



# Weighted one-dimensional Poincaré inequalities in Banach function spaces

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## Abstract

We characterize some weighted one-dimensional Poincaré inequalities on Banach function spaces, generalizing in this way Chua and Wheeden's theorem in the case  $p \leq q$ . As particular cases, we get the characterizations of some weighted Orlicz-Poincaré inequalities and Poincaré inequalities in variable Lebesgue spaces. We also get a new result on weighted weak-type inequalities of this kind.

**Keywords** Hardy inequalities · Orlicz-Poincaré inequalities · Poincaré inequalities · Variable Lebesgue spaces · Weak-type inequalities · Weighted inequalities

**Mathematics Subject Classification** 26D10 · 26D15

## 1 Introduction

Let  $-\infty < a < b < \infty$  and let  $\mu$  be a finite positive Borel measure on  $[a, b]$ . S.-K. Chua and R. Wheeden characterized in [5] the pairs of positive weights  $w, v$  on  $[a, b]$  such that for all absolutely continuous functions  $f$  on  $[a, b]$  and some constant  $C > 0$  independent of  $f$  the weighted one-dimensional Poincaré inequality

$$\left\| f - \frac{1}{\mu[a, b]} \int_a^b f d\mu \right\|_{L^q(w)} \leq C \|f'\|_{L^p(v)} \quad (1.1)$$

holds, where  $1 \leq p, q < \infty$  and  $\|g\|_{L^p(v)} = \left( \int_a^b |g|^p v \right)^{\frac{1}{p}}$ . Their results are the next ones.

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**Theorem A** [5] *Let  $-\infty < a < b < \infty$  and let  $\mu$  be a finite positive Borel measure on  $[a, b]$ . Let  $v, w$  be two positive measurable functions on  $[a, b]$ . Then, inequality (1.1) holds for  $1 \leq p \leq q < \infty$  if and only if*

$$C_1(p, q) = \sup_{a \leq x \leq b} \left( \int_x^b w \right)^{\frac{1}{q}} \left( \int_a^x \mu[a, y]^{p'} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} < \infty$$

and

$$C_2(p, q) = \sup_{a \leq x \leq b} \left( \int_a^x w \right)^{\frac{1}{q}} \left( \int_x^b \mu[y, b]^{p'} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} < \infty.$$

Moreover, the best constant  $C$  in (1.1) satisfies

$$\frac{\max\{C_1(p, q), C_2(p, q)\}}{\mu[a, b]} \leq C \leq \frac{C_1(p, q) + C_2(p, q)}{\mu[a, b]} K(p, q),$$

where  $K(q, p) = \left(1 + \frac{q}{p'}\right)^{\frac{1}{q}} \left(1 + \frac{p'}{q}\right)^{\frac{1}{p'}}$  for  $p > 1$  and  $K(q, 1) = 1$ .

**Theorem B** [5] *In the conditions of Theorem A, inequality (1.1) holds for  $q < p$  if and only if*

$$C_1(p, q) = \left( \int_a^b \left( \int_x^b w \right)^{\frac{r}{q}} \left( \int_a^x \mu[a, y]^{p'} v^{1-p'}(y) dy \right)^{\frac{r}{q'}} \mu[a, x]^{p'} v^{1-p'}(x) dx \right)^{\frac{1}{r}} < \infty$$

and

$$C_2(p, q) = \left( \int_a^b \left( \int_a^x w \right)^{\frac{r}{q}} \left( \int_x^b \mu[y, b]^{p'} v^{1-p'}(y) dy \right)^{\frac{r}{q'}} \mu[x, b]^{p'} v^{1-p'}(x) dx \right)^{\frac{1}{r}} < \infty,$$

where  $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ . Moreover, the best constant  $C$  in (1.1) satisfies

$$C \leq \frac{C_1(p, q) + C_2(p, q)}{\mu[a, b]} q^{\frac{1}{q}} (p')^{\frac{1}{q'}}.$$

These results show that there is a close relationship between one-dimensional Poincaré inequalities with weights and weighted Hardy inequalities, since the conditions  $C_1(p, q) < \infty$  and  $C_2(p, q) < \infty$  characterize the pairs of weights  $(v, w)$  such that the Hardy-type operators  $T_\mu$  and  $T_\mu^*$ , defined by

$$T_\mu f(x) = \int_a^x \mu[a, t] f(t) dt \quad \text{and} \quad T_\mu^* f(x) = \int_x^b \mu[t, b] f(t) dt$$

are bounded from  $L^p(v)$  to  $L^q(w)$ , respectively (see [4] and [15] for the case  $p \leq q$  and [12] for  $q < p$ ).

Very recently, Korobenko, Milshstein and Yong have studied in [10] the weighted inequality

$$\left\| f - \frac{1}{\mu[a, b]} \int_a^b f d\mu \right\|_{L^\Phi(w)} \leq C \|f'\|_{L^p(v)}, \tag{1.2}$$

where  $\Phi$  is a submultiplicative Young function such that  $\Phi(t^{1/p})$  is convex and  $L^\Phi(w)$  is the weighted Orlicz space whose norm is defined by

$$\|f\|_{L^\Phi(w)} = \inf \left\{ K > 0 : \int_a^b \Phi\left(\frac{|f(t)|}{K}\right) w(t) dt \leq 1 \right\}.$$

Specifically, they have given, when  $p > 1$ , necessary conditions on the weights for (1.2) to hold and different sufficient conditions, which turn to be necessary and sufficient when  $\Phi(t) = t^q$  with  $p \leq q$ . In this way, they get a result that generalizes Theorem A. The main obstacle to achieving the necessary and sufficient conditions on the weights so that inequality (1.2) holds is the fact that, to the best of our knowledge, the boundedness of the Hardy-type operators  $T_\mu$  and  $T_\mu^*$  from  $L^p(v)$  to  $L^\Phi(w)$  has not yet been characterized. However, there is another possibility, which is to consider the weight as a multiplier in the Orlicz space, instead of as a measure. This is common when working with function spaces other than  $L^p$  spaces (see, for instance, [1, 7, 11]). If we use this approach to the problem, we have the advantage that the inequalities with weights for the operators  $T_\mu$  and  $T_\mu^*$  are already characterized (see [3, 11]).

One of our purposes is to characterize the weights for the inequality

$$\left\| f - \frac{1}{\mu[a, b]} \int_a^b f d\mu \right\|_{L^\Phi_w} \leq C \|f'\|_{L^p(v)}, \tag{1.3}$$

to hold, where

$$\|f\|_{L^\Phi_w} = \inf \left\{ K > 0 : \int_a^b \Phi\left(\frac{|f(t)|w(t)}{K}\right) dt \leq 1 \right\}.$$

Really, we will work in a much more general setting, since we will characterize the good weights for the inequality

$$\left\| f - \frac{1}{\mu[a, b]} \int_a^b f d\mu \right\|_{Y_w} \leq C \|f'\|_{X_v} \tag{1.4}$$

to hold, where  $(X, Y)$  is a pair of Banach function spaces. The pairs  $(X, Y)$  we work with satisfy a suitable condition so that if  $X = L^p$  and  $Y = L^q$ , then the condition holds for  $p \leq q$ . In this way, we will get a generalization of Theorem A. In the weighted spaces  $X_v$  and  $Y_w$ , the weights act as multipliers, i. e.,  $X_v = \{f : f v \in X\}$  and  $\|f\|_{X_v} = \|f v\|_X$ . All this will be done in section 2, where, as a consequence of our general result, we will also characterize some weighted one-dimensional Poincaré inequalities in variable Lebesgue spaces.

Our second objective is to characterize some weighted weak-type Poincaré inequalities and study their relationship with inequality (1.1). By a weak-type Poincaré inequality we mean an inequality of the form

$$\left\| f(x) - \frac{1}{\mu[a, b]} \int_a^b f d\mu \right\|_{L^{q,\infty}(w)} \leq C \|f'\|_{L^p(v)}, \tag{1.5}$$

where  $\|\cdot\|_{L^{q,\infty}(w)}$  is defined by

$$\|g\|_{L^{q,\infty}(w)} = \sup_{\lambda > 0} \lambda \left( \int_{\{x \in [a,b] : |g(x)| > \lambda\}} w \right)^{\frac{1}{q}}.$$

Comparing our result in inequality (1.5) with Theorems A and B, we will see that (1.5) and (1.1) are equivalent if  $p \leq q$ , but it does not hold in the case  $q < p$ . This will be done in Sect. 3.

## 2 Weighted Poincaré inequalities in Banach function spaces

In order to state and prove our results, let us recall some concepts from the theory of Banach function spaces (see [2] for the basic definitions). Let  $X$  be a Banach function space on  $[a, b]$  and let  $\ell$  be a Banach sequence space. We will say that  $X$  is  $\ell$ -concave if there exists  $d_1 > 0$  such that for every sequence of pairwise disjoint intervals  $\{I_k\}$  with  $[a, b] = \cup_k I_k$  and all  $f \in X$  the inequality

$$\left\| \sum_k e_k \|f \chi_{I_k}\|_X \right\|_{\ell} \leq d_1 \|f\|_X \tag{2.1}$$

holds, where  $e_k$  stands for the sequence whose terms are all null except the one in position  $k$  which is 1. In the same way, a Banach function space  $Y$  is said to be  $\ell$ -convex if there exists  $d_2 > 0$  such that for all sequences of pairwise disjoint intervals  $\{I_k\}$  with  $[a, b] = \cup_k I_k$  and all  $g \in Y$  the inequality

$$\|g\|_Y \leq d_2 \left\| \sum_k e_k \|g \chi_{I_k}\|_Y \right\|_{\ell} \tag{2.2}$$

holds. It is well known that  $Y$  is  $\ell$ -convex if and only if  $Y'$  is  $\ell'$ -concave with the same constant, where  $Y'$  and  $\ell'$  stand for the associate spaces of  $Y$  and  $\ell$ , respectively. Given a Banach function space  $X$  on  $[a, b]$ , its associate space  $X'$  consists of the measurable functions  $g$  such that  $\|g\|_{X'} < \infty$ , where

$$\|g\|_{X'} = \sup \left\{ \int_a^b |fg| : f \in X, \|f\| \leq 1 \right\}.$$

We will say that a pair  $(X, Y)$  of Banach function spaces on  $[a, b]$  satisfies a Bereznoi  $\ell$ -condition (see [3, 11]) if there exists a Banach sequence space  $\ell$  such that  $X$  is  $\ell$ -concave and  $Y$  is  $\ell$ -convex. Observe that  $(L^p, L^q)$  verify an  $\ell$ -condition if

$p \leq q$ . For these pairs of Banach function spaces, Bereznoi characterized the weights  $w, v$  such that the Hardy-type operators  $T_g$  and  $T_g^*$ , defined by

$$T_g f(x) = \int_a^x g(t)f(t)dt \quad \text{and} \quad T_g^* f(x) = \int_x^b g(t)f(t)dt$$

are bounded from  $X_v$  to  $Y_w$ . The result is the next one.

**Theorem C** [3, 11] *Let  $\ell$  be a Banach sequence space and let  $X$  and  $Y$  be Banach function spaces on  $[a,b]$  such that  $(X,Y)$  satisfies the  $\ell$ -condition. Let  $w,v$  and  $g$  be positive measurable functions on  $[a,b]$ .*

- (i) The operator  $T_g$  is bounded from  $X_v$  to  $Y_w$  if and only if

$$B_1 = \sup_{a \leq x \leq b} \|\chi_{[x,b]} w\|_Y \|\chi_{[a,x]} g v^{-1}\|_{X'} < \infty.$$

- (ii) The operator  $T_g^*$  is bounded from  $X_v$  to  $Y_w$  if and only if

$$B_2 = \sup_{a \leq x \leq b} \|\chi_{[a,x]} w\|_Y \|\chi_{[x,b]} g v^{-1}\|_{X'} < \infty.$$

Moreover, the norms of the operators  $T_g$  and  $T_g^*$  satisfy  $B_1 \leq \|T_g\|_{X_v \rightarrow Y_w} \leq 4d_1 d_2 B_1$  and  $B_2 \leq \|T_g^*\|_{X_v \rightarrow Y_w} \leq 4d_1 d_2 B_2$ , respectively.

By means of these results, we can state and prove the next theorem, which extends Theorem A to the setting of Banach function spaces.

**Theorem 1** *Let  $-\infty < a < b < \infty$  and let  $(X,Y)$  be a pair of Banach function spaces on  $[a,b]$  which satisfies an  $\ell$ -condition. Let  $w,v$  be positive measurable functions on  $[a,b]$  and let  $\mu$  be a finite positive Borel measure on  $[a,b]$ . Then inequality (1.4) holds for all absolutely continuous functions  $f$  on  $[a,b]$  with a constant  $C > 0$  independent of  $f$  if and only if*

$$P_1 = \sup_{a \leq x \leq b} \|\chi_{[x,b]} w\|_Y \|\chi_{[a,x]} \mu[a, \cdot] v^{-1}\|_{X'} < \infty \tag{2.3}$$

and

$$P_2 = \sup_{a \leq x \leq b} \|\chi_{[a,x]} w\|_Y \|\chi_{[x,b]} \mu[\cdot, b] v^{-1}\|_{X'} < \infty. \tag{2.4}$$

Moreover, the best constant  $C$  in inequality (1.4) satisfies  $\frac{\max\{P_1, P_2\}}{\mu[a,b]} \leq C \leq \frac{4d_1 d_2 (P_1 + P_2)}{\mu[a,b]}$ .

**Proof** Assume that (1.4) holds. Let  $g$  be a nonnegative function on  $[a, b]$  such that  $g v \in X$  and  $\beta \in [a, b]$ . Let, for each  $n \in \mathbb{N}$ ,  $g_n$  be the function defined on  $[a, b]$  by  $g_n(x) = \min\{g(x), n\}$ . Then  $g_n$  is integrable on  $[a, b]$  and  $\{g_n\} \nearrow g$ . Let  $f_n$  be the function defined by  $f_n(x) = \int_a^x g_n(t) \chi_{[a,\beta]}(t) dt$ . The function  $f_n$  is absolutely continuous and satisfies  $f_n'(x) = g_n(x) \chi_{[a,\beta]}(x)$ . Then, by identity (1.9) in [5],

$$\begin{aligned} \left\| f_n - \frac{1}{\mu[a, b]} \int_a^b f_n d\mu \right\|_{Y_w} &= \frac{1}{\mu[a, b]} \left\| \int_a^x \mu[a, z] f'_n(z) dz - \int_x^b \mu[z, b] f'_n(z) dz \right\|_{Y_w} \\ &\geq \frac{1}{\mu[a, b]} \left\| \chi_{[\beta, b]}(x) \left( \int_a^x \mu[a, z] f'_n(z) dz - \int_x^b \mu[z, b] f'_n(z) dz \right) \right\|_{Y_w} \\ &= \frac{1}{\mu[a, b]} \left\| \chi_{[\beta, b]}(x) \int_a^x \mu[a, t] g_n(t) \chi_{[a, \beta]}(t) dt \right\|_{Y_w} \\ &= \frac{1}{\mu[a, b]} \left( \int_a^\beta \mu[a, t] g_n(t) dt \right) \|\chi_{[\beta, b]} w\|_Y. \end{aligned}$$

On the other hand,

$$\|f'_n\|_{X_v} = \|g_n \chi_{[a, \beta]}\|_{X_v} \leq \|g_n v\|_{X'}.$$

Therefore, by (1.4),

$$\frac{1}{\mu[a, b]} \left( \int_a^\beta \mu[a, t] g_n(t) dt \right) \|\chi_{[\beta, b]} w\|_Y \leq C \|g_n v\|_{X'}. \tag{2.5}$$

Letting  $n$  tend to infinity in (2.5), we get

$$\frac{1}{\mu[a, b]} \left( \int_a^\beta \mu[a, t] v^{-1}(t) \frac{g(t)v(t)}{\|g v\|_{X'}} dt \right) \|\chi_{[\beta, b]} w\|_Y \leq C.$$

Since the inequality above holds for all  $g$ , by the definition of the norm in the associate space we have

$$\frac{1}{\mu[a, b]} \|\mu[a, \cdot] v^{-1} \chi_{[a, \beta]}\|_{X'} \|\chi_{[\beta, b]} w\|_Y \leq C,$$

which shows (2.3) and  $P_1 \leq C\mu[a, b]$ . In a similar way, we get  $P_2 \leq C\mu[a, b]$ .

The sufficiency of the conditions follows immediately from the fact that, by Theorem C,  $P_1 < \infty$  and  $P_2 < \infty$  characterize the Hardy inequalities

$$\left\| \int_a^x \mu[a, z] g(z) dz \right\|_{Y_w} \leq K_1 \|g\|_{X_v} \quad \text{and} \quad \left\| \int_x^b \mu[z, b] g(z) dz \right\|_{Y_w} \leq K_2 \|g\|_{X_v},$$

respectively. Indeed,

$$\begin{aligned} \left\| f - \frac{1}{\mu[a, b]} \int_a^b f d\mu \right\|_{Y_w} &= \frac{1}{\mu[a, b]} \left\| \int_a^x \mu[a, z] f'(z) dz - \int_x^b \mu[z, b] f'(z) dz \right\|_{Y_w} \\ &\leq \frac{1}{\mu[a, b]} \left\| \int_a^x \mu[a, z] f'(z) dz \right\|_{Y_w} + \frac{1}{\mu[a, b]} \left\| \int_x^b \mu[z, b] f'(z) dz \right\|_{Y_w} \\ &\leq \frac{4d_1 d_2 (P_1 + P_2)}{\mu[a, b]} \|f'\|_{X_v}. \end{aligned}$$

This proves (1.4) and shows that the best constant  $C$  in (1.4) satisfies  $C \leq \frac{4d_1 d_2 (P_1 + P_2)}{\mu[a, b]}$ . □

As we mentioned in the introduction, as a particular case of the previous theorem we get the characterization of the Orlicz-Poincaré inequality (1.3), where  $\Phi$  satisfies that  $h(t) = \Phi(t^{\frac{1}{p}})$  is convex. This property guarantees that the space  $L^\Phi$  is  $p$ -convex (i. e.  $\ell^p$ -convex) with constant 1 (see [9], Theorem 5.1). Then,  $(L^p, L^\Phi)$  satisfies the  $\ell^p$ -condition and, by Theorem 1, the next result follows immediately.

**Corollary 1** *Let  $-\infty < a < b < \infty$  and let  $\mu$  be a finite positive measure on  $[a, b]$ . Let  $v, w$  be positive measurable functions on  $[a, b]$ . Let  $p \geq 1$  and let  $\Phi$  be a Young function such that  $\Phi(t^{\frac{1}{p}})$  is convex. Then, inequality (1.3) holds for all absolutely continuous functions  $f$  on  $[a, b]$  with a constant  $C > 0$  independent of  $f$  if and only if*

$$O_1 = \sup_{a \leq x \leq b} \|\chi_{[x, b]} w\|_{L^\Phi} \left( \int_a^x \mu[a, t]^{p'} v^{1-p'}(t) dt \right)^{\frac{1}{p}} < \infty$$

and

$$O_2 = \sup_{a \leq x \leq b} \|\chi_{[a, x]} w\|_{L^\Phi} \left( \int_x^b \mu[t, b]^{p'} v^{1-p'}(t) dt \right)^{\frac{1}{p}} < \infty.$$

Furthermore, the best constant  $C$  satisfies  $\frac{\max\{O_1, O_2\}}{\mu[a, b]} \leq C \leq \frac{4(O_1 + O_2)}{\mu[a, b]}$ .

Observe that, unlike [10], we do not assume that the function  $\Phi$  is submultiplicative. Therefore, our theorem is different from that in [10], and it is not easy to compare them, but both generalize Theorem A. However, if  $\Phi$  is submultiplicative, then we can see that  $\|f\|_{L^{\Phi \circ \Phi^{-1} \circ w}} \leq \|f\|_{L^{\Phi(w)}}$ , so that the sufficient conditions given in [10] for (1.2) to hold are also sufficient for (1.3) to hold with the weights  $v$  and  $\Phi^{-1} \circ w$ . In the same way, our conditions for (1.3) to hold with the weights  $v$  and  $\Phi^{-1} \circ w$  are necessary for (1.2).

In the rest of the section, we focus on variable Lebesgue spaces  $L^{p(\cdot)}$ . If  $p : [a, b] \rightarrow [1, \infty)$  is Lebesgue measurable,  $L^{p(\cdot)}$  is defined as the set of all measurable functions  $f : [a, b] \rightarrow \mathbb{R}$  satisfying that there exists  $\lambda > 0$  such that  $\int_a^b \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty$ . With this definition,  $L^{p(\cdot)}$  is a vector space on which a norm is defined as follows:

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ K > 0 : \int_a^b \left( \frac{|f(x)|}{K} \right)^{p(x)} dx \leq 1 \right\}.$$

In this way, the normed space  $(L^{p(\cdot)}, \|\cdot\|_{L^{p(\cdot)}})$  is a Banach function space. It is also well known that if  $1 < p_- \leq p_+ < \infty$ , where  $p_- = \inf\{p(x) : x \in [a, b]\}$  and  $p_+ = \sup\{p(x) : x \in [a, b]\}$ , then the associate space of  $L^{p(\cdot)}$  is  $L^{p'(\cdot)}$ , with  $p'(x)$  given by  $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ . For a detailed study of variable Lebesgue spaces, see [6].

In order to prove our result, we will need the following lemma. For a proof, see [6, Proposition 2.21 and Corollary 2.23].

**Lemma 1** Let  $p : [a, b] \rightarrow [1, \infty)$  such that  $p_+ < \infty$  and  $\Omega \subset [a, b]$ .

- (i) If  $p_+(\Omega) = \sup\{p(x) : x \in \Omega\}$ ,  $p_-(\Omega) = \inf\{p(x) : x \in \Omega\}$  and  $\|f\chi_\Omega\|_{L^{p(\cdot)}} \leq 1$ , then

$$\|f\chi_\Omega\|_{L^{p(\cdot)}}^{p_+(\Omega)} \leq \int_\Omega |f(x)|^{p(x)} dx \leq \|f\chi_\Omega\|_{L^{p(\cdot)}}^{p_-(\Omega)}.$$

- (ii) If  $\|f\|_{L^{p(\cdot)}} = 1$ , then  $\int_a^b |f(x)|^{p(x)} dx = 1$ .

Our result on weighted one-dimensional Poincaré inequalities on variable Lebesgue spaces reads as follows.

**Corollary 2** Let  $-\infty < a < b < \infty$  and let  $\mu$  be a finite positive measure on  $[a, b]$ . Let  $v, w$  be positive measurable functions on  $[a, b]$ . Let  $p, q : [a, b] \rightarrow [1, \infty)$  satisfying that there exists  $r \geq 1$  such that  $p(x) \leq r \leq q(x)$  for all  $x \in [a, b]$ ,  $p_- > 1$  and  $q_+ < \infty$ . Then there exists  $C > 0$  such that

$$\left\| f - \frac{1}{\mu[a, b]} \int_a^b f d\mu \right\|_{L_w^{p(\cdot)}} \leq C \|f'\|_{L_v^{q(\cdot)}} \tag{2.6}$$

holds for all absolutely continuous functions  $f$  on  $[a, b]$  if and only if

$$V_1 = \sup_{a \leq x \leq b} \|\chi_{[x, b]} w\|_{L^{q(\cdot)}} \|\chi_{[a, x]} \mu[a, \cdot] v^{-1}\|_{L^{p(\cdot)}} < \infty \tag{2.7}$$

and

$$V_2 = \sup_{a \leq x \leq b} \|\chi_{[a, x]} w\|_{L^{q(\cdot)}} \|\chi_{[x, b]} \mu[\cdot, b] v^{-1}\|_{L^{p(\cdot)}} < \infty. \tag{2.8}$$

Moreover, the best constant  $C$  in inequality (2.6) satisfies  $\frac{\max\{V_1, V_2\}}{\mu[a, b]} \leq C \leq \frac{4(V_1 + V_2)}{\mu[a, b]}$ .

**Proof** Let us prove that the pair  $(L^{p(\cdot)}, L^{q(\cdot)})$  satisfies the  $\ell^r$ -condition. Let us see first that  $L^{p(\cdot)}$  is  $\ell^r$ -concave with constant 1. Let  $\{I_k\}$  be a sequence of pairwise disjoint intervals with  $[a, b] = \cup_k I_k$ ,  $f \in L^{p(\cdot)}$  and for each  $k$ , let  $p_+^k = \sup\{p(x) : x \in I_k\}$ . Then, by Lemma 1 we have

$$\begin{aligned} \left( \sum_k \left\| \frac{f\chi_{I_k}}{\|f\|_{L^{p(\cdot)}}} \right\|_{L^{p(\cdot)}}^r \right)^{\frac{1}{r}} &\leq \left( \sum_k \left\| \frac{f\chi_{I_k}}{\|f\|_{L^{p(\cdot)}}} \right\|_{L^{p(\cdot)}}^{p_+^k} \right)^{\frac{1}{r}} \leq \left( \sum_k \int_{I_k} \left( \frac{|f(x)|}{\|f\|_{L^{p(\cdot)}}} \right)^{p(x)} dx \right)^{\frac{1}{r}} \\ &= \left( \int_a^b \left( \frac{|f(x)|}{\|f\|_{L^{p(\cdot)}}} \right)^{p(x)} dx \right)^{\frac{1}{r}} = 1. \end{aligned}$$

In the same way, we can prove that  $L^{q(\cdot)}$  is  $\ell^r$ -convex. Then the result follows immediately from Theorem 1. □

### 3 Weighted weak-type Poincaré inequalities

In order to characterize the good weights for the weak-type inequality (1.5) to hold we need the next result on weighted weak-type inequalities for the operators  $T_g$  and  $T_g^*$  defined previously. We state it as a lemma and include its proof for completeness. We assume that the function  $g$  is bounded, since later  $g(t) = \mu[a, t]$  or  $g(t) = \mu[t, b]$ . Similar, more general results can be found, for example, in [8, 13, 14].

**Lemma 2** *Let  $-\infty < a < b < \infty$  and  $1 \leq p, q < \infty$ . Let  $g$  be a nonnegative, measurable, and bounded function on  $[a, b]$  and let  $w, v$  be positive measurable functions on  $[a, b]$ .*

- (i) Then there exists a positive constant  $C$  such that inequality

$$\|T_g f\|_{L^{q,\infty}(w)} \leq C \|f\|_{L^p(v)} \tag{3.1}$$

holds for all nonnegative functions  $f$  on  $[a, b]$  if and only if

$$A_L = \sup_{a \leq x \leq b} \left( \int_x^b w \right)^{\frac{1}{q}} \left( \int_a^x g^{p'} v^{1-p'} \right)^{\frac{1}{p'}} < \infty.$$

- (ii) Then there exists a positive constant  $C$  such that inequality

$$\|T_g^* f\|_{L^{q,\infty}(w)} \leq C \|f\|_{L^p(v)} \tag{3.2}$$

holds for all nonnegative functions  $f$  on  $[a, b]$  if and only if

$$A_R = \sup_{a \leq x \leq b} \left( \int_a^x w \right)^{\frac{1}{q}} \left( \int_x^b g^{p'} v^{1-p'} \right)^{\frac{1}{p'}} < \infty.$$

Moreover, the best constants for (3.1) and (3.2) are  $A_L$  and  $A_R$ , respectively.

**Proof** We will only prove (i), since the proof of (ii) is similar. Assume first that  $A_L < \infty$ . Let  $f$  be a nonnegative function and  $\lambda > 0$ . Since the function  $\int_a^x g f$  is nondecreasing,  $O_\lambda = \{x \in [a, b] : \int_a^x g f > \lambda\} = (c, b]$ , where  $c$  verifies  $\int_a^c g f = \lambda$ . Then, by Hölder's inequality,

$$\int_{O_\lambda} w = \frac{1}{\lambda^q} \left( \int_a^c g f \right)^q \int_c^b w \leq \frac{1}{\lambda^q} \left( \int_a^c f^p v \right)^{\frac{q}{p}} \left( \int_a^c g^{p'} v^{1-p'} \right)^{\frac{q}{p'}} \int_c^b w \leq \frac{A_L^q}{\lambda^q} \left( \int_a^b f^p v \right)^{\frac{q}{p}},$$

which gives (3.1) and shows that the best constant is less than  $A_L$ .

Conversely, if (3.1) holds,  $c \in [a, b]$  and  $f_n = \chi_{\{t \in [a, c] : v(t) > \frac{1}{n}\}} g^{p'-1} v^{1-p'}$ , then  $f_n \in L^p(v)$ ,  $f_n^p v = g f_n$ ,  $\{f_n\} \nearrow \chi_{[a, c]} g^{p'-1} v^{1-p'}$ , and

$$\begin{aligned} \left(\int_c^b w\right)^{\frac{1}{q}} \int_a^c g f_n &= \|\chi_{[c,b]}\|_{L^{q,\infty}(w)} \int_a^c g f_n = \left\| \chi_{[c,b]} \int_a^c g f_n \right\|_{L^{q,\infty}(w)} \leq \left\| \int_a^x g f_n \right\|_{L^{q,\infty}(w)} \\ &\leq C \left(\int_a^c f_n^p v\right)^{\frac{1}{p}} = C \left(\int_a^c g f_n\right)^{\frac{1}{p}}, \end{aligned}$$

which implies

$$\left(\int_c^b w\right)^{\frac{1}{q}} \left(\int_a^c g f_n\right)^{\frac{1}{p'}} \leq C.$$

Letting  $n$  tend to  $\infty$ , by monotone convergence we have

$$\left(\int_c^b w\right)^{\frac{1}{q}} \left(\int_a^c g^{p'} v^{1-p'}\right)^{\frac{1}{p'}} \leq C,$$

and taking supremum on  $c \in [a, b]$  we obtain  $A_L \leq C$ .  $\square$

Our result on weighted weak-type one-dimensional Poincaré inequalities reads as follows.

**Theorem 2** *Let  $1 \leq p, q < \infty$  and let  $v, w$  be positive measurable functions on  $[a, b]$ . Then, inequality (1.5) holds for all absolutely continuous functions  $f$  on  $[a, b]$  with a constant  $C > 0$  independent of  $f$  if and only if*

$$P_L = \sup_{a \leq x \leq b} \left(\int_x^b w\right)^{\frac{1}{q}} \left(\int_a^x \mu[a, y]^{p'} v^{1-p'}(y) dy\right)^{\frac{1}{p'}} < \infty \tag{3.3}$$

and

$$P_R = \sup_{a \leq x \leq b} \left(\int_a^x w\right)^{\frac{1}{q}} \left(\int_x^b \mu[y, b]^{p'} v^{1-p'}(y) dy\right)^{\frac{1}{p'}} < \infty. \tag{3.4}$$

Moreover, the best constant  $C$  in (1.5) satisfies  $\frac{1}{\mu[a,b]} \max\{P_L, P_R\} \leq C \leq \frac{2}{\mu[a,b]} (P_L + P_R)$ .

**Proof** Let  $f$  be absolutely continuous on  $[a, b]$ . Then, following [5],

$$f(x) - \frac{1}{\mu[a,b]} \int_a^b f d\mu = \frac{1}{\mu[a,b]} \left[ \int_a^x \mu[a, z] f'(z) dz - \int_x^b \mu[z, b] f'(z) dz \right]. \tag{3.5}$$

Then

$$\left\| f(x) - \frac{1}{\mu[a, b]} \int_a^b f d\mu \right\|_{L^{q, \infty}(w)} \leq \frac{2}{\mu[a, b]} (I + II),$$

where  $I = \left\| \int_a^x \mu[a, z] f'(z) dz \right\|_{L^{q, \infty}(w)}$  and  $II = \left\| \int_x^b \mu[z, b] f'(z) dz \right\|_{L^{q, \infty}(w)}$ . Applying Lemma 2, conditions (3.3) and (3.4) imply

$$I \leq P_L \left( \int_a^b |f'|^p v d\mu \right)^{\frac{1}{p}} \quad \text{and} \quad II \leq P_R \left( \int_a^b |f'|^p v d\mu \right)^{\frac{1}{p}},$$

respectively, and then we get (1.5) with constant  $\frac{2}{\mu[a, b]} (P_L + P_R)$ .

Let us now prove the necessity of the conditions. Let  $\beta \in [a, b]$  and, for each natural  $n$ , let  $(v^{1-p'})_n(x) = \min\{v(x)^{1-p'}, n\}$ . Let us define

$$f_1(x) = \int_a^x \mu[a, t]^{p'-1} (v^{1-p'})_n(t) \chi_{[a, \beta]}(t) dt.$$

Observe that  $f_1$  is absolutely continuous on  $[a, b]$  and  $f_1'(x) = \mu[a, x]^{p'-1} (v^{1-p'})_n(x) \chi_{[a, \beta]}(x)$ . Then  $f_1'(x) = 0$  for  $x > \beta$ . Let us estimate the left-hand side of (1.5) for  $f_1$ . By (3.5), we have

$$\begin{aligned} & \left\| f_1(x) - \frac{1}{\mu[a, b]} \int_a^b f_1 d\mu \right\|_{L^{q, \infty}(w)} \\ &= \frac{1}{\mu[a, b]} \left\| \int_a^x \mu[a, z] f_1'(z) dz - \int_x^b \mu[z, b] f_1'(z) dz \right\|_{L^{q, \infty}(w)} \\ &\geq \frac{1}{\mu[a, b]} \left\| \chi_{[\beta, b]}(x) \left[ \int_a^x \mu[a, z] f_1'(z) dz - \int_x^b \mu[z, b] f_1'(z) dz \right] \right\|_{L^{q, \infty}(w)} \\ &= \frac{1}{\mu[a, b]} \left\| \chi_{[\beta, b]}(x) \int_a^\beta \mu[a, z] f_1'(z) dz \right\|_{L^{q, \infty}(w)} \\ &= \frac{1}{\mu[a, b]} \left( \int_a^\beta \mu[a, z]^{p'} (v^{1-p'})_n(z) dz \right) \left( \int_\beta^b w \right)^{\frac{1}{q}}. \end{aligned}$$

Then (1.5) yields

$$\frac{1}{\mu[a, b]} \left( \int_a^\beta \mu[a, z]^{p'} (v^{1-p'})_n(z) dz \right)^{\frac{1}{p'}} \left( \int_\beta^b w \right)^{\frac{1}{q}} \leq C,$$

which, letting  $n$  tend to  $\infty$ , implies  $\frac{P_L}{\mu[a, b]} \leq C$ . For condition (3.4), we work in the same way with the function  $f_2(x) = \int_x^b \mu[t, b]^{p'-1} (v^{1-p'})_n(t) \chi_{[\beta, b]}(t) dt$ , getting

$$\frac{P_R}{\mu[a, b]} \leq C.$$

□

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## Declarations

**Conflict of interest** The authors declare that they do not have conflict of interest.

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