# Adaptive Global WASF-GA to Handle Many-objective Optimization Problems

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**Abstract**

In this paper, a new version of the aggregation-based evolutionary algorithm Global WASF-GA (GWASF-GA) for many-objective optimization is proposed, called *Adaptive Global WASF-GA* (A-GWASF-GA). The fitness function of GWASF-GA is defined by an achievement scalarizing function (ASF) based on the Tchebychev distance, which considers two reference points (the nadir and utopian points) and a set of weight vectors. Despite of the benefits of using these two reference points simultaneously and a well-distributed set of weight vectors, it is necessary to go a step further to get better approximations in problems with complicated Pareto optimal fronts. For this, in A-GWASF-GA, some of the weight vectors are re-calculated during the optimization process based on the sparsity of the solutions found so far, and taking into account some theoretical results demonstrated in this paper regarding the ASF considered. Different strategies are carried out to accelerate the convergence and to maintain the diversity. The computational results, carried out in comparison with RVEA, NSGA-III, and different versions of MOEA/D, show the potential of A-GWASF-GA in well-known but also in novel many-objective optimization benchmark problems.

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## 1. Introduction

In many real applications, a *multiobjective optimization problem* (MOPs) arises, consisting of the simultaneous optimization of several objective functions, subject to several constraints that determine the feasible set of solutions. As finding a single solution optimizing all the objectives at the same time is usually impossible because of the conflict existing among the objectives, the interest is focused on the so-called *Pareto optimal solutions*. At them, an objective function improvement can only be achieved at the expense of worsening, at least, one of the others. Problems handling more than three objectives, referred to as *many-objective optimization problems* (MaOPs), are more difficult to solve and lately they have drawn much attention of the multiobjective optimization community [1, 2].

Along the years, *Evolutionary Multiobjective Optimization* (EMO) algorithms have demonstrated their ability for solving MOPs [3, 4] and MaOPs [1, 2, 5, 6]. They are aimed at finding a subset of non-dominated solutions approximating the *Pareto optimal front* (PF) (the set of all Pareto optimal solutions in the objective space). The approximation set must be formed by solutions as evenly distributed as possible in the PF (diversity), and as close as possible to the PF (convergence). However, achieving convergence and diversity simultaneously is not always easy for an EMO algorithm, specially when handling MaOPs or problems with complicated PFs (such as e.g. non-convex, degenerated, or discontinuous). Actually, several works propose to modify the dominance relationship in order to enhance the convergence, such as e.g. the fuzzy dominance suggested in [7] or the $\epsilon$-dominance proposed in [8].

Based on the selection strategies used, EMO algorithms can be categorized into three main classes: indicator-based approaches (whose selection strategy is based on the computation of a performance indicator metric), dominance-based approaches (which use a dominance relation -the Pareto dominance or any other one- for the comparison of solutions), and aggregation-based or decomposition-based approaches (which transform the original problem into a set of single-objective optimization subproblems).

In the indicator-based class, we can mention SMS-MOEA [9], in which each solution is evaluated by its hypervolume contribution, distance-based algorithms such as e.g. IBEA [10], or algorithms defined according to the R2 indicator such as e.g. MOMBI [11]. Other indicator-based algorithms are HypE [12], an hypervolume-based evolutionary algorithm, and TwoArch2 [13], based on the $I_{\epsilon^+}$ indicator.

A well-known dominance-based algorithm is NSGA-II [14], which has successfully solved many real-life MOPs [4, 15]. It uses an elite-preserving strategy and a diversity preserving mechanism, and it has stood out by its fast non-dominated sorting procedure to rank the solutions into several non-dominated fronts for the selection of the best individuals. The new version NSGA-III [16, 17] also ranks individuals according to the Pareto dominance relation, but includes a diversity strategy that considers a set of reference points and emphasizes the closest individuals to each of these reference points. In practice, the distribution of reference points affects both the convergence and the diversity of the approximation generated by NSGA-III. Recently, new versions of NSGA-III have been proposed [18, 19] to obtain better approximations for MaOPs.

Within the aggregation-based class, one of the most promising algorithms is MOEA/D [20, 21]. In MOEA/D, a set of well-spread normalized weight vectors is used to formulate a set of single-objective subproblems, each of them using the same scalarizing function with a different weight vector. At each generation, the new population is composed by the best solutions found so far for each of the subproblems, relying on the assumption that neighbor weight vectors (i.e. neighbor subproblems) will produce neighbor solutions. Different versions of MOEA/D have been proposed in the last years [22, 23, 24]. One of the most recent versions is MOEA/DD [25], which combines the Pareto dominance and aggregation techniques to achieve a balance between convergence and diversity. The Tchebychev metric is one of the most used approaches to decompose a multiobjective optimization problem into a set of scalar optimization subproblems. The geometric properties of the subproblem objective functions after this decomposition have been studied in [26].

As stated in e.g. [27, 28], the distribution of the weight vectors used in aggregation-based EMO algorithms may hinder the generation of a population properly covering the whole PF. Their performance strongly depends on the shapes of PFs. For MOPs with an hyperplane shaped PF, many existing decomposition-based algorithms perform well using a pre-defined set of evenly distributed weight vectors. However, MaOPs and MOPs with more complicated PFs, such as e.g. non-convex or discontinuous, become more difficult to solve using a fixed set of weight vectors [27]. To enhance this drawback, several studies [29, 30] propose strategies, for example, to adapt the weight vectors used in MOEA/D while the population converges to the PF. In this line, MOEA/D-AWA [31] is a variant of MOEA/D that includes new parameters to dynamically adjust the weight vectors during the optimization process. Also, other algorithms like RVEA [32] and DMOEA/D [33] are based on a dynamic adjustment of reference vectors. We understand by reference vectors or reference directions in the metric Tchebychev (the most used fitness function), as the directions obtained by taking a point (in most cases it is the ideal point although it can also be the nadir point as in our algorithm) and each of the vectors obtained by considering the inverse components of each vector of the set of weight vectors. Another approach proposed in this line is AR-MOEA [34], which adapts different reference points during the search. There also exist other kinds of EMO algorithms such as PICEA [35], which uses multiple sets of hypothetical preferences simultaneously to provide multiple comparison perspectives, thus allowing a good representation of the whole PF.

Global WASF-GA (GWASF-GA) [36] is another aggregation-based EMO algorithm which works with an evenly-distributed set of weight vectors, two reference points and the achievement scalarizing function (ASF) proposed in [37]. When this function is minimized over the feasible set, the Pareto optimality of the solution found is assured (and not only the weakly Pareto optimality, as it occurs with the Tchebychev distance [38]). At each generation, GWASF-GA classifies individuals into fronts according to their ASF values for a particular weight vector and considering the nadir and the utopian points. Following a similar idea to MOEA/D-AWA, we propose to enhance the performance of the algorithm GWASF-GA using a dynamic adjustment of the weight vectors. We refer to the new algorithm as *Adaptive Global WASF-GA* (A-GWASF-GA). The main goal is to re-direct some of the search directions defined by the weight vectors based on the distribution of the solutions obtained. So that, weight vectors generating solutions in overcrowded areas of the PF are replaced by new ones pushing the search towards regions of the PF with a lack of solutions. This adjustment takes into account the fact of considering both the nadir and the utopian points as reference points at the same time, unlike other algorithms such as e.g. MOEA/D-AWA, which only considers the ideal point.

Next, Section 2 introduces the main concepts and notations used in multiobjective optimization. A-GWASF-GA is motivated and described in

Section 3. The computational experiments carried out are shown in Section 4, where we compare A-GWASF-GA with respect to RVEA, NSGA-III and different versions of MOEA/D. Finally, the conclusions are drawn in Section 5.

## 2. Background concepts

A *multiobjective optimization problem* (MOP) can be defined as[1]:

$$\begin{array}{ll} \text{minimize} & \{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})\} \\ \text{subject to} & \mathbf{x} \in S, \end{array} \tag{1}$$

where $k \geq 2$ is the number of *objective functions* $f_i : S \rightarrow \mathbb{R}$ $(i = 1, \ldots, k)$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is the vector of *decision variables*, and $S \subset \mathbb{R}^n$ is the *feasible set*. The image of the feasible set in the objective space $\mathbb{R}^k$ is called the *feasible objective region* $Z = \mathbf{f}(S)$, and it is formed by *objective vectors* $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x}))^T$, with $\mathbf{x} \in S$. A *many-objective optimization problem* (MaOP) is a particular case where $k > 3$.

When minimizing all the objective functions simultaneously, the degree of conflict existing among them makes it impossible to find a single solution where all of them can reach their individual optima. Because of this, the interest is focused on the so-called *Pareto optimal solutions*, at which no objective function can be improved without deteriorating, at least, one of the others. A solution $\mathbf{x} \in S$ is said to be *Pareto optimal* if and only if there is no other $\bar{\mathbf{x}} \in S$ such that $f_i(\bar{\mathbf{x}}) \leq f_i(\mathbf{x})$ for all $i = 1, \ldots, k$, and $f_j(\bar{\mathbf{x}}) < f_j(\mathbf{x})$ for, at least, one index $j$. The corresponding objective vector $\mathbf{f}(\mathbf{x})$ is referred to as a *Pareto optimal objective vector*. The set of all Pareto optimal solutions is called the *Pareto optimal set* (PS), denoted by $E$, and the set of all Pareto optimal objective vectors is called the *Pareto optimal front* (PF), denoted by $\mathbf{f}(E)$. Besides, a solution $\mathbf{x} \in S$ is *weakly Pareto optimal* if there does not exist another $\bar{\mathbf{x}} \in S$ such that $f_i(\bar{\mathbf{x}}) < f_i(\mathbf{x})$ for all $i = 1, \ldots, k$. Also, there are also *properly Pareto optimal solutions*, which are Pareto optimal solutions with bounded trade-offs between the objectives.

In addition, given two vectors $\mathbf{z}, \bar{\mathbf{z}} \in \mathbb{R}^k$, we say that $\mathbf{z}$ *dominates* $\bar{\mathbf{z}}$ if and only if $z_i \leq \bar{z}_i$ for all $i = 1, \ldots, k$, and $z_j < \bar{z}_j$ for, at least, one index $j$. In this

[1] For simplicity, all objectives are assumed to be minimized. In case any of them must be maximized, it can be transformed into the minimization form by multiplying by -1.

paper, we refer as a *nondominated set* to a set of solutions whose objective vectors are not dominated by those of the rest of solutions in the set. A relaxation of the strict dominance is the $\varepsilon$-dominance, for a small positive real value $\varepsilon > 0$. We say that $\mathbf{z}$ *$\varepsilon$-dominates* $\bar{\mathbf{z}}$ if and only if $z_i \leq \bar{z}_i + \varepsilon$ for all $i = 1, \ldots, k$, and $z_j < \bar{z}_j + \varepsilon$ for, at least, one index $j$.

In multiobjective optimization, the ideal and the nadir points are important because they provide upper and lower bounds for the objective function values in the Pareto optimal set. Besides, the utopian point, defined as a slight modification of the ideal point, is also very useful since it is a point strictly dominating the ideal point. The *ideal point* $z^\star = (z_1^\star, \ldots, z_k^\star)^T$ is calculated as $z_i^\star = \min_{\mathbf{x} \in E} f_i(\mathbf{x}) = \min_{\mathbf{x} \in S} f_i(\mathbf{x})$ $(i = 1, \ldots, k)$, and thus, the *utopian point* $\mathbf{z}^{\star\star} = (z_1^{\star\star}, \ldots, z_k^{\star\star})^T$ can be obtained by $z_i^{\star\star} = z_i^\star - \epsilon$ $(i = 1, \ldots, k)$, where $\epsilon > 0$ is a relatively small real value. The *nadir point* $\mathbf{z}^{\mathrm{nad}} = (z_1^{\mathrm{nad}}, \ldots, z_k^{\mathrm{nad}})^T$ is defined as $z_i^{\mathrm{nad}} = \max_{\mathbf{x} \in E} f_i(\mathbf{x})$ $(i = 1, \ldots, k)$.

In A-GWASF-GA, for any approximation set of feasible solutions $P \subset S$, we need to measure the diversity around each solution in $P$ in the objective space. This is required to be able to detect which regions of the PF are well-covered and which are not (according to the set $P$). Then, we define the *scattering level* of each $\mathbf{x} \in P$, and denote it by $s(\mathbf{x})$, as follows:

$$s(\mathbf{x}) = \prod_{j=1}^{k} \sqrt{\sum_{i=1}^{k} (f_i(\mathbf{x}) - f_i(\mathbf{x}^j))^2}, \tag{2}$$

where $\mathbf{x}^1, \ldots, \mathbf{x}^k \in P$ are the $k$ solutions with the closest objective vectors to $\mathbf{f}(\mathbf{x})$ regarding the $L_2$-distance. The objective vectors are assumed to be normalized to avoid scale problems[2]. The higher (respectively, the lower) the value of $s(\mathbf{x})$, the less (respectively, the more) crowded is the region where $\mathbf{f}(\mathbf{x})$ lies. Thus, we can identify areas of the PF that are poorly approximated (i.e. with a lack of solutions) by means of the solutions with the highest scattering levels, and overcrowded regions as these containing the solutions with the lowest scattering levels.

### *2.1. Brief description of GWASF-GA*

GWASF-GA [36] is based on an *achievement scalarizing function* (ASF) proposed by Wierbicki [37]. This function is formulated as followed,

[2] If the ranges of the objectives are in very different scales, the differences $f_i(\mathbf{x}) - f_i(\mathbf{x}^j)$ can be normalized dividing it by $z_i^{\mathrm{nad}} - z_i^{\star\star}$ $(i = 1, \ldots, k)$.

establishing a relationship between the objective functions and a reference point $\mathbf{q} = (q_1, \ldots, q_k)^T$ (formed by desirable values $q_i$ for the objective functions $f_i$, for $i = 1, \ldots, k$):

$$s(\mathbf{q}, \mathbf{f}(\mathbf{x}), \mu) = \max_{i=1,\ldots,k} \{ \mu_i(f_i(\mathbf{x}) - q_i) \} + \rho \sum_{i=1}^{k} \mu_i(f_i(\mathbf{x}) - q_i), \qquad (3)$$

where $\mu = (\mu_1, \ldots, \mu_k)$ is a vector of strictly positive weights. This ASF has to be minimized over $S$ to generate a Pareto optimal solution [38]:

$$\begin{array}{ll} \text{minimize} & s(\mathbf{q}, \mathbf{f}(\mathbf{x}), \mu) \\ \text{subject to} & \mathbf{x} \in S. \end{array} \qquad (4)$$

The real value $\rho \geq 0$ is the so-called *augmentation coefficient*. The augmentation term (i.e. the term $\rho \sum_{i=1}^{k} \mu_i(f_i(\mathbf{x}) - q_i)$ in (3)) is added to assure that the optimal solution to (4) is a Pareto optimal solution to the original problem (1) for any reference point. In practice, solving problem (4) means to project the reference point onto the PF in the projection direction determined by the inverse of the weights considered, that is, $(\frac{1}{\mu_1}, \ldots, \frac{1}{\mu_k})$.

In GWASF-GA, the nadir and the utopian point are simultaneously used as reference points to formulate the scalarized subproblems according to (4). Furthermore, these subproblems are generated using a predefined set with $N_\mu$ weight vectors, verifying that the vectors formed by their inverse components (their projection directions) are as uniformly distributed as possible. A half of the weight vectors are used to build subproblems with the nadir point, and the other half are used with the utopian point. Owing to this, the weight vectors are key parameters in GWASF-GA since they set the search directions for new non-dominated solutions in the objective space, either from the nadir or from the utopian point.

To select the best individuals, at each generation of GWASF-GA, the population of parents and offspring is divided into several fronts. This front classification is done according to the values that each individual takes on the ASF given in (3), for each one of the weight vectors, and using the nadir or the utopian. Because of this, each individual in the population produced at each generation is associated either with the nadir or with the utopian point, and with a weight vector. Finally, those individuals in the lowest level fronts are selected for the next generation. To some extent, these solutions can be considered as the best individuals at the current generation for minimizing the ASF (3) with respect to the $N_\mu$ weight vectors, using the utopian or the nadir point. Besides, in practice, we can say that the PF is approximated by projecting the nadir and the utopian points simultaneously, taking into account the set of projection directions (or search directions) defined by the set of weight vectors.

## 3. Adaptive Global WASF-GA (A-GWASF-GA)

In this section, we first motivates in Section 3.1 the performance improvement we propose for GWASF-GA. Later, Section 3.2 presents some theoretical results about the adaptation of weight vectors suggested. Finally, the new algorithm A-GWASF-GA is described in details in Section 3.3.

### *3.1. Motivation*

The performance of GWASF-GA mainly relies on two facts. First, it uses both the nadir and the utopian points simultaneously to formulate the subproblems, enabling to better approximate the PF than when using just one of them (or just the ideal point). Second, as stated in [36], the set of weight vectors considered must hold that the projection directions they determine in the objective space (i.e. the vectors formed by their inverse components) are uniformly distributed, instead of imposing that the original weight vectors themselves are well-distributed. The benefits of this feature of GWASF-GA are supported by the mathematical properties of the Wierzbicki's ASF given in (3), which is the Tchebychev metric with the augmentation term.[3] To clarify this second property of GWASF-GA, let us consider a two-objective minimization problem where the PF is defined by $\{(z, 1-z) \mid z \in [0,1]\}$ and whose utopian and nadir points are $(0,0)$ and $(1,1)$, respectively, as shown in Figure 1. Let us also consider e.g. 14 weight vectors divided into two subsets: $W^{\star\star} = \{(\mu_1^1, \mu_2^1), (\mu_1^3, \mu_2^3), \ldots, (\mu_1^{13}, \mu_2^{13})\}$ to be used with the utopian point, and $W^{\text{nad}} = \{(\mu_1^2, \mu_2^2), (\mu_1^4, \mu_2^4), \ldots, (\mu_1^{14}, \mu_2^{14})\}$ to be used with the nadir point. Thanks to the properties of the ASF (3), in practice, for each $(\mu_1^j, \mu_2^j) \in W^{\star\star}$, the utopian point is projected onto the PF using the projection direction defined by the vector $\left(\frac{1}{\mu_1^j}, \frac{1}{\mu_2^j}\right)$ along the generations of GWASF-GA. Similarly, for any $(\mu_1^j, \mu_2^j) \in W^{\text{nad}}$, the nadir point is progressively projected onto the PF using the projection direction

[3] For more details about the properties of the ASF proposed in [37], see [38, 39].

$\left(\frac{-1}{\mu_1^j}, \frac{-1}{\mu_2^j}\right)$. These projection directions are also depicted in Figure 1. Thus, if the projection directions defined by the weight vectors used are widely-distributed, it is likely that the solutions generated are evenly distributed along the PF, as said in [36].

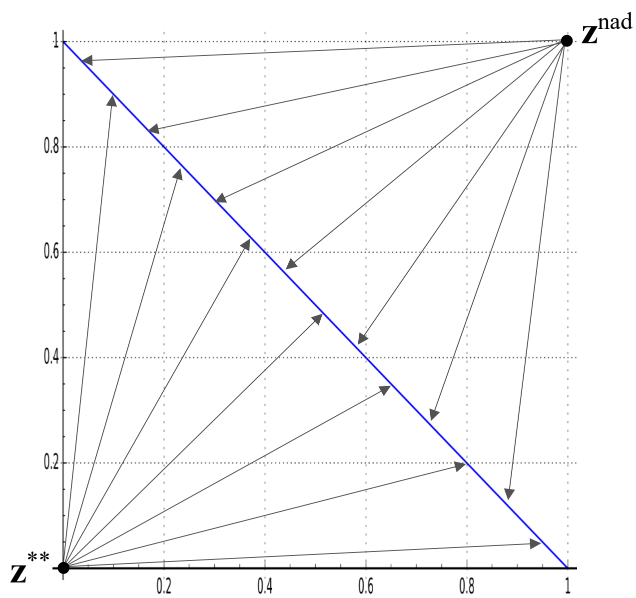



Figure 1: Projection directions from $\mathbf{z}^{\text{nad}}$ and $\mathbf{z}^{\star\star}$ in GWASF-GA.

From an operational point of view, in GWASF-GA, the same importance is given to the projection from the utopian point as to that from the nadir point given that, as said, a half of the weight vectors are used with the utopian point and the other half with the nadir point at each generation. However, a key issue affecting the performance of the algorithm is the use of a pre-fixed set of weight vectors. In GWASF-GA, the set of weight vectors are pre-generated and are not updated along the whole algorithm procedure. Thus, in certain problems, when classifying the solutions into several fronts according to the ASF values in the subproblems, a single non-dominated solution (or very close ones) may be shared by several subproblems due to the uniformly spread of the projection directions. For example, this may occur if the PF is discontinuous and several projection directions are directing the search into a gap among several disconnected regions where there are no Pareto optimal solutions to approximate, resulting in a waste of effort. This fact does not contribute to the convergence of the algorithm and produces a lack of the diversity among the solutions generated.

According to this, it seems appropriate to dynamically adjust the predefined set of weight vectors as the population evolves towards the PF, so that the search directions for new non-dominated solutions better adapt to the shape of the PF to enhance the performance of GWASF-GA. Then, using some of the suggestions in [31], the idea behind A-GWASF-GA is to dynamically change some of the weight vectors as the algorithm converges. To achieve this, after some generations have been performed, we re-calculate the weight vectors according to the solutions generated. These individuals have survived along the generations and, therefore, they can be assumed to be the best solutions found so far to approximate the PF until the procedure has been stopped. Based on them, some weight vectors providing solutions in overcrowded areas (i.e. subsets of the current population with a high density of solutions) are replaced by new weight vectors directing the search towards areas with less solutions. This adjustment of the weight vectors is supported by the mathematical properties of ASF used [37, 38] and by the theoretical results presented in Section 3.2.

Note that the fact of considering both the utopian and nadir points, in addition to pushing the search for new non-dominated solutions towards areas with few solutions (by means of the adjustment of the weight vectors), may enable A-GWASF-GA to better adapt the convergence and diversity to the shape of the PF. As explained hereafter, as the adjustment is dynamically performed, a higher number of weight vectors may be assigned to the utopian point to re-orientate some projection directions towards parts of the PF with a convex shape (if any). At the same time, more weight vectors may be used with the nadir point to better cover those parts of the PF which are concave. With this procedure, the approximation found by A-GWASF-GA is luckily to be improved, specially in problems with complex shapes (with concave and convex parts, discontinuous, degenerate ones, etc.). Technical parameters, such as when the adjustment starts, the number of weight vectors to be changed, the number of adjustments to be carried out, and how the weight vectors are re-calculated, will be described in Section 3.3.

### *3.2. Theoretical results*

Let us see two theorems supporting the benefits of the adjustment of the weight vectors carried out in A-GWASF-GA.

For any feasible solution $\mathbf{x} \in S$, we can built two weigh vectors, denoted by $\mu^U = (\mu_1^U, \ldots, \mu_k^U)^T$ and $\mu^N = (\mu_1^N, \ldots, \mu_k^N)^T$, as follows:

$$\mu_i^U = \tfrac{1}{f_i(\mathbf{x}) - z_i^{\star\star}}, \quad \text{for each } i = 1, \ldots, k. \tag{5}$$

$$\mu_i^N = \tfrac{1}{z_i^{\text{nad}} - f_i(\mathbf{x})}, \quad \text{for each } i = 1, \ldots, k. \tag{6}$$

Since $\mathbf{z}^{\star\star}$ strictly dominates $\mathbf{f}(\mathbf{x})$, for any $\mathbf{x} \in S$, it is assured that $\mu_i^U > 0$, for every $i = 1, \ldots, k$. Similarly, since $\mathbf{z}^{\text{nad}}$ is strictly dominated by $\mathbf{f}(\mathbf{x})$, for any $\mathbf{x} \in S$, it is also verified that $\mu_i^N > 0$, for every $i = 1, \ldots, k$. Note that, in many EMO algorithms (as in GWASF-GA), the nadir point is estimated using the maximum objective function values achieved among all the solutions found, which are later slightly modified with a small positive value. Thus, the objective vector of any feasible solution generated is guaranteed to strictly dominate $\mathbf{z}^{\text{nad}}$, enabling us to assure that $\mu_i^N > 0$, for every $i = 1, \ldots, k$.

The following results show that, if the solution $\mathbf{x}$ considered for generating the weight vector $\mu^U$ (respectively, $\mu^N$) is weakly Pareto optimal, then it is an optimal solution to (4) using $\mu^U$ as the weight vector and $\mathbf{z}^{\star\star}$ as the reference point (respectively, using $\mu^N$ and $\mathbf{z}^{\text{nad}}$). And, in case $\mathbf{x}$ is not weakly Pareto optimal, it is assured that the objective vector of any optimal solution to (4) strictly dominates or $\varepsilon$-dominates $\mathbf{f}(\mathbf{x})$.

**Theorem 1.** *Given a feasible solution $\mathbf{x} \in S$, then, either $\mathbf{x}$ is an optimal solution to problem* (4) *considering $\mathbf{q} = \mathbf{z}^{\star\star}$ and the weight vector $\mu^U$ defined by* (5)*, or the objective vector of any optimal solution to problem* (4) *with these parameters strictly dominates or $\varepsilon$-dominates $\mathbf{f}(\mathbf{x})$, for some $\varepsilon > 0$.*

Proof. See Appendix.

**Corollary 1.** *If $\mathbf{x} \in S$ is a weakly Pareto optimal solution, then $\mathbf{x}$ is an optimal solution to problem* (4) *considering $\mathbf{q} = \mathbf{z}^{\star\star}$ and using the weight vector $\mu^U$ defined by* (5)*.*

Proof. See Appendix.

**Theorem 2.** *Given a feasible solution $\mathbf{x} \in S$, then, either $\mathbf{x}$ is an optimal solution to problem* (4) *considering $\mathbf{q} = \mathbf{z}^{\text{nad}}$ and the weight vector $\mu^N$ defined by* (6)*, or the objective vector of any optimal solution to problem* (4) *with these parameters strictly dominates or $\varepsilon$-dominates $\mathbf{f}(\mathbf{x})$, for some $\varepsilon > 0$.*

Proof. See Appendix.

**Corollary 2.** *If* $\mathbf{x} \in S$ *is a weakly Pareto optimal solution, then* $\mathbf{x}$ *is an optimal solution to problem* (4) *considering* $\mathbf{q} = \mathbf{z}^{\text{nad}}$ *and using the weight vector* $\mu^N$ *defined by* (6)*.*

Proof. See Appendix.

Based on these theoretical results, any solution generated by A-GWASF-GA enables us to calculate two weight vectors ($\mu^N$ and $\mu^U$) that, in practice, can be used to project either the utopian or the nadir point towards the PF, in a direction directly pointing to the solution considered (in the objective space). This enables the convergence process to push the search for new non-dominated solutions to the region where the solution considered lies. This is so because, either in the case $\mu^N$ is used or if $\mu^U$ is considered, the solution used to build them is the optimal solution to (4), or its objective vector is strictly dominated (or $\varepsilon$-dominated) by the objective vector of any optimal solution to (4), i.e. there is a feasible solution which improves all its objective values (or which improves all of them with a threshold $\varepsilon$). Thus, regions of the PF which are poorly approximated can be better covered by re-defining some of the subproblems in A-GWASF-GA with these new weight vectors (which re-direct the search towards them), using the utopian or the nadir point (depending on how the solutions considered for calculating the new weight vectors have been selected, as described later). Besides, solutions which are not yet enough closed to the PF can be improved and brought closer to the PF based on these theoretical results.

### *3.3. Description of A-GWASF-GA*

Initially, the first generations of A-GWASF-GA are performed as in the original GWASF-GA [36], using $N_\mu$ weight vectors generated as indicated in [36] (pp. 321-323), which produce evenly scattered projection directions in the objective space. Let us denote by $W = \{\mu^1, \ldots, \mu^{N_\mu}\}$ this set of weight vectors. If $NG_T$ is the total number of generations, let us consider that $NG_p$ denotes the number of generations initially performed. Actually, $NG_p$ can be defined as a percentage of $NG_T$, that is, $NG_p = p \cdot NG_T$ where $0 < p < 1$. The purpose of executing these $NG_p$ generations is to produce an initial approximation of the PF. The solutions obtained at the generation $NG_p$ may not have converged closely enough to the PF, but their distribution helps us to have an initial insight of the true PF. Besides, regions of the PF that are complicated to converge to may still have not been well-covered at this generation. Thus, $NG_p$ cannot be too small in order not to stop the algorithm too prematurely, neither too large so that there are still generations to improve the search directions before the process finalizes.

Along these $NG_p$ generations of A-GWASF-GA, the front classification of the individuals is done according to their ASF values considering a half of the $N_\mu$ weight vectors (the odd ordered ones) for the utopian point and the other half (the even ordered ones) for the nadir point. That is, if $W^{\star\star}$ refers to the set of weight vectors used with the utopian point and $W^{\text{nad}}$ is the set of weight vectors considered for the nadir point, we have $W^{\star\star} = \{\mu^1, \mu^3, \mu^5, \dots\}$ and $W^{\text{nad}} = \{\mu^2, \mu^4, \mu^6, \dots\}$.

Once the first $NG_p$ generations are carried out, the adjustment of the weight vectors will be started in A-GWASF-GA. Let $N_a$ be the number of weight vectors to be re-calculated ($N_a < N_\mu/2$) and $NG_a = (1-p) \cdot NG_T$ the number of generations that are left for the dynamical adjustment (i.e. $NG_p + NG_a = NG_T$). In the next $NG_a$ generations, we perform $n_a$ adjustments of the weight vectors (with $n_a \in \{1, \dots, NG_a\}$). This implies that the weight vectors are adjusted each $E(NG_a/n_a)$ generations, starting at generation $NG_p$ (where $E(\cdot)$ denotes the integer part of a number). That is, if $NG_a^r$ denotes the generation at which the $r$-th adjustment is done ($r = 1, \dots, n_a$), we have $NG_a^r = NG_p + (r-1) \cdot E(NG_a/n_a)$. For each $r = 1, \dots, n_a$, let us denote by $P^r$ the population generated at generation $NG_a^r$, and by $W_r^{\star\star}$ and $W_r^{\text{nad}}$ the sets of weight vectors for the utopian and the nadir points obtained at the $r$-th adjustment. Note that the $r$-th adjustment of the weight vectors is done as explained hereafter once the generation $NG_a^r$ has concluded and before running the generation $NG_a^r + 1$.

Firstly, at the $r$-th adjustment, we identify the weight vectors to be replaced as indicated in Algorithm 1, by selecting the $N_a$ solutions in $P^r$ with the lowest scattering level calculated according to (2). These solutions are referred to as $\{\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^{N_a}\}$. For each $j = 1, \dots, N^a$, we can assume that there are close enough individuals in $P^r$ around each $\bar{\mathbf{x}}^j$ (in the objective space). This means that the area of the PF where these solutions lye have been covered enough at generation $NG_a^r$ in comparison to other regions. As previously said, in A-GWASF-GA, each solution is associated with a weight vector, as well as with the nadir or with the utopian point. Then, we can consider that the weight vector corresponding to each $\bar{\mathbf{x}}^j$ is directing the search towards an overcrowded area of the PF, where other weight vectors are also orientating the search towards (those of the solutions around $\bar{\mathbf{x}}^j$). In view of this, the $N_a$ weight vectors corresponding to the solutions $\{\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^{N_a}\}$ are the candidates to be replaced by new ones. Thus, they are removed (either from $W_r^{\star\star}$ or from $W_r^{\mathrm{nad}}$) and only $N_\mu - N_a$ weight vectors are left from the original ones.

**Algorithm 1** Weight vectors' removal

**Require:** The population $P^r$ produced at generation $NG_a^r$, the sets of weight vectors $W^{\star\star}$ and $W^{\mathrm{nad}}$, and the number of weight vectors to be removed $N_a$.
**Ensure:** The sets $W_r^{\star\star}$ and $W_r^{\mathrm{nad}}$.

```
Initialize W_r** = W** and W_r^nad = W^nad.
Define A = P^r as an auxiliary set.
For every x ∈ A, calculate the scattering level s(x) according to (2).
for j = 1, ..., N_a do
    Let x̄^j be the solution from A with the lowest value of s(x).
    Identify the weight vector μ̄^j ∈ {W** ∪ W^nad} associated to x̄^j.
    If μ̄^j ∈ W**, remove it from W_r**, else (i.e. μ̄^j ∈ W^nad) remove it from W_r^nad.
    Remove x̄^j from A.
end for
```

Secondly, to generate the new $N_a$ weight vectors as described in Algorithm 2, we identify the $N_a$ solutions in $P^r$ with the highest scattering level obtained according to (2), referred to as $\{\hat{\mathbf{x}}^1, \ldots, \hat{\mathbf{x}}^{N_a}\}$. The objective vectors of these solutions enable the search to be focused on parts that have not been well-covered by the population $P^r$. To this, for every $j = 1, \ldots, N_a$, a new weight vector $\hat{\mu}^j = (\hat{\mu}_1^j, \ldots, \hat{\mu}_k^j)$ is calculated: (a) if the solution $\hat{\mathbf{x}}^j$ has been obtained in the front classification process as one minimizing the ASF with the utopian point, then $\hat{\mu}^j = \mu^U$ following equation (5) using $\mathbf{x} = \hat{\mathbf{x}}^j$; (b) in case $\hat{\mathbf{x}}^j$ has been obtained with the nadir point, then $\hat{\mu}^j = \mu^N$ following equation (6) using $\mathbf{x} = \hat{\mathbf{x}}^j$. Once $W_r^{\star\star}$ and $W_r^{\mathrm{nad}}$ are determined, the sets of weight vectors $W^{\star\star}$ and $W^{\mathrm{nad}}$ are updated accordingly and the next generations are run using them, until the next weight vectors' adjustment needs to be carried out at generation $NG_a^{r+1}$.

**Algorithm 2** Calculation of new weight vectors

**Require:** The population $P^r$ produced at generation $NG_a^r$, the set of weight vectors $W_r^{\star\star}$ and $W_r^{\text{nad}}$ retrieved by Algorithm 1, and the number of weight vectors $N_a$ to be adapted.
**Ensure:** The updated set of weight vectors $W^{\star\star}$ and $W^{\text{nad}}$.
1: Define $A = P^r$ as an auxiliary set.
2: For every $\mathbf{x} \in A$, calculate the scattering level $s(\mathbf{x})$ according to (2).
3: **for** $j = 1, \ldots, N_a$ **do**
4: Let $\hat{\mathbf{x}}^j$ be the solution from $A$ with the highest value of $s(\mathbf{x})$.
5: Calculate the new weight vector $\hat{\mu}^j$ as follows:

(a) If $\hat{\mathbf{x}}^j$ has been obtained in the front classification using $\mathbf{z}^{\star\star}$, then use $\hat{\mathbf{x}}^j$ to calculate $\mu^U$ following equation (5). Set $\hat{\mu}^j = \mu^U$ and update $W_r^{\star\star} = W_r^{\star\star} \cup \{\hat{\mu}^j\}$.

(b) If $\hat{\mathbf{x}}^j$ has been obtained in the front classification using $\mathbf{z}^{\text{nad}}$, then use $\hat{\mathbf{x}}^j$ to calculate $\mu^N$ following equation (6). Set $\hat{\mu}^j = \mu^N$ and update $W_r^{\text{nad}} = W_r^{\text{nad}} \cup \{\hat{\mu}^j\}$.

6: Remove $\hat{\mathbf{x}}^j$ from $A$.
7: **end for**
8: Update $W^{\star\star} = W_r^{\star\star}$ and $W^{\text{nad}} = W_r^{\text{nad}}$.

Figure 2 graphically shows how Algorithms 1 and 2 would perform for a two-objective minimization problem with a discontinuous PF. The black points represent the individuals generated when the weight vectors are adjusted in A-GWASF-GA. As shown, two projection directions (highlighted as bold arrows) defined by two of the weight vectors used are pointing towards an overcrowded area. Then, based on Algorithm 1, their corresponding weight vectors would be selected and, according to Algorithm 2, they would be replaced by new ones whose projection directions (represented with dashed arrows) direct the search to other areas which are not so well-covered.

Note that, to classify the individuals into different fronts after each adjustment is done, each new weight vector $\hat{\mu}^j$ is considered to select individuals with the lowest ASF values either with the utopian point (case (a) in Algorithm 2) or with the nadir point (case (b) in Algorithm 2), depending on the way the solution $\hat{\mathbf{x}}^j$ has been previously selected. Thus, due to the formulas (5) and (6), each individual $\hat{\mathbf{x}}^j$ (lying in a not-well approximated region) is used to calculate a new $\hat{\mu}^j$, taking into account whether this solution has been elicited in A-GWASF-GA using the utopian or the nadir point. Based on the Theorems 1 and 2, when using the utopian point, $\hat{\mu}^j$ determines a projection direction towards the PF which directly points to $\mathbf{f}(\hat{\mathbf{x}}^j)$ from $\mathbf{z}^{\star\star}$. But if the nadir point has been employed, the projection direction set by $\hat{\mu}^j$ is orientated to $\mathbf{f}(\hat{\mathbf{x}}^j)$ from $\mathbf{z}^{\text{nad}}$. In this way, the region of the PF where each $\mathbf{f}(\hat{\mathbf{x}}^j)$ lies may be better covered in subsequent generations than before, since a new weight vector is directing the search directly towards it, from the utopian or from the nadir point, assuring that the best one of them is used.

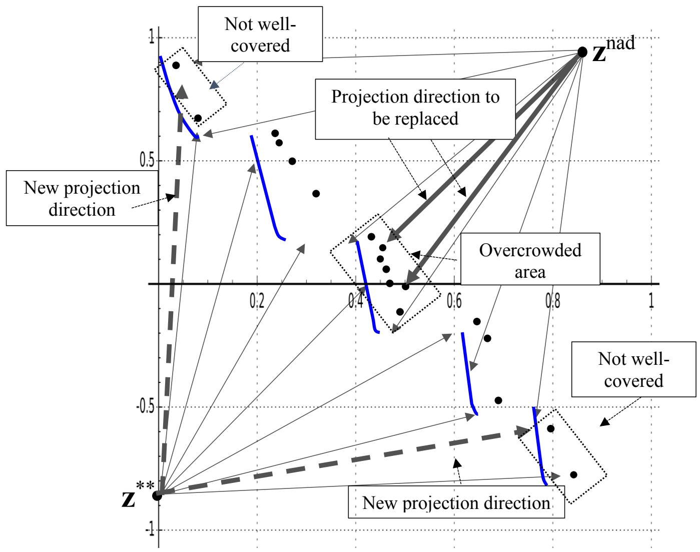



Figure 2: Weight vectors' removal and calculation of new weight vectors in A-GWASF-GA.

After the first adjustment in A-GWASF-GA, observe that the number of weight vectors used for the utopian point is no longer equal to that for the nadir point, so as in the original GWASF-GA. In case the PF is convex, it is likely that the process automatically assigns a larger amount of weight vectors to the nadir point, giving more importance to the projection from it. But if it is concave, more new weight vectors may be likely to be associated with the utopian point, meaning that its projection is more suitable.

To summarize, a general description of A-GWASF-GA is given in Algorithm 3, and every basic generation is described in Algorithm 4. In these two algorithms, the following notation is followed: at each generation $h$, $P^h$ is the population of $N$ individuals, $Q^h$ is the population of the $N$ offspring produced, $Z^h$ is the combined population (of size $2N$), and $F_n^h$ represents the $n$-th front. Besides, we denote by $\#(A)$ the number of elements in a set $A$.

## 4. Computational experiments

In this section, we demonstrate the effectiveness of A-GWASF-GA through computational experiments. We compare A-GWASF-GA with five of the most representative EMO algorithms, two of them based on reference directions, for solving novel challenging many-objective optimization bench-

**Algorithm 3** A-GWASF-GA

**Require:** The total number of generations ($NG_T$), the population size ($N$), the number of weight vectors ($N_\mu$), the number of adjustments to be performed ($n_a$), the percentage of generations before the first adjustment is applied ($p$), the number of weight vectors to be adjusted ($N_a$).

**Ensure:** An approximation of the the PF, $P_{final}$.

Set $NG_p = p \cdot NG_T$ and $NG_a = (1-p) \cdot NG_T$.
Generate $N_\mu$ weight vectors following [36] and initialize $W^{\star\star}$ and $W^{\text{nad}}$ with the odd ordered and the even ordered weight vectors, respectively.
Initialize $h = 1$ and randomly create an initial population $P^1$ of $N$ individuals.
**for** $i = 1, \dots, k$ **do**
    $z_i^\star = \min_{\mathbf{x} \in P^1} f_i(\mathbf{x})$ and $z_i^{\text{nad}} = \max_{\mathbf{x} \in P^1} f_i(\mathbf{x})$.
    $z_i^{\star\star} = z_i^\star - \epsilon$ and $z_i^{\text{nad}} = z_i^{\text{nad}} + \epsilon$.
**end for**
$r = 1$.
**while** $h \leq NG_T$ **do**
    **if** $h \geq NG_p$ and $h = NG_p + (r-1) \cdot E(NG_a/n_a)$ **then**
        Run Algorithm 1 to generate the sets $W_r^{\star\star}$ and $W_r^{\text{nad}}$ once $N_a$ weight vectors are deleted from $W^{\star\star}$ and $W^{\text{nad}}$ using the population $P^h$.
        Run Algorithm 2 to update $W^{\star\star}$ and $W^{\text{nad}}$, using the sets $W_r^{\star\star}$ and $W_r^{\text{nad}}$ and the population $P^h$ to generate $N_a$ new weight vectors.
        $r = r + 1$.
    **end if**
    Execute a new generation $h$ following Algorithm 4 to obtain $P^{h+1}$, using the population $P^h$ and the sets of weight vectors $W^{\star\star}$ and $W^{\text{nad}}$.
    $h = h + 1$.
**end while**
$P_{final} = F_1^{h-1}$.

mark problems and also well-known multiobjective optimization benchmark problems.

### *4.1. Experimental Design*

For the comparison, the EMO algorithms considered are RVEA [32], NSGA-III [16] and three variants of MOEA/D: MOEA/D-DE [20], MOEA/D-DE-AWA[4], and MOEA/DD [25]. We choose these algorithms because they have shown good results in test problems with complicated Pareto sets, but also in many-objective optimization problems.

In our empirical study, we use a family of 10 novel many and multiobjective bound constrained benchmark problems proposed in [40], that is, we consider the problems named as MaOP1-MaOP10 with $n = 20$ and $n = 50$ decision variables and 3, 5, 8, and 10 objective functions. We also use the well-known DTLZ [41] and WFG [42] benchmark problems with 3, 5,

[4]MOEA/D-DE-AWA is similar to MOEA/D-AWA [31] but using differential evolution (DE) as offspring reproducing mechanism.

**Algorithm 4** Basic generation of A-GWASF-GA

**Require:** The population size $N$, the sets $W_r^{\star\star}$ and $W_r^{\mathrm{nad}}$ and the population $P^h$ at generation $h$.
**Ensure:** The population $P^{h+1}$ for the generation $h+1$.

Apply the selection, crossover and mutation operators to $P^h$ and create a population $Q^h$ of $N$ new individuals.
Set $Z^h = P^h \cup Q^h$.
**for** $i = 1, \dots, k$ **do**
  If there is $\mathbf{x} \in Q^h$ with $f_i(\mathbf{x}) < z_i^\star$, update $z_i^\star = f_i(\mathbf{x})$. Subsequently, update $z_i^{\star\star} = z_i^\star - \epsilon$.
  If $h > 0$, for every $i = 1, \dots, k$, if there is $\mathbf{x} \in Q^h$ with $f_i(\mathbf{x}) > z_i^{\mathrm{nad}}$, update $z_i^{\mathrm{nad}} = f_i(\mathbf{x}) + \epsilon$.
**end for**
For every $\mathbf{x} \in Z^h$ and $j \in W^{\star\star}$, compute $s(\mathbf{z}^{\star\star}, \mathbf{f}(\mathbf{x}), \mu^j)$ as in (3).
For every $\mathbf{x} \in Z^h$ and $j \in W^{\mathrm{nad}}$, compute $s(\mathbf{z}^{\mathrm{nad}}, \mathbf{f}(\mathbf{x}), \mu^j)$ as in (3).
$n = 1$.
**while** $Z^h \neq \emptyset$ **do**
  $F_n^h = \emptyset$.
  **for** $j \in W^{**}$ **do**
    Find $\mathbf{x}_j^n \in Z^h$ such that $s(\mathbf{z}^{\star\star}, \mathbf{f}(\mathbf{x}_j^n), \mu^j) = \min_{\mathbf{x} \in Z^h} s(\mathbf{z}^{\star\star}, \mathbf{f}(\mathbf{x}), \mu^j)$.
    Update $F_n^h = F_n^h \cup \{\mathbf{x}_j^n\}$ and remove $\mathbf{x}_j^n$ from $Z^h$.
  **end for**
  **for** $j \in W^{\mathrm{nad}}$ **do**
    Find $\mathbf{x}_j^n \in Z^h$ such that $s(\mathbf{z}^{\mathrm{nad}}, \mathbf{f}(\mathbf{x}_j^n), \mu^j) = \min_{\mathbf{x} \in Z^h} s(\mathbf{z}^{\mathrm{nad}}, \mathbf{f}(\mathbf{x}), \mu^j)$.
    Update $F_n^h = F_n^h \cup \{\mathbf{x}_j^n\}$ and remove $\mathbf{x}_j^n$ from $Z^h$.
  **end for**
  $n = n + 1$.
**end while**
$P^{h+1} = \emptyset$ and $m = 1$.
**while** $\#(P^{h+1} \cup F_m^h) \leq N$ **do**
  $P^{h+1} = P^{h+1} \cup F_m^h$.
  $m = m + 1$.
**end while**
Let $m'$ be the value of $m$ for which $\#(P^{h+1} \cup F_{m'}^h) > N$. Consider the set $V = \{s(\mathbf{z}^{\star\star}, \mathbf{f}(\mathbf{x}_i^{m'}), \mu^i), s(\mathbf{z}^{\mathrm{nad}}, \mathbf{f}(\mathbf{x}_j^{m'}), \mu^j)\}_{i \in W^{**},\, j \in W^{\mathrm{nad}}}$ formed by the ASF values attained by the individuals in $F_{m'}^h$. Until $\#(P^{h+1}) = N$, insert into $P^{h+1}$ the individuals of $F_{m'}^h$ reaching the lowest values in $V$.

8, and 10 objective functions. We consider the DTLZ1, DTLZ2-DTLZ4, and DTLZ7 problems with $k+4$, $k+9$, and $k+19$ decision variables (where $k$ is the number of objective functions), respectively. For the WFG problems, we set the position- and distance-related parameters to $k-1$ and 10, respectively. Thus, we consider 34 problems with 3, 5, 8, and 10 objective functions ($34 \cdot 4 = 136$ different problems, in total). MaOP, DTLZ, and WFG problems have been chosen because they take into account challenging difficulties in multiobjective optimization, such as objective scalability, multi-modality, complicated Pareto sets, disconnectedness, degeneracy, and bias.

Taking into account that $k$ denotes the number of objectives, the population size $N$ has been set to $100 \cdot k$ and the maximum number of evaluations to $50,000 \cdot k$ (termination criterion). In A-GWASF-GA and NSGA-III, we use the SBX crossover operator [43] with a distribution index $\eta_c = 30$ and a probability $P_c = 0.9$. For these two algorithms, the polynomial mutation operator [43] is used with a distribution index $\eta_m = 20$ and a probability $P_m = 1/n$, where $n$ is the number of variables. In A-GWASF-GA, we consider $n_a = 6$ and $p = 0.8$, and we set $N_a$ to $\frac{N}{4}, \frac{N}{4}, \frac{N}{20}$, and $\frac{N}{10}$ for 3, 5, 8, and 10 objective function problems, respectively. In RVEA, we set the parameters $\alpha = 2$ (which is the rate of change of the penalty function internally used in this method) and $f_r = 0.1$ (which is the frequency to employ the reference vector adaptation), as suggested by the authors. For MOEA/D-DE, MOEA/D-DE-AWA, and MOEA/DD, the size of the neighborhood is set to $0.1 \cdot N$ and the probability of choosing the mate subproblem from their neighborhood is set as 0.9. The crossover ratio and the scale factor are set to 0.5 for the DE operator. In addition, according to [31], MOEA/D-DE-AWA has been configured as follows: 80% of computing resources are devoted to MOEA/D-DE while the remaining 20% are assigned to the adaptive weight vector adjustment; the rate update weight is set to 0.05 and the maximal number of subproblem adjusted is set to $0.05 \cdot N$. To perform a fair comparison, A-GWASF-GA, MOEA/D-DE, MOEA/D-DE-AWA, and MOEA/DD use initially the same set of weight vectors.

We execute 30 independent runs for each algorithm and each test problem. A-GWASF-GA, MOEA/D-DE-AWA, and the test problems have been implemented in Java using the jMetal framework [44], where implementations of the other algorithms are available.[5] jMetal is an object-oriented Java-based framework for multiobjective optimization using metaheuristics. We conduct our experiments in a cluster of 21 computers offering a total of 172 cores and 190 GB of memory. The cluster is managed by HTCondor, a specialized workload management system for compute-intensive jobs.[6] We use HTCondor because it provides a job queuing mechanism, scheduling policy, and resource management that allow users to submit parallel jobs to HTCondor.

[5] Given that RVEA is not included in jMetal, the experiments for this algorithm have been run in PlatEMO [45].

[6] `https://research.cs.wisc.edu/htcondor/index.html`

*4.2. Data Analysis*

To evaluate the performance of the algorithms, we use the Inverted Generational Distance (IGD) [46] and the Hypervolume (HV) indicator [47].

IGD provides combined information about convergence and diversity of the solutions obtained by each algorithm. This metric is widely used to evaluate the approximated solution sets for both MOPs and MaOPs and it is defined as:

$$IGD(S_{pf}, R) = \frac{1}{|S_{pf}|} \sum_{y \in S_{pf}} dist(y, R),$$

where $S_{pf}$ is the approximation of the PF found by an algorithm for a given problem, $R$ is a reference set of points uniformly distributed over the PF in the objective space, and $dist(y, R) = \min_{z \in R} \|y - z\|_2$. The lower the IGD value is, the better the algorithm performs.

The HV metric measures the volume of the objective space which is dominated by the solutions in a population $P$ and is bounded by a reference point dominated by all the Pareto-optimal objective vectors. The larger the HV value, the better the quality of $P$ for approximating the true PF. The hypervolume is calculated by specifying the reference point such as e.g. the nadir point. To obtain the HV, we use the WFG calculation method of the exact hypervolume value suggested in [48] for all test problems with 3, 5, and 8 objective functions. Note that due to the computational cost needed to calculate this indicator, we exclude from the study the HV metric for ten-objective problems.

For the computation of the IGD and HV metrics a representative set of the true PF is required. For the DTLZ and WFG problems, we generate them using an open-source tool.[7] For the MaOP benchmark problems we use the approximations of the PF available on the Internet.[8]

In order to check whether the IGD and the HV achieved by A-GWASF-GA are significantly different to that of the others algorithms, we apply a Wilcoxon rank-sum test [49]. For each problem, the null hypothesis is that the distribution of the IGD average values (respectively, of the HV average values) in the 30 runs differ by a location shift of $\alpha$. We consider the difference by a location shift of $\alpha$. We consider the difference

[7]https://research.cs.wisc.edu/htcondor/index.html

[8]http://web.mysites.ntu.edu.sg/epnsugan/PublicSite/Shared%20Documents/SWEVO-SI-MOEA-2018/POF.zip.

to be significant if the obtained $p$-value is lower than $\alpha = 0.05$. To compute the Wilcoxon rank-sum test, we use the `wilcox.test` function from the R software.[9]

### *4.3. Results*

We have run experimental tests to compare the performance of A-GWASF-GA with the original GWASF-GA, although we have not included the results obtained here to reduce the paper length. All the results are available upon request to the authors. According to our experiments, A-GWASF-GA and GWASF-GA generate similar results for three-objective problems (their means are not statistically different). However, in problems with five-, eight-, and ten-objectives, A-GWASF-GA performs significantly better or equal than GWASF-GA in most of the cases.

Hereafter, we include tables that show, for each algorithm and each problem, the IGD and HV mean and standard deviation values over the 30 independent runs. For the MaOP problems, these tables contain the results for both 20 and 50 decision variables ($n = 20$ and $n = 50$, respectively). For each test problem, the algorithm that has obtained the best mean value of each metric is highlighted in dark gray color and, similarly, the algorithm with the second best mean value is highlighted in light gray color. We also indicate in the same tables the number of problems for which each of the metrics (IGD or HV) mean value achieved by A-GWASF-GA is significantly better than (▲), equal to (⊙), or worse than (▽) the metric mean value obtained by each of the algorithms under comparison with a level of significance $\alpha = 0.05$.

Next, we analyze the computational results for the MaOP problems [40] in Section 4.3.1 and we show the results for the problems of the DTLZ [41] and the WFG [42] families in Subsection 4.3.2.

#### *4.3.1. MaOP test problems*

Tables 1 and 2 depict the IGD metric values obtained for the MaOP problems using $n = 20$ and $n = 50$, respectively.

According to them, A-GWASF-GA performs better than RVEA in most of the cases regarding the IGD metric, since A-GWASF-GA wins RVEA in, at least, 7 of the 10 problems (indicated as 7/10 in what follows) with five, eight, and ten objectives for both $n = 20$ and $n = 50$. However, for

[9] `https://stat.ethz.ch/R-manual/R-devel/library/stats/html/wilcox.test.html`

Table 1: IGD mean and standard deviation in 30 independent runs for the MaOP problems with $n = 20$.

| $k$ | Prob. | A-GWASF-GA | RVEA | NSGA-III | MOEA/D-DE | MOEA/DD | MOEA/D-DE-AWA |
|---|---|---|---|---|---|---|---|
| 3 | MaOP1 | $0.074461_{0.000154}$ | $0.171652_{0.082138}+$ | $0.082507_{0.009809}+$ | $0.087203_{0.003751}+$ | $0.420689_{0.114401}+$ | $0.082633_{0.002787}+$ |
| | MaOP2 | $0.006190_{0.001090}$ | $0.003840_{0.001583}-$ | $0.007317_{0.000081}+$ | $0.002184_{0.000056}-$ | $0.002062_{0.000171}-$ | $0.001552_{0.000062}-$ |
| | MaOP3 | $0.646605_{0.084677}$ | $1.447856_{0.057937}+$ | $0.979013_{0.096495}+$ | $0.019720_{0.007616}-$ | $2.021674_{0.075647}+$ | $0.097447_{0.061558}-$ |
| | MaOP4 | $0.020904_{0.000000}$ | $0.020917_{0.000001}+$ | $0.020905_{0.000001}+$ | $0.020903_{0.000000}-$ | $0.020921_{0.000002}+$ | $0.020903_{0.000000}-$ |
| | MaOP5 | $0.008894_{0.000573}$ | $0.006747_{0.000447}-$ | $0.008201_{0.000372}-$ | $0.008211_{0.000323}-$ | $0.004415_{0.000380}-$ | $0.007771_{0.000862}-$ |
| | MaOP6 | $0.008540_{0.001882}$ | $0.008473_{0.000638}=$ | $0.013366_{0.001686}+$ | $0.006014_{0.001502}-$ | $0.004029_{0.000355}-$ | $0.007111_{0.001806}-$ |
| | MaOP7 | $0.015173_{0.004086}$ | $0.005892_{0.001027}-$ | $0.011092_{0.001773}-$ | $0.002144_{0.000230}-$ | $0.006111_{0.002017}-$ | $0.002114_{0.000196}-$ |
| | MaOP8 | $0.014000_{0.002957}$ | $0.006018_{0.001886}-$ | $0.012721_{0.001276}-$ | $0.003239_{0.000571}-$ | $0.006796_{0.002557}-$ | $0.002710_{0.000539}-$ |
| | MaOP9 | $0.013571_{0.001619}$ | $0.005209_{0.000784}-$ | $0.014339_{0.002386}+$ | $0.010619_{0.003346}-$ | $0.006077_{0.002672}-$ | $0.009824_{0.003367}-$ |
| | MaOP10 | $0.008359_{0.001363}$ | $0.005528_{0.001251}-$ | $0.011201_{0.003068}+$ | $0.005703_{0.003414}-$ | $0.006389_{0.002370}-$ | $0.005623_{0.003447}-$ |
| | ▲ ⊙ ▽ | | **▲3 ⊙1 ▽6** | **▲7 ⊙0 ▽3** | **▲1 ⊙0 ▽9** | **▲3 ⊙0 ▽7** | **▲1 ⊙0 ▽9** |
| 5 | MaOP1 | $0.044081_{0.001047}$ | $1.858189_{0.994651}+$ | $0.076271_{0.027495}+$ | $0.330009_{0.062571}+$ | $1.747646_{0.326408}+$ | $0.227667_{0.107990}+$ |
| | MaOP2 | $0.002056_{0.000033}$ | $0.001528_{0.000079}-$ | $0.001666_{0.000044}-$ | $0.002809_{0.000050}+$ | $0.003353_{0.000063}+$ | $0.001980_{0.000053}-$ |
| | MaOP3 | $0.139616_{0.013336}$ | $0.497889_{0.015367}+$ | $1.190402_{0.031224}+$ | $0.193949_{0.060131}+$ | $0.616248_{0.028638}+$ | $0.236425_{0.079543}+$ |
| | MaOP4 | $0.009758_{0.000006}$ | $0.009477_{0.000710}=$ | $0.009628_{0.000003}-$ | $0.009932_{0.000005}+$ | $0.009467_{0.000330}-$ | $0.009657_{0.000007}-$ |
| | MaOP5 | $0.005778_{0.000233}$ | $0.009295_{0.003473}+$ | $0.013091_{0.000624}+$ | $0.006095_{0.000295}+$ | $0.005182_{0.000340}-$ | $0.005912_{0.000235}+$ |
| | MaOP6 | $0.005643_{0.000976}$ | $0.016434_{0.003940}+$ | $0.018345_{0.004664}+$ | $0.002814_{0.000590}-$ | $0.006586_{0.001026}+$ | $0.003041_{0.000608}-$ |
| | MaOP7 | $0.003263_{0.000357}$ | $0.005080_{0.000380}+$ | $0.004043_{0.000722}+$ | $0.002640_{0.000224}-$ | $0.004397_{0.000557}+$ | $0.003287_{0.000313}=$ |
| | MaOP8 | $0.003728_{0.000721}$ | $0.004324_{0.000269}+$ | $0.004078_{0.000780}+$ | $0.002441_{0.000254}-$ | $0.004088_{0.000329}+$ | $0.003302_{0.000211}-$ |
| | MaOP9 | $0.002940_{0.001215}$ | $0.005023_{0.000877}+$ | $0.002810_{0.000469}=$ | $0.002692_{0.000812}-$ | $0.004512_{0.000913}+$ | $0.003003_{0.000174}+$ |
| | MaOP10 | $0.002898_{0.000063}$ | $0.004280_{0.000142}+$ | $0.003460_{0.001275}+$ | $0.002550_{0.000119}-$ | $0.003803_{0.000329}+$ | $0.002923_{0.000182}=$ |
| | ▲ ⊙ ▽ | | **▲8 ⊙1 ▽1** | **▲7 ⊙1 ▽2** | **▲5 ⊙0 ▽5** | **▲8 ⊙0 ▽2** | **▲4 ⊙2 ▽4** |
| 8 | MaOP1 | $0.048414_{0.001640}$ | $2.825349_{0.225191}+$ | $0.230251_{0.161118}+$ | $0.155229_{0.070106}+$ | $2.958254_{0.217839}+$ | $0.315601_{0.223430}+$ |
| | MaOP2 | $0.002014_{0.000019}$ | $0.001853_{0.000074}-$ | $0.002684_{0.001030}=$ | $0.003696_{0.000038}+$ | $0.003664_{0.000076}+$ | $0.002516_{0.000154}+$ |
| | MaOP3 | $0.038932_{0.005468}$ | $0.253490_{0.010653}+$ | $1.128126_{0.038557}+$ | $0.561596_{0.134120}+$ | $0.434729_{0.037908}+$ | $0.139731_{0.055437}+$ |
| | MaOP4 | $0.007556_{0.000014}$ | $0.006462_{0.000340}-$ | $0.007172_{0.000008}-$ | $0.008187_{0.000010}+$ | $0.006936_{0.000115}-$ | $0.007465_{0.000046}-$ |
| | MaOP5 | $0.006180_{0.000219}$ | $0.008304_{0.001621}+$ | $0.013740_{0.002235}+$ | $0.006306_{0.000198}+$ | $0.005867_{0.000550}-$ | $0.005846_{0.000171}-$ |
| | MaOP6 | $0.004241_{0.000589}$ | $0.010721_{0.002216}+$ | $0.019508_{0.003604}+$ | $0.002510_{0.000299}-$ | $0.010213_{0.000903}+$ | $0.002823_{0.000133}-$ |
| | MaOP7 | $0.008842_{0.000869}$ | $0.021727_{0.004462}+$ | $0.011790_{0.000652}+$ | $0.134259_{0.047392}+$ | $0.686993_{0.075057}+$ | $0.030482_{0.005052}+$ |
| | MaOP8 | $0.009673_{0.001823}$ | $0.022877_{0.004053}+$ | $0.011056_{0.001196}+$ | $0.121303_{0.051810}+$ | $0.709876_{0.083601}+$ | $0.030004_{0.003069}+$ |
| | MaOP9 | $0.007903_{0.000205}$ | $0.021938_{0.003191}+$ | $0.012763_{0.001361}+$ | $0.114324_{0.035566}+$ | $0.584171_{0.176906}+$ | $0.029495_{0.003412}+$ |
| | MaOP10 | $0.008771_{0.000278}$ | $0.019226_{0.002104}+$ | $0.009357_{0.000929}+$ | $0.124706_{0.041580}+$ | $0.563928_{0.135957}+$ | $0.029733_{0.003808}+$ |
| | ▲ ⊙ ▽ | | **▲8 ⊙0 ▽2** | **▲8 ⊙1 ▽1** | **▲9 ⊙0 ▽1** | **▲8 ⊙0 ▽2** | **▲7 ⊙0 ▽3** |
| 10 | MaOP1 | $0.031865_{0.002118}$ | $2.760569_{0.005160}+$ | $1.063300_{0.412552}+$ | $0.266254_{0.062277}+$ | $2.876382_{0.033470}+$ | $2.285280_{0.782512}+$ |
| | MaOP2 | $0.001560_{0.000032}$ | $0.001302_{0.000023}-$ | $0.002611_{0.000609}+$ | $0.003072_{0.000063}+$ | $0.003011_{0.000043}+$ | $0.002962_{0.000128}+$ |
| | MaOP3 | $2.013976_{0.054984}$ | $0.003710_{0.000006}-$ | $7.061385_{0.622855}+$ | $7.045042_{0.181081}+$ | $0.291078_{0.027966}-$ | $5.754696_{0.268042}+$ |
| | MaOP4 | $0.005701_{0.000011}$ | $0.003889_{0.000033}-$ | $0.005962_{0.000735}=$ | $0.005917_{0.000015}+$ | $0.004966_{0.000030}-$ | $0.005802_{0.000041}+$ |
| | MaOP5 | $0.005111_{0.000009}$ | $0.005680_{0.001092}+$ | $0.010705_{0.002869}+$ | $0.004600_{0.000391}-$ | $0.004350_{0.000264}-$ | $0.004720_{0.000202}-$ |
| | MaOP6 | $0.003257_{0.000325}$ | $0.005618_{0.000988}+$ | $0.011055_{0.002080}+$ | $0.004845_{0.000146}+$ | $0.007681_{0.000622}+$ | $0.004493_{0.000258}+$ |
| | MaOP7 | $0.006041_{0.000382}$ | $0.016583_{0.000535}+$ | $0.012287_{0.001559}+$ | $0.329374_{0.042144}+$ | $0.917217_{0.145716}+$ | $0.279564_{0.041484}+$ |
| | MaOP8 | $0.007612_{0.000389}$ | $0.015338_{0.001154}+$ | $0.011199_{0.002856}+$ | $0.315177_{0.048033}+$ | $0.940283_{0.148661}+$ | $0.288607_{0.027758}+$ |
| | MaOP9 | $0.005430_{0.000411}$ | $0.015813_{0.001378}+$ | $0.012012_{0.001988}+$ | $0.310281_{0.045262}+$ | $0.739585_{0.119373}+$ | $0.267444_{0.023215}+$ |
| | MaOP10 | $0.006071_{0.000133}$ | $0.014244_{0.000526}+$ | $0.009844_{0.001792}+$ | $0.341292_{0.038123}+$ | $0.703974_{0.095329}+$ | $0.272385_{0.030043}+$ |
| | ▲ ⊙ ▽ | | **▲7 ⊙0 ▽3** | **▲9 ⊙1 ▽0** | **▲9 ⊙1 ▽0** | **▲7 ⊙0 ▽3** | **▲9 ⊙0 ▽1** |

Table 2: IGD mean and standard deviation in 30 independent runs for the MaOP problems with $n = 50$.

| $k$ | Prob. | A-GWASF-GA | RVEA | NSGA-III | MOEA/D-DE | MOEA/DD | MOEA/D-DE-AWA |
|---|---|---|---|---|---|---|---|
| 3 | MaOP1 | $0.074964_{0.000556}$ | $0.209986_{0.071267}$ + | $0.082622_{0.011330}$ + | $0.092252_{0.002186}$ + | $0.440462_{0.107199}$ + | $0.089466_{0.002381}$ + |
| | MaOP2 | $0.006622_{0.000756}$ | $0.025173_{0.004522}$ + | $0.007341_{0.000017}$ + | $0.004199_{0.000519}$ − | $0.002881_{0.000134}$ − | $0.004045_{0.000518}$ − |
| | MaOP3 | $4.349181_{0.675887}$ | $16.523352_{0.547482}$ + | $9.302088_{1.028217}$ + | $3.622851_{0.349504}$ − | $12.831934_{0.310307}$ + | $4.918757_{0.514149}$ + |
| | MaOP4 | $0.020904_{0.000000}$ | $0.020916_{0.000001}$ + | $0.020906_{0.000001}$ + | $0.020903_{0.000000}$ − | $0.020922_{0.000002}$ + | $0.020903_{0.000000}$ − |
| | MaOP5 | $0.009590_{0.000424}$ | $0.016915_{0.002143}$ + | $0.008966_{0.000065}$ − | $0.008677_{0.000024}$ − | $0.006173_{0.000243}$ − | $0.008621_{0.000030}$ − |
| | MaOP6 | $0.010601_{0.001098}$ | $0.038542_{0.003629}$ + | $0.016188_{0.000087}$ + | $0.010778_{0.002850}$ = | $0.006568_{0.000971}$ − | $0.009436_{0.002200}$ − |
| | MaOP7 | $0.017266_{0.002201}$ | $0.013241_{0.001753}$ − | $0.011212_{0.001268}$ − | $0.010892_{0.001227}$ − | $0.010588_{0.001879}$ − | $0.009682_{0.001488}$ − |
| | MaOP8 | $0.015551_{0.001936}$ | $0.016277_{0.000934}$ = | $0.013304_{0.001893}$ − | $0.011104_{0.000983}$ − | $0.013484_{0.002607}$ − | $0.010385_{0.001013}$ − |
| | MaOP9 | $0.014553_{0.001941}$ | $0.015950_{0.002686}$ + | $0.015492_{0.001589}$ + | $0.011577_{0.000208}$ − | $0.013236_{0.003537}$ = | $0.011375_{0.000242}$ − |
| | MaOP10 | $0.009327_{0.000776}$ | $0.015300_{0.003117}$ + | $0.012133_{0.003595}$ + | $0.011530_{0.000663}$ + | $0.017216_{0.006891}$ + | $0.010969_{0.000746}$ + |
| | ▲ ⊙ ▽ | | **▲8 ⊙1 ▽1** | **▲7 ⊙0 ▽3** | **▲2 ⊙1 ▽7** | **▲4 ⊙1 ▽5** | **▲3 ⊙0 ▽7** |
| 5 | MaOP1 | $0.044808_{0.000113}$ | $2.064042_{0.861184}$ + | $0.072967_{0.026877}$ + | $0.367036_{0.064518}$ + | $1.970991_{0.420707}$ + | $0.288488_{0.124753}$ + |
| | MaOP2 | $0.002068_{0.000043}$ | $0.003128_{0.000457}$ + | $0.001713_{0.000080}$ − | $0.002785_{0.000040}$ + | $0.003144_{0.000063}$ + | $0.002432_{0.000120}$ + |
| | MaOP3 | $1.129358_{0.081782}$ | $7.742707_{0.141425}$ + | $8.545175_{0.111020}$ + | $5.272199_{0.315726}$ + | $4.772130_{0.201870}$ + | $7.529462_{0.496328}$ + |
| | MaOP4 | $0.009751_{0.000005}$ | $0.010125_{0.000042}$ + | $0.009631_{0.000003}$ − | $0.009929_{0.000003}$ + | $0.009681_{0.000002}$ − | $0.009657_{0.000006}$ − |
| | MaOP5 | $0.006379_{0.000881}$ | $0.019255_{0.004362}$ + | $0.012904_{0.001556}$ + | $0.005921_{0.000146}$ − | $0.006582_{0.000668}$ = | $0.005935_{0.000217}$ − |
| | MaOP6 | $0.009218_{0.001707}$ | $0.031688_{0.006702}$ + | $0.024175_{0.005078}$ + | $0.005769_{0.000353}$ − | $0.013549_{0.001769}$ + | $0.005208_{0.000341}$ − |
| | MaOP7 | $0.003965_{0.001420}$ | $0.004610_{0.000320}$ + | $0.004564_{0.000595}$ + | $0.003083_{0.000465}$ − | $0.007229_{0.001671}$ + | $0.003243_{0.000496}$ − |
| | MaOP8 | $0.004330_{0.000715}$ | $0.004202_{0.000619}$ = | $0.005234_{0.000681}$ + | $0.003742_{0.000592}$ − | $0.006926_{0.000908}$ + | $0.003938_{0.000587}$ = |
| | MaOP9 | $0.003076_{0.001088}$ | $0.005548_{0.000872}$ + | $0.005877_{0.003150}$ + | $0.007547_{0.001785}$ + | $0.006622_{0.001732}$ + | $0.008171_{0.001646}$ + |
| | MaOP10 | $0.002977_{0.000369}$ | $0.004639_{0.000581}$ + | $0.004967_{0.002548}$ + | $0.008797_{0.001715}$ + | $0.005526_{0.001169}$ + | $0.009753_{0.001495}$ + |
| | ▲ ⊙ ▽ | | **▲9 ⊙1 ▽ 0** | **▲8 ⊙0 ▽ 2** | **▲6 ⊙0 ▽ 4** | **▲8 ⊙1 ▽ 1** | **▲5 ⊙1 ▽ 4** |
| 8 | MaOP1 | $0.045840_{0.003148}$ | $2.875514_{0.051745}$ + | $0.325896_{0.168972}$ + | $0.197836_{0.064443}$ + | $3.338031_{0.092532}$ + | $0.762856_{0.453539}$ + |
| | MaOP2 | $0.002008_{0.000020}$ | $0.001862_{0.000073}$ − | $0.002838_{0.000853}$ = | $0.003604_{0.000044}$ + | $0.003735_{0.000084}$ + | $0.002932_{0.000170}$ + |
| | MaOP3 | $0.826009_{0.030784}$ | $5.675290_{0.079396}$ + | $9.378405_{0.581249}$ + | $9.881269_{0.206099}$ + | $5.457098_{0.356362}$ + | $6.675712_{0.742397}$ + |
| | MaOP4 | $0.007507_{0.000012}$ | $0.007308_{0.000027}$ − | $0.007177_{0.000008}$ − | $0.008154_{0.000008}$ + | $0.007020_{0.000005}$ − | $0.007432_{0.000022}$ − |
| | MaOP5 | $0.006530_{0.000040}$ | $0.010701_{0.001505}$ + | $0.016489_{0.001632}$ + | $0.005881_{0.000189}$ − | $0.009412_{0.001661}$ + | $0.006027_{0.000281}$ − |
| | MaOP6 | $0.005641_{0.000676}$ | $0.020332_{0.005547}$ + | $0.025226_{0.003273}$ + | $0.004979_{0.000339}$ − | $0.019099_{0.001883}$ + | $0.005138_{0.000234}$ − |
| | MaOP7 | $0.009596_{0.001603}$ | $0.222366_{0.103992}$ + | $0.013014_{0.001966}$ + | $0.602553_{0.095720}$ + | $4.143050_{0.694599}$ + | $0.281130_{0.035833}$ + |
| | MaOP8 | $0.011399_{0.000747}$ | $0.137879_{0.069018}$ + | $0.011816_{0.001312}$ + | $0.556211_{0.070767}$ + | $4.091687_{0.630396}$ + | $0.259887_{0.036393}$ + |
| | MaOP9 | $0.008111_{0.000651}$ | $0.040096_{0.013159}$ + | $0.015104_{0.005188}$ + | $0.480545_{0.054326}$ + | $3.121726_{0.566888}$ + | $0.252103_{0.012617}$ + |
| | MaOP10 | $0.008907_{0.000033}$ | $0.036115_{0.009771}$ + | $0.010250_{0.001073}$ + | $0.520233_{0.073543}$ + | $3.528247_{0.471903}$ + | $0.246276_{0.029443}$ + |
| | ▲ ⊙ ▽ | | **▲8 ⊙0 ▽2** | **▲8 ⊙1 ▽1** | **▲8 ⊙0 ▽2** | **▲9 ⊙0 ▽1** | **▲7 ⊙0 ▽3** |
| 10 | MaOP1 | $0.037959_{0.002478}$ | $2.761394_{0.048381}$ + | $1.199738_{0.404902}$ + | $0.271786_{0.037836}$ + | $3.252496_{0.080453}$ + | $1.770051_{0.637717}$ + |
| | MaOP2 | $0.001519_{0.000014}$ | $0.001557_{0.000055}$ + | $0.002268_{0.000468}$ + | $0.003094_{0.000038}$ + | $0.003039_{0.000044}$ + | $0.002325_{0.000153}$ + |
| | MaOP3 | $0.597269_{0.018326}$ | $3.478112_{0.057711}$ + | $5.385264_{0.772540}$ + | $6.678029_{0.191810}$ + | $4.943396_{0.165430}$ + | $4.913196_{0.305745}$ + |
| | MaOP4 | $0.005656_{0.000005}$ | $0.005186_{0.000033}$ − | $0.005844_{0.000636}$ − | $0.005923_{0.000011}$ + | $0.004983_{0.000006}$ − | $0.005510_{0.000023}$ − |
| | MaOP5 | $0.005113_{0.000085}$ | $0.008046_{0.001736}$ + | $0.008956_{0.001565}$ + | $0.004400_{0.000154}$ − | $0.011819_{0.002648}$ + | $0.004754_{0.000136}$ − |
| | MaOP6 | $0.003136_{0.000322}$ | $0.013437_{0.003792}$ + | $0.010484_{0.001499}$ + | $0.003686_{0.000206}$ + | $0.020088_{0.003833}$ + | $0.003992_{0.000207}$ + |
| | MaOP7 | $0.006168_{0.000012}$ | $0.020683_{0.013283}$ + | $0.011275_{0.001266}$ + | $0.318817_{0.059121}$ + | $4.066297_{0.374962}$ + | $0.336528_{0.040840}$ + |
| | MaOP8 | $0.007777_{0.000672}$ | $0.034538_{0.033635}$ + | $0.010250_{0.000463}$ + | $0.311987_{0.045755}$ + | $3.928390_{0.329327}$ + | $0.300307_{0.043601}$ + |
| | MaOP9 | $0.005390_{0.000149}$ | $0.020200_{0.003575}$ + | $0.011506_{0.000619}$ + | $0.308962_{0.041327}$ + | $2.876545_{0.356471}$ + | $0.290947_{0.023480}$ + |
| | MaOP10 | $0.006028_{0.000029}$ | $0.023399_{0.014193}$ + | $0.009054_{0.000804}$ + | $0.322660_{0.049207}$ + | $0.318384_{0.04216}$ + | $0.280874_{0.016912}$ + |
| | ▲ ⊙ ▽ | | **▲9 ⊙0 ▽1** | **▲9 ⊙0 ▽1** | **▲9 ⊙0 ▽1** | **▲9 ⊙0 ▽1** | **▲8 ⊙0 ▽2** |

$n = 20$, A-GWASF-GA performs worse than RVEA in the majority of the three-objective problems, although this is not the case for $n = 50$.

In comparison to NSGA-III, A-GWASF-GA shows a statistically better performance regarding the IGD metric. Observe that it wins NSGA-III in, at least, 7/10 problems for both $n = 20$ and $n = 50$, regardless the number of objective functions.

With respect to MOEA/D-DE, MOEA/DD, and MOEA/D-DE-AWA, these algorithms provide significantly better results than A-GWASF-GA in most of the three-objective problems for $n = 20$ and $n = 50$. For the five-objective problems, A-GWASF-GA performs very similar to MOEA/D-DE and MOEA/D-DE-AWA, but it does better than MOEA/DD. In the problems with the highest numbers of objectives, A-GWASF-GA shows a significant improvement over MOEA/D-DE, MOEA/DD, and MOEA/D-DE-AWA. To be more precise, for the eight-objective problems, A-GWASF-GA is better than MOEA/D-DE in 9/10 (when $n = 20$) and 8/10 (when $n = 50$) problems, it performs better than MOEA/DD in 8/10 (when $n = 20$) and 9/10 (when $n = 50$) problems, and it improves MOEA/D-DE-AWA in 7/10 problems (for both $n = 20$ and $n = 50$). Regarding the ten-objective problems, A-GWASF-GA wins in 9/10 (for both $n = 20$ and $n = 50$) problems against MOEA/D-DE, in 7/10 (when $n = 20$) and 9/10 (when $n = 50$) problems with respect to MOEA/DD, and in 9/10 (when $n = 20$) and 8/10 (when $n = 50$) problems in comparison to MOEA/D-DE-AWA.

Next, we analyze the HV mean values obtained for the MaOP problems with $n = 20$ and $n = 50$, which can be seen in Tables 3 and 4, respectively. As shown, A-GWASF-GA gets better results than RVEA in the majority of the problems for both $n = 20$ and $n = 50$, regardless the number of objectives. Note that it performs statistically better than RVEA in, at least, 8/10 problems (when $n = 20$) and 9/10 problems (when $n = 50$) for all dimensions.

Concerning NSGA-III, A-GWASF-GA also obtains significantly better results for every $k = 3, 5, 8$ and both $n = 20$ and $n = 50$. Actually, it wins in, at least, 7/10 problems in all the cases, except for the three-objective problems with $n = 50$, when it only wins to NSGA-III in 5/10 problems.

With respect to MOEA/D-DE, MOEA/DD, and MOEA/D-DE-AWA, A-GWASF-GA provides statistically better HV values than these algorithms for both $n = 20$ and $n = 50$ and nearly all the numbers of objectives. Only in the problems with three objectives MOEA/D-DE improves A-GWASF-GA in

Table 3: HV mean and standard deviation in 30 independent runs for the MaOP problems with $n = 20$.

| $k$ | Prob. | A-GWASF-GA | RVEA | NSGA-III | MOEA/D-DE | MOEA/DD | MOEA/D-DE-AWA |
|---|---|---|---|---|---|---|---|
| 3 | MaOP1 | $0.773559_{0.001554}$ | $0.737752_{0.011369}$+ | $0.764213_{0.004319}$+ | $0.756885_{0.006408}$+ | $0.731543_{0.004689}$+ | $0.759094_{0.005384}$+ |
| | MaOP2 | $1.000000_{0.000000}$ | $0.999996_{0.000004}$+ | $0.999999_{0.000002}$+ | $0.999988_{0.000018}$+ | $0.999996_{0.000003}$+ | $0.999985_{0.000015}$+ |
| | MaOP3 | $0.997815_{0.000939}$ | $0.978325_{0.002586}$+ | $0.993925_{0.001685}$+ | $0.999999_{0.000001}$− | $0.949612_{0.003998}$+ | $0.999977_{0.000028}$− |
| | MaOP4 | $1.000000_{0.000001}$ | $0.999909_{0.000141}$+ | $1.000000_{0.000000}$+ | $0.999999_{0.000005}$+ | $1.000000_{0.000000}$+ | $1.000000_{0.000000}$+ |
| | MaOP5 | $0.999997_{0.000003}$ | $0.997606_{0.001875}$+ | $0.999997_{0.000002}$ = | $0.999697_{0.000413}$+ | $0.999916_{0.000065}$+ | $0.999831_{0.000174}$+ |
| | MaOP6 | $1.000000_{0.000000}$ | $0.999979_{0.000041}$+ | $1.000000_{0.000000}$+ | $0.999999_{0.000001}$+ | $0.999996_{0.000002}$+ | $0.999984_{0.000018}$+ |
| | MaOP7 | $0.999913_{0.000077}$ | $0.999721_{0.000232}$+ | $0.999382_{0.000311}$+ | $0.999993_{0.000005}$− | $0.999623_{0.000312}$+ | $0.999989_{0.000004}$− |
| | MaOP8 | $0.999806_{0.000085}$ | $0.999701_{0.000223}$ = | $0.999031_{0.000356}$+ | $0.999935_{0.000019}$− | $0.999608_{0.000254}$+ | $0.999935_{0.000018}$− |
| | MaOP9 | $0.999775_{0.000173}$ | $0.999769_{0.000078}$ = | $0.998937_{0.000561}$+ | $0.999849_{0.000236}$− | $0.999314_{0.000389}$+ | $0.999976_{0.000104}$− |
| | MaOP10 | $0.999933_{0.000076}$ | $0.999881_{0.000174}$+ | $0.999368_{0.000332}$+ | $0.999904_{0.000183}$ = | $0.999705_{0.000254}$+ | $0.999958_{0.000127}$− |
| | ▲ ⊙ ▽ | | **▲9 ⊙1 ▽0** | **▲7 ⊙2 ▽1** | **▲9 ⊙1 ▽0** | **▲10 ⊙0 ▽0** | **▲8 ⊙2 ▽0** |
| 5 | MaOP1 | $0.395742_{0.001021}$ | $0.255557_{0.026616}$+ | $0.367484_{0.004571}$+ | $0.339999_{0.002367}$+ | $0.199070_{0.007763}$+ | $0.334726_{0.006045}$+ |
| | MaOP2 | $1.000000_{0.000000}$ | $0.999926_{0.000093}$+ | $1.000000_{0.000000}$+ | $0.998890_{0.000354}$+ | $0.993482_{0.002460}$+ | $0.999996_{0.000002}$+ |
| | MaOP3 | $0.999998_{0.000001}$ | $0.999537_{0.000056}$+ | $0.967137_{0.003403}$+ | $0.999989_{0.000015}$+ | $0.997894_{0.000383}$+ | $0.999963_{0.000056}$+ |
| | MaOP4 | $1.000000_{0.000000}$ | $1.000000_{0.000001}$+ | $1.000000_{0.000000}$+ | $0.999982_{0.000043}$− | $1.000000_{0.000000}$+ | $0.999999_{0.000001}$+ |
| | MaOP5 | $0.995070_{0.001699}$ | $0.992140_{0.010703}$ = | $0.996957_{0.000024}$− | $0.993643_{0.000645}$ = | $0.980944_{0.007836}$+ | $0.994400_{0.000677}$ = |
| | MaOP6 | $0.999589_{0.000078}$ | $0.994154_{0.004900}$+ | $0.999430_{0.000062}$+ | $0.997258_{0.000079}$+ | $0.990057_{0.002387}$+ | $0.998644_{0.000813}$+ |
| | MaOP7 | $0.959291_{0.003609}$ | $0.948618_{0.015050}$+ | $0.896824_{0.037850}$+ | $0.958434_{0.003705}$ = | $0.894565_{0.022169}$+ | $0.953255_{0.019373}$ = |
| | MaOP8 | $0.990713_{0.006367}$ | $0.992056_{0.007602}$ = | $0.973363_{0.011386}$+ | $0.991742_{0.002884}$+ | $0.973726_{0.009154}$+ | $0.995189_{0.000390}$ = |
| | MaOP9 | $0.940887_{0.000666}$ | $0.934694_{0.006306}$+ | $0.922574_{0.020778}$+ | $0.935457_{0.013051}$+ | $0.850847_{0.032311}$+ | $0.940173_{0.000644}$+ |
| | MaOP10 | $0.651279_{0.001382}$ | $0.616093_{0.003461}$+ | $0.620453_{0.026398}$+ | $0.636296_{0.003045}$+ | $0.562482_{0.021416}$+ | $0.638304_{0.003610}$+ |
| | ▲ ⊙ ▽ | | **▲8 ⊙2 ▽0** | **▲8 ⊙0 ▽2** | **▲8 ⊙2 ▽0** | **▲10 ⊙0 ▽0** | **▲7 ⊙3 ▽0** |
| 8 | MaOP1 | $0.061482_{0.000410}$ | $0.020083_{0.002082}$+ | $0.053091_{0.001782}$+ | $0.047250_{0.000634}$+ | $0.008329_{0.000908}$+ | $0.029240_{0.001214}$+ |
| | MaOP2 | $1.000000_{0.000000}$ | $0.999918_{0.000060}$+ | $1.000000_{0.000001}$ = | $0.999841_{0.000074}$+ | $0.997832_{0.001207}$+ | $1.000000_{0.000000}$+ |
| | MaOP3 | $1.000000_{0.000000}$ | $1.000000_{0.000000}$+ | $0.987648_{0.001385}$+ | $0.999427_{0.000654}$+ | $0.999629_{0.000233}$+ | $1.000000_{0.000000}$ = |
| | MaOP4 | $1.000000_{0.000000}$ | $0.999999_{0.000001}$+ | $1.000000_{0.000000}$− | $0.999552_{0.000119}$+ | $0.999794_{0.000086}$+ | $0.999879_{0.000064}$+ |
| | MaOP5 | $0.988962_{0.006556}$ | $0.992945_{0.014489}$ = | $0.995980_{0.000191}$ = | $0.989427_{0.000025}$ = | $0.980359_{0.008620}$+ | $0.989854_{0.000361}$ = |
| | MaOP6 | $0.999436_{0.000007}$ | $0.989975_{0.010453}$+ | $0.998663_{0.000690}$+ | $0.995869_{0.000055}$+ | $0.974258_{0.003892}$+ | $0.999079_{0.000161}$+ |
| | MaOP7 | $0.542989_{0.002342}$ | $0.425215_{0.047608}$+ | $0.084507_{0.009956}$+ | $0.528305_{0.004301}$+ | $0.495519_{0.005009}$+ | $0.531094_{0.007049}$+ |
| | MaOP8 | $0.484948_{0.001012}$ | $0.356638_{0.031682}$+ | $0.140113_{0.008660}$+ | $0.469744_{0.001197}$+ | $0.423853_{0.006843}$+ | $0.463297_{0.006786}$+ |
| | MaOP9 | $0.542842_{0.002802}$ | $0.236411_{0.048197}$+ | $0.043969_{0.008958}$+ | $0.515658_{0.003315}$+ | $0.433795_{0.006672}$+ | $0.504263_{0.006153}$+ |
| | MaOP10 | $0.591515_{0.003662}$ | $0.262098_{0.080698}$+ | $0.049311_{0.012004}$+ | $0.570477_{0.002512}$+ | $0.516377_{0.008738}$+ | $0.545480_{0.013119}$+ |
| | ▲ ⊙ ▽ | | **▲9 ⊙1 ▽0** | **▲7 ⊙2 ▽1** | **▲9 ⊙1 ▽0** | **▲10 ⊙0 ▽0** | **▲8 ⊙2 ▽0** |

Table 4: HV mean and standard deviation in 30 independent runs for the MaOP problems with $n = 50$.

| $k$ | Prob. | A-GWASF-GA | RVEA | NSGA-III | MOEA/D-DE | MOEA/DD | MOEA/D-DE-AWA |
|---|---|---|---|---|---|---|---|
| 3 | MaOP1 | $0.602398_{0.003679}$ | $0.537672_{0.016460}$+ | $0.590877_{0.002978}$+ | $0.539020_{0.012079}$+ | $0.564792_{0.003700}$+ | $0.543765_{0.010691}$+ |
| | MaOP2 | $1.000000_{0.000000}$ | $0.999979_{0.000011}$+ | $1.000000_{0.000000}$+ | $1.000000_{0.000000}$+ | $0.999999_{0.000002}$+ | $1.000000_{0.000000}$+ |
| | MaOP3 | $0.988365_{0.005799}$ | $0.394752_{0.047747}$+ | $0.904438_{0.031637}$+ | $0.994251_{0.001557}$− | $0.762093_{0.010199}$+ | $0.986101_{0.004271}$+ |
| | MaOP4 | $0.209325_{0.000034}$ | $0.195972_{0.000470}$+ | $0.206573_{0.000896}$+ | $0.211140_{0.000002}$− | $0.195315_{0.000003}$+ | $0.211912_{0.000110}$− |
| | MaOP5 | $0.986075_{0.000155}$ | $0.298767_{0.357958}$+ | $0.984841_{0.003307}$− | $0.968536_{0.004244}$+ | $0.979850_{0.003219}$+ | $0.977114_{0.003873}$+ |
| | MaOP6 | $0.999941_{0.000060}$ | $0.997950_{0.000951}$+ | $0.999990_{0.000000}$− | $0.999993_{0.000001}$− | $0.999938_{0.000051}$ = | $0.999951_{0.000036}$ = |
| | MaOP7 | $0.831551_{0.028393}$ | $0.832725_{0.012366}$ = | $0.866380_{0.028348}$− | $0.895315_{0.004303}$− | $0.862168_{0.012116}$− | $0.903793_{0.005723}$− |
| | MaOP8 | $0.871587_{0.034011}$ | $0.835633_{0.010615}$+ | $0.869947_{0.023333}$ = | $0.896986_{0.006741}$− | $0.863225_{0.011069}$+ | $0.904218_{0.004998}$− |
| | MaOP9 | $0.765536_{0.036786}$ | $0.637793_{0.052136}$+ | $0.767451_{0.035181}$ = | $0.748014_{0.020996}$ = | $0.674146_{0.059899}$+ | $0.766922_{0.033864}$ = |
| | MaOP10 | $0.722770_{0.018576}$ | $0.552658_{0.048503}$+ | $0.687172_{0.030925}$+ | $0.659193_{0.019377}$+ | $0.602350_{0.059814}$+ | $0.682335_{0.027026}$+ |
| | ▲ ⊙ ▽ | | **▲9 ⊙1 ▽ 0** | **▲5 ⊙2 ▽ 3** | **▲4 ⊙1 ▽ 5** | **▲8 ⊙1 ▽ 1** | **▲5 ⊙2 ▽ 3** |
| 5 | MaOP1 | $0.123640_{0.001241}$ | $0.063120_{0.007581}$+ | $0.105324_{0.002372}$+ | $0.061259_{0.006303}$+ | $0.034919_{0.003491}$+ | $0.054140_{0.004286}$+ |
| | MaOP2 | $0.999999_{0.000000}$ | $0.999747_{0.000299}$+ | $0.999999_{0.000000}$+ | $0.998716_{0.000298}$+ | $0.991789_{0.002324}$+ | $0.999984_{0.000005}$+ |
| | MaOP3 | $0.999997_{0.000001}$ | $0.968253_{0.002256}$+ | $0.950056_{0.002363}$+ | $0.994238_{0.001739}$+ | $0.996059_{0.000903}$+ | $0.958507_{0.012490}$+ |
| | MaOP4 | $0.959207_{0.000236}$ | $0.866359_{0.063705}$+ | $0.961585_{0.000075}$− | $0.936646_{0.001145}$+ | $0.939774_{0.002257}$+ | $0.938224_{0.008902}$+ |
| | MaOP5 | $0.991208_{0.008954}$ | $0.993682_{0.006584}$+ | $0.997347_{0.000057}$+ | $0.978476_{0.000137}$+ | $0.925750_{0.040528}$+ | $0.983020_{0.001825}$+ |
| | MaOP6 | $0.999504_{0.000016}$ | $0.971870_{0.018696}$+ | $0.999168_{0.000124}$+ | $0.997265_{0.000169}$+ | $0.984739_{0.002728}$+ | $0.998713_{0.000693}$+ |
| | MaOP7 | $0.857533_{0.004445}$ | $0.843679_{0.005643}$+ | $0.810285_{0.015189}$+ | $0.833442_{0.008161}$+ | $0.742786_{0.040267}$+ | $0.829624_{0.007228}$+ |
| | MaOP8 | $0.607045_{0.011374}$ | $0.578505_{0.017349}$+ | $0.535605_{0.022714}$+ | $0.564377_{0.020099}$+ | $0.346995_{0.040122}$+ | $0.551138_{0.020815}$+ |
| | MaOP9 | $0.625029_{0.008794}$ | $0.570311_{0.014005}$+ | $0.539094_{0.054098}$+ | $0.454146_{0.045510}$+ | $0.399083_{0.056044}$+ | $0.426726_{0.049646}$+ |
| | MaOP10 | $0.852442_{0.003167}$ | $0.823569_{0.005581}$+ | $0.803191_{0.079556}$+ | $0.761644_{0.031502}$+ | $0.754703_{0.022681}$+ | $0.738081_{0.036577}$+ |
| | ▲ ⊙ ▽ | | **▲10 ⊙0 ▽0** | **▲9 ⊙0 ▽1** | **▲10 ⊙0 ▽0** | **▲10 ⊙0 ▽0** | **▲10 ⊙0 ▽0** |
| 8 | MaOP1 | $0.026181_{0.000304}$ | $0.007159_{0.001093}$+ | $0.018859_{0.000895}$+ | $0.007991_{0.000454}$+ | $0.000811_{0.000209}$+ | $0.004972_{0.000366}$+ |
| | MaOP2 | $1.000000_{0.000000}$ | $0.999559_{0.000214}$+ | $0.999997_{0.000007}$+ | $0.999686_{0.000111}$+ | $0.993352_{0.002083}$+ | $0.999998_{0.000001}$+ |
| | MaOP3 | $1.000000_{0.000000}$ | $0.999384_{0.000078}$+ | $0.969370_{0.005654}$+ | $0.906171_{0.007059}$+ | $0.995918_{0.002560}$+ | $0.976170_{0.012229}$+ |
| | MaOP4 | $0.998435_{0.000003}$ | $0.978027_{0.020319}$+ | $0.998684_{0.000006}$− | $0.939485_{0.003331}$+ | $0.986536_{0.000988}$+ | $0.977194_{0.011592}$+ |
| | MaOP5 | $0.985703_{0.007128}$ | $0.993618_{0.000234}$ = | $0.991889_{0.000898}$ = | $0.986717_{0.000018}$ = | $0.943910_{0.024534}$+ | $0.987020_{0.000341}$ = |
| | MaOP6 | $0.999422_{0.000017}$ | $0.936308_{0.041559}$+ | $0.996974_{0.000866}$+ | $0.995487_{0.000076}$+ | $0.956550_{0.003421}$+ | $0.998624_{0.000773}$+ |
| | MaOP7 | $0.544776_{0.003211}$ | $0.280137_{0.039869}$+ | $0.045002_{0.004204}$+ | $0.526534_{0.007061}$+ | $0.468424_{0.008928}$+ | $0.524814_{0.004861}$+ |
| | MaOP8 | $0.541766_{0.002381}$ | $0.243721_{0.044455}$+ | $0.040370_{0.003240}$+ | $0.530940_{0.003156}$+ | $0.478659_{0.008591}$+ | $0.527536_{0.005361}$+ |
| | MaOP9 | $0.619750_{0.004269}$ | $0.122978_{0.041565}$+ | $0.046283_{0.007846}$+ | $0.607223_{0.004887}$+ | $0.497113_{0.009952}$+ | $0.571467_{0.005718}$+ |
| | MaOP10 | $0.635509_{0.004577}$ | $0.106143_{0.040335}$+ | $0.046320_{0.003714}$+ | $0.620241_{0.004907}$+ | $0.529020_{0.011345}$+ | $0.594993_{0.004150}$+ |
| | ▲ ⊙ ▽ | | **▲9 ⊙1 ▽0** | **▲8 ⊙1 ▽1** | **▲9 ⊙1 ▽0** | **▲10 ⊙0 ▽0** | **▲9 ⊙1 ▽0** |

a higher number of cases when $n = 50$ (5/10 problems). Observe that there are many cases in which A-GWASF-GA performs statistically better than these three algorithms in 10/10 problems, e.g. in the five-objective problems with $n = 50$. This shows the promising results that can be generated with A-GWASF-GA.

*4.3.2. DTLZ and WFG test problems*

Table 5 contains the comparison of A-GWASF-GA against the other algorithms for the DTLZ and WFG problems regarding the IGD metric. As shown, RVEA gets significantly better results than A-GWASF-GA in all dimensions. On the other hand, NSGA-III performs significantly better than A-GWASF-GA in the three-, five- and eight-objective problems. However, both algorithms perform similarly for the ten-objective problems.

With respect to MOEA/D-DE, A-GWASF-GA achieves better results in all the problems, regardless the number of objectives. Regarding MOEA/DD, A-GWASF-GA achieves a better performance for the three- and ten-objective problems, but this does not happen for the cases with five and eight objectives. In comparison to MOEA/D-DE-AWA, A-GWASF-GA retrieves better IGD mean values for the eight- and ten-objective problems.

Although the performance of A-GWASF-GA is not as satisfactory for the DTLZ and WFG problems regarding the IGD metric than for the MaOP problems, note that it shows better results for the higher dimensions. To be more specific, for the eight-objective problems, A-GWASF-GA wins in 3/14 with respect RVEA, in 5/14 problems against NSGA-III, in 14/14 problems in comparison to MOEA/D-DE, in 5/14 problems against MOEA/DD, and in 7/14 problems with respect to MOEA/D-DE-AWA. For the ten-objective problems, A-GWASF-GA is better in 2/14 against RVEA, in 7/14 problems in comparison to NSGA-III, in 14/14 problems against MOEA/D-DE, in 9/14 problems with respect to MOEA/DD, and in 8/10 problems against MOEA/D-DE-AWA, respectively.

Finally, Table 6 shows the results obtained with respect to the HV metric for the DTLZ and WFG problems. Note that A-GWASF-GA is better than RVEA in 12/14 of the three-objective problems and in 8/14 of the eight-objective problems, while RVEA performs better than A-GWASF-GA in 7/14 of the five-objective problems. In comparison to NSGA-III, A-GWASF-GA wins in 9/14 problems with five objectives and 11/14 of the eight-objective problems, but this is not the case for the three-objective problems, since NSGA-III performs better in 7/14

Table 5: IGD mean and standard deviation in 30 independent runs for DTLZ and WFG problems.

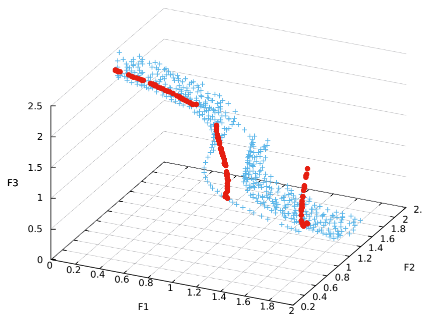
| $k$ | Prob. | A-GWASF-GA | RVEA | NSGA-III | MOEA/D-DE | MOEA/DD | MOEA/D-DE-AWA |
|---|---|---|---|---|---|---|---|
| 3 | DTLZ1 | $3.55e-04_{2.1e-05}$ | $2.31e-04_{1.5e-07}$ − | $2.32e-04_{1.3e-06}$ − | $3.48e-04_{1.0e-06}$ − | $2.75e-04_{1.6e-06}$ − | $2.76e-04_{3.2e-06}$ − |
| | DTLZ2 | $4.45e-04_{1.2e-05}$ | $3.12e-04_{7.6e-07}$ − | $3.12e-04_{5.9e-07}$ − | $4.30e-04_{1.2e-06}$ − | $3.40e-04_{1.0e-06}$ − | $3.86e-04_{6.2e-06}$ − |
| | DTLZ3 | $7.28e-04_{1.2e-05}$ | $5.11e-04_{4.7e-06}$ − | $5.17e-04_{1.9e-05}$ − | $1.73e-03_{3.8e-03}$ − | $5.37e-04_{2.6e-06}$ − | $1.16e-03_{2.8e-03}$ − |
| | DTLZ4 | $6.96e-04_{7.3e-05}$ | $3.60e-04_{1.1e-07}$ − | $3.61e-04_{1.9e-06}$ − | $9.07e-04_{2.5e-05}$ + | $5.51e-04_{6.6e-06}$ − | $4.69e-04_{2.3e-05}$ − |
| | DTLZ7 | $2.40e-03_{9.3e-05}$ | $1.72e-03_{1.5e-04}$ − | $1.07e-03_{1.9e-05}$ − | $2.67e-03_{1.2e-04}$ + | $2.32e-03_{6.2e-05}$ − | $1.77e-03_{4.7e-05}$ − |
| | WFG1 | $1.89e-03_{2.6e-04}$ | $1.81e-03_{1.5e-04}$ = | $2.85e-03_{4.2e-04}$ + | $4.71e-03_{7.8e-04}$ + | $4.72e-03_{9.5e-04}$ + | $2.95e-03_{6.0e-04}$ + |
| | WFG2 | $6.90e-04_{3.4e-05}$ | $6.87e-04_{3.2e-05}$ = | $6.53e-04_{2.3e-05}$ − | $1.11e-03_{1.8e-05}$ + | $2.08e-03_{1.5e-05}$ + | $7.55e-04_{8.4e-05}$ + |
| | WFG3 | $7.35e-05_{6.1e-06}$ | $5.16e-04_{1.8e-05}$ + | $3.32e-04_{3.0e-05}$ + | $1.49e-04_{1.7e-06}$ + | $1.02e-03_{2.2e-04}$ + | $1.67e-04_{1.2e-05}$ + |
| | WFG4 | $5.66e-04_{6.1e-05}$ | $3.21e-04_{4.2e-06}$ − | $2.98e-04_{7.5e-07}$ − | $7.42e-04_{2.6e-05}$ + | $5.82e-04_{1.9e-05}$ + | $5.96e-04_{2.4e-05}$ + |
| | WFG5 | $5.64e-04_{1.8e-05}$ | $4.11e-04_{4.8e-06}$ − | $3.84e-04_{5.5e-07}$ − | $6.37e-04_{6.0e-06}$ + | $6.12e-04_{2.3e-05}$ + | $5.45e-04_{1.2e-05}$ − |
| | WFG6 | $5.82e-04_{4.4e-05}$ | $4.93e-04_{4.3e-05}$ − | $4.26e-04_{3.1e-05}$ − | $7.25e-04_{1.8e-04}$ + | $6.02e-04_{2.9e-05}$ + | $6.09e-04_{1.6e-04}$ − |
| | WFG7 | $4.84e-04_{1.4e-05}$ | $3.21e-04_{2.2e-06}$ − | $3.09e-04_{4.5e-07}$ − | $5.12e-04_{4.7e-06}$ + | $5.22e-04_{1.4e-05}$ + | $4.83e-04_{1.5e-05}$ = |
| | WFG8 | $7.97e-04_{1.6e-05}$ | $7.29e-04_{6.1e-05}$ − | $6.68e-04_{1.2e-05}$ − | $8.73e-04_{5.2e-05}$ + | $7.92e-04_{3.3e-05}$ = | $7.86e-04_{8.4e-06}$ − |
| | WFG9 | $5.20e-04_{3.1e-05}$ | $3.37e-04_{5.3e-06}$ − | $3.28e-04_{3.6e-06}$ − | $6.26e-04_{9.6e-06}$ + | $5.67e-04_{2.4e-05}$ + | $5.34e-04_{2.2e-05}$ + |
| | ▲ ⊙ ▽ | | **▲1 ⊙2 ▽11** | **▲2 ⊙0 ▽12** | **▲11 ⊙0 ▽3** | **▲8 ⊙1 ▽5** | **▲5 ⊙1 ▽8** |
| 5 | DTLZ1 | $3.67e-03_{3.4e-05}$ | $3.49e-03_{1.9e-06}$ − | $3.48e-03_{2.2e-05}$ − | $4.09e-03_{1.4e-05}$ + | $2.78e-03_{1.5e-05}$ − | $3.99e-03_{8.4e-05}$ + |
| | DTLZ2 | $5.72e-03_{6.9e-05}$ | $4.50e-03_{9.8e-07}$ − | $4.51e-03_{5.1e-06}$ − | $6.21e-03_{2.5e-05}$ + | $4.43e-03_{2.1e-05}$ − | $5.47e-03_{6.1e-05}$ − |
| | DTLZ3 | $6.32e-03_{1.2e-04}$ | $4.58e-03_{1.5e-05}$ − | $6.70e-03_{3.4e-03}$ − | $6.41e-03_{1.6e-04}$ + | $4.34e-03_{2.5e-05}$ − | $5.50e-03_{9.9e-05}$ − |
| | DTLZ4 | $3.11e-03_{7.8e-05}$ | $2.50e-03_{3.7e-07}$ − | $2.50e-03_{5.3e-06}$ − | $5.04e-03_{7.6e-05}$ + | $4.47e-03_{1.5e-04}$ + | $3.11e-03_{8.8e-05}$ = |
| | DTLZ7 | $4.72e-03_{8.7e-05}$ | $6.40e-03_{9.1e-05}$ + | $5.24e-03_{1.5e-03}$ + | $7.53e-03_{5.7e-05}$ + | $9.97e-03_{9.6e-04}$ + | $5.92e-03_{1.1e-04}$ + |
| | WFG1 | $4.70e-03_{5.1e-04}$ | $2.79e-03_{1.9e-04}$ − | $9.87e-03_{1.7e-03}$ + | $1.68e-02_{1.4e-03}$ + | $9.72e-03_{5.8e-04}$ + | $1.44e-02_{1.4e-03}$ + |
| | WFG2 | $3.35e-03_{9.9e-05}$ | $2.53e-03_{5.3e-05}$ − | $2.78e-03_{3.9e-05}$ − | $5.06e-03_{2.7e-04}$ + | $5.60e-03_{1.9e-04}$ + | $3.74e-03_{2.0e-04}$ + |
| | WFG3 | $4.00e-04_{4.1e-05}$ | $4.78e-03_{5.6e-04}$ + | $5.09e-03_{8.8e-04}$ + | $1.48e-03_{9.4e-05}$ + | $2.70e-02_{4.6e-03}$ + | $1.47e-03_{2.1e-04}$ + |
| | WFG4 | $7.32e-03_{8.2e-05}$ | $4.93e-03_{1.5e-05}$ − | $4.89e-03_{9.0e-06}$ − | $7.65e-03_{1.1e-04}$ + | $6.39e-03_{1.3e-04}$ − | $6.89e-03_{9.4e-05}$ − |
| | WFG5 | $6.78e-03_{7.1e-05}$ | $4.67e-03_{1.9e-05}$ − | $4.61e-03_{4.8e-06}$ − | $6.83e-03_{1.0e-04}$ + | $6.45e-03_{9.9e-05}$ − | $6.46e-03_{1.1e-04}$ − |
| | WFG6 | $6.98e-03_{9.4e-05}$ | $4.81e-03_{2.5e-05}$ − | $4.73e-03_{2.2e-05}$ − | $7.24e-03_{1.4e-04}$ + | $6.76e-03_{1.2e-04}$ − | $6.68e-03_{1.3e-04}$ − |
| | WFG7 | $6.99e-03_{9.2e-05}$ | $4.69e-03_{2.1e-05}$ − | $4.62e-03_{1.2e-05}$ − | $6.93e-03_{6.1e-05}$ − | $6.27e-03_{8.5e-05}$ − | $6.66e-03_{1.2e-04}$ − |
| | WFG8 | $8.55e-03_{1.1e-04}$ | $6.59e-03_{2.8e-05}$ − | $6.51e-03_{5.2e-05}$ − | $8.46e-03_{1.2e-04}$ − | $6.90e-03_{4.2e-04}$ − | $7.58e-03_{9.2e-05}$ − |
| | WFG9 | $7.81e-03_{1.3e-04}$ | $4.98e-03_{2.7e-05}$ − | $4.98e-03_{3.2e-05}$ − | $7.65e-03_{1.3e-04}$ − | $7.56e-03_{1.4e-04}$ − | $7.15e-03_{1.4e-04}$ − |
| | ▲ ⊙ ▽ | | **▲2 ⊙0 ▽12** | **▲3 ⊙0 ▽11** | **▲11 ⊙0 ▽3** | **▲5 ⊙0 ▽9** | **▲5 ⊙1 ▽8** |
| 8 | DTLZ1 | $1.57e-02_{1.6e-04}$ | $1.64e-02_{1.0e-04}$ + | $2.36e-02_{8.5e-03}$ + | $1.67e-02_{1.4e-04}$ + | $9.78e-03_{9.2e-05}$ − | $1.96e-02_{5.0e-04}$ + |
| | DTLZ2 | $1.89e-02_{1.0e-04}$ | $1.66e-02_{5.1e-06}$ − | $1.72e-02_{3.0e-03}$ − | $1.94e-02_{6.3e-05}$ + | $1.39e-02_{1.3e-04}$ − | $2.00e-02_{3.4e-04}$ + |
| | DTLZ3 | $2.04e-02_{2.4e-04}$ | $1.78e-02_{3.7e-03}$ − | $2.78e-02_{2.1e-02}$ + | $2.12e-02_{1.5e-03}$ + | $1.39e-02_{1.2e-04}$ − | $2.01e-02_{3.6e-04}$ − |
| | DTLZ4 | $2.00e-02_{2.0e-04}$ | $1.66e-02_{1.5e-05}$ − | $1.66e-02_{1.9e-05}$ − | $2.09e-02_{1.6e-04}$ + | $1.65e-02_{3.2e-04}$ − | $2.09e-02_{2.0e-04}$ + |
| | DTLZ7 | $1.64e-02_{2.9e-04}$ | $3.18e-02_{6.7e-04}$ + | $6.57e-02_{6.0e-03}$ + | $2.13e-02_{3.8e-04}$ + | $4.19e-02_{1.9e-03}$ + | $1.57e-02_{1.5e-03}$ − |
| | WFG1 | $7.40e-03_{7.0e-04}$ | $5.43e-03_{4.0e-04}$ − | $2.33e-02_{2.4e-03}$ + | $2.60e-02_{2.6e-03}$ + | $1.58e-02_{1.1e-03}$ + | $2.89e-02_{2.0e-03}$ + |
| | WFG2 | $6.18e-03_{3.4e-04}$ | $5.56e-03_{2.9e-04}$ − | $7.14e-03_{3.0e-03}$ = | $1.33e-02_{1.1e-04}$ + | $1.48e-02_{6.6e-04}$ + | $6.84e-03_{2.6e-04}$ + |
| | WFG3 | $1.67e-03_{1.4e-04}$ | $1.65e-01_{8.8e-02}$ + | $3.78e-02_{7.7e-03}$ + | $4.95e-02_{1.5e-02}$ + | $2.12e-01_{6.1e-02}$ + | $1.68e-02_{2.0e-03}$ + |
| | WFG4 | $1.32e-02_{1.3e-04}$ | $9.92e-03_{6.5e-05}$ − | $9.84e-03_{2.3e-05}$ − | $1.80e-02_{5.5e-04}$ + | $1.31e-02_{3.8e-04}$ = | $1.11e-02_{1.1e-04}$ − |
| | WFG5 | $1.38e-02_{7.0e-05}$ | $1.04e-02_{2.9e-05}$ − | $1.04e-02_{1.3e-05}$ − | $1.64e-02_{1.5e-04}$ + | $1.28e-02_{2.3e-04}$ − | $1.36e-02_{1.5e-04}$ − |
| | WFG6 | $1.38e-02_{1.6e-04}$ | $1.03e-02_{4.7e-05}$ − | $1.03e-02_{3.3e-05}$ − | $1.63e-02_{2.7e-04}$ + | $1.38e-02_{2.4e-04}$ = | $1.33e-02_{1.9e-04}$ − |
| | WFG7 | $1.42e-02_{1.1e-04}$ | $1.05e-02_{7.9e-05}$ − | $1.06e-02_{1.1e-05}$ − | $1.70e-02_{1.5e-04}$ + | $1.28e-02_{1.4e-04}$ − | $1.45e-02_{2.8e-04}$ + |
| | WFG8 | $1.74e-02_{4.3e-04}$ | $1.35e-02_{5.6e-05}$ − | $1.31e-02_{7.3e-04}$ − | $1.95e-02_{4.5e-04}$ + | $1.27e-02_{2.4e-04}$ − | $1.31e-02_{3.2e-04}$ − |
| | WFG9 | $1.45e-02_{2.0e-04}$ | $1.14e-02_{3.6e-04}$ − | $1.02e-02_{5.4e-05}$ − | $1.69e-02_{3.7e-04}$ + | $2.26e-02_{1.1e-03}$ + | $1.41e-02_{3.0e-04}$ − |
| | ▲ ⊙ ▽ | | **▲3 ⊙0 ▽11** | **▲5 ⊙1 ▽8** | **▲14 ⊙0 ▽0** | **▲5 ⊙2 ▽7** | **▲7 ⊙0 ▽7** |
| 10 | DTLZ1 | $1.11e-02_{1.2e-04}$ | $8.94e-03_{4.3e-06}$ − | $1.50e-02_{1.4e-03}$ + | $1.27e-02_{1.1e-04}$ + | $9.31e-03_{1.3e-04}$ − | $1.34e-02_{5.9e-04}$ + |
| | DTLZ2 | $1.56e-02_{8.6e-05}$ | $1.17e-02_{1.1e-06}$ − | $2.00e-02_{3.8e-03}$ + | $1.59e-02_{1.2e-04}$ + | $1.12e-02_{8.4e-05}$ − | $1.75e-02_{3.6e-04}$ + |
| | DTLZ3 | $1.64e-02_{1.1e-04}$ | $1.17e-02_{1.4e-06}$ − | $2.36e-02_{1.9e-03}$ + | $1.72e-02_{4.4e-04}$ + | $1.10e-02_{6.4e-05}$ − | $1.79e-02_{3.5e-04}$ + |
| | DTLZ4 | $7.46e-03_{9.1e-05}$ | $6.03e-03_{1.0e-07}$ − | $6.04e-03_{3.7e-06}$ − | $1.62e-02_{2.7e-04}$ + | $1.19e-02_{2.6e-04}$ + | $7.42e-03_{1.7e-04}$ = |
| | DTLZ7 | $1.92e-02_{1.3e-04}$ | $2.45e-02_{2.7e-04}$ + | $5.05e-02_{2.2e-03}$ + | $1.94e-02_{5.0e-05}$ + | $3.20e-02_{8.3e-04}$ + | $2.01e-02_{2.9e-04}$ + |
| | WFG1 | $1.66e-02_{7.6e-04}$ | $8.25e-03_{3.4e-04}$ − | $9.72e-02_{1.3e-02}$ + | $8.76e-02_{1.2e-02}$ + | $3.51e-02_{3.8e-03}$ + | $1.24e-01_{1.0e-02}$ + |
| | WFG2 | $9.41e-03_{8.8e-04}$ | $8.30e-03_{9.2e-04}$ − | $1.48e-02_{2.7e-03}$ + | $1.60e-02_{1.7e-04}$ + | $2.23e-02_{9.1e-04}$ + | $1.25e-02_{8.0e-04}$ + |
| | WFG3 | $2.23e-03_{3.0e-04}$ | $2.15e-01_{2.5e-01}$ + | $9.40e-02_{2.9e-02}$ + | $3.30e-01_{9.0e-02}$ + | $1.14e+00_{1.3e-01}$ + | $8.47e-02_{4.4e-02}$ + |
| | WFG4 | $1.60e-02_{1.8e-04}$ | $1.31e-02_{6.5e-05}$ − | $1.39e-02_{1.6e-03}$ − | $2.52e-02_{1.7e-03}$ + | $1.70e-02_{4.5e-04}$ + | $1.35e-02_{1.8e-04}$ − |
| | WFG5 | $1.67e-02_{6.9e-05}$ | $1.17e-02_{4.9e-05}$ − | $1.44e-02_{1.7e-05}$ − | $2.08e-02_{1.2e-04}$ + | $1.72e-02_{3.0e-04}$ + | $1.54e-02_{1.5e-04}$ − |
| | WFG6 | $1.62e-02_{9.6e-05}$ | $1.15e-02_{6.2e-05}$ − | $1.39e-02_{1.5e-03}$ − | $2.09e-02_{2.1e-04}$ + | $1.79e-02_{3.5e-04}$ + | $1.50e-02_{1.7e-04}$ − |
| | WFG7 | $1.82e-02_{1.1e-04}$ | $1.32e-02_{1.9e-04}$ − | $1.65e-02_{1.1e-03}$ − | $2.14e-02_{2.8e-04}$ + | $1.81e-02_{3.9e-04}$ = | $1.63e-02_{2.0e-04}$ − |
| | WFG8 | $1.90e-02_{2.7e-04}$ | $1.33e-02_{7.1e-05}$ − | $1.55e-02_{1.4e-03}$ − | $2.12e-02_{4.9e-04}$ + | $1.45e-02_{2.5e-04}$ − | $1.48e-02_{1.9e-04}$ − |
| | WFG9 | $1.67e-02_{3.6e-04}$ | $1.37e-02_{5.7e-04}$ − | $1.31e-02_{7.7e-04}$ − | $2.19e-02_{8.7e-04}$ + | $2.67e-02_{1.1e-03}$ + | $1.77e-02_{3.5e-04}$ + |
| | ▲ ⊙ ▽ | | **▲2 ⊙0 ▽12** | **▲7 ⊙0 ▽7** | **▲14 ⊙0 ▽0** | **▲9 ⊙1 ▽4** | **▲8 ⊙1 ▽5** |

problems. Lately, A-GWASF-GA achieves significantly better results than MOEA/D-DE, MOEA/DD, and MOEA/D-DE-AWA in nearly all the three-, five- and eight-objective problems, given that it wins to each of them in, at least, 12/14 problems at each dimension. Overall, we can say that A-GWASF-GA seems to achieve promising results, specially as the number of objective functions increases. In general, A-GWASF-GA has reached better IGD and HV values than the algorithms considered in the eight- and ten-objective problems used. Actually, for the MaOP problems, A-GWASF-GA obtains better results for the numbers of variables considered ($n = 20$ and $n = 50$) and for all objectives (three, five, eight and ten).

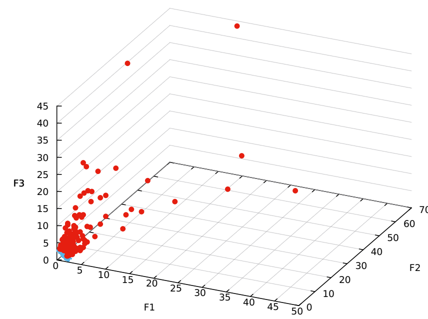
In Figures 3, 4, 5 and 6, we show the approximation of the Pareto fronts generated by the algorithms considered for the three-objective MaOP10 (with $n = 50$), WFG3 and WFG6 problems, and for the ten-objective MaOP6 (with $n = 20$) problem, respectively. Despite the difficulty to obtain a good approximation of their fronts, we can see that our algorithm performs well compared to the other algorithms. However, there are cases, like the WFG6 problem, where the approximations generated by RVEA and NSGAIII are better distributed than the one generated by A-GWASFGA. As it was pointed out previously, it should be noted that our algorithm works better for many objectives (5, 8 and 10) than for three-objective problems.

Table 6: HV mean and standard deviation in 30 independent runs for the DTLZ and WFG problems.

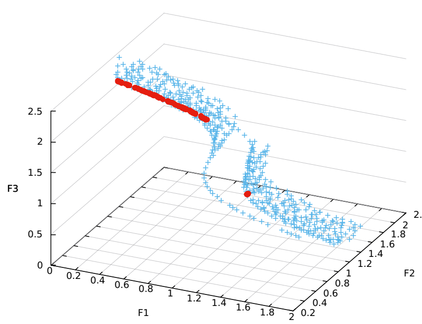
| $k$ | Prob. | A-GWASF-GA | RVEA | NSGA-III | MOEA/D-DE | MOEA/DD | MOEA/D-DE-AWA |
|---|---|---|---|---|---|---|---|
| 3 | DTLZ1 | $9.70e-01_{1.0e-03}$ | $9.59e-01_{8.4e-03}$+ | $9.72e-01_{3.4e-04}$− | $9.67e-01_{2.4e-04}$+ | $9.64e-01_{1.6e-03}$+ | $9.61e-01_{5.5e-03}$+ |
| | DTLZ2 | $9.38e-01_{8.5e-04}$ | $9.16e-01_{3.8e-03}$+ | $9.39e-01_{3.1e-05}$− | $9.31e-01_{3.8e-03}$+ | $9.36e-01_{3.3e-04}$+ | $9.24e-01_{4.1e-03}$+ |
| | DTLZ3 | $9.90e-01_{3.2e-05}$ | $9.87e-01_{1.8e-03}$+ | $9.90e-01_{1.4e-04}$− | $9.84e-01_{1.8e-02}$+ | $9.89e-01_{6.7e-04}$+ | $9.84e-01_{1.4e-02}$+ |
| | DTLZ4 | $9.47e-01_{5.2e-03}$ | $9.51e-01_{2.4e-05}$− | $9.46e-01_{3.8e-03}$+ | $9.48e-01_{4.1e-04}$ = | $9.49e-01_{4.3e-04}$ = | $9.48e-01_{1.2e-03}$ = |
| | DTLZ7 | $9.13e-01_{2.4e-04}$ | $8.91e-01_{1.4e-02}$+ | $9.14e-01_{1.8e-04}$− | $9.10e-01_{7.9e-04}$+ | $8.65e-01_{9.8e-03}$+ | $8.80e-01_{1.6e-02}$+ |
| | WFG1 | $9.51e-01_{1.9e-03}$ | $9.03e-01_{1.3e-02}$+ | $9.02e-01_{1.3e-02}$+ | $7.22e-01_{4.1e-02}$+ | $8.54e-01_{2.3e-02}$+ | $8.32e-01_{3.6e-02}$+ |
| | WFG2 | $9.23e-01_{7.1e-04}$ | $9.14e-01_{1.3e-03}$+ | $9.22e-01_{5.4e-04}$+ | $9.05e-01_{2.7e-03}$+ | $9.17e-01_{2.1e-03}$+ | $9.19e-01_{6.8e-04}$+ |
| | WFG3 | $7.13e-01_{4.0e-04}$ | $6.87e-01_{1.2e-03}$+ | $7.01e-01_{1.6e-03}$+ | $7.09e-01_{1.1e-03}$+ | $6.57e-01_{8.3e-03}$+ | $7.08e-01_{1.4e-03}$+ |
| | WFG4 | $4.88e-01_{1.3e-03}$ | $4.82e-01_{1.5e-03}$+ | $4.94e-01_{3.2e-04}$− | $4.54e-01_{2.8e-03}$+ | $4.80e-01_{1.5e-03}$+ | $4.68e-01_{2.1e-03}$+ |
| | WFG5 | $5.43e-01_{1.7e-03}$ | $5.46e-01_{7.7e-04}$− | $5.51e-01_{1.3e-04}$− | $5.14e-01_{1.8e-03}$+ | $5.24e-01_{5.7e-03}$+ | $5.40e-01_{2.3e-03}$+ |
| | WFG6 | $4.69e-01_{1.2e-02}$ | $4.59e-01_{1.1e-02}$+ | $4.69e-01_{8.7e-03}$ = | $4.57e-01_{5.7e-02}$− | $4.63e-01_{8.8e-03}$+ | $4.72e-01_{5.4e-02}$− |
| | WFG7 | $4.83e-01_{9.3e-04}$ | $4.81e-01_{7.5e-04}$+ | $4.88e-01_{3.4e-04}$− | $4.56e-01_{1.4e-03}$+ | $4.69e-01_{2.9e-03}$+ | $4.72e-01_{1.3e-03}$+ |
| | WFG8 | $6.47e-01_{8.2e-04}$ | $6.34e-01_{6.6e-03}$+ | $6.45e-01_{8.6e-04}$+ | $6.38e-01_{1.6e-03}$+ | $6.30e-01_{3.9e-03}$+ | $6.40e-01_{1.1e-03}$+ |
| | WFG9 | $5.67e-01_{1.9e-03}$ | $5.60e-01_{1.9e-03}$+ | $5.64e-01_{1.6e-03}$+ | $5.41e-01_{1.9e-03}$+ | $5.42e-01_{4.4e-03}$+ | $5.54e-01_{2.5e-03}$+ |
| | ▲ ⊙ ▽ | | **▲12 ⊙0 ▽2** | **▲6 ⊙1 ▽7** | **▲12 ⊙1 ▽1** | **▲13 ⊙1 ▽0** | **▲12 ⊙1 ▽1** |
| 5 | DTLZ1 | $9.99e-01_{1.0e-06}$ | $9.99e-01_{1.1e-04}$+ | $9.99e-01_{1.3e-07}$− | $9.99e-01_{1.8e-04}$+ | $9.96e-01_{1.2e-03}$+ | $9.99e-01_{2.2e-04}$+ |
| | DTLZ2 | $9.74e-01_{1.2e-04}$ | $9.53e-01_{7.4e-03}$+ | $9.76e-01_{4.2e-05}$− | $9.59e-01_{1.2e-03}$+ | $9.61e-01_{9.2e-04}$+ | $9.51e-01_{4.7e-03}$+ |
| | DTLZ3 | $1.00e+00_{3.6e-07}$ | $1.00e+00_{7.0e-07}$− | $1.00e+00_{1.4e-05}$+ | $1.00e+00_{3.0e-05}$+ | $9.98e-01_{6.2e-04}$+ | $1.00e+00_{1.1e-06}$+ |
| | DTLZ4 | $9.95e-01_{5.3e-05}$ | $9.95e-01_{4.1e-05}$− | $9.95e-01_{4.3e-04}$+ | $9.94e-01_{6.9e-05}$+ | $9.91e-01_{6.9e-04}$+ | $9.95e-01_{6.2e-05}$+ |
| | DTLZ7 | $8.82e-01_{2.5e-04}$ | $7.66e-01_{4.5e-02}$+ | $8.80e-01_{1.2e-03}$+ | $3.01e-01_{4.6e-03}$+ | $4.40e-01_{1.7e-02}$+ | $6.08e-01_{5.5e-02}$+ |
| | WFG1 | $9.11e-01_{4.2e-04}$ | $9.53e-01_{2.1e-02}$− | $7.32e-01_{5.1e-02}$+ | $5.29e-01_{3.4e-02}$+ | $7.47e-01_{2.5e-02}$+ | $5.78e-01_{3.9e-02}$+ |
| | WFG2 | $9.98e-01_{1.6e-04}$ | $9.93e-01_{9.5e-04}$+ | $9.94e-01_{6.3e-04}$+ | $9.84e-01_{5.5e-04}$+ | $9.26e-01_{4.8e-03}$+ | $9.87e-01_{1.3e-03}$+ |
| | WFG3 | $7.00e-01_{5.4e-04}$ | $6.57e-01_{3.7e-03}$+ | $6.60e-01_{5.1e-03}$+ | $6.92e-01_{1.0e-03}$+ | $5.65e-01_{1.1e-02}$+ | $6.87e-01_{2.4e-03}$+ |
| | WFG4 | $7.56e-01_{1.3e-03}$ | $7.56e-01_{1.2e-03}$ = | $7.57e-01_{1.1e-03}$− | $6.35e-01_{2.9e-03}$+ | $6.05e-01_{1.0e-02}$+ | $6.78e-01_{3.5e-03}$+ |
| | WFG5 | $7.77e-01_{1.0e-03}$ | $7.88e-01_{6.6e-04}$− | $7.87e-01_{9.0e-04}$− | $6.72e-01_{6.7e-03}$+ | $6.10e-01_{1.2e-02}$+ | $7.46e-01_{3.5e-03}$+ |
| | WFG6 | $7.35e-01_{8.9e-03}$ | $7.40e-01_{1.0e-02}$− | $7.30e-01_{9.0e-03}$+ | $5.77e-01_{6.6e-02}$+ | $5.97e-01_{1.5e-02}$+ | $6.76e-01_{6.4e-02}$+ |
| | WFG7 | $7.64e-01_{1.1e-03}$ | $7.70e-01_{5.2e-04}$− | $7.68e-01_{8.7e-04}$− | $6.49e-01_{3.0e-03}$+ | $5.90e-01_{1.1e-02}$+ | $7.18e-01_{4.1e-03}$+ |
| | WFG8 | $7.70e-01_{1.4e-03}$ | $7.72e-01_{9.1e-04}$− | $7.65e-01_{2.8e-03}$+ | $7.11e-01_{3.3e-03}$+ | $6.87e-01_{3.2e-02}$+ | $7.28e-01_{3.1e-03}$+ |
| | WFG9 | $7.82e-01_{3.6e-03}$ | $7.67e-01_{2.8e-03}$+ | $7.65e-01_{2.7e-03}$+ | $6.54e-01_{5.7e-02}$+ | $6.24e-01_{7.4e-03}$+ | $6.94e-01_{6.2e-02}$+ |
| | ▲ ⊙ ▽ | | **▲6 ⊙1 ▽7** | **▲9 ⊙0 ▽5** | **▲14 ⊙0 ▽0** | **▲14 ⊙0 ▽0** | **▲14 ⊙0 ▽0** |
| 8 | DTLZ1 | $9.99e-01_{0.0e+00}$ | $9.99e-01_{0.0e+00}$ = | $9.99e-01_{1.2e-08}$+ | $9.99e-01_{9.1e-11}$+ | $9.99e-00_{6.2e-05}$+ | $1.00e+00_{0.0e+00}$ = |
| | DTLZ2 | $9.99e-01_{3.9e-06}$ | $9.99e-01_{1.8e-05}$+ | $9.99e-01_{3.2e-04}$− | $9.94e-01_{5.4e-04}$+ | $9.92e-01_{9.8e-04}$+ | $9.97e-01_{4.6e-04}$+ |
| | DTLZ3 | $9.99e-01_{0.0e+00}$ | $9.99e-01_{0.0e+00}$ = | $9.99e-01_{3.7e-08}$+ | $9.99e-01_{0.0e+00}$ = | $9.99e-01_{7.3e-07}$+ | $9.99e-01_{0.0e+00}$ = |
| | DTLZ4 | $9.99e-01_{2.0e-06}$ | $9.99e-01_{1.0e-06}$− | $9.99e-01_{1.7e-06}$− | $9.99e-01_{1.3e-04}$+ | $9.98e-01_{5.5e-04}$+ | $9.99e-01_{2.7e-06}$+ |
| | DTLZ7 | $8.54e-01_{2.5e-04}$ | $5.45e-01_{8.5e-02}$+ | $6.57e-01_{2.3e-01}$+ | $1.24e-02_{5.6e-04}$+ | $1.91e-01_{1.6e-02}$+ | $3.99e-01_{1.0e-01}$+ |
| | WFG1 | $9.18e-01_{1.6e-04}$ | $9.89e-01_{1.7e-02}$− | $7.60e-01_{2.4e-02}$+ | $5.55e-01_{4.6e-02}$+ | $7.73e-01_{3.7e-02}$+ | $4.67e-01_{3.0e-02}$+ |
| | WFG2 | $1.00e+00_{2.0e-04}$ | $9.94e-01_{1.2e-03}$+ | $9.97e-01_{2.6e-03}$+ | $9.92e-01_{4.9e-04}$+ | $9.52e-01_{8.4e-03}$+ | $9.83e-01_{1.9e-03}$+ |
| | WFG3 | $7.13e-01_{1.8e-03}$ | $5.77e-01_{3.3e-02}$+ | $6.42e-01_{8.0e-03}$+ | $6.87e-01_{7.5e-03}$+ | $5.53e-01_{1.3e-02}$+ | $6.82e-01_{5.6e-03}$+ |
| | WFG4 | $9.18e-01_{8.1e-04}$ | $9.05e-01_{2.0e-03}$+ | $9.06e-01_{2.3e-03}$+ | $7.01e-01_{3.5e-03}$+ | $6.06e-01_{1.2e-02}$+ | $8.19e-01_{3.8e-03}$+ |
| | WFG5 | $9.30e-01_{4.7e-04}$ | $9.31e-01_{6.9e-04}$− | $9.28e-01_{1.3e-03}$+ | $7.51e-01_{6.5e-03}$+ | $6.00e-01_{7.9e-03}$+ | $8.71e-01_{5.5e-03}$+ |
| | WFG6 | $9.17e-01_{1.0e-02}$ | $8.98e-01_{1.2e-02}$+ | $8.94e-01_{1.1e-02}$+ | $6.55e-01_{2.9e-02}$+ | $6.13e-01_{1.5e-02}$+ | $7.65e-01_{2.2e-02}$+ |
| | WFG7 | $9.25e-01_{5.4e-04}$ | $9.20e-01_{1.1e-03}$+ | $9.18e-01_{8.7e-04}$+ | $7.19e-01_{2.8e-03}$+ | $5.71e-01_{7.7e-03}$+ | $8.32e-01_{6.3e-03}$+ |
| | WFG8 | $8.50e-01_{2.1e-03}$ | $9.39e-01_{1.6e-03}$− | $9.00e-01_{3.3e-02}$− | $7.66e-01_{3.2e-02}$+ | $7.16e-01_{9.1e-02}$+ | $8.17e-01_{2.4e-02}$+ |
| | WFG9 | $9.07e-01_{3.7e-03}$ | $8.74e-01_{6.2e-03}$+ | $8.82e-01_{2.9e-03}$+ | $6.07e-01_{4.4e-02}$+ | $5.62e-01_{1.4e-02}$+ | $6.85e-01_{2.0e-02}$+ |
| | ▲ ⊙ ▽ | | **▲8 ⊙2 ▽4** | **▲11 ⊙0 ▽3** | **▲13 ⊙1 ▽0** | **▲14 ⊙0 ▽0** | **▲12 ⊙2 ▽0** |



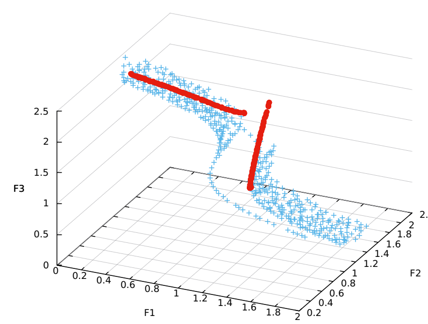

(a) A-GWASF-GA



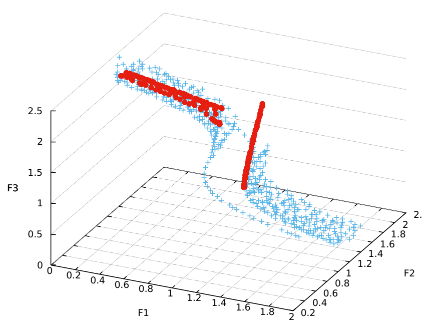

(b) RVEA



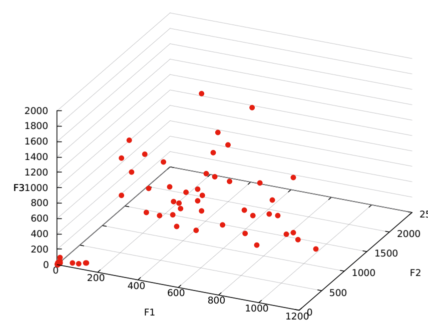

(c) NSGA-III




(d) MOEA/D-DE

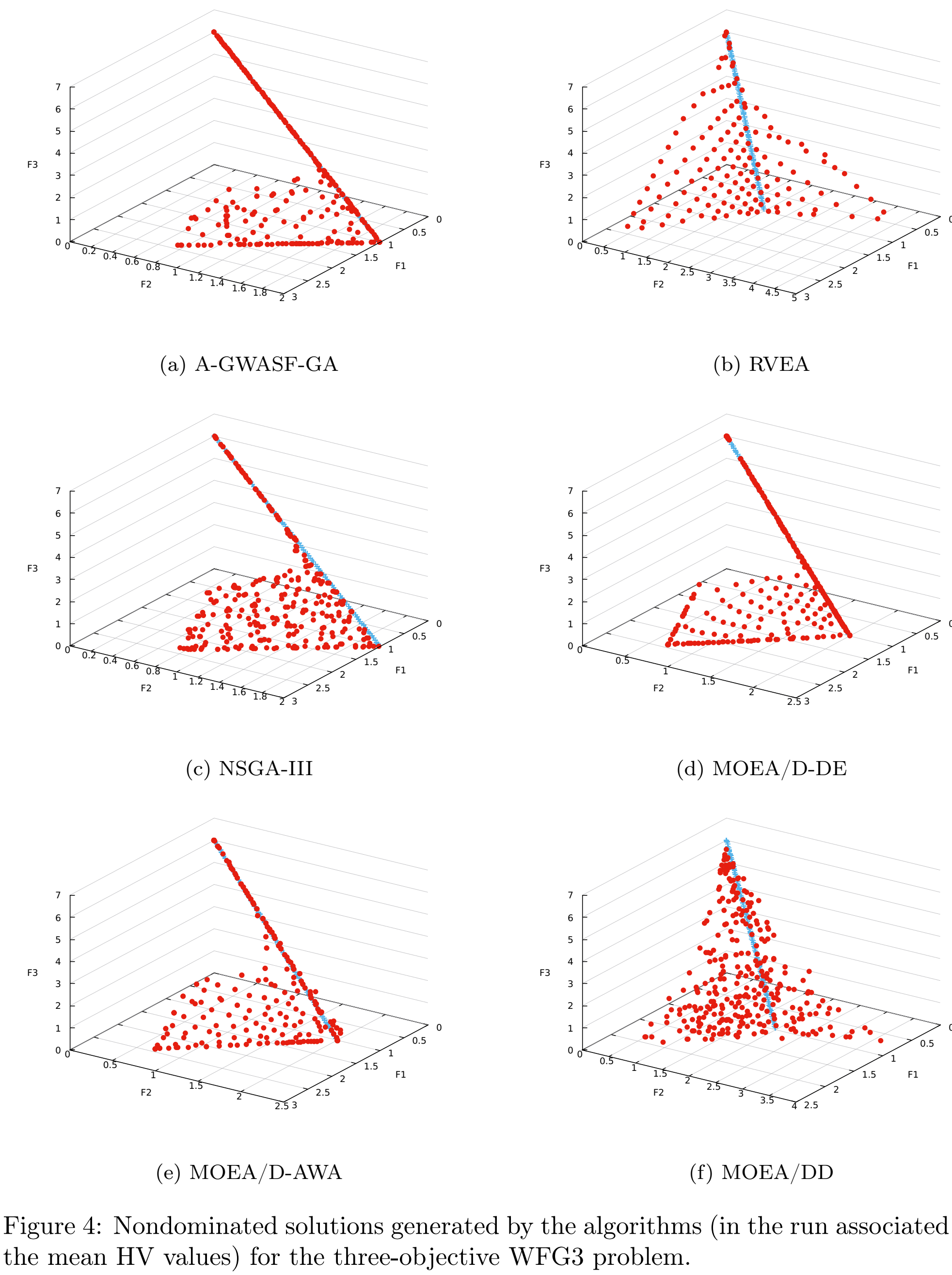



(e) MOEA/D-AWA




(f) MOEA/DD

Figure 3: Nondominated solutions generated by the algorithms (in the run associated with the mean HV values) for the three-objective MaOP10 problem with 50 decision variables.

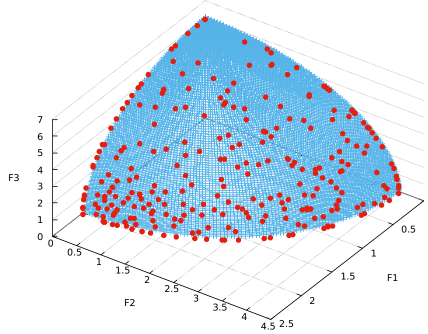



(a) A-GWASF-GA (b) RVEA

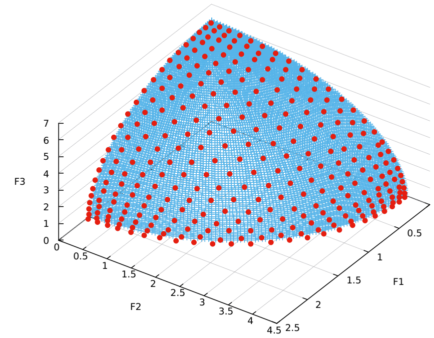
(c) NSGA-III (d) MOEA/D-DE

(e) MOEA/D-AWA (f) MOEA/DD

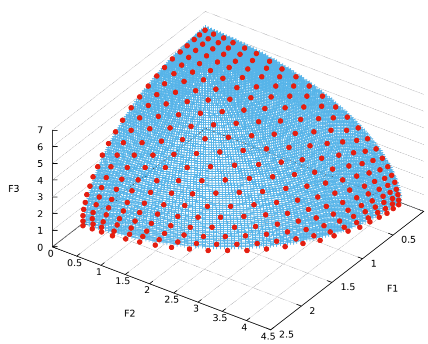
Figure 4: Nondominated solutions generated by the algorithms (in the run associated with the mean HV values) for the three-objective WFG3 problem.



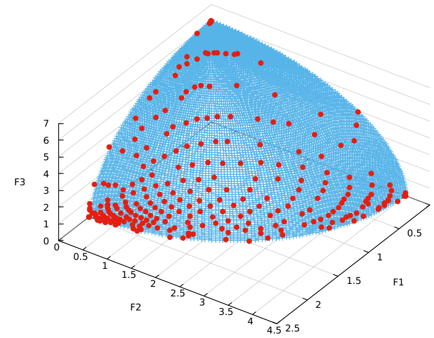

(a) A-GWASF-GA



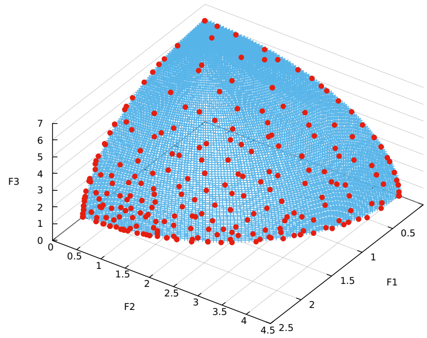

(b) RVEA



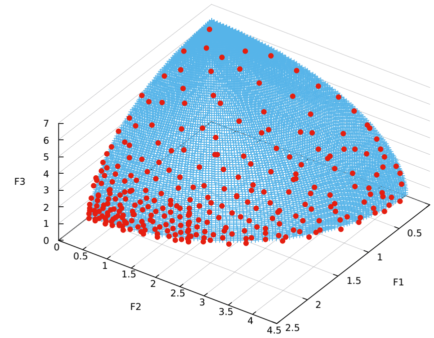

(c) NSGA-III




(d) MOEA/D-DE

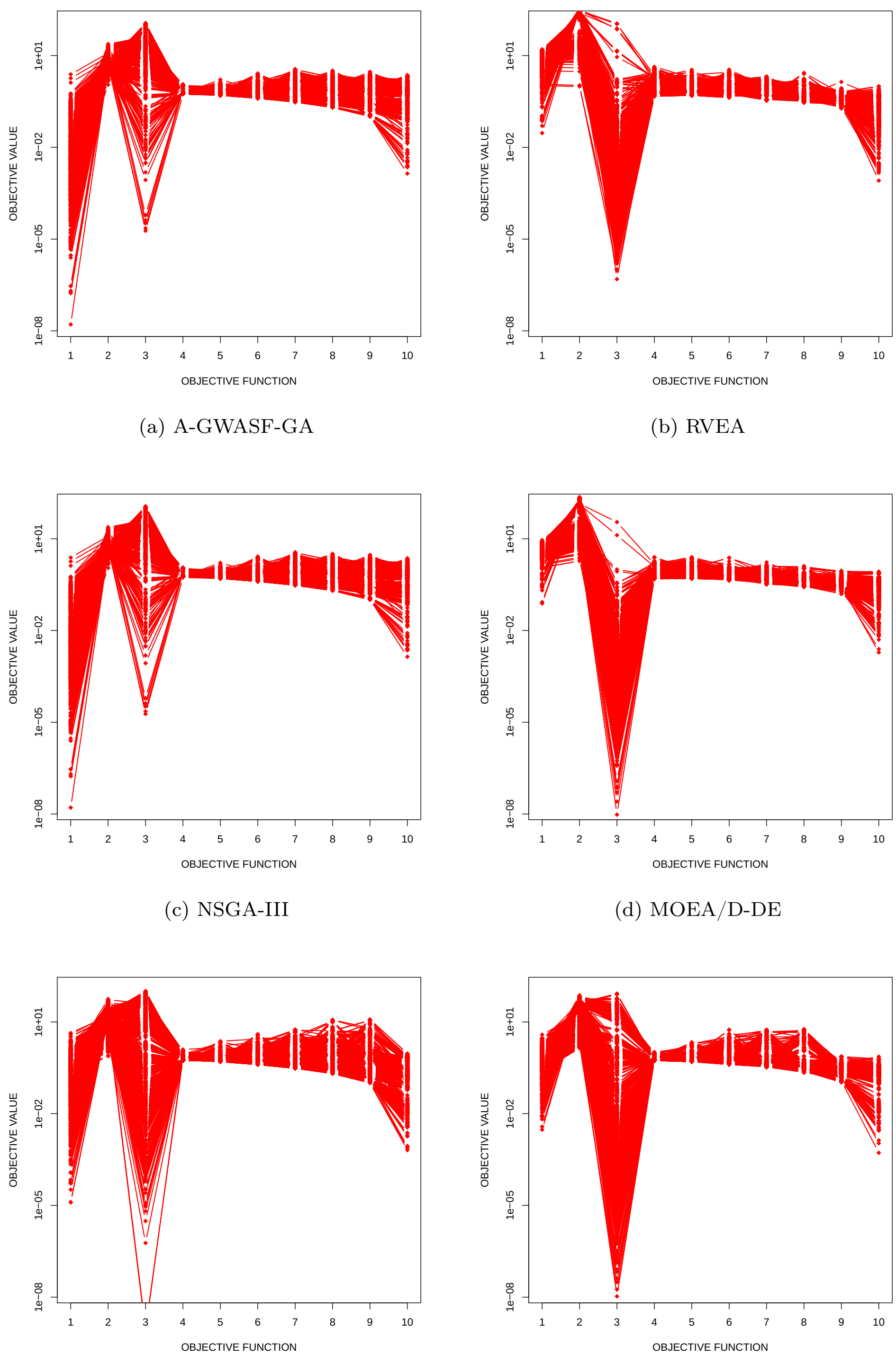



(e) MOEA/D-AWA




(f) MOEA/DD

Figure 5: Nondominated solutions generated by the algorithms (in the run associated with the mean HV values) for the three-objective WFG6 problem.




(a) A-GWASF-GA

(b) RVEA

(c) NSGA-III

(d) MOEA/D-DE

(e) MOEA/D-AWA

(f) MOEA/DD

Figure 6: Nondominated solutions generated by the algorithms (in the run associated with the mean IGD values) for the ten-objective MaOP6 problem with 20 decision variables, using a logarithmic scale in the y-axes.

## 5. Conclusion

In this paper, we have proposed A-GWASF-GA, an enhanced version of the aggregation-based EMO algorithm GWASF-GA that includes a weight vectors' adjustment strategy to handle many-objective optimization problems.

In A-GWASF-GA (as well as in GWASF-GA), a set of weight vectors is used to classify the individuals into several fronts at each generation, considering at the same time the utopian and the nadir points. These weight vectors determine, in practice, the search directions for new non-dominated solutions from the utopian and the nadir points. To improve the convergence and the diversity, a dynamic adjustment of the weight vectors (i.e. of the search directions) is performed in A-GWASF-GA based on the distribution of the solutions generated. The regions of the PF that are overcrowded and these that are poorly approximated (according to the current population) are identified. The idea is to replace the weight vectors whose search directions are pointing to the overcrowded regions by new ones that re-direct the search towards the parts with a lack of solutions. While the algorithm converges, the weight vector adjustment is performed several times. As a result, the number of weight vectors assigned to the utopian and to the nadir points is automatically adjusted in order to better approximate either convex or concave parts, depending on the complexity of the PF.

According to our knowledge, A-GWASF-GA is the first MOEA that adapts the weights while it takes into account both the utopian and nadir points simultaneously, in order to get the highest benefit from the properties of the ASF proposed by Wierzbicki. Note that this ASF takes positive values for the utopian point, but it retrieves negative values for the nadir point (since this point is always achievable). This means that, in practice, adapting the weights while using both the nadir and the utopian points in A-GWASF-GA contributes to converge to the PF from two different points progressively improving, at the same time, the projection directions. We have tested the performance of A-GWASF-GA against well-known EMO algorithms in three-, five-, eight-, and ten-objective novel benchmark problems and in problems belonging to the DTLZ and the WFG families. According to our experiments, A-GWASF-GA has generated very promising results, specially as the number of objective functions increases, achieving better IGD and HV values in comparison to the rest of algorithms in the eight- and ten-objective problems considered. Thus, the computational experiments have shown that our proposal has been able to improve the approximation of the PF in the highest dimension problems used. This is also supported by the theoretical results demonstrated here regarding the ASF considered and the new weight vectors calculated. Based on all of this, this paper helps to cover the existing demand for EMO algorithms able to handle many-objective optimization problems.

As future research, we plan to study how the algorithm can take advantage of using a set of reference points well-distributed in the objective space, besides using different weight vectors for each of them.

## Acknowledgements

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## Appendix

**Theorem 1.** *Given a feasible solution* $\mathbf{x} \in S$, *then, either* $\mathbf{x}$ *is an optimal solution to problem* (4) *considering* $\mathbf{q} = \mathbf{z}^{\star\star}$ *and the weight vector* $\mu^U$ *defined by* (5), *or the objective vector of any optimal solution to problem* (4) *with these parameters strictly dominates or* $\varepsilon$*-dominates* $\mathbf{f}(\mathbf{x})$, *for some* $\varepsilon > 0$.

Proof. If $\mathbf{x}$ is an optimal solution to (4) using $\mathbf{z}^{\star\star}$ as reference point and $\mu^U$ as weight vector, then the proof is trivial. Let us suppose that $\mathbf{x}$ is not an optimal solution to (4) and let $\bar{\mathbf{x}}$ be an optimal solution to it. Next, let us prove that either $\mathbf{f}(\bar{\mathbf{x}})$ strictly dominates $\mathbf{f}(\mathbf{x})$, or there exists a real value $\varepsilon > 0$ such that $\mathbf{f}(\bar{\mathbf{x}})$ strictly $\varepsilon$-dominates $\mathbf{f}(\mathbf{x})$.

Since $\bar{\mathbf{x}}$ is an optimal solution to (4), we have:

$$s(\mathbf{z}^{\star\star}, \mathbf{f}(\bar{\mathbf{x}}), \mu^U) < s(\mathbf{z}^{\star\star}, \mathbf{f}(\mathbf{x}), \mu^U). \tag{7}$$

Replacing $\mu^U$ by the expression given in (5) and calculating $s(\mathbf{z}^{\star\star}, \mathbf{f}(\mathbf{x}), \mu^U)$, we have:

$$s(\mathbf{z}^{\star\star}, \mathbf{f}(\mathbf{x}), \mu^U) = \max_{i=1,\dots,k} \left\{ \frac{f_i(\mathbf{x}) - z_i^{\star\star}}{f_i(\mathbf{x}) - z_i^{\star\star}} \right\} + \rho \sum_{i=1}^{k} \frac{f_i(\mathbf{x}) - z_i^{\star\star}}{f_i(\mathbf{x}) - z_i^{\star\star}} =$$

$$= \max_{i=1,\dots,k} \{1\} + \rho \sum_{i=1}^{k} 1 = 1 + \rho \cdot k,$$

which, according to (7), implies that:

$$s(\mathbf{z}^{\star\star}, \mathbf{f}(\bar{\mathbf{x}}), \mu^U) < 1 + \rho \cdot k. \tag{8}$$

If we now evaluate $s(\mathbf{z}^{\star\star}, \mathbf{f}(\bar{\mathbf{x}}), \mu^U)$ in (8):

$$s(\mathbf{z}^{\star\star}, \mathbf{f}(\bar{\mathbf{x}}), \mu^U) = \max_{i=1,\dots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\star\star}}{f_i(\mathbf{x}) - z_i^{\star\star}} \right\} + \rho \sum_{i=1}^{k} \frac{f_i(\bar{\mathbf{x}}) - z_i^{\star\star}}{f_i(\mathbf{x}) - z_i^{\star\star}} < 1 + \rho \cdot k. \tag{9}$$

In the ASF given in (3), the augmentation term (i.e. $\rho$ term) is added to assure the Pareto optimality of the solution found [38]. However, this term can made as small as desired, given that $\rho$ may be set as 0 or as any other positive real value.

If, on the one hand, $\rho = 0$ (meaning that the ASF (3) matches the Tchebychev metric), (9) implies that:

$$\max_{i=1,\ldots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\star\star}}{f_i(\mathbf{x}) - z_i^{\star\star}} \right\} < 1 \Rightarrow \frac{f_i(\bar{\mathbf{x}}) - z_i^{\star\star}}{f_i(\mathbf{x}) - z_i^{\star\star}} < 1, \quad \forall i = 1, \ldots, k, \tag{10}$$

and thus:

$$f_i(\bar{\mathbf{x}}) - z_i^{\star\star} < f_i(\mathbf{x}) - z_i^{\star\star} \Rightarrow f_i(\bar{\mathbf{x}}) < f_i(\mathbf{x}), \quad \forall i = 1, \ldots, k,$$

which means that $\bar{\mathbf{x}}$ strictly dominates $\mathbf{x}$.

On the other hand, when $\rho > 0$, let us assign a value to $\rho$ such that[10]:

$$0 < \rho \leq \frac{\delta \cdot \epsilon}{k \cdot \left( \max_{j=1,\ldots,k} \{ z_j^{\text{nad}} - z_j^{\star\star} \} \right)^2}, \tag{11}$$

where $\delta > 0$ is a small real value (e.g. $\delta = 0.001$) and $\epsilon$ is the value used to calculate the utopian point from the ideal point (as indicated in Section 2). Taking into account (7) and the expression of the ASF (3), we have:

$$\max_{i=1,\ldots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\star\star}}{f_i(\mathbf{x}) - z_i^{\star\star}} \right\} + \rho \sum_{i=1}^{k} \frac{f_i(\bar{\mathbf{x}}) - z_i^{\star\star}}{f_i(\mathbf{x}) - z_i^{\star\star}} < 1 + \rho \sum_{i=1}^{k} \frac{f_i(\mathbf{x}) - z_i^{\star\star}}{f_i(\mathbf{x}) - z_i^{\star\star}} \Rightarrow$$

$$\Rightarrow \max_{i=1,\ldots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\star\star}}{f_i(\mathbf{x}) - z_i^{\star\star}} \right\} < 1 + \rho \sum_{i=1}^{k} \frac{f_i(\mathbf{x}) - f_i(\bar{\mathbf{x}})}{f_i(\mathbf{x}) - z_i^{\star\star}}. \tag{12}$$

Given that $z_i^{\star\star} = z_i^{\star} - \epsilon$ and $z_i^{\star} \leq f_i(\mathbf{x}) \leq z_i^{\text{nad}}$, for every $i = 1, \ldots, k$, it is easy to see that $f_i(\mathbf{x}) - z_i^{\star\star} \geq \epsilon$ and $f_i(\mathbf{x}) - f_i(\bar{\mathbf{x}}) \leq z_i^{\text{nad}} - z_i^{\star\star}$, for every $i = 1, \ldots, k$. Then, from the inequality (12), we have:

$$\max_{i=1,\ldots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\star\star}}{f_i(\mathbf{x}) - z_i^{\star\star}} \right\} < 1 + \rho \sum_{i=1}^{k} \frac{z_i^{\text{nad}} - z_i^{\star\star}}{\epsilon}, \quad \forall i = 1, \ldots, k \Rightarrow$$

$$\max_{i=1,\ldots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\star\star}}{f_i(\mathbf{x}) - z_i^{\star\star}} \right\} < 1 + \rho \sum_{i=1}^{k} \frac{\max_{j=1,\ldots,k} \{ z_j^{\text{nad}} - z_j^{\star\star} \}}{\epsilon} \Rightarrow$$

[10] At this point in the demonstration, we are implicitly assuming that this value can be assigned to $\rho$ at the beginning of the process; in case that $\mathbf{z}^{\star}$ and/or $\mathbf{z}^{\text{nad}}$ are not known a priori, estimations of both can be calculated and $\rho$ would be redefined at this moment.

$$\Rightarrow \max_{i=1,\ldots,k}\left\{\frac{f_i(\bar{\mathbf{x}})-z_i^{\star\star}}{f_i(\mathbf{x})-z_i^{\star\star}}\right\} < 1+\rho\cdot k\cdot\frac{\max_{j=1,\ldots,k}\{z_j^{\text{nad}}-z_j^{\star\star}\}}{\epsilon}. \tag{13}$$

Taking into account the upper bound defined for $\rho$ in (11), we obtain from (21) that:

$$\max_{i=1,\ldots,k}\left\{\frac{f_i(\bar{\mathbf{x}})-z_i^{\star\star}}{f_i(\mathbf{x})-z_i^{\star\star}}\right\} < 1+\frac{\delta}{\max_{j=1,\ldots,k}\{z_j^{\text{nad}}-z_j^{\star\star}\}} \Rightarrow$$

$$\Rightarrow \frac{f_i(\bar{\mathbf{x}})-z_i^{\star\star}}{f_i(\mathbf{x})-z_i^{\star\star}} < 1+\frac{\delta}{\max_{j=1,\ldots,k}\{z_j^{\text{nad}}-z_j^{\star\star}\}}, \quad \forall i=1,\ldots,k \Rightarrow$$

$$\Rightarrow f_i(\bar{\mathbf{x}})-z_i^{\star\star} < f_i(\mathbf{x})-z_i^{\star\star}+\delta\frac{f_i(\mathbf{x})-z_i^{\star\star}}{\max_{j=1,\ldots,k}\{z_j^{\text{nad}}-z_j^{\star\star}\}}, \quad \forall i=1,\ldots,k \Rightarrow$$

$$\Rightarrow f_i(\bar{\mathbf{x}}) < f_i(\mathbf{x})+\delta\frac{f_i(\mathbf{x})-z_i^{\star\star}}{\max_{j=1,\ldots,k}\{z_j^{\text{nad}}-z_j^{\star\star}\}}, \quad \forall i=1,\ldots,k. \tag{14}$$

Besides, for every $i=1,\ldots,k$, we have:

$$f_i(\mathbf{x}) \le z_i^{\text{nad}} \Rightarrow f_i(\mathbf{x})-z_i^{\star\star} \le z_i^{\text{nad}}-z_i^{\star\star} \Rightarrow$$

$$\Rightarrow f_i(\mathbf{x})-z_i^{\star\star} \le \max_{j=1,\ldots,k}\{z_j^{\text{nad}}-z_j^{\star\star}\} \Rightarrow \frac{f_i(\mathbf{x})-z_i^{\star\star}}{\max_{j=1,\ldots,k}\{z_j^{\text{nad}}-z_j^{\star\star}\}} \le 1. \tag{15}$$

And finally, using (15) in inequality (14), it is followed that:

$$f_i(\bar{\mathbf{x}}) < f_i(\mathbf{x})+\delta, \quad \forall i=1,\ldots,k,$$

meaning that, if we consider $\varepsilon=\delta$, $\mathbf{f}(\mathbf{x})$ is strictly $\varepsilon$-dominated by $\mathbf{f}(\bar{\mathbf{x}})$.

**Corollary 1.** *If $\mathbf{x}\in S$ is a weakly Pareto optimal solution, then $\mathbf{x}$ is an optimal solution to problem* (4) *considering $\mathbf{q}=\mathbf{z}^{\star\star}$ and using the weight vector $\mu^U$ defined by* (5).

Proof. This result is trivial from Theorem 1 since, if $\mathbf{x}$ is weakly Pareto optimal, there is no feasible solution whose objective vector strictly dominates $\mathbf{f}(\mathbf{x})$.

**Theorem 2.** *Given a feasible solution* $\mathbf{x} \in S$*, then, either* $\mathbf{x}$ *is an optimal solution to problem* (4) *considering* $\mathbf{q} = \mathbf{z}^{\text{nad}}$ *and the weight vector* $\mu^N$ *defined by* (6)*, or the objective vector of any optimal solution to problem* (4) *with these parameters strictly dominates or* $\varepsilon$*-dominates* $\mathbf{f}(\mathbf{x})$*, for some* $\varepsilon > 0$*.*

Proof. If $\mathbf{x}$ is an optimal solution to (4) using $\mathbf{z}^{\text{nad}}$ as reference point and $\mu^N$ as weight vector, then the theorem is hold. Assuming that $\mathbf{x}$ is not an optimal solution to (4), let $\bar{\mathbf{x}}$ be an optimal solution to it. Let us prove that either $\mathbf{f}(\bar{\mathbf{x}})$ strictly dominates $\mathbf{f}(\mathbf{x})$, or there exists a real value $\varepsilon > 0$ such that $\mathbf{f}(\bar{\mathbf{x}})$ strictly $\varepsilon$-dominates $\mathbf{f}(\mathbf{x})$.

Given that $\bar{\mathbf{x}}$ is an optimal solution to (4), we have:

$$s(\mathbf{z}^{\text{nad}}, \mathbf{f}(\bar{\mathbf{x}}), \mu^N) < s(\mathbf{z}^{\text{nad}}, \mathbf{f}(\mathbf{x}), \mu^N). \tag{16}$$

Replacing $\mu^N$ by the expression given in (6) and calculating $s(\mathbf{z}^{\text{nad}}, \mathbf{f}(\mathbf{x}), \mu^N)$, we have:

$$s(\mathbf{z}^{\text{nad}}, \mathbf{f}(\mathbf{x}), \mu^N) = \max_{i=1,\dots,k} \left\{ \frac{f_i(\mathbf{x}) - z_i^{\text{nad}}}{z_i^{\text{nad}} - f_i(\mathbf{x})} \right\} + \rho \sum_{i=1}^{k} \frac{f_i(\mathbf{x}) - z_i^{\text{nad}}}{z_i^{\text{nad}} - f_i(\mathbf{x})} =$$

$$= \max_{i=1,\dots,k} \{-1\} + \rho \sum_{i=1}^{k} (-1) = -1 - \rho \cdot k,$$

which, according to (16), means that:

$$s(\mathbf{z}^{\text{nad}}, \mathbf{f}(\bar{\mathbf{x}}), \mu^N) < -(1 + \rho \cdot k). \tag{17}$$

If we now evaluate $s(\mathbf{z}^{\text{nad}}, \mathbf{f}(\bar{\mathbf{x}}), \mu^N)$ in (17):

$$s(\mathbf{z}^{\text{nad}}, \mathbf{f}(\bar{\mathbf{x}}), \mu^N) = \max_{i=1,\dots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\text{nad}}}{z_i^{\text{nad}} - f_i(\mathbf{x})} \right\} + \rho \sum_{i=1}^{k} \frac{f_i(\bar{\mathbf{x}}) - z_i^{\text{nad}}}{z_i^{\text{nad}} - f_i(\mathbf{x})} < -(1+\rho \cdot k). \tag{18}$$

As previously said, the augmentation parameter $\rho$ may be 0 or any other positive real value. If, on the one hand, $\rho = 0$ (i.e. the ASF (3) matches the Tchebychev metric), (18) implies that:

$$\max_{i=1,\dots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\text{nad}}}{z_i^{\text{nad}} - f_i(\mathbf{x})} \right\} < -1 \Rightarrow \frac{f_i(\bar{\mathbf{x}}) - z_i^{\text{nad}}}{z_i^{\text{nad}} - f_i(\mathbf{x})} < -1, \quad \forall i = 1, \dots, k, \tag{19}$$

and thus:

$$f_i(\bar{\mathbf{x}}) - z_i^{\mathrm{nad}} < f_i(\mathbf{x}) - z_i^{\mathrm{nad}} \Rightarrow f_i(\bar{\mathbf{x}}) < f_i(\mathbf{x}), \quad \forall i = 1, \ldots, k,$$

which means that $\mathbf{f}(\bar{\mathbf{x}})$ strictly dominates $\mathbf{f}(\mathbf{x})$.

On the other hand, in case $\rho > 0$, let us consider a value for $\rho$ upper bounded as in (11), using a small real value $\delta$ (e.g. $\delta = 0.001$) and the value $\epsilon$ used to calculate the utopian point from the ideal point. Taking this into account, and based on (16) and on the expression of the ASF (3), we have:

$$\max_{i=1,\ldots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\mathrm{nad}}}{z_i^{\mathrm{nad}} - f_i(\mathbf{x})} \right\} + \rho \sum_{i=1}^{k} \frac{f_i(\bar{\mathbf{x}}) - z_i^{\mathrm{nad}}}{z_i^{\mathrm{nad}} - f_i(\mathbf{x})} < -1 + \rho \sum_{i=1}^{k} \frac{f_i(\mathbf{x}) - z_i^{\mathrm{nad}}}{z_i^{\mathrm{nad}} - f_i(\mathbf{x})} \Rightarrow$$

$$\Rightarrow \max_{i=1,\ldots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\mathrm{nad}}}{z_i^{\mathrm{nad}} - f_i(\mathbf{x})} \right\} < -1 + \rho \sum_{i=1}^{k} \frac{f_i(\mathbf{x}) - f_i(\bar{\mathbf{x}})}{z_i^{\mathrm{nad}} - f_i(\mathbf{x})} \leq -1 + \rho \sum_{i=1}^{k} \frac{z_i^{\mathrm{nad}} - z_i^{\star\star}}{z_i^{\mathrm{nad}} - f_i(\mathbf{x})} \tag{20}$$

For every $i = 1, \ldots, k$, it is assured[11] that $z_i^{\mathrm{nad}} - f_i(\mathbf{x}) \geq \epsilon$. Then, from the inequality (20), we have:

$$\max_{i=1,\ldots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\mathrm{nad}}}{z_i^{\mathrm{nad}} - f_i(\mathbf{x})} \right\} < -1 + \rho \sum_{i=1}^{k} \frac{z_i^{\mathrm{nad}} - z_i^{\star\star}}{\epsilon}, \quad \forall i = 1, \ldots, k \Rightarrow$$

$$\Rightarrow \max_{i=1,\ldots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\mathrm{nad}}}{z_i^{\mathrm{nad}} - f_i(\mathbf{x})} \right\} < -1 + \rho \sum_{i=1}^{k} \frac{\max_{j=1,\ldots,k} \{z_j^{\mathrm{nad}} - z_j^{\star\star}\}}{\epsilon} \Rightarrow$$

$$\Rightarrow \max_{i=1,\ldots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\mathrm{nad}}}{z_i^{\mathrm{nad}} - f_i(\mathbf{x})} \right\} < -1 + \rho \cdot k \cdot \frac{\max_{j=1,\ldots,k} \{z_j^{\mathrm{nad}} - z_j^{\star\star}\}}{\epsilon}. \tag{21}$$

Taking into account the upper bound defined for $\rho$ in (11), it is hold from (21) that:

$$\max_{i=1,\ldots,k} \left\{ \frac{f_i(\bar{\mathbf{x}}) - z_i^{\mathrm{nad}}}{z_i^{\mathrm{nad}} - f_i(\mathbf{x})} \right\} < -1 + \frac{\delta}{\max_{j=1,\ldots,k} \{z_j^{\mathrm{nad}} - z_j^{\star\star}\}} \Rightarrow$$

[11] Note that, as previously said and as indicated in Algorithm 4 (line 5), the nadir point components are commonly updated as $z_i^{\mathrm{nad}} = f_i(\mathbf{x}') + \epsilon$, if there is a new generated solution $\mathbf{x}'$ with $f_i(\mathbf{x}') > z_i^{\mathrm{nad}}$, which assures that the estimation of $\mathbf{z}^{\mathrm{nad}}$ is always dominated by any feasible solution obtained.

$$\Rightarrow \frac{f_i(\bar{\mathbf{x}}) - z_i^{\text{nad}}}{z_i^{\text{nad}} - f_i(\mathbf{x})} < -1 + \frac{\delta}{\max\limits_{j=1,\ldots,k}\{z_j^{\text{nad}} - z_j^{\star\star}\}}, \quad \forall i = 1, \ldots, k \Rightarrow$$

$$\Rightarrow f_i(\bar{\mathbf{x}}) - z_i^{\text{nad}} < f_i(\mathbf{x}) - z_i^{\text{nad}} + \delta \frac{z_i^{\text{nad}} - f_i(\mathbf{x})}{\max\limits_{j=1,\ldots,k}\{z_j^{\text{nad}} - z_j^{\star\star}\}}, \quad \forall i = 1, \ldots, k \Rightarrow$$

$$\Rightarrow f_i(\bar{\mathbf{x}}) < f_i(\mathbf{x}) + \delta \frac{z_i^{\text{nad}} - f_i(\mathbf{x})}{\max\limits_{j=1,\ldots,k}\{z_j^{\text{nad}} - z_j^{\star\star}\}}, \quad \forall i = 1, \ldots, k \Rightarrow$$

$$\Rightarrow f_i(\bar{\mathbf{x}}) < f_i(\mathbf{x}) + \delta \frac{z_i^{\text{nad}} - z_i^{\star\star}}{\max\limits_{j=1,\ldots,k}\{z_j^{\text{nad}} - z_j^{\star\star}\}} \leq f_i(\mathbf{x}) + \delta \cdot 1, \quad \forall i = 1, \ldots, k,$$

$$\Rightarrow f_i(\bar{\mathbf{x}}) < f_i(\mathbf{x}) + \delta, \quad \forall i = 1, \ldots, k,$$

meaning that, if we consider $\varepsilon = \delta$, $\mathbf{f}(\mathbf{x})$ is strictly $\varepsilon$-dominated by $\mathbf{f}(\bar{\mathbf{x}})$.

**Corollary 2.** *If* $\mathbf{x} \in S$ *is a weakly Pareto optimal solution, then* $\mathbf{x}$ *is an optimal solution to problem* (4) *considering* $\mathbf{q} = \mathbf{z}^{\text{nad}}$ *and using the weight vector* $\mu^N$ *defined by* (6).

Proof. This result is trivial from Theorem 2 since, if $\mathbf{x}$ is weakly Pareto optimal, there is no feasible solution whose objective vector strictly dominates $\mathbf{f}(\mathbf{x})$.

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