi^n\}$ have a limit, i.e., $\exists u \in \overset{*}{H}^2(D), \Psi \in \overset{\circ}{H}^2(D\backslash\gamma_0) \cap \overset{\circ}{H}^1(D)$ such that $u = \lim_{n\to\infty} u^n$ and $\Psi = \lim_{n\to\infty} \Psi^n$.

On the other hand, since

$$\|u\|^2_{\overset{*}{H}^2(D)} = \|u\|^2_{L_2(D)} + \|u_x\|^2_{L_2(D)} + \|u_y\|^2_{L_2(D)} + 2\|u_{xy}\|^2_{L_2(D)} + \|u_{xx}\|^2_{L_2(D)} + \|u_{yy}\|^2_{L_2(D)},$$

one has

$$\|u^k - u\|^2_{\overset{*}{H}^2(D)} \geq \|u^k_{xx} - u_{xx}\|^2_{L_2(D)}.$$

Since $\lim_{k\to 0} \|u^k - u\|_{\overset{*}{H}^2(D)} = 0$, then one gets

$$0 \leq \lim_{k\to\infty} \|u^k_{xx} - u_{xx}\|_{L_2(D)} \leq \lim_{k\to\infty} \|u^k - u\|_{\overset{*}{H}^2(D)} = 0$$

Hence $\lim_{k\to\infty} \|u^k_{xx} - u_{xx}\|_{L_2(D)} = 0$, which means $\lim_{k\to\infty} u^k_{xx} = u_{xx}$ in $L_2(D)$. Analogously one gets $\lim_{k\to\infty} u^k_{yy} = u_{yy}$ in $L_2(D)$. Taking limits in (21) as $k\to\infty$ in $L_2(D)$ one obtains

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - q(x,y)u = \frac{a^2(x,y)}{2N(x,y)}\Psi + b(x,y), \qquad (x,y)\in D$$

Similarly, as above and taking limits in (22) as $k\to\infty$ in $H^2(D\backslash\gamma_0)$ one gets

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} - q(x,y)\Psi = -2c(x,y)\left(c(x,y)u - z_d(x,y)\right), \qquad (x,y)\in D\backslash\gamma_0$$

Now, let the boundary conditions be considered. Let us consider the following mapping

$$\nu : [x_0, l_1]\times[0,l_2] \to I\!R$$

where $v(x,y) = u(x,y) - \sigma u(x_0,y)$.

Let the iterative process $u^k(x,y) - \sigma u^k(x_0,y) = \nu^k(x,y)$ be considered.

Since $\lim_{k\to\infty} \|u^k - u\|_{\overset{*}{H}^2(D)} = 0$, then according to the embedding theory one obtains $\lim_{k\to\infty} \nu^k(x,y) = \nu(x,y)$. The function $\nu(x,y)$ is continuous and $\nu(x,y)|_{\partial D} = 0$. Consequently $u(x,y) = \sigma u(x_0,y)$. For $x = l_1$ one gets that $u(l_1,y) = \sigma u(x_0,y)$.

Uniqueness . Let (u_1,Ψ_1) and (u_2,Ψ_2) be two pairs of solutions to system (19)-(20). Then their difference $\bar{u} = u_1 - u_2$, $\bar{\Psi} = \Psi_1 - \Psi_2$ satisfies system of equations

$$\begin{cases} \dfrac{\partial^2 \bar{u}}{\partial x^2} + \dfrac{\partial^2 \bar{u}}{\partial y^2} - q(x,y)\bar{u} = \bar{\Psi}\dfrac{a^2(x,y)}{2N(x,y)}, & (x,y)\in D, \\ \bar{u}(x,y) = 0, & (x,y)\in\partial D\backslash\gamma, \\ \bar{u}(l_1,y) = \sigma\bar{u}(x_0,y), & 0\leq y\leq l_2, \end{cases} \tag{46}$$

$$\begin{cases} \dfrac{\partial^2 \bar{\Psi}}{\partial x^2} + \dfrac{\partial^2 \bar{\Psi}}{\partial y^2} - q(x,y)\bar{\Psi} = -2c^2(x,y)\bar{u} - \delta(x_0 - x)\sigma\bar{\Psi}_x(l_1,y), & (x,y)\in D \\ \bar{\Psi}(x,y) = 0, & (x,y)\in\partial D \end{cases} \tag{47}$$

Let $\bar{\Psi} = \bar{\Psi}_1 + \bar{\Psi}_2$, where $\bar{\Psi}_1$ is the solution to the following problem:

$$\begin{cases} \dfrac{\partial^2 \bar{\Psi}_1}{\partial x^2} + \dfrac{\partial^2 \bar{\Psi}_1}{\partial y^2} - q(x,y)\bar{\Psi}_1 = -2c^2(x,y)\bar{u}, & (x,y) \in D \\ \bar{\Psi}_1(x,y) = 0, & (x,y) \in \partial D \end{cases} \tag{48}$$

and $\bar{\Psi}_2$ is the solution to the following problem:

$$\begin{cases} \dfrac{\partial^2 \bar{\Psi}_2}{\partial x^2} + \dfrac{\partial^2 \bar{\Psi}_2}{\partial y^2} - q(x,y)\bar{\Psi}_2 = -\delta(x_0 - x)\delta\bar{\Psi}_x(l_1,y), & (x,y) \in D \\ \bar{\Psi}_2(x,y) = 0, & (x,y) \in \partial D \end{cases} \tag{49}$$

One obtains the following estimations for the solutions to problem (46), (48) [21]:

$$\begin{aligned} \|\bar{u}\|_{\overset{*}{H}^2(D)} \le M' \left(\|\sigma\bar{u}\|_{\overset{*}{H}^2(D)} + \left\| \frac{a^2(x,y)}{2N(x,y)} \bar{\Psi} \right\|_{L_2(D)} \right) \le \\ \le M'(\sigma\|\bar{u}\|_{\overset{*}{H}^2(D)} + \mu_1\|\bar{\Psi}\|_{L_2(D)}) \le \\ \le M' \left(\sigma\|\bar{u}\|_{\overset{*}{H}^2(D)} + \mu_1(\|\bar{\Psi}_1\|_{H^2(D)} + \|\bar{\Psi}_2\|_{H^1(D)}) \right), \end{aligned} \tag{50}$$

$$\|\bar{\Psi}_1\|_{H^2(D)} \le M'\|2c^2(x,y)\bar{u}\|_{L_2(D)} \le M'\mu_2\|\bar{u}\|_{\overset{*}{H}^2(D)}. \tag{51}$$

where $M' = \min(\mu_0/2, 1 - 1/(2\mu_0))$. Let $\|\bar{\Psi}_2\|_{H1(D)}$ be estimated.

Taking into account the properties of Green's function, the solution to problem (49) can be presented in the following form:

$$\bar{\Psi}_2(x,y) = \int_0^{l_1}\!\!\int_0^{l_2} G(x,y,\xi,\eta)\delta(x - x_0)\sigma\Psi_x(l_1,\eta)\,d\xi\,d\eta = \int_0^{l_2} G(x,y,x_0,\eta)\sigma\Psi_x(l_1,\eta)\,d\eta \tag{52}$$

Similarly as above, the following estimations can be obtained:

$$\|\bar{\Psi}_2\|_{H^1(D)} \le \Theta\left(\sigma\|\bar{T}\|_{L_2[0,l_2]} + M\sigma\|\bar{\Psi}_1\|_{H^2(D)}\right) \tag{53}$$

$$\|\bar{T}\|_{L_2[0,l_2]} \le \widetilde{\Theta}\left(\sigma\|\bar{T}\|_{L_2[0,l_2]} + M\sigma\|\bar{\Psi}_1\|_{H^2(D)}\right) \tag{54}$$

where

$$\begin{aligned} &\widetilde{\Theta} = \|G_x(l_1,\cdot,x_0,\cdot)\|_{L_2[0,l_2]\times[0,l_2]}, \\ &M = \text{const} > 0, \qquad \Theta = \|G(\cdot,\cdot,x_0,\cdot)\|_{H^1(D\times[0,l_2])}, \qquad \bar{T} = \bar{\Psi}_{2x}(l_1,\cdot). \end{aligned}$$

and G is Green's function of problem (49)

Let system of inequalities (50), (51), (53), (54) be rewritten in the following form: $X \le AX$, where

$$A = \begin{pmatrix} M'\sigma & M'\mu_1 & M'\mu_1 & 0 \\ M'\mu_2 & 0 & 0 & 0 \\ 0 & \Theta\sigma M & 0 & \Theta\sigma \\ 0 & \widetilde{\Theta}\sigma M & 0 & \widetilde{\Theta}\sigma \end{pmatrix}$$

$$X = (\|\bar{u}\|_{\overset{*}{H}^2(D)}, \|\bar{\Psi}_1\|_{H^2(D)}, \|\bar{\Psi}_2\|_{H^1(D)}, \|\bar{T}\|_{L_2[0,l_2]})^T$$

Let norm $\|A\| = \sup\limits_{\|X\| \le 1} \|AX\|_{R_4}$ be computed. Let us show that $\|A\| < 1$, so $\|X\|_{R_4} = 0$.

$$\|AX\|_{R_4} = \big((M'\sigma x_1 + M'\mu_1 x_2 + M'\mu_1 x_3)^2 + (M'\mu_2 x_1)^2 + \\ +(\Theta\sigma M x_2 + \Theta\sigma x_4)^2 + (\widetilde{\Theta}\sigma M x_2 + \widetilde{\Theta}\sigma x_4)^2\big)^{\frac{1}{2}}$$

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Hence, taking into account $0 \leq a_{ij} < \bar{q} < 1/4$ it is not difficult to show $\|A\| < 1$. If $\|X\| \neq 0$ one gets $\|X\| < \|X\|$, since $\|A\| < 1$. Hence $\|X\| = 0$ and thus the uniqueness is proved.

Theorem 2
Suppose the following conditions hold:

A1: $a \in L_\infty(D)$, $b \in L_2(D)$, $q \in L_\infty(D)$, $c \in L_\infty(D)$, $z_d \in L_2(D)$, $N \in L_\infty(D)$.
A2: $a^2/(2N) \in L_\infty(D)$, $0 \leq q(x,y) \leq \mu_0$, $\frac{a^2(x,y)}{2N(x,y)} \leq \mu_1$, $2c^2(x,y) \leq \mu_2$, $0 < \sigma < 1$, $\sigma, \mu_i =$ const, $i = 0, 1, 2$.
A3: Let σ, M, M', μ_1, μ_2 be such that a_{ij} (elements of matrix A) satisfy the condition: $0 \leq a_{ij} < \bar{q}$, $0 < \bar{q} < 1/4$

$$A = \begin{pmatrix} M'\sigma & M'\mu_1 & M'\mu_1 & 0 \\ M'\mu_2 & 0 & 0 & 0 \\ 0 & \Theta\sigma M & 0 & \Theta\sigma \\ 0 & \widetilde{\Theta}\sigma M & 0 & \widetilde{\Theta}\sigma \end{pmatrix}$$

where

$$\begin{aligned} M' &= \min(\mu_0/2; 1 - 1/(2\mu_0)), \\ M &= \text{const} > 0, \\ \Theta &= \|G(\cdot,\cdot,x_0,\cdot)\|_{H^1(D\times[0,l_2])}, \\ \widetilde{\Theta} &= \|G_x(l_1,\cdot,x_0,\cdot)\|_{L_2[0,l_2]\times[0,l_2])} \end{aligned}$$

and G is the Green's function of the problem:

$$\begin{cases} \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} - q(x,y)z = F(x,y), & (x,y) \in D, \\ z(x,y) = 0, & (x,y) \in \partial D \end{cases}$$

Then there exists a unique solution (u, Ψ) to system (19)-(20) such that $\exists u \in \overset{*}{H}{}^2(D)$, $\exists \Psi \in \overset{\circ}{H}{}^2(D \backslash \gamma_0) \cap \overset{\circ}{H}{}^1(D)$.

Thus, the existence and uniqueness of the solution to problem (1)-(5) is proved under conditions A1, A2 and

$$\|A\| \leq \sqrt{\sum_{i,j=1}^{4} a_{ij}^2} < 1.$$

4. REMARKS

The upshot of this paper in the control theory is that under certain conditions a maximality condition is satisfied (cf. maximum principle of Pontrjagin [24]).

4.1. *Maximum principle*

Let us consider the increment of the functional obtained in (14):

$$\widetilde{I} = \iint_D (2N(x,y)v_0 - a(x,y)\Psi)\, \widetilde{v}\, dx\, dy + \iint_D \left(c^2(x,y)\widetilde{u}^2 + N(x,y)\widetilde{v}^2\right) dx\, dy$$

where $\tilde{v} = v_\epsilon - v_0$ and Ψ is the solution to the adjoint problem (12)-(13).

Let us show that the following relationship:

$$\iint_D \big(2N(x,y)v_0 - a(x,y)\Psi\big)(v_\epsilon - v_0)\, dx\, dy \geq 0, \qquad \forall v_\epsilon \in V_{ad}, \qquad \text{for a.e. } (x,y) \in D, \quad (55)$$

and

$$\inf_{v\in V}\Big[\big(2N(x,y)v_0 - a(x,y)\Psi\big)v\Big] = \Big[2N(x,y)v_0 - a(x,y)\Psi\Big]v_0, \qquad \text{for a. e. } (x,y)\in D \quad (56)$$

are equivalent.

Proof: (55)⇒(56) Let notations

$$T = 2N(x,y)v_0 - a(x,y)\Psi$$

be introduced.

Define

$$D_\delta = \{(x,y)\in D : |x-x_0|<\delta, \quad |y-y_0|<\delta\}, \qquad \delta>0$$

where (x_0,y_0) is an fixed arbitrary point in D.

Let us consider admissible control:

$$v_\delta(x,y) = \begin{cases} v_0(x,y), & (x,y)\in D\backslash D_\delta \\ v, & (x,y)\in D_\delta \end{cases}$$

where v is arbitrary point from V.

From (55) it follows that

$$\iint_D T(x,y)v_\delta\,dx\,dy \geq \iint_D T(x,y)v_0\,dx\,dy$$

Since $D = D_\delta\cup(D\backslash D_\delta)$, one has

$$\iint_{D_\delta} T(x,y)v\,dx\,dy + \iint_{D\backslash D_\delta} T(x,y)v_0\,dx\,dy \geq \iint_{D_\delta} T(x,y)v_0(x,y)\,dx\,dy + \iint_{D\backslash D_\delta} T(x,y)v_0(x,y)\,dx\,dy$$

Hence,

$$\iint_{D_\delta} T(x,y)v\,dx\,dy \geq \iint_{D_\delta} T(x,y)v_0(x,y)\,dx\,dy$$

$$\text{mes}D_\delta \leq 4\delta^2$$

Hence, one obtains

$$\lim_{\delta\to 0}\frac{1}{\text{mes}D_\delta}\iint_{D_\delta} T(x,y)v\,dx\,dy \geq \lim_{\delta\to 0}\frac{1}{\text{mes}D_\delta}\iint_{D_\delta} T(x,y)v_0(x,y)\,dx\,dy$$

According to Lebesgue's theorem

$$T(x_0,y_0)v \geq T(x_0,y_0)v_0(x_0,y_0)$$

Hence,

$$\inf_{v\in V} T(x_0,y_0)v \geq T(x_0,y_0)v_0(x,y)$$

therefore, one has

$$\inf_{v\in V} T(x,y)v = T(x,y)v_0(x,y), \quad \text{almost everywhere on } D$$

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Proof: (56)⇒(55) Assuming that

$$\inf_{v\in V} T(x,y)v = T(x,y)v_0(x,y)$$

the following expressions can be obtained

$$T(x,y)v(x,y) \geq \inf_{w\in V} T(x,y)w = T(x,y)v_0(x,y)$$
$$T(x,y)v(x,y) \geq T(x,y)v_0(x,y), \qquad \forall v \in V_{ad}$$

Therefore

$$\iint\limits_D T(x,y)(v(x,y) - v_0(x,y))\,dx\,dy \geq 0, \qquad \forall v \in V_{ad}.$$

Hence, given that by hypothesis $N > 0$, one obtains that

$$\widetilde{I} = \iint\limits_D (2N(x,y)v_0 - a(x,y)\Psi)\,\widetilde{v}\,dx\,dy + \iint\limits_D \left(c^2(x,y)\widetilde{u}^2 + N(x,y)\widetilde{v}^2\right)\,dx\,dy \geq 0, \qquad \forall v_\epsilon \in V_{ad}$$

The maximum principle is obtained, which can be formulated as the following theorem:

Theorem 3
Let the cost functional I given by formula (5) and Ψ be the solution to adjoin problem (12)-(13). Then, for optimality of pair (u_0, v_0) it is necessary and sufficient that the following relationship is satisfied:

$$\inf_{v\in V} \left[2N(x,y)v_0 - a(x,y)\Psi\right] v(x,y) = \left[2N(x,y)v_0 - a(x,y)\Psi\right] v_0(x,y)$$

almost everywhere on D.

Given that D is open, by Theorem 1 and Theorem 3 and equivalence of expressions (55) and (56), the following expressions are equivalent

- $2N(x,y)v_0 - a(x,y)\Psi = 0$, for a.e. $(x,y) \in D$,
- $\inf_{v\in V}[2N(x,y)v_0 - a(x,y)\Psi]v(x,y) = [2N(x,y)v_0 - a(x,y)\Psi]v_0(x,y)$, for a.e. $(x,y) \in D$,
- $\iint_D [2N(x,y)v_0 - a(x,y)\Psi](v - v_0)\,dx\,dy \geq 0$, $\forall v \in V_{ad}$

and provide the optimality of (u_0, v_0).

5. ON A PROBLEM OF OPTIMAL CONTROL CONNECTED WITH ELASTIC EQUILIBRIUM OF ISOTROPIC BODIES UNDER PURE BENDING

5.1. Statement of the problem

Let $\bar{\Gamma}$ be a elastic body, $\bar{\Gamma} = [0, l_1] \times [0, l_2] \times [0, l_3]$, and $U(u, v, w)$ be a displacement vector by elastic deformation with u, v, w components along the x, y, z respectively.

Furthermore, let X_X, Y_Y, Z_Z be normal stresses on the corresponding squares, X_Y, Y_Z, X_Z tangential stresses on the squares indicated by index, as shown in Figure 2, and Z be the mass force acting at the unit volume [25].

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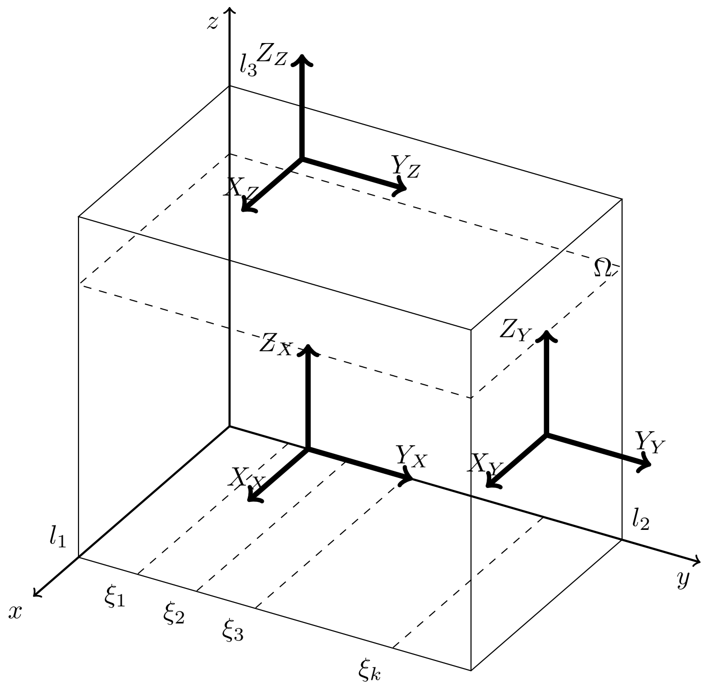

Figure 2. Normal and tangential stresses

The following boundary conditions:

$$
\begin{aligned}
z = 0, l_3: &\quad Z_Z = 0; \quad u = 0; \quad v = 0 && (57)\\
x = 0: &\quad Y_X = 0 \text{ or } v = 0; \quad X_X = 0 \text{ or } u = 0; && (58)\\
&\quad Z_X = f_1(y) \text{ or } w = f_1(y) &&\\
x = l_1: &\quad Y_X = 0 \text{ or } v = 0; \quad X_X = 0 \text{ or } u = 0; && (59)\\
&\quad Z_X = f_2(y) \text{ or } w = f_2(y) &&\\
&\quad \text{or } w(l_1, y) = \sum_{i=1}^{n} \sigma_i w(\xi_i, y) + f_2(y) &&\\
y = 0: &\quad X_Y = 0 \text{ or } u = 0; \quad Y_Y = 0 \text{ or } v = 0; && (60)\\
&\quad Z_Y = f_3(x) \text{ or } w = f_3(x) &&\\
y = l_2: &\quad X_Y = 0 \text{ or } u = 0; \quad Y_Y = 0 \text{ or } v = 0; && (61)\\
&\quad Z_Y = f_4(x) \text{ or } w = f_4(x) &&
\end{aligned}
$$

will be considered, where f_i $(i = 1, \ldots, 4)$ are surface forces, σ_i $(i = 1, \ldots, n)$ positive constants such that $\sum_{i=1}^{n} \sigma_i < 1$ and ξ_i $(i = 1, \ldots, n)$ fixed points of interval $(0, l_1)$.

The potential energy of the deformable elastic body has the form [26]

$$I = \iiint_\Gamma \frac{1}{2\mu} \left[Z_X^2 + Z_Y^2 + 2\mu Z^2 \right] \, dx \, dy \, dz \tag{62}$$

The following problem concern us: Defining a mass force Z from interval $V \subset I\!R$ for which the body equilibrium satisfies (57)-(61) and minimises the potential energy (deformability).

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The equilibrium equations for the elastic body has the following form [25]:

$$\begin{aligned}
\frac{\partial X_X}{\partial x} + \frac{\partial X_Y}{\partial y} + \frac{\partial X_Z}{\partial z} &= 0, \\
\frac{\partial Y_X}{\partial x} + \frac{\partial Y_Y}{\partial y} + \frac{\partial Y_Z}{\partial z} &= 0, \\
\frac{\partial Z_X}{\partial x} + \frac{\partial Z_Y}{\partial y} + \frac{\partial Z_Z}{\partial z} &= -Z(x,y)
\end{aligned} \tag{63}$$

The equations of Hook's Law have the form

$$\begin{aligned}
X_X &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial u}{\partial x}, \\
Y_Y &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial v}{\partial y}, \\
Z_Z &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z},
\end{aligned} \tag{64}$$

$$\begin{aligned}
X_Y = Y_X \quad &= \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \\
X_Z = Z_X \quad &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \\
Y_Z = Z_Y \quad &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),
\end{aligned} \tag{65}$$

where E is Young's modulus, $\mu(x,y)$ is modulus of shift, $\nu(x,y)$ is Poisson's ratio,

$$\lambda(x,y) = \mu\frac{E}{(1-2\nu)(1+\nu)}.$$

By virtue of (65) the boundary conditions (57) can be written in the following way:

$$z = 0, l_3: \qquad u = 0, \quad v = 0, \quad \frac{\partial w}{\partial z} = 0. \tag{66}$$

Let us find the solution to problem (57)-(61), (63)-(65) in the form:

$$u \equiv 0, \quad v \equiv 0, \quad w \equiv w(x,y) \tag{67}$$

By virtue of independence of the displacement vector from z, stemming from (63)-(65) we obtain:

$$\begin{gathered}
\frac{\partial Z_X}{\partial x} + \frac{\partial Z_Y}{\partial y} = -Z(x,y), \\
X_X = 0, \quad Y_Y = 0, \quad Z_Z = 0, \quad X_Y = Y_X = 0, \\
Z_X = X_Z = \mu\frac{\partial w}{\partial x} \\
Z_Y = Y_Z = \mu\frac{\partial w}{\partial y}
\end{gathered} \tag{68}$$

With the aim of defining the function $w(x,y)$ the expression

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) = -Z(x,y) \tag{69}$$

can be obtained from (68),

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By virtue of independence of the displacement vector from z, let us consider the arbitrary perpendicular section Ω of domain Γ to Z-axis. The boundary conditions for this section will have the form:

$$
\begin{aligned}
x = 0: &\quad \mu\frac{\partial w}{\partial x} = f_1(y) \quad \text{or} \quad w(0,y) = f_1(y),\\
x = l_1: &\quad \mu\frac{\partial w}{\partial x} = f_2(y) \quad \text{or} \quad w(l_1,y) = f_2(y) \quad \text{or} \quad w(l_1,y) = \sum_{i=1}^{n}\sigma_i w(\xi_i,y) + f_2(y),\\
y = 0: &\quad \mu\frac{\partial w}{\partial y} = f_3(y) \quad \text{or} \quad w(x,0) = f_3(x),\\
y = l_2: &\quad \mu\frac{\partial w}{\partial y} = f_4(y) \quad \text{or} \quad w(x,l_2) = f_4(x)
\end{aligned}
\tag{70}
$$

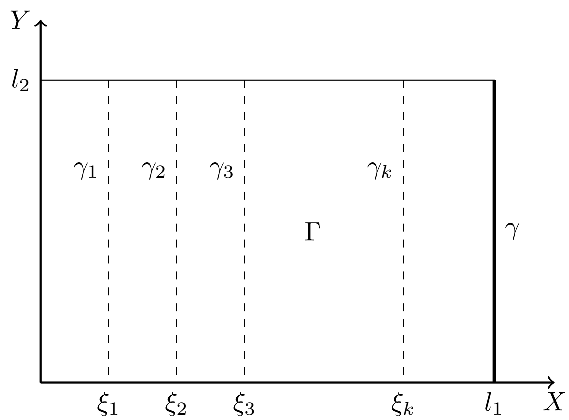

Figure 3. Horizontal section, Ω

Solving equation (69) with boundary conditions (70), all components of displacement vector from (67) and consequently all component of stress tensor from (64)-(65) can be defined. Thus, the stress tensor will be defined in an arbitrary point of elastic body $\bar{\Gamma}$.

The considered problem is connected with the investigation of optimal control problem with cost function (62) for equation (69) with non-local boundary conditions. Equation (69) is a mathematical model of elastic pure bending of tetrahedral prism.

It's important to note that if body $\bar{\Gamma}$ is homogeneous then $\mu(x,y) = \text{const}$ and equation (69) amounts to Poisson equation.

Connected with the previous problem and with reference to an application of this paper, let $\bar{D}$ be a rectangle, $\bar{D} = [0,l_1]\times[0,l_2]$, ∂D the boundary of the rectangular domain, $\gamma = \{(l_1,y), 0 \le y \le l_2\}$), $\gamma_i = \{(\xi_i,y) : 0 \le y \le l_2\}$, $i = 1,\dots,n$, ξ_i $(i = 1,\dots,n)$ are the fixed points of interval $(0,l_1)$, V some open subset in $I\!R$, $Z_{ad} = \{Z : D \to V, Z \in L_2(D)\}$ the set of control functions, which correspond to mass forces, μ is modulus of shift, $\mu = \text{const}$.

Let us consider the non-local boundary problem for Poisson equation for each fixed $Z \in V_{ad}$ in the domain D

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{1}{\mu}Z, \qquad (x,y) \in D, \tag{71}$$

$$
\begin{aligned}
w(x,y) &= 0, && (x,y) \in \Gamma\backslash\gamma\\
w(l_1,y) &= \sum_{i=1}^{n}\sigma_i w(\xi_i,y) + f_2(y), && 0 \le y \le l_2,
\end{aligned}
\tag{72}
$$

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where $0 < \sigma_i = \text{const}$, $\sum_{i=1}^{n} \sigma_i < 1$.

The potential energy of the deformable elastic body has the form [26]:

$$I(Z) = \iint_D \frac{\mu}{2} \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dx\, dy + \iint_D Z^2\, dx\, dy \tag{73}$$

So, we have to minimise the functional (73), governed by the Poisson equation (71) and the non-local boundary conditions (72).

The technique shown in this paper can be used to solve this problem as follows: find function $Z_0 \in Z_{ad}$ whose corresponding solution ω_0 to boundary value problem (71)-(72) together with Z_0 results in a minimal potential energy (73). The solution is written as:

$$\inf_{Z \in Z_{ad}} \left[2Z_0 + \frac{1}{\mu} \psi_0 \right] Z = \left[2Z_0 + \frac{1}{\mu} \psi_0 \right] Z_0 \qquad \text{a.e. on } D$$

being ψ_0 the solution to the adjoint problem

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -Z_0, \qquad (x, y) \in D \backslash \sum_{i=1}^{k} \gamma_i$$

$$\psi(x, y) = 0, \qquad (x, y) \in \partial(D)$$

$$\psi_x(\xi_i^+, y) - \psi_x(\xi_i^-, y) - \sigma_i \psi_x(l_1, y) = \mu \sigma_i Z_{0x}(l_1, y), \qquad i = 1, 2, \ldots, n, \quad 0 \le y \le l_1.$$

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