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A weighted weak-type multilinear gradient inequality

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Abstract. We characterize the weights $w, v_1, v_2, \dots, v_m$ for which the weak-type multilinear gradient inequality

$$\left\|\prod_{i=1}^{m} f_i\right\|_{p,\infty;w} \leq C \prod_{i=1}^{m} \|x \cdot \nabla f_i(x)\|_{p_i,v_i}$$

holds for all $f_1, f_2, \dots, f_m \in C_c^\infty(\mathbb{R}^n)$ in the case $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$.

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1. Introduction and results. Hardy's inequality in $\mathbb{R}^n$ is a fundamental result in mathematical analysis, with many applications (see the monographs [2,4]). It asserts that if $p \geq 1$, $p \neq n$, then

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p} dx \leq \left|\frac{p}{n-p}\right|^p \int_{\mathbb{R}^n} |\nabla f(x)|^p dx \tag{1.1}$$

for all $f \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$.

It is also possible to prove the next inequality, which is stronger than (1.1) in the case $p < n$ (see [2]): if $p \geq 1$ and $f \in C_c^\infty(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} |f(x)|^p dx \leq \left(\frac{p}{n}\right)^p \int_{\mathbb{R}^n} |x \cdot \nabla f(x)|^p dx. \tag{1.2}$$

Sinnamon studied in [5] the weighted counterpart of inequality (1.2). Specifically, by means of the methods of the theory of weighted Hardy inequalities,

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he characterized the nonnegative measurable functions v, w on $\mathbb{R}^n$ such that the gradient inequality

$$\int_{\mathbb{R}^n} |f(x)|^p w(x)dx \le C \int_{\mathbb{R}^n} |x \cdot \nabla f(x)|^p v(x)dx \tag{1.3}$$

holds for each $f \in C_c^\infty(\mathbb{R}^n)$ with a positive constant C independent of f. His result is the next one.

Theorem 1.1 ([5]). *Let $1 < p < \infty$ and let v, w be nonnegative locally integrable functions on $\mathbb{R}^n$ with $v \neq 0$. Then inequality (1.3) holds for all $f \in C_c^\infty(\mathbb{R}^n)$ if and only if*

$$\sup_{x\in\mathbb{R}^n} \left(\int_0^1 w(tx)t^{n-1}dt\right)^{\frac{1}{p}} \left(\int_1^\infty v(tx)^{1-p'} t^{n(1-p')}\frac{dt}{t}\right)^{\frac{1}{p'}} < \infty,$$

where p' stands for the conjugate exponent of p.

More recently, Ortega [3] applied some results on weighted bilinear Hardy inequalities (see [1]) in order to characterize the weights w, v_1, v_2 such that the inequality

$$\left(\int_{\mathbb{R}^n} |f_1 f_2|^p w\right)^{\frac{1}{p}} \le C \prod_{i=1}^{2} \left(\int_{\mathbb{R}^n} |x \cdot \nabla f_i(x)|^{p_i} v_i(x)dx\right)^{\frac{1}{p_i}} \tag{1.4}$$

holds for all $f_1, f_2 \in C_c^\infty(\mathbb{R}^n)$ with a constant $C > 0$ independent of f_1 and f_2, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. The result is the following one.

Theorem 1.2 ([3]). *Let $p, p_1, p_2 > 1$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Let w, v_1, v_2 be nonnegative locally integrable functions defined on $\mathbb{R}^n$ with $v_1, v_2 \neq 0$. Let $V_i(t,x) = v_i(tx)^{1-p_i'} t^{n(1-p_i')}$ for $i = 1, 2$. Inequality (1.4) then holds for all $f_1, f_2 \in C_c^\infty(\mathbb{R}^n)$ with a positive constant C independent of f_1 and f_2 if and only if the following conditions hold:*

(i)

$$\sup_{x\in\mathbb{R}^n} \left(\int_0^1 w(tx)t^{n-1}dt\right)^{\frac{1}{p}} \prod_{i=1}^{2} \left(\int_1^\infty V_i(t,x)\frac{dt}{t}\right)^{\frac{1}{p_i'}} < \infty,$$

(ii)

$$\sup_{x\in\mathbb{R}^n} \left(\int_1^\infty V_1(t,x)\frac{dt}{t}\right)^{\frac{1}{p_1'}} \times \left(\int_0^1 \left(\int_0^z w(tx)t^{n-1}dt\right)^{\frac{p_1}{p}} \left(\int_z^\infty V_2(t,x)\frac{dt}{t}\right)^{\frac{p_1}{p'}} V_2(z,x)\frac{dz}{z}\right)^{\frac{1}{p_1}} < \infty,$$

(iii)

$$\sup_{x\in\mathbb{R}^n}\left(\int_1^\infty V_2(t,x)\frac{dt}{t}\right)^{\frac{1}{p_2'}}$$
$$\times\left(\int_0^1\left(\int_0^z w(tx)t^{n-1}dt\right)^{\frac{p_2}{p}}\left(\int_z^\infty V_1(t,x)\frac{dt}{t}\right)^{\frac{p_2}{p'}}V_1(z,x)\frac{dz}{z}\right)^{\frac{1}{p_2}}<\infty.$$

In this paper, our purpose will be to characterize the weights $w, v_1, v_2, \dots, v_m$ for which the multilinear weak-type gradient inequality

$$\left\|\prod_{i=1}^m f_i\right\|_{p,\infty;w}\le C\prod_{i=1}^m \|x\cdot\nabla f_i(x)\|_{p_i,v_i} \tag{1.5}$$

holds for all $f_1, f_2, \dots, f_m \in C_c^\infty(\mathbb{R}^n)$ in the case $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}$, where for a measurable function g on $\mathbb{R}^n$,

$$\|g\|_{p,\infty;w}=\sup_{\lambda>0}\lambda\left(\int_{\{x\in\mathbb{R}^n:|g(x)|>\lambda\}}w\right)^{\frac{1}{p}}$$

and

$$\|g\|_{p;v}=\left(\int_{\mathbb{R}^n}|g|^p v\right)^{\frac{1}{p}}.$$

In order to characterize inequality (1.5), we will need the next theorem.

Theorem 1.3. *Let $p, p_1, p_2, \dots, p_m \in \mathbb{R}$ with $0<p<\infty$, $p_1, p_2, \dots, p_m \ge 1$, and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}$. Let $w, v_1, v_2, \dots, v_m$ be positive measurable functions on $\mathbb{R}^n$. Then, there exists a positive constant C such that the weighted multilinear weak-type inequality*

$$\left\|\prod_{i=1}^m\int_1^\infty\frac{f_i(tx)}{t}dt\right\|_{p,\infty;w}\le C\prod_{i=1}^m\|f_i\|_{p_i,v_i} \tag{1.6}$$

holds for all functions $f_1, f_2, \dots, f_m$, with $f_i \in L^{p_i}(v_i)$ if and only if

$$B=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left(\int_0^1 w(sx)s^{n-1}ds\right)^{\frac{1}{p}}\prod_{i=1}^m\left\|\frac{\chi_{(1,\infty)}(s)v_i(sx)^{\frac{-1}{p_i}}s^{\frac{-(n-1)}{p_i}}}{s}\right\|_{p_i'}<\infty.$$

Furthermore, the best constant C in inequality (1.6) satisfies $C=B$.

The connection between inequalities (1.6) and (1.5) is established in the next result.

Theorem 1.4. *Let $p, p_1, p_2, \dots p_m \in \mathbb{R}$ with $0 < p < \infty$, $p_1, p_2, \dots, p_m \geq 1$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$. Let $w, v_1, v_2, \dots, v_m$ be positive locally integrable functions on $\mathbb{R}^n$ and $C > 0$. The following statements are equivalent:*

(i) Inequality (1.6) holds with constant C for all $f_1, f_2, \dots, f_m$ with $f_i \in L^{p_i}(v_i)$.
(ii) Inequality (1.6) holds with constant C for all $f_1, f_2, \dots, f_m \in C_c^\infty(\mathbb{R}^n)$.
(iii) Inequality (1.5) holds with constant C for all $f_1, f_2, \dots, f_m \in C_c^\infty(\mathbb{R}^n)$.

In this way, we immediately get the result which characterizes (1.5). It reads as follows.

Corollary 1.5. *Let $p, p_1, p_2, \dots p_m \in \mathbb{R}$ with $0 < p < \infty$, $p_1, p_2, \dots, p_m \geq 1$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$. Let $w, v_1, v_2, \dots, v_m$ be positive locally integrable functions on $\mathbb{R}^n$. Then, there exists $C > 0$ such that (1.5) holds for every $f_1, f_2, \dots f_m \in C_c^\infty(\mathbb{R}^n)$ if and only if $B < \infty$. Moreover, the best constant C in inequality (1.5) satisfies $C = B$.*

As far as we know, this result is new, even in the linear case. It is worth noting that for $m = 1$ the weak and strong-type gradient inequalities are equivalent (see Theorem 1.1), whereas it does not hold for $m > 1$ (see Theorem 1.2). This observation applies in the linear case even when $p = 1$ since, although Sinnamon's statement of Theorem 1.1 only includes the case $p > 1$, the theorem actually also holds for $p = 1$.

The condition $B < \infty$ is easily verifiable for power weights. In this case, we get the next result.

Corollary 1.6. *Let $p, p_1, p_2, \dots p_m \in \mathbb{R}$ with $0 < p < \infty$, $p_1, p_2, \dots, p_m \geq 1$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$. If $w(x) = |x|^\alpha$ and $v_i = |x|^{\beta_i}$ with $\alpha, \beta_1, \dots, \beta_m > -n$, then inequality (1.5) holds if and only if $\sum_{i=1}^m \frac{\alpha - \beta_i}{p_i} = 0$. Moreover, the best constant for (1.5) is*

$$B = \left(\frac{1}{n+\alpha}\right)^{\frac{1}{p}} \prod_{i \in S} \left(\frac{p_i - 1}{\beta_i + n}\right)^{\frac{1}{p_i'}},$$

where $S = \{i : p_i > 1\}$.

Observe that inequality (1.5) holds without weights and Corollary 1.6 shows that in such case the best constant is

$$B = \left(\frac{1}{n}\right)^{\frac{1}{p}} \prod_{i \in S} \left(\frac{p_i - 1}{n}\right)^{\frac{1}{p_i'}},$$

which is the value of B when $w = v_1 = v_2 = \cdots = v_m = 1$. In particular, for $m = 1$ and without weights, we get the sharp inequalities

$$\|f\|_{p,\infty} \leq \frac{(p-1)^{\frac{1}{p'}}}{n} \|x \cdot \nabla f(x)\|_p$$

if $p > 1$, and

$$\|f\|_{1,\infty} \leq \frac{1}{n} \|x \cdot \nabla f(x)\|_1.$$

2. Proof of Theorem 1.3. In order to prove Theorem 1.3, we will need the following lemma.

Lemma 2.1. *Let $p, p_1, p_2, \ldots, p_m \in \mathbb{R}$ with $0 < p < \infty$ and $p_1, p_2, \ldots, p_m \geq 1$. Let $w, v_1, v_2, \ldots, v_m$ be positive measurable functions on $(0, \infty)$. Then the weighted multilinear weak-type inequality*

$$\left\| \prod_{i=1}^{m} \int_{x}^{\infty} \frac{f_i(t)}{t} dt \right\|_{p,\infty;w} \leq C \prod_{i=1}^{m} \|f_i\|_{p_i,v_i} \tag{2.1}$$

holds if and only if

$$A = \sup_{b>0} \left(\int_0^b w \right)^{\frac{1}{p}} \prod_{i=1}^{m} \left\| \frac{\chi_{(b,\infty)}(t) v_i(t)^{\frac{-1}{p_i}}}{t} \right\|_{p_i'} < \infty.$$

Furthermore, the best constant in (2.1) is $C = A$.

Proof. Let us suppose first that $A < \infty$. Let $f_1, f_2, \ldots, f_m$ be positive functions on $(0, \infty)$ and let $\lambda > 0$. The function $h(x) = \prod_{i=1}^{m} \int_x^\infty \frac{f_i(t)}{t} dt$ decreases, which implies that

$$O_\lambda = \left\{ x \in (0,\infty) : \prod_{i=1}^{m} \int_{x}^{\infty} \frac{f_i(t)}{t} dt > \lambda \right\} = (0, b),$$

with $\lambda = \prod_{i=1}^{m} \int_b^\infty \frac{f_i(t)}{t} dt$. Then, applying Hölder's inequality and taking into account the definition of A, we get

$$\begin{aligned} \int_{O_\lambda} w = \int_0^b w &= \left(\int_0^b w \right) \frac{1}{\lambda^p} \prod_{i=1}^{m} \left(\int_b^\infty \frac{f_i(t)}{t} dt \right)^p \\ &\leq \frac{1}{\lambda^p} \left(\int_0^b w \right) \prod_{i=1}^{m} \left(\int_b^\infty f_i^{p_i} v_i \right)^{\frac{p}{p_i}} \left\| \frac{\chi_{(b,\infty)}(t) v_i(t)^{\frac{-1}{p_i}}}{t} \right\|_{p_i'}^p \\ &\leq \frac{A^p}{\lambda^p} \prod_{i=1}^{m} \left(\int_b^\infty f_i^{p_i} v_i \right)^{\frac{p}{p_i}} \leq \frac{A^p}{\lambda^p} \prod_{i=1}^{m} \|f_i\|_{p_i,v_i}^p, \end{aligned}$$

which proves (2.1) and shows that the best constant C in (2.1) satisfies $C \leq A$.

Conversely, assume that (2.1) holds. Let $b > 0$ and define the function $f_i(t) = \chi_{(b,\infty)}(t) \frac{v_i(t)^{1-p_i'}}{t^{p_i'-1}}$ for $i \in \{1, 2, \ldots, m\}$ such that $p_i > 1$. If $p_i = 1$, let $a_i > \operatorname{ess\,inf}_{t \in (b,\infty)} t v_i(t)$, $M_{a_i} = \{t \in (b,\infty) : t v_i(t) < a_i\}$, which has positive measure, and $f_i(t) = t \chi_{M_{a_i}}(t)$. Then, if $\varepsilon > 1$ and $x \in (0, b)$, we have that

$$\prod_{i=1}^{m} \int_{x}^{\infty} \frac{f_i(t)}{t} dt = \left(\prod_{p_i > 1} \int_b^\infty \frac{v_i(t)^{1-p_i'}}{t^{p_i'}} dt \right) \left(\prod_{p_i = 1} |M_{a_i}| \right) > \lambda_0,$$

where $\lambda_0 = \frac{1}{\varepsilon}\left(\prod_{p_i>1}\int_b^\infty \frac{v_i(t)^{1-p_i'}}{t^{p_i'}}dt\right)\left(\prod_{p_i=1}|M_{a_i}|\right)$. This implies that

$$(0,b)\subset\left\{x\in(0,\infty):\prod_{i=1}^m\int_x^\infty \frac{f_i(t)}{t}dt>\lambda_0\right\}.$$

Now, by the weak-type inequality,

$$\left(\int_0^b w\right)^{\frac{1}{p}} \le \frac{C}{\lambda_0}\prod_{p_i>1}\left(\int_b^\infty \frac{v_i(t)^{1-p_i'}}{t^{p_i'}}dt\right)^{\frac{1}{p_i}}\left(\prod_{p_i=1}a_i|M_{a_i}|\right),$$

or equivalently,

$$\left(\int_0^b w\right)^{\frac{1}{p}}\prod_{p_i>1}\left(\int_b^\infty \frac{v_i(t)^{1-p_i'}}{t^{p_i'}}dt\right)^{\frac{1}{p_i'}} \le C\varepsilon\prod_{p_i=1}a_i.$$

Finally, letting a_i tend to $\operatorname{ess\,inf}_{t\in(b,\infty)} tv_i(t)$ and ε to 1, we get $A\le C$. □

Let us already prove Theorem 1.3. Let us see first that $B<\infty$ is sufficient for (1.6) to hold. Let $\lambda>0$ and let $f_1, f_2, \ldots, f_m$ be positive functions on $\mathbb{R}^n$. Changing the variable to polar coordinates, we have

$$\begin{aligned}
&\int\limits_{\{x\in\mathbb{R}^n:\prod_{i=1}^m\int_1^\infty f_i(sx)\frac{ds}{s}>\lambda\}} w\\
&= \int\limits_{S^{n-1}}\int\limits_{\{t\in(0,\infty):\prod_{i=1}^m\int_1^\infty f_i(st\alpha)\frac{ds}{s}>\lambda\}} w(t\alpha)t^{n-1}dtd\alpha\\
&= \int\limits_{S^{n-1}}\int\limits_{\{t\in(0,\infty):\prod_{i=1}^m\int_t^\infty f_i(s\alpha)\frac{ds}{s}>\lambda\}} w(t\alpha)t^{n-1}dtd\alpha.
\end{aligned}$$

Observe that $B=\operatorname{ess\,sup}_{\alpha\in S^{n-1}}A_\alpha$, where

$$A_\alpha=\sup_{b>0}\left(\int_0^b w(t\alpha)t^{n-1}dt\right)^{\frac{1}{p}}\prod_{i=1}^m\left\|\frac{\chi_{(b,\infty)}(t)v_i(t\alpha)^{\frac{-1}{p_i}}t^{\frac{-(n-1)}{p_i}}}{t}\right\|_{p_i'}.$$

Then, by the lemma, for almost all $\alpha\in S^{n-1}$, we get the inequality

$$\int\limits_{\{t\in(0,\infty):\prod_{i=1}^m\int_t^\infty f_i(z\alpha)\frac{dz}{z}>\lambda\}} w(t\alpha)t^{n-1}dt \le \frac{B^p}{\lambda^p}\prod_{i=1}^m\left(\int_0^\infty f_i^{p_i}(t\alpha)v_i(t\alpha)t^{n-1}dt\right)^{\frac{p}{p_i}}. \tag{2.2}$$

Integrating (2.2) on S^{n-1} and applying Hölder's inequality, we have

$$\int\limits_{\{x\in\mathbb{R}^n:\prod_{i=1}^m\int_1^\infty f_i(sx)\frac{ds}{s}>\lambda\}} w$$
$$\le \frac{B^p}{\lambda^p}\int\limits_{S^{n-1}}\prod_{i=1}^m\left(\int\limits_0^\infty f_i^{p_i}(t\alpha)v_i(t\alpha)t^{n-1}dt\right)^{\frac{p}{p_i}}d\alpha$$
$$\le \frac{B^p}{\lambda^p}\prod_{i=1}^m\left(\int\limits_{S^{n-1}}\int\limits_0^\infty f_i^{p_i}(t\alpha)v_i(t\alpha)t^{n-1}dtd\alpha\right)^{\frac{p}{p_i}}$$
$$\le \frac{B^p}{\lambda^p}\prod_{i=1}^m\|f_i\|_{p_i,v_i}^p,$$

which is inequality (1.6). This shows that the best constant C satisfies $C\le B$.

Now, let us prove the necessity of the condition. Assume that the weak-type inequality (1.6) holds. If we rewrite (1.6) in polar coordinates, we have that

$$\begin{aligned}\int\limits_{S^{n-1}}\int\limits_{\{t\in(0,\infty):\prod_{i=1}^m\int_t^\infty f_i(s\alpha)\frac{ds}{s}>\lambda\}} w(t\alpha)t^{n-1}dtd\alpha\\ \le \frac{C^p}{\lambda^p}\prod_{i=1}^m\left(\int\limits_{S^{n-1}}\int\limits_0^\infty f_i(t\alpha)^{p_i}v_i(t\alpha)t^{n-1}dtd\alpha\right)^{\frac{p}{p_i}}.\end{aligned}\tag{2.3}$$

Let $A\subset S^{n-1}$ with $\alpha(A)>0$ and $f_i(t\alpha)=\chi_A(\alpha)h_i(t)$, where $h_1,h_2,\dots,h_m$ are nonnegative measurable functions on $(0,\infty)$. Applying inequality (2.3) to these functions f_i, we have

$$\int\limits_A\int\limits_{\{t\in(0,\infty):\prod_{i=1}^m\int_t^\infty h_i(s)\frac{ds}{s}>\lambda\}} w(t\alpha)t^{n-1}dtd\alpha$$
$$\le \frac{C^p}{\lambda^p}\prod_{i=1}^m\left(\int\limits_A\int\limits_0^\infty h_i(t)^{p_i}v_i(t\alpha)t^{n-1}dtd\alpha\right)^{\frac{p}{p_i}}.$$

Since $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, by differentiation, the inequality above implies that

$$\int\limits_{\{t\in(0,\infty):\prod_{i=1}^m\int_t^\infty h_i(s)\frac{ds}{s}>\lambda\}} w(t\alpha)t^{n-1}dt$$
$$\le \frac{C^p}{\lambda^p}\prod_{i=1}^m\left(\int\limits_0^\infty h_i(t)^{p_i}v_i(t\alpha)t^{n-1}dt\right)^{\frac{p}{p_i}}$$

for almost every $\alpha \in S^{n-1}$, where the constant C is independent of $\alpha, h_1, h_2, \dots, h_m$. Applying the lemma, we have that

$$\operatorname{ess}\sup_{\alpha\in S^{n-1}} \sup_{b>0} \left(\int_0^b w(t\alpha)t^{n-1}dt\right)^{\frac{1}{p}} \prod_{i=1}^m \left\| \frac{\chi_{(b,\infty)}(t)v_i(t\alpha)^{\frac{-1}{p_i}} t^{\frac{-(n-1)}{p_i}}}{t} \right\|_{p_i'} \le C,$$

which is equivalent to $B \le C$.

3. Proof of Theorem 1.4. It is clear that (ii) follows from (i) since $C_c^\infty(\mathbb{R}^n) \subset L^{p_i}(v_i)$ because the functions v_i are locally integrable. Let us prove now that (ii) implies (i). Let $f_1, f_2, \dots, f_m$ be simple, nonnegative functions with bounded supports. Let $R > 0$ be such that $\operatorname{supp}(f_i) \subset B(0, R)$ for all i. Then, by standard arguments, for each $i \in \{1, 2, \dots, m\}$, there exists a sequence $\{g_k^i\} \subset C_c^\infty(\mathbb{R}^n)$ and $M_i > 0$ such that $g_k^i \ge 0$, $\max\{f_i(x), g_k^i(x)\} \le M_i$ for all x and all k, $\operatorname{supp}(g_k^i) \subset B(0, 2R)$, and $\{g_k^i\}$ converges to f_i almost everywhere and in $L^{p_i}(\mathbb{R}^n)$. By (ii), we have

$$\left\| \prod_{i=1}^m \int_1^\infty \frac{g_k^i(tx)}{t} dt \right\|_{p,\infty;w} \le C \prod_{i=1}^m \|g_k^i\|_{p_i,v_i} \tag{3.1}$$

and by the dominated convergence theorem,

$$\lim_{k\to\infty} \prod_{i=1}^m \|g_k^i\|_{p_i,v_i} = \prod_{i=1}^m \|f_i\|_{p_i,v_i}. \tag{3.2}$$

On the other hand, for each i, the sequence $\left\{\int_1^\infty \frac{g_k^i(tx)}{t} dt\right\}$ converges to the function $\int_1^\infty \frac{f_i(tx)}{t} dt$ in $L^{p_i}(\mathbb{R}^n)$. Indeed, by the Minkowski integral inequality,

$$\left\| \int_1^\infty \frac{g_k^i(tx)}{t} dt - \int_1^\infty \frac{f_i(tx)}{t} dt \right\|_{p_i} = \left\| \int_1^\infty \frac{g_k^i(tx) - f_i(tx)}{t} dt \right\|_{p_i}$$
$$\le \int_1^\infty \frac{1}{t} \|g_k^i(tx) - f_i(tx)\|_{p_i} dt = \frac{p_i}{n} \|g_k^i - f_i\|_{p_i} \to 0.$$

Then, for each i, there exists a subsequence, which we also call $\{g_k^i\}$, such that

$$\lim_{k\to\infty} \int_1^\infty \frac{g_k^i(tx)}{t} dt = \int_1^\infty \frac{f_i(tx)}{t} dt$$

for almost all x. This clearly implies that

$$\lim_{k\to\infty} \prod_{i=1}^m \int_1^\infty \frac{g_k^i(tx)}{t} dt = \prod_{i=1}^m \int_1^\infty \frac{f_i(tx)}{t} dt \quad \text{a.e.}$$

Let $\lambda > 0$, $O_\lambda = \{x \in \mathbb{R}^n : \prod_{i=1}^m \int_1^\infty \frac{f_i(tx)}{t} dt > \lambda\}$, and $O_\lambda^k = \{x \in \mathbb{R}^n : \prod_{i=1}^m \int_1^\infty \frac{g_k^i(tx)}{t} dt > \lambda\}$. Then $\chi_{O_\lambda}(x) \le \liminf_{k\to\infty} \chi_{O_\lambda^k}(x)$ for almost all x and Fatou's lemma gives

$$\int_{O_\lambda} w = \int_{\mathbb{R}^n} \chi_{O_\lambda} w \le \liminf_{k\to\infty} \int_{\mathbb{R}^n} \chi_{O_\lambda^k} w = \liminf_{k\to\infty} \int_{O_\lambda^k} w,$$

which, by definition of $\|\cdot\|_{p,\infty;w}$, yields

$$\left\| \prod_{i=1}^m \int_1^\infty \frac{f_i(tx)}{t} dt \right\|_{p,\infty;w} \le \liminf_{k\to\infty} \left\| \prod_{i=1}^m \int_1^\infty \frac{g_k^i(tx)}{t} dt \right\|_{p,\infty;w}. \tag{3.3}$$

Now, by (3.3), (3.1), and (3.2), we get

$$\left\| \prod_{i=1}^m \int_1^\infty \frac{f_i(tx)}{t} dt \right\|_{p,\infty;w} \le C \prod_{i=1}^m \|f_i\|_{p_i,v_i}.$$

Finally, let $f_1, f_2, \ldots, f_m$ be nonnegative functions with $f_i \in L^{p_i}(v_i)$ for all i. For each i, there exists an increasing sequence $\{f_k^i\}$ of simple, nonnegative functions with bounded supports that converges pointwise to f_i. Then, the inequality

$$\left\| \prod_{i=1}^m \int_1^\infty \frac{f_k^i(tx)}{t} dt \right\|_{p,\infty;w} \le C \prod_{i=1}^m \|f_k^i\|_{p_i,v_i}$$

holds for all k and letting k tend to ∞, we get (i) by monotone convergence.

The equivalence between (ii) and (iii) is essentially proved in [5]. We include it for completeness. Assume that (ii) holds and let $g_1, g_2, \ldots, g_m \in C_c^\infty(\mathbb{R}^n)$. Let, for each i, $f_i(x) = x \cdot \nabla g_i(x)$. Then, $f_i \in C_c^\infty(\mathbb{R}^n)$, $\int_1^\infty \frac{f_i(tx)}{t} dt = -g_i(x)$, and (ii) gives

$$\left\| \prod_{i=1}^m g_i \right\|_{p,\infty;w} \le C \prod_{i=1}^m \|x \cdot \nabla g_i(x)\|_{p_i,v_i}.$$

Conversely, if (iii) holds and $f_1, f_2, \ldots, f_m \in C_c^\infty(\mathbb{R}^n)$, then the functions $g_i(x) = \int_1^\infty \frac{f_i(tx)}{t} dt$ satisfy that $g_i \in C_c^\infty(\mathbb{R}^n)$ and $x \cdot \nabla g_i(x) = -f_i(x)$. Therefore, by (iii), the inequality

$$\left\| \prod_{i=1}^m \int_1^\infty \frac{f_i(tx)}{t} dt \right\|_{p,\infty;w} \le C \prod_{i=1}^m \|f_i\|_{p_i,v_i}$$

holds.

4. Proof of Corollary 1.6. Let us compute B. On the one hand, for $x \in \mathbb{R}^n$,

$$\left(\int_0^1 w(sx)s^{n-1}ds\right)^{\frac{1}{p}} = \left(\int_0^1 |sx|^{\alpha} s^{n-1}ds\right)^{\frac{1}{p}} = |x|^{\frac{\alpha}{p}}\left(\frac{1}{n+\alpha}\right)^{\frac{1}{p}}.$$

On the other hand, if $p_i > 1$,

$$\left\|\frac{\chi_{(1,\infty)}(s)v_i(sx)^{\frac{-1}{p_i}}s^{\frac{-(n-1)}{p_i}}}{s}\right\|_{p_i'} = |x|^{\frac{-\beta_i}{p_i}}\left(\frac{p_i-1}{\beta_i+n}\right)^{\frac{1}{p_i'}}$$

and if $p_i = 1$,

$$\left\|\frac{\chi_{(1,\infty)}(s)v_i(sx)^{\frac{-1}{p_i}}s^{\frac{-(n-1)}{p_i}}}{s}\right\|_{p_i'} = \operatorname{ess}\sup_{s\in(1,\infty)} |sx|^{-\beta_i}s^{-n} = |x|^{-\beta_i}.$$

Then,

$$B = \sup_{x\in\mathbb{R}^n} |x|^{\frac{\alpha}{p}-\sum_{i=1}^m \frac{\beta_i}{p_i}}\left(\frac{1}{n+\alpha}\right)^{\frac{1}{p}}\prod_{i\in S}\left(\frac{p_i-1}{\beta_i+n}\right)^{\frac{1}{p_i'}},$$

which is finite if and only if $\sum_{i=1}^m \frac{\alpha-\beta_i}{p_i} = 0$ and, in that case,

$$B = \left(\frac{1}{n+\alpha}\right)^{\frac{1}{p}}\prod_{i\in S}\left(\frac{p_i-1}{\beta_i+n}\right)^{\frac{1}{p_i'}}.$$

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References

[1] Aguilar Cañestro, M.I., Ortega Salvador, P., Ramírez Torreblanca, C.: Weighted bilinear Hardy inequalities. J. Math. Anal. Appl. **387**(1), 320–334 (2012)

[2] Balinsky, A.A., Evans, W.D., Lewis, R.T.: The Analysis and Geometry of Hardy's Inequality. Springer, Berlin (2015)

[3] Ortega Salvador, P.: A weighted bilinear gradient inequality. Proc. Roy. Soc. Edinb. Sect. A **144**(4), 819–829 (2014)

[4] Opic, B., Kufner, A.: Hardy-Type Inequalities. Longman, Harlow (1990)

[5] Sinnamon, G.: A weighted gradient inequality. Proc. Roy. Soc. Edinb. Sect. A **111**(3–4), 329–335 (1989)

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