Weighted inequalities for multilinear potential operators and its commutator

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Abstract

We prove weighted strong inequalities for the multilinear potential operator $\mathcal{T}_\phi$ and its commutator, where the kernel ϕ satisfies certain growth condition. For these operators we also obtain Fefferman-Stein type inequalities and Coifman type estimates. On the other hand we prove weighted weak type inequalities for the multilinear maximal operator $\mathcal{M}_{\varphi,B}$ associated to a essentially nondecreasing function φ and to a submultiplicative Young function B. This result allows us to obtain a weighted weak type inequality for the operator $\mathcal{T}_\phi$

1 Introduction

In [Pe], C. Pérez proved weighted norm inequalities for general potential operators. For a given nonnegative locally integrable function Φ defined in $\mathbb{R}^n$, the potential operator T_Φ is defined by

$$T_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy. \tag{1.1}$$

In a previous article E. Sawyer and R. Wheeden obtained Fefferman-Phong type conditions on the weights and proved weighted boundedness results for the fractional integral operator I_α between Lebesgue spaces (see [SW]). Motivated by this paper, Pérez considered weaker norms than those involved in the Fefferman-Phong type conditions in [SW] and obtained weighted boundedness results for the potential operator T_Φ, where the kernel Φ satisfies certain growth condition. These norms are defined in terms of certain mapping properties of appropriate maximal operators associated to each norm. In the same article, the author considered the corresponding problem for the maximal operator related to the potential operator T_Φ defined by

$$M_{\tilde{\Phi}} f(x) = \sup_{x\in Q} \frac{\tilde{\Phi}(l(Q))}{|Q|} \int_Q |f(y)|\, dy,$$

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where $l(Q)$ is the sidelength of Q and

$$\tilde{\Phi}(t) = \int_{|z|\le t} \Phi(z)\, dz. \tag{1.2}$$

Other authors also proved weighted norm inequalities involving the potential operator defined in (1.1).

In [CUMP] the authors proved a Coifman type inequality by using extrapolation arguments. More precisely, they showed that if w is a weight in the A_∞ class, the $L^p(w)$ norm of T_ϕ is bounded by the $L^p(w)$ norm of $M_{\tilde{\Phi}}$. They also obtained the corresponding result in the context of Lorentz spaces.

On the other hand, in [LQY] and [L1] the authors obtained weighted L^p inequalities of Fefferman-Stein type for T_Φ and for the higher order commutators associated to this operator, respectively whenever $1 < p < \infty$. These estimates involve pairs of weights of the form $(w, \mathcal{M}w)$, where $\mathcal{M}$ are suitable maximal operators.

Two weighted norm inequalities in the spirit of those in [Pe] were proved in [L2] for the higher order commutators associated to T_ϕ.

In [PW] and [LYY], weighted inequalities like the ones described above were proved in the general setting of the spaces of homogeneous type.

Motivated by the work in [LOPTT], K. Moen [M] considered the multilinear fractional integral. For $0 < \alpha < mn$ and the collection $\vec{f} = (f_1, \dots, f_m)$ of m functions on $\mathbb{R}^n$ this operator is defined by

$$\mathcal{I}_\alpha \vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{1}{(\sum_{i=1}^m |x - y_i|)^{mn-\alpha}} \prod_{i=1}^m f_i(y_i)\, d\vec{y}.$$

For this operator, Moen obtained two weighted $L^p - L^q$ estimates and a Coifman type inequality, generalizing to the multilineal context the results given in [Pe] and [CUMP]. As a consequence of the two weighted inequalities, he obtained a generalized version of the well known characterization proved in [MW]. He also obtained weighted weak and strong inequalities for the multi(sub)linear fractional maximal operator.

A weighted weak type inequality for $\mathcal{I}_\alpha$ and the control of the $L^p(w)$ norm of $\mathcal{I}_\alpha$ by the $L^p(Mw)$ norm of the multi(sub)linear fractional maximal operator, where M is the Hardy-Littlewood maximal operator, were obtained in [P]. In this article the author also obtained weighted inequalities for multi(sub)linear fractional maximal operators associated to Young functions.

On the other hand, weighted strong boundedness and weighted endpoint estimates for the multilinear commutator of $\mathcal{I}_\alpha$ were proved in [CX]. The authors also studied the multilinear fractional integral with homogeneous kernels.

In this paper we shall consider the multilinear potential operator defined as

$$\mathcal{T}_\phi(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \phi(x - y_1, \dots, x - y_m) \prod_{i=1}^m f_i(y_i)\, d\vec{y}, \tag{1.3}$$

where ϕ is a nonnegative function defined on $(\mathbb{R}^n)^m$. We shall also deal with the commutator associated to this operator, given by

$$\mathcal{T}_{\vec{b},\phi}(\vec{f})(x) = \sum_{j=1}^{m} \mathcal{T}_{b_j,\phi}(\vec{f})(x), \tag{1.4}$$

where

$$\mathcal{T}_{b_j,\phi}(\vec{f})(x) = b_j(x)\mathcal{T}_\phi(\vec{f})(x) - \mathcal{T}_\phi(f_1, \cdots, b_j f_j, \cdots, f_m)(x).$$

The aim of this paper is to prove weighted strong and weak type inequalities for these multilinear operators like the ones described above. As in the case $m = 1$ we assume that the function ϕ satisfies a growth condition. More precisely, we say that a non-negative locally integrable function ϕ defined in $(\mathbb{R}^n)^m$ satisfies a $\mathfrak{D}$-condition or that $\phi \in \mathfrak{D}$ if there exist two positive constants δ and ϵ such that the inequality

$$\sup_{\vec{w} \in \mathcal{A}_{(2^k,1,0)}} \phi(\vec{w}) \leq \frac{C}{2^{knm}} \int_{\mathcal{A}_{(2^k,\delta,\epsilon)}} \phi(\vec{y})\, d\vec{y}$$

holds for every $k \in \mathbb{Z}$, where

$$\mathcal{A}_{(t,\delta,\epsilon)} = \{\vec{y} = (y_1, \cdots, y_m) : \ \delta(1-\epsilon)t < \sum_{i=1}^{m} |y_i| \leq \delta(1+\epsilon)2t\}, \quad t > 0. \tag{1.5}$$

Although the basic example of operators of this type is provided by the multilinear fractional integral operator defined by the kernel $\phi(w_1, \ldots, w_m) = \left(\sum_{i=1}^m |w_i|\right)^{\alpha - nm}$, for $0 < \alpha < nm$, another important example is the multilinear Bessel potential. For $\alpha > 0$ the kernel of this operator is given by

$$G_\alpha(x_1, \ldots, x_m) = C_{\alpha,n,m} \int_0^\infty e^{-t} e^{-\frac{(\sum_{i=1}^m |x_i|)^2}{4t}} t^{(\alpha - nm)/2} \frac{dt}{t},$$

where the constant $C_{\alpha,n,m} = \frac{1}{2^{nm}\Gamma(\alpha/2)\pi^{nm/2}}$. If $\vec{\xi} = (\xi_1, \ldots, \xi_m)$, $\xi_i \in \mathbb{R}^n$, it is easy to see that the Fourier transform of the kernel G_α valuated in $\vec{\xi}$ is given by

$$\widehat{G_\alpha}(\vec{\xi}) = \left(1 + 4\pi^2|\vec{\xi}|\right)^{-\alpha/2}.$$

As in the case $m = 1$ (see [G, 418]), we can prove that if α is a positive number and $\vec{x} = (x_1, \ldots, x_m)$ with $x_i \in \mathbb{R}^n$ then $G_\alpha(\vec{x}) > 0$ for every $\vec{x} \in (\mathbb{R}^n)^m$. Moreover, there exist positive constants C and c that only depend on α, n and m such that

$$G_\alpha(\vec{x}) \leq Ce^{-|\vec{x}|/2}, \qquad \text{when} \quad |\vec{x}| \geq 2,$$

and

$$c^{-1}H_\alpha(\vec{x}) \leq G_\alpha(\vec{x}) \leq cH_\alpha(\vec{x}), \qquad \text{for} \quad |\vec{x}| \leq 2,$$

where H_α is a function that satisfies

$$H_\alpha(\vec{x}) = \begin{cases} |\vec{x}|^{\alpha-nm} + 1 + o(|\vec{x}|^{\alpha-nm+2}) & \text{for } 0 < \alpha < nm, \\ \log \frac{1}{|\vec{x}|} + 1 + o(|\vec{x}|^2) & \text{if } \alpha = nm, \\ 1 + o(|\vec{x}|^{\alpha-nm}) & \text{if } \alpha > nm, \end{cases}$$

as $|\vec{x}| \to 0$. As in the linear case, G_α satisfies condition $\mathfrak{D}$.

In the setting of the spaces of homogeneous type, D. Maldonado, K. Moen and V. Naibo [MMN] defined the multilinear version of the potential type operators studied in [PW]. They proved two weighted strong inequalities like the ones in [PW] for the case $m = 1$. Observe that in the euclidean context and whenever $K(x, y) = \phi(x - y)$, the growth condition in [MMN] is more restrictive than the condition $\mathfrak{D}$. It is easy to check that condition $\mathfrak{D}$ is satisfied by any radial increasing or decreasing function, and functions ϕ essentially constants on annuli. Since we also work with commutator operators, our results are not included in the results in [MMN].

2 Preliminaries

Let X be a Banach function space over $\mathbb{R}^n$ with respect to the Lebesgue measure. Examples of these spaces are given by the Lebesgue L^p spaces, Lorentz spaces, Orlicz spaces and so on. Given any measurable function $f \in X$ and a cube $Q \subset \mathbb{R}^n$, we define the X average of f over Q to be

$$\|f\|_{X,Q} = \|\delta_{l(Q)}(f\chi_Q)\|_X,$$

where $\delta_a f(x) = f(ax)$ for $a > 0$ and χ_A denotes the characteristic function of the set A. In particular, when $X = L^{\mathcal{B}}$, the Orlicz space associated to a Young function $\mathcal{B}$,

$$\|f\|_{X,Q} = \|f\|_{L^{\mathcal{B}},Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \mathcal{B}\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1\right\}.$$

Particularly, when $X = L^r$, $r \geq 1$, we have that

$$\|f\|_{X,Q} = ||f||_{L^r,Q} = \left(\frac{1}{|Q|}\int_Q |f(y)|^r\right)^{1/r}.$$

Given a Banach function space X there exists an associated Banach function space X' for which the generalized Hölder's inequality,

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq \|f\|_X \|g\|_{X'},$$

holds. Notice that if we apply the above inequality to $\delta_{l(Q)}(f\chi_Q)$ and $\delta_{l(Q)}(g\chi_Q)$ we get that

$$||fg||_{L^1,Q} \leq \|f\|_{X,Q}\|g\|_{X',Q}.$$

On the other hand, when we deal with Orlicz spaces a further generalization of Hölder's inequality that will be useful later is the following: If $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are Young functions and

$$\mathcal{A}^{-1}(t)\mathcal{B}^{-1}(t) \leq \mathcal{C}^{-1}(t)$$

then

$$\|fg\|_{L^{\mathcal{C}},Q} \leq 2\|f\|_{L^{\mathcal{A}},Q}\|g\|_{L^{\mathcal{B}},Q},$$

when $\mathcal{C}(t) = t$, the function $\mathcal{B}$ is called the complementary function of $\mathcal{A}$.

In general, if $\mathcal{C}$ and $\mathcal{C}_i$ are Young functions such that

$$\prod_{i=1}^{m} \mathcal{C}_i^{-1}(t) \leq \mathcal{C}^{-1}(t)$$

then

$$\| \prod_{i=1}^{m} f_i\|_{L^{\mathcal{C}},Q} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{\mathcal{C}_i},Q} .$$

For more information about Orlicz spaces see [O].

We shall say that the function $\varphi : (0,\infty) \to (0,\infty)$ is essentially nondecreasing if there exists a positive constant ρ such that, if $t \leq s$ then $\varphi(t) \leq \rho\varphi(s)$. We shall also suppose that $\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0$. Let $X_1, \dots, X_m$ be m Banach function spaces and $\vec{X} = (X_1, \dots, X_m)$. If φ is a essentially nondecreasing function, the multilinear maximal operator $\mathcal{M}_{\varphi,\vec{X}}$ associated to φ and $\vec{X}$ is defined by

$$\mathcal{M}_{\varphi,\vec{X}}\vec{f}(x) = \sup_{Q\ni x} \varphi(|Q|) \prod_{i=1}^{m} \|f_i\|_{X_i;Q}.$$

If $\vec{X} = (X, \dots, X)$ we simply write $\mathcal{M}_{\varphi,X}$ and $M_{\varphi,X}$ if $m = 1$. If $\varphi \equiv 1$ we write $\mathcal{M}_{\vec{X}}$, $\mathcal{M}_X$ or M_X respectively. When X is an Orlicz space $L^{\mathcal{B}}$, sometimes we shall denote with $|| \cdot ||_{\mathcal{B}}$ the norm $|| \cdot ||_{L^{\mathcal{B}}}$ and with $\mathcal{M}_{\varphi,\mathcal{B}}$ the operator $\mathcal{M}_{\varphi,L^{\mathcal{B}}}$. In particular, when $\mathcal{B}(t) = t^p(1 + \log^+ t)^\alpha$, for any $\alpha > 0$ and $p \geq 1$, we shall use the notation $|| \cdot ||_{L^p(\log L)^\alpha,Q}$ and $\mathcal{M}_{\varphi,L^p(\log L)^\alpha}$ to denote the corresponding norm and maximal operator.

Associated to the kernel ϕ we denote Φ_θ, $0 < \theta \leq 1$, to be the positive function defined for $t > 0$ by

$$\Phi_\theta(t) = \left[\sum_{\nu\in\mathbb{Z},\ \nu\geq\log_{1/2} t} \left(\int_{\mathcal{A}(2^{-\nu},\delta,\epsilon)} \phi(\vec{y})\, d\vec{y} \right)^\theta \right]^{1/\theta} . \tag{2.1}$$

For $\varphi(t) = \Phi_\theta(t^{1/n})$ the corresponding multilinear maximal operator $\mathcal{M}_{\varphi,X}$ will be denoted by $\mathcal{M}_{\Phi_\theta,X}$.

Let $1 < p_1, \dots, p_m < \infty$ such that $1/p = \sum_{i=1}^m 1/p_i$ and suppose that $M_{X_i} : L^{p_i} \to L^{p_i}$. From the fact that $\mathcal{M}_{\vec{X}}\vec{f}(x) \leq \prod_{i=1}^m M_{X_i} f_i(x)$ and applying Hölder's inequality we obtain that

$$\mathcal{M}_{\vec{X}} : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

When we deal with the commutator defined in (1.4) we suppose that the vector of symbols $\vec{b} = (b_1, \ldots, b_m) \in BMO^m$ and we omit this hypothesis. Moreover, we denote $\|\vec{b}\|_* = \sup_{1\le j\le m} \|b_j\|_*$.

If $\mathcal{A}(t) = e^t - 1$, let $||\cdot||_{\exp L,Q}$ denote the average $||\cdot||_{L^{\mathcal{A}},Q}$. By the John Niremberg inequality we get that $||b - b_Q||_{\exp L,Q} \le C||b||_*$ for any BMO function b. In the proof of the results we shall use frecuently the following inequality

$$\frac{1}{|Q|}\int |b - b_Q||f| \le C||b - b_Q||_{\exp L,Q}||f||_{L(\log L),Q} \le C||b||_*||f||_{L(\log L),Q}$$

which can be proved by Hölder inequality.

In this paper we shall use the following properties of the weights. We say that a weight w satisfies a Reverse Hölder's inequality with exponent $s > 1$, or equivalently, we say that $w \in RH(s)$, if there exists a positive constant C such that

$$\left(\frac{1}{|Q|}\int_Q w^s\right)^{1/s} \le C\left(\frac{1}{|Q|}\int_Q w\right),$$

for all cube Q. We say that $w \in RH_\infty$ if there exists a positive constant C such that the inequality

$$\sup_{x\in Q} w(x) \le \frac{C}{|Q|}\int_Q w$$

holds for every $Q \subset \mathbb{R}^n$.

In all the results of this paper we shall use the following notation: with $\mathcal{T}_{\vec{b}^\ell,\phi}$ we shall denote the operator $\mathcal{T}_\phi$ when $\ell = 0$ and the commutator $\mathcal{T}_{\vec{b},\phi}$ when $\ell = 1$. We also use C_ℓ to denote a constant that depend on the norm $\|\vec{b}\|_*$ if $\ell = 1$.

Sometimes we shall only estimate the term $\mathcal{T}_{\vec{b_j},\phi}$ in the definition of the commutator given in (1.4) since the final estimate goes straightforward.

3 Statement of the main results

3.1 Strong type results

In the next theorem we give Fefferman-Phong type conditions on the weights in order to obtain the boundedness of the operator $\mathcal{T}_\phi$ and its commutator between weighted Lebesgue spaces.

For $1 \le i, j \le m$, $\delta_{i,j}$ will be 1 if $i = j$ and will be 0 if $i \ne j$. If $u, v_1, \ldots, v_m$ are weights we write $(u, \vec{v}) = (u, v_1, \ldots, v_m)$. Let $Y_{i,j}$ and $\tilde{Y}_{i,j}$, $i, j = 1, \ldots, m$ be Banach function spaces. We denote $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ the matrix whose components are $Y_{i,j}$ and $\tilde{Y}_{i,j}$ respectively.

(3.1) **Theorem:** *Let $\ell \in \{0, 1\}$ be given. Let $1 < p_1, \ldots, p_m < \infty$, with $1/p = \sum_{i=1}^m 1/p_i$, let q be an exponent satisfying $1/m < p \le q < \infty$ and let $\phi \in \mathfrak{D}$. For*

$k \in \{0,1\}$, $k \le \ell$, and $i,j \in \{1,\cdots,m\}$, let X^k, $\tilde{X}^k$, $Y^k_{i,j}$ and $\tilde{Y}^k_{i,j}$ be Banach function spaces such that

$$||fg||_{L(\log L)^k,Q} \le 2||f||_{X^k,Q}||g||_{\tilde{X}^k,Q},$$
$$||fg||_{L(\log L)^{k\delta_{i,j}},Q} \le 2||f||_{Y^k_{i,j},Q}||g||_{\tilde{Y}^k_{i,j},Q},$$

for every cube Q. If $\vec{\tilde{Y}}^k_j = (\tilde{Y}^k_{1,j},\ldots,\tilde{Y}^k_{m,j})$ denotes the j-th column in $\tilde{\mathbf{Y}}^k$ suppose that for every $k \in \{0,1\}$, $k \le \ell$,

$$\mathcal{M}_{\vec{\tilde{Y}}^k_j} : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

For the weights $(u,\vec{v})$ let us write

$$\mathbb{W}(\theta,\gamma,X,\mathbf{Y}) = \max_{1\le j\le m} \sup_Q \; \Phi_\theta(l(Q)) \; |Q|^{1/q-1/p} \; ||u^\gamma||^{1/\gamma}_{X,Q} \; \prod_{i=1}^m ||v_i^{-1}||_{Y_{i,j},Q}. \tag{3.2}$$

Suppose that one of the following two conditions holds

i) $q > 1$, $M_{\tilde{X}^k} : L^{q'}(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n)$, for every $k \in \{0,1\}$, $k \le \ell$, and $(u,\vec{v})$ are weights that satisfy

$$\max\{\mathbb{W}(1,1,X^\ell,\mathbf{Y}^0), \ell\; \mathbb{W}(1,1,X^0,\mathbf{Y}^1)\} < \infty \tag{3.3}$$

ii) $q \le 1$ and $(u,\vec{v})$ are weights that satisfy

$$\max\{\mathbb{W}(q,q,L(\log L)^{\ell q},\mathbf{Y}^0), \ell\; \mathbb{W}(q,1,L,\mathbf{Y}^1)\} < \infty \tag{3.4}$$

Then there exists a constant C_ℓ such that the inequality

$$\left(\int_{\mathbb{R}^n} (|\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}|\, u)^q\right)^{1/q} \le C_\ell \prod_{i=1}^m \left(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i}\right)^{1/p_i} \tag{3.5}$$

holds for every $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

Notice that, in the case $q > 1$, the factor involving the function ϕ in the conditions on the weights is given by $\Phi_1(l(Q))$. Also notice that $\Phi_1(t)$ is equivalent to $\tilde{\phi}(\delta(1+\epsilon)t)$, where δ and ϵ are the constant in condition $\mathfrak{D}$ and $\tilde{\phi}$ is the function defined in (1.2). Then, when $q > 1$ we get the corresponding conditions on the weights that the ones previously obtained in the linear cases or for the multilinear fractional integral. On the other hand, when the function ϕ is any radial increasing or decreasing function, or ϕ is essentially constant on annuli, then the function $\Phi_\theta(t)$, $0 < \theta \le 1$, is equivalent to the function $\tilde{\phi}(\delta(1+\epsilon)t)$. In particular, when $\phi(\vec{y}) = (\sum_{i=1}^m |y_i|)^{\alpha-nm}$ we recover the results in [M] for the case $\ell = 0$.

In the following remarks we give some examples of Banach function spaces that can be used in conditions (3.3) and (3.4).

(3.6) **Remark:** When we consider the operator $\mathcal{T}_\phi$ ($\ell = 0$), we get the theorem above for $X^0 = L^{qr}$ and $Y^0_{i,j} = L^{rp_i'}$ for every $j = 1, \dots, m$ and for some $r > 1$ (with $\tilde{X}^0 = L^{(qr)'}$ and $\tilde{Y}^0_{i,j} = L^{(rp_i')'}$). Using Orlicz spaces we get a better result. In fact, applying the examples of Young functions give in [CUP] (see page 828) we can take $X^0 = L^q(\log L)^{q-1+\delta}$ and $Y^0_{i,j} = L^{p_i'}(\log L)^{p_i'-1+\delta}$, for every $j = 1, \dots, m$ and for some $\delta > 0$.

(3.7) **Remark:** When $\ell = 1$, that is, when we consider the operator $\mathcal{T}_{\vec{b},\phi}$, we get the theorem above with $X^0 = L^q(\log L)^{q-1+\delta}$, $X^1 = L^q(\log L)^{2q-1+\delta}$, $Y^0_{i,j} = L^{p_i'}(\log L)^{p_i'-1+\delta}$ for every $j = 1, \dots, m$, $Y^1_{i,j} = L^{p_i'}(\log L)^{p_i'-1+\delta}$ for $i \neq j$ and $Y^1_{j,j} = L^{p_j'}(\log L)^{2p_j'-1+\delta}$. The case $m = 1$ of Theorem 3.1 improves the results in [L2].

As a consequence of Theorem 3.1 we obtain the following Fefferman-Stein type results.

(3.8) **Corollary:** *Let $1 < p_1, \dots, p_m < \infty$ with $1/p = \sum_{i=1}^m 1/p_i$, $\ell \in \{0, 1\}$ and let $\phi \in \mathfrak{D}$.*

i) If $p > 1$, for each $0 < \delta < 1$ there exists a constant $C_{\ell,\delta}$ such that

$$\left(\int_{\mathbb{R}^n} |\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}|^p\, \Pi_{i=1}^m u_i^{p/p_i}\right)^{1/p} \leq C_\ell \prod_{i=1}^m \left(\int_{\mathbb{R}^n} |f_i|^{p_i} M_{\Phi_1^p, L(\log L)^{p(1+\ell)-1+\delta}}(u_i)\right)^{1/p_i}.$$

ii) If $p \leq 1$ and $\ell = 0$, there exists a constant C_ℓ such that

$$\left(\int_{\mathbb{R}^n} |\mathcal{T}_\phi\vec{f}|^p\, \Pi_{i=1}^m u_i^{p/p_i}\right)^{1/p} \leq C_\ell \prod_{i=1}^m \left(\int_{\mathbb{R}^n} |f_i|^{p_i} M_{\Phi_p^p, L}(u_i)\right)^{1/p_i}.$$

iii) If $p \leq 1$ and $\ell = 1$ there exists a constant C_ℓ such that

$$\left(\int_{\mathbb{R}^n} |\mathcal{T}_{\vec{b},\phi}\vec{f}|^p\, \Pi_{i=1}^m u_i^{p/p_i}\right)^{1/p} \leq C_\ell \prod_{i=1}^m \left(\int_{\mathbb{R}^n} |f_i|^{p_i} M_{\Phi_p^p, L^{1/p}}(u_i)\right)^{1/p_i}.$$

(3.9) **Remark:** If we take $m = 1$, $\ell = 0$ and $\delta = [p] + 1 - p$ in the estimate in i) of the Corollary above we recover a result obtained in [LQY]. On the other hand, by taking $m = 1$, $\ell = 1$ and $\delta = [2p] + 1 - 2p$, the estimate in i) improves the result given in [L1] for the first order commutator.

In the following result we obtain Coifman type inequalities for the multilinear potential operators and its commutators.

(3.10) **Theorem:** *Let $\phi \in \mathfrak{D}$, $\ell \in \{0, 1\}$ and $w \in A_\infty$. Then, for every $\vec{f} = (f_1, \dots, f_m)$ with f_i bounded with compact support, we get the following inequalities.*

i) If $0 < p \leq 1$ and $\ell = 0$, there exists a constant C_ℓ such that

$$\int_{\mathbb{R}^n} |\mathcal{T}_\phi\vec{f}(x)|^p\, w(x)\, dx \leq C_\ell \int_{\mathbb{R}^n} [\mathcal{M}_{\Phi_p, L}\vec{f}(x)]^p\, w(x)\, dx.$$

ii) If $0 < p \leq 1$, $\ell = 1$ and $w \in RH(1/p)$, there exists a constant C_ℓ such that

$$\int_{\mathbb{R}^n} |\mathcal{T}_{\vec{b},\phi}\vec{f}(x)|^p\, w(x)\, dx \leq C_\ell \int_{\mathbb{R}^n} [\mathcal{M}_{\Phi_p, L(\log L)}\vec{f}(x)]^p\, w(x)\, dx.$$

iii) If $p > 1$, there exists a constant C_ℓ such that

$$\int_{\mathbb{R}^n} |\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}(x)|^p\, w(x)\, dx \leq C_\ell \int_{\mathbb{R}^n} [\mathcal{M}_{\Phi_1, L(\log L)^\ell}\vec{f}(x)]^p\, w(x)\, dx.$$

For $\ell = 0$, the result above was proved in [M] for the case of the multilinear fractional integral. On the other hand, when $m = 1$ an analogous result was obtained in [LYY] on spaces of homogeneous type.

The following theorem contains other weighted inequalities for the operators $\mathcal{T}_{\vec{b}^\ell,\phi}$ that we shall use in the next section to prove weak type inequalities.

(3.11) **Theorem:** *Let $\phi \in \mathfrak{D}$, $\ell \in \{0, 1\}$, and u a weight.*

i) If $0 < p \leq 1$, there exists a constant C_ℓ such that

$$\int_{\mathbb{R}^n} |\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}(x)|^p\, u(x)\, dx \leq C_\ell \int_{\mathbb{R}^n} [\mathcal{M}_{\Phi_1, L(\log L)^\ell}\vec{f}(x)]^p\, M_{L(\log L)^\ell}u(x)\, dx.$$

ii) If $p > 1$, there exists a constant C_ℓ such that

$$\int_{\mathbb{R}^n} |\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}(x)|^p\, u(x)\, dx \leq C_\ell \int_{\mathbb{R}^n} [\mathcal{M}_{\Phi_1, L(\log L)^\ell}\vec{f}(x)]^p M_{L(\log L)^{[\ell p + p]}}u(x)\, dx.$$

(3.12) **Remark:** For $m = 1$ and $p > 1$ the result above was proved in [Pe1] for the fractional integral operator I_α.

3.2 Weak type results

The following theorem is a weighted endpoint estimate for the multilinear maximal operator $\mathcal{M}_{\varphi,\mathcal{B}}$.

(3.13) **Theorem:** *Let φ be a essencially nondecreasing function, $\mathcal{B}$ a submultiplicative Young function, $\mathcal{B}^m = \overbrace{\mathcal{B} \circ \cdots \circ \mathcal{B}}^{m}$, $\psi = \mathcal{B}^m \circ \varphi^{1/m}$ and $u = \prod_{i=1}^m u_i^{1/m}$. Then*

$$\sup_{\lambda > 0} \frac{1}{\mathcal{B}^m(1/\lambda)} u\left(\left\{x \in \mathbb{R}^n : \mathcal{M}_{\varphi,\mathcal{B}}\vec{f}(x) > \lambda^m\right\}\right)^m \leq C \prod_{i=1}^m \int_{\mathbb{R}^n} \mathcal{B}^m(|f_i|) M_{\psi,L} u_i.$$

Particularly, if $0 \leq \alpha < nm$ and $\varphi(t) = t^\alpha$, the result above was proved in [P]. On the other hand, when $\varphi(t) = 1$ and $\mathcal{B}(t) = t(1 + \log^+ t)$ an analogous result was proved in [PPTT]. Similar estimates for several maximal operators can be found in [LOPTT] in the multilineal context, and, for example, in [FS], [Pe3] in the case $m = 1$.

The next "control type" result follows by applying similar techniques to those in [P] for the case of multilinear fractional operators.

(3.14) **Theorem:** *Let $\phi \in \mathfrak{D}$, $\ell \in \{0,1\}$, δ_0, $\delta_1 > 0$ and let u be a weight. Then there exists a constant C_ℓ such that*

$$\|\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}\|_{L^{1/m,\infty}(u)} \le C_\ell \|\mathcal{M}_{\Phi_1,L(\log L)^\ell}\vec{f}\|_{L^{1/m,\infty}(M_{L(\log L)^{\ell+\delta_\ell}}u)}.$$

As an easy consequence of Theorem 3.14, Theorem 3.13 applied to the case $\mathcal{B}(t) = t$, and the inequality $M_{L(\log L)^\delta}(u) \le \prod_{i=1}^m [M_{L(\log L)^\delta}(u_i)]^{1/m}$, we obtain the following result for the multilinear potential operator.

(3.15) **Corollary:** *Let $\phi \in \mathfrak{D}$, $\delta > 0$ and let $u = \prod_{i=1}^m u_i^{1/m}$. Then*

$$\|\mathcal{T}_\phi\vec{f}\|_{L^{1/m,\infty}(u)} \le C\prod_{i=1}^m \int_{\mathbb{R}^n} |f_i| M_{\Phi_1^{1/m},L}(M_{L(\log L)^\delta}(u_i)).$$

The case $m = 1$ of theorem above was proved in [CPSS] for $\phi(t) = t^{\alpha-n}$ and in [P] for $m > 1$ and $\phi(t) = t^{\alpha-nm}$.

4 Proofs of strong type results

We begin this section proving the following lemma based in the discretization method developed in [Pe] (see [M] for the case of the multilinear fractional integral).

(4.1) **Lemma:** *Let f_i, $i = 1,\dots,m$, be positive functions with compact support, let u be a weight and $\phi \in \mathfrak{D}$. Let $\ell \in \{0,1\}$, $0 < q \le 1$ and, for $j \in \{1,\dots,m\}$ let b_j be a BMO function. Then there exist a constant C_ℓ, two family of dyadic cubes $\{Q^0_{k,\eta}\}_{k,\eta\in\mathbb{Z}}$ and $\{Q^j_{k,\eta}\}_{k,\eta\in\mathbb{Z}}$ and two family of pairwise disjoint subsets $\{E^0_{k,\eta}\}$ and $\{E^j_{k,\eta}\}$, $E^0_{k,\eta} \subset Q^0_{k,\eta}$ and $E^j_{k,\eta} \subset Q^j_{k,\eta}$ with*

$$|Q^0_{k,\eta}| \le C|E^0_{k,\eta}| \quad \text{and} \quad |Q^j_{k,\eta}| \le C|E^j_{k,\eta}|$$

for some positive constant C and for every k, η, such that

$$\begin{aligned}\int_{\mathbb{R}^n} [|\mathcal{T}_{b_j^\ell,\phi}(\vec{f})|u]^q \quad &\le \quad C_\ell \sum_{k,\eta} \Phi_q(l(Q^0_{k,\eta}))^q ||u^q||_{L(\log L)^{\ell q},3Q^0_{k,\eta}} \prod_{i=1}^m ||f_i||^q_{L,3Q^0_{k,\eta}} |E^0_{k,\eta}| \\ &\quad + \ell C_\ell \sum_{k,\eta} \Phi_q(l(Q^j_{k,\eta}))^q ||u||^q_{L,3Q^j_{k,\eta}} \prod_{i=1}^m ||f_i||^q_{L(\log L)^{\delta_{i,j}},3Q^j_{k,\eta}} |E^j_{k,\eta}|.\end{aligned}$$

Proof. Proceeding as in the proof of Proposition 3.4 in [CUMP])we can assume that the weight u has compact support. For each $t > 0$ we set $\bar{\phi}(t) = \sup_{\vec{y}\in\mathcal{A}_{(t,1,0)}} \phi(\vec{y})$, where

$\mathcal{A}_{(t,\delta,\epsilon)}$ is the set defined in (1.5). Then, if $\hat{x}-\vec{y}=(x-y_1,\dots,x-y_m)$, we get

$$
\begin{aligned}
|\mathcal{T}_{b_j^\ell,\phi}(\vec{f})(x)| &\leq \sum_{\nu\in\mathbb{Z}} \int_{\hat{x}-\vec{y}\in\mathcal{A}_{(2^{\nu-1},1,0)}} |b_j(x)-b_j(y_j)|^\ell \phi(\hat{x}-\vec{y}) \prod_{i=1}^m f_i(y_i)\,d\vec{y} \\
&\leq \sum_{\nu\in\mathbb{Z}} \bar{\phi}(2^{\nu-1}) \int_{\sum_{i=1}^m |x-y_i|<2^\nu} |b_j(x)-b_j(y_j)|^\ell \prod_{i=1}^m f_i(y_i)\,d\vec{y} \\
&\leq \sum_{\nu\in\mathbb{Z}} \sum_{\{Q:l(Q)=2^\nu\}} \bar{\phi}\left(\frac{l(Q)}{2}\right) \chi_Q(x) \int_{(3Q)^m} |b_j(x)-b_j(y_j)|^\ell \prod_{i=1}^m f_i(y_i)\,d\vec{y}
\end{aligned}
$$

(4.2)

$$
\begin{aligned}
&\leq \sum_{Q\in\mathcal{D}} \bar{\phi}\left(\frac{l(Q)}{2}\right) |b_j(x)-(b_j)_Q|^\ell \chi_Q(x) \prod_{i=1}^m \int_{3Q} f_i(y_i)\,dy_i \\
&+ \ell \sum_{Q\in\mathcal{D}} \bar{\phi}\left(\frac{l(Q)}{2}\right) \chi_Q(x) \prod_{i=1,\ i\neq j}^m \left(\int_{3Q} f_i(y_i)\,dy_i\right) \\
&\quad \times \left(\int_{3Q} |b_j(y_j)-(b_j)_Q|^\ell f_j(y_j)\,dy_j\right).
\end{aligned}
$$

From the above estimate, using the generalized Hölder inequality and the estimate $||(b-b_Q)^q||_{expL^{1/q},Q} \leq C||b||_*^q$ that holds for any BMO function b, we get

$$
\begin{aligned}
\int_{\mathbb{R}^n} \left[|\mathcal{T}_{b_j^\ell,\phi}(\vec{f})|u\right]^q &\leq \sum_{Q\in\mathcal{D}} \bar{\phi}\left(\frac{l(Q)}{2}\right)^q \left(\int_Q |b_j-(b_j)_Q|^{\ell q} u^q\right) |3Q|^{mq} \prod_{i=1}^m ||f_i||_{L,3Q}^q \\
&+ \ell \sum_{Q\in\mathcal{D}} \bar{\phi}\left(\frac{l(Q)}{2}\right)^q \left(\int_Q u^q\right) \prod_{i\neq j} \left(\int_{3Q} f_i\right)^q \left(\int_{3Q} |b_j-(b_j)_Q|^\ell f_j\right)^q \\
&\leq C||b_j||_*^{\ell q} \sum_{Q\in\mathcal{D}} \bar{\phi}\left(\frac{l(Q)}{2}\right)^q ||u^q||_{L(\log L)^{\ell q},Q} |3Q|^{mq+1} \prod_{i=1}^m ||f_i||_{L,3Q}^q \\
&+ \ell C||b_j||_*^{\ell q} \sum_{Q\in\mathcal{D}} \bar{\phi}\left(\frac{l(Q)}{2}\right)^q ||u^q||_{L,Q} |3Q|^{mq+1} \prod_{i=1}^m ||f_i||_{L(\log L)^{\delta_{i,j}},3Q}^q \\
&= C||b_j||_*^{\ell q} \sum_{Q\in\mathcal{D}} \bar{\phi}\left(\frac{l(Q)}{2}\right)^q ||u^q||_{L(\log L)^{\ell q},Q} |3Q|^{mq+1} \prod_{i=1}^m ||f_i||_{L,3Q}^q \\
&+ \ell C||b_j||_*^{\ell q} \sum_{Q\in\mathcal{D}} \bar{\phi}\left(\frac{l(Q)}{2}\right)^q ||u||_{L,Q}^q |3Q|^{mq+1} \prod_{i=1}^m ||f_i||_{L(\log L)^{\delta_{i,j}},3Q}^q.
\end{aligned}
$$

Notice that, since $q\leq 1$ in the last line of the inequalities above we have used that $||u^q||_{L,Q} \leq ||u||_{L,Q}^q$.

From the results in [M] we get the following Calderón-Zygmund decomposition. For $\vec{h}=(h_1,\dots,h_m)$ let

$$
\mathcal{M}_{3\mathcal{D}}\vec{h}(x) = \sup_{x\in Q\in\mathcal{D}} \prod_{i=1}^m ||h_i||_{L,3Q}
$$

and

$$\mathcal{D}_k = \{x \in \mathbb{R}^n : \ \mathcal{M}_{3\mathcal{D}}\vec{h}(x) > a^k\}$$

with $a > 6^n||\mathcal{M}_L||$, where $||\mathcal{M}_L||$ is the constant from the $L^1 \times \cdots \times L^1 \to L^{1/m,\infty}$ inequality for $\mathcal{M}_L$. It was proved in [M] that there exists a family $Q_{k,\eta}$ of maximal, disjoint and dyadic cubes such that $\mathcal{D}_k = \bigcup_{\eta\in\mathbb{Z}} Q_{k,\eta}$, and

$$a^k < \prod_{i=1}^m ||h_i||_{L,3Q_{k,\eta}} \le 2^{nm}a^k.$$

Moreover, if $E_{k,\eta} = Q_{k,\eta} \setminus \mathcal{D}_{k+1}$ then $\{E_{k,\eta}\}_{k,\eta}$ is a disjoint family of sets that satisfies $|Q_{k,\eta}| \le C|E_{k,\eta}|$, for some positive constant C.

Observe that the cubes $Q_{k,\eta}$ and the sets $E_{k,\eta}$ depend on the function $\vec{h}$. We shall apply the above results to two different functions. In one case we shall use $\vec{h} = \vec{f}$ and in this case we shall denote with $Q^0_{k,\eta}$ and $E^0_{k,\eta}$ the corresponding cubes and sets. In the other case we shall apply the above decomposition with $\vec{h} = (h^j_1, \dots, h^j_m)$, where $h^j_i = f_i$ if $i \ne j$ and $h^j_j = u$. In this second case we shall denote with $Q^j_{k,\eta}$ and $E^j_{k,\eta}$ the corresponding cubes and sets. On the other hand, let

$$\mathcal{F}^j_k = \{Q \in \mathcal{D} : a^k < \prod_{i=1}^m ||h^j_i||_{L,3Q} \le a^{k+1}\}$$

and let $\mathcal{F}^0_k$ be the above set with f_i instead of h^j_i. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} \left[|\mathcal{T}_{b^\ell_j,\phi}(\vec{f})|u\right]^q \ \le\ & C_\ell \sum_{k\in\mathbb{Z}} a^{(k+1)q} \sum_{Q\in\mathcal{F}^0_k} \bar{\phi}\left(\frac{l(Q)}{2}\right)^q |3Q|^{mq+1} ||u^q||_{L(\log L)^{\ell q},Q} \\
& + \ell C_\ell \sum_{k\in\mathbb{Z}} a^{(k+1)q} \sum_{Q\in\mathcal{F}^j_k} \bar{\phi}\left(\frac{l(Q)}{2}\right)^q |3Q|^{mq+1} ||f_j||^q_{L(\log L),3Q} \\
\le\ & C_\ell \sum_{k\in\mathbb{Z}} a^{(k+1)q} \sum_\eta \sum_{Q\in\mathcal{F}^0_k : Q\subset Q^0_{k,\eta}} \bar{\phi}\left(\frac{l(Q)}{2}\right)^q |3Q|^{mq+1} ||u^q||_{L(\log L)^{\ell q},3Q} \\
& + \ell C_\ell \sum_{k\in\mathbb{Z}} a^{(k+1)q} \sum_\eta \sum_{Q\in\mathcal{F}^j_k : Q\subset Q^j_{k,\eta}} \bar{\phi}\left(\frac{l(Q)}{2}\right)^q |3Q|^{mq+1} ||f^q_j||_{L^{1/q}(\log L),3Q}
\end{aligned}$$

It is easy to prove (see for example [L2]) that for any dyadic cube Q_0 with $l(Q_0) = 2^{-\nu_0}$ and any Young function ψ, there exists a positive constant C such that

$$\sum_{\{Q\subset Q_0, l(Q)=2^{-\nu}\}} |3Q| \ ||f||_{L^\psi,3Q} \le C|3Q_0| \ ||f||_{L^\psi,3Q_0}.$$

On the other hand, since $\phi \in \mathfrak{D}$,

$$\begin{aligned}
\sum_{\nu\ge\nu_0} [2^{-\nu nm}\bar{\phi}(2^{-\nu-1})]^q \ &\le\ C \sum_{\nu\ge\nu_0} \left(\int_{\mathcal{A}_{(2^{-\nu-1},\delta,\epsilon)}} \phi(\vec{y})\, d\vec{y}\right)^q \\
&=\ C\Phi_q(l(Q_0))^q.
\end{aligned}$$

From the results above we get that there exists $C>0$ such that

$$\begin{aligned}\sum_{\{Q:Q\subset Q_0\}} \bar{\phi}\left(\frac{l(Q)}{2}\right)^q |3Q|^{mq+1}||f||_{L^\psi,3Q} &= 3^{nmq}\sum_{\nu\geq\nu_0}[2^{-\nu nm}\bar{\phi}(2^{-\nu-1})]^q \\ &\quad\times \sum_{\{Q\subset Q_0, l(Q)=2^{-\nu}\}} |3Q|||f||_{L^\psi,3Q} \\ &\leq C\Phi_q(l(Q_0))^q|3Q_0|||f||_{L^\psi,3Q_0}\end{aligned}$$

where C depends on δ, ϵ in condition $\mathfrak{D}$. Using the inequality above with $\psi(t)=t(1+\log^+t)^{\ell q}$ and $\psi(t)=t^{1/q}(1+\log^+t)$ we get

$$\begin{aligned}\int_{\mathbb{R}^n}\left[|\mathcal{T}_{b_j^\ell,\phi}(\vec{f})|u\right]^q &\leq C_\ell\sum_{k,\eta}\Phi_q(l(Q_{k,\eta}^0))^q||u^q||_{L(\log L)^{\ell q},3Q_{k,\eta}^0}\prod_{i=1}^m||f_i||^q_{L,3Q^0_{k,\eta}}|E^0_{k,\eta}| \\ &\quad+\ell C_\ell\sum_{k,\eta}\Phi_q(l(Q^j_{k,\eta}))^q||u||^q_{L,3Q^j_{k,\eta}}\prod_{i=1}^m||f_i||^q_{L(\log L)^{\delta_{i,j}},3Q^j_{k,\eta}}|E^j_{k,\eta}|.\end{aligned}$$

□

Proof of theorem 3.1: Let us first consider the case $q>1$. It is enough to prove that, for every $j=1,\ldots,m$,

$$\int_{\mathbb{R}^n}|\mathcal{T}_{b_j^\ell,\phi}(\vec{f})(x)|u(x)g(x)\,dx\leq C_\ell\|g\|_{q'}\prod_{i=1}^m\|f_iv_i\|_{p_i},$$

for all $g\in L^{q'}(\mathbb{R}^n)$, with $g\geq 0$, and for all positive function f_i bounded with compact support. From Lemma 4.1 with $q=1$, the generalized Hölder's inequality and condition (3.3) we have that

$$\begin{aligned}\int_{\mathbb{R}^n}|\mathcal{T}_{b_j^\ell,\phi}\vec{f}|ug &\leq C_\ell\sum_{k,\eta}\Phi_1(3l(Q^0_{k,\eta}))||ug||_{L(\log L)^\ell,Q^0_{k,\eta}}\prod_{i=1}^m||f_i||_{L,3Q^0_{k,\eta}}|E^0_{k,\eta}| \\ &\quad+\ell C_\ell\sum_{k,\eta}\Phi_1(3l(Q^j_{k,\eta}))||ug||_{L,3Q^j_{k,\eta}}\prod_{i=1}^m||f_i||_{L(\log L)^{\delta_{i,j}},3Q^j_{k,\eta}}|E^j_{k,\eta}| \\ &\leq C_\ell\sum_{k,\eta}||g||_{\tilde{X}^\ell,3Q^0_{k,\eta}}\prod_{i=1}^m||f_iv_i||_{\tilde{Y}^0_{i,j},3Q^0_{k,\eta}}|E^0_{k,\eta}|^{1/p+1/q'} \\ &\quad+\ell C_\ell\sum_{k,\eta}||g||_{\tilde{X}^0,3Q^j_{k,\eta}}\prod_{i=1}^m||f_iv_i||_{\tilde{Y}^1_{i,j},3Q^j_{k,\eta}}|E^j_{k,\eta}|^{1/p+1/q'}\end{aligned}$$

By Hölder's inequality and the fact that $p \le q$ we obtain that

$$
\begin{aligned}
\int_{\mathbb{R}^n} |\mathcal{T}_{b_j^\ell\phi}\vec{f}|ug \;&\le\; C_\ell \left(\sum_{k,\eta} ||g||^{q'}_{\tilde{X}^\ell,3Q^0_{k,\eta}} |E^0_{k,\eta}|\right)^{1/q'} \left(\sum_{k,\ell} |E^0_{k,\eta}|^{q/p} \prod_{i=1}^m ||f_i v_i||^q_{\tilde{Y}^0_{i,j},3Q^0_{k,\eta}}\right)^{1/q} \\
&\quad + \ell C_\ell \left(\sum_{k,\ell} ||g||^{q'}_{\tilde{X}^0,3Q^j_{k,\eta}} |E^j_{k,\eta}|\right)^{1/q'} \left(\sum_{k,\ell} |E^j_{k,\eta}|^{q/p} \prod_{i=1}^m ||f_i v_i||^q_{\tilde{Y}^1_{i,j},3Q^j_{k,\eta}}\right)^{1/q} \\
&\le\; C_\ell \left(\int_{\mathbb{R}^n} [M_{\tilde{X}^\ell}(g)]^{q'}\right)^{1/q'} \left(\sum_{k,\eta} |E^0_{k,\eta}| \prod_{i=1}^m ||f_i v_i||^p_{\tilde{Y}^0_{i,j},3Q^0_{k,\eta}}\right)^{1/p} \\
&\quad + \ell C_\ell \left(\int_{\mathbb{R}^n} [M_{\tilde{X}^0}(g)]^{q'}\right)^{1/q'} \left(\sum_{k,\eta} |E^j_{k,\eta}| \prod_{i=1}^m ||f_i v_i||^p_{\tilde{Y}^1_{i,j},3Q^j_{k,\eta}}\right)^{1/p} \\
&\le\; C_\ell \left(\int_{\mathbb{R}^n} M_{\tilde{X}^\ell}(g)^{q'}\right)^{1/q'} \left(\int_{\mathbb{R}^n} \left[\mathcal{M}_{\vec{\tilde{Y}}^0_j}(\vec{fv})\right]^p\right)^{1/p} \\
&\quad + \ell C_\ell \left(\int_{\mathbb{R}^n} M_{\tilde{X}^0}(g)^{q'}\right)^{1/q'} \left(\int_{\mathbb{R}^n} \left[\mathcal{M}_{\vec{\tilde{Y}}^1_j}(\vec{fv})\right]^p\right)^{1/p},
\end{aligned}
$$

where $\vec{fv} = (f_1v_1, \ldots, f_mv_m)$. Then by the hypotheses on the boundedness properties of $\mathcal{M}_{\vec{\tilde{Y}}^k_j}$ and $M_{\tilde{X}^k}$, $k \in \{0,1\}$, $k \le \ell$, we conclude the proof of the case $q > 1$.

Suppose now that $1/m < p \le q \le 1$. From Lemma 4.1, by the generalized Hölder's inequality, condition (3.4) and the fact that $p \le q$ we obtain that

$$
\begin{aligned}
\left(\int_{\mathbb{R}^n} \left[|\mathcal{T}_{b_j^\ell,\phi}(\vec{f})(x)|\, u(x)\right]^q dx\right)^{1/q} \;&\le\; C_\ell \left(\sum_{k,\eta} \prod_{i=1}^m ||f_i v_i||^q_{\tilde{Y}^0_{i,j},3Q^{j,0}_{k,\eta}} |E^{j,0}_{k,\eta}|^{q/p}\right)^{1/q} \\
&\quad + \ell C_\ell \left(\sum_{k,\eta} \prod_{i=1}^m ||f_i v_i||^q_{\tilde{Y}^1_{i,j},3Q^j_{k,\eta}} |E^j_{k,\eta}|^{q/p}\right)^{1/q} \\
&\le\; C_\ell \left(\int_{\mathbb{R}^n} [\mathcal{M}_{\vec{\tilde{Y}}^0_j}(\overrightarrow{fv})]^p\right)^{1/p} \\
&\quad + \ell C_\ell \left(\int_{\mathbb{R}^n} [\mathcal{M}_{\vec{\tilde{Y}}^1_j}(\overrightarrow{fv})]^p\right)^{1/p},
\end{aligned}
$$

and from the hypotheses on the boundedness of the multilinear maximal operator $\mathcal{M}_{\vec{\tilde{Y}}^k_j}$, $k \in \{0,1\}$, $k \le \ell$, we finally get the theorem. □

Proof of Corollary 3.8: For the case $p > 1$ and $\ell = 0$, the proof follows from Theorem 3.1 and Remark 3.6 by taking $p = q$, $X^0 = L^p(\log L)^{p-1+\delta}$ for some $0 < \delta < 1$, $Y^0_{i,j}$ any Orlicz space such that the maximal operator associated to the space $\tilde{Y}^0_{i,j}$ is bounded from L^{p_i} into L^{p_i}, $i = 1, \ldots, m$, $u = \Pi_{i=1}^m u_i^{1/p_i}$ and $v_i = [M_{\Phi_1^p, L(\log L)^{p-1+\delta}}(u_i)]^{1/p_i}$. In fact, we

use the generalized Hölder's inequality with $\mathcal{C}(t)=t^p(1+\log^+t)^{p-1+\delta}$ and $\mathcal{C}_i(t)=t^{p_i}(1+\log^+t)^{p-1+\delta}$, $1\le i\le m$ to estimate $||u||_{X^0}$ and the inequality $M_{\Phi_1^p,L(\log L)^{p-1+\delta}}(u_i)(x)\ge \Phi_1(l(Q))^p||u_i||_{L(\log L)^{p-1+\delta},Q}$, for all $x\in Q$, to estimate $||v_i^{-1}||_{Y^0_{i,j}}$. In a similar way we can prove the result for $p\le 1$ and $\ell=0$.
For the case $p>1$ and $\ell=1$, as in the previous case, the proof follows from Theorem 3.1 and Remark 3.7 by taking $p=q$, $Y^0_{i,j}$ and $Y^1_{i,j}$ be any Orlicz spaces such that the maximal operators associated to the spaces $\tilde{Y}^0_{i,j}$ and $\tilde{Y}^1_{i,j}$ are bounded from L^{p_i} into L^{p_i}, $i=1,\dots,m$, $X^0=L^p(\log L)^{p-1+\delta}$, $X^1=L^p(\log L)^{2p-1+\delta}$ and the weights $u=\Pi_{i=1}^m u_i^{1/p_i}$ and $v_i=[M_{\Phi_1^p,L(\log L)^{2p-1+\delta}}(u_i)]^{1/p_i}$. For the case $p\le 1$ and $\ell=1$ take $u=\Pi_{i=1}^m u_i^{1/p_i}$ and $v_i=[M_{\Phi_p^p,L^{1/p}(\log L)^p}(u_i)]^{1/p_i}$.

Proof of theorem 3.10 : We start proving the case $p\le 1$. By taking $q=p$ and $u=w^{1/p}$ in Lemma 4.1 we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}|\mathcal{T}_{b_j^\ell,\phi}(\vec f)|^p\,w &\le C_\ell\sum_{k,\eta}\Phi_p(l(Q^0_{k,\eta}))^p||w||_{L(\log L)^{\ell p},3Q^0_{k,\eta}}\prod_{i=1}^m||f_i||^p_{L,3Q^0_{k,\eta}}|E^0_{k,\eta}|\\ &\quad+\ell C_\ell\sum_{k,\eta}\Phi_p(l(Q^j_{k,\eta}))^p||w^{1/p}||^p_{L,3Q^j_{k,\eta}}\prod_{i=1}^m||f_i||^p_{L(\log L)^{\delta_{i,j}},3Q^j_{k,\eta}}|E^j_{k,\eta}|.\end{aligned}$$

Since $w\in A_\infty$ then w satisfies the reverse Hölder's condition $RH(s)$ for some $s>1$. Then, from the fact that $\|.\|_{L(\log L)^{\ell p},Q}\le\|.\|_{L^s,Q}$ for any $s>1$, the condition $RH(s)$ and the condition $RH(1/p)$ for the case $\ell=1$, we get that

$$\begin{aligned}\int_{\mathbb{R}^n}|\mathcal{T}_{b_j^\ell,\phi}(\vec f)|^p\,w &\le C_\ell\sum_{k,\eta}\Phi_p(l(Q^0_{k,\eta}))^p||w||_{L,3Q^0_{k,\eta}}\prod_{i=1}^m||f_i||^p_{L,3Q^0_{k,\eta}}|E^0_{k,\eta}|\\ &\quad+\ell C_\ell\sum_{k,\eta}\Phi_p(l(Q^j_{k,\eta}))^p||w||_{L,3Q^j_{k,\eta}}\prod_{i=1}^m||f_i||^p_{L(\log L)^{\delta_{i,j}},3Q^j_{k,\eta}}|E^j_{k,\eta}|.\end{aligned}$$

Since $w\in A_\infty$ and $|Q_{k,\eta}|\le C|E_{k,\eta}|$ we obtain that $w(Q_{k,\eta})\le Cw(E_{k,\eta})$, so that

$$\begin{aligned}\int_{\mathbb{R}^n}|\mathcal{T}_{b_j^\ell,\phi}(\vec f)|^p\,w &\le C_\ell\sum_{k,\eta}\Phi_p(l(Q^0_{k,\eta}))^p\prod_{i=1}^m||f_i||^p_{L,3Q^0_{k,\eta}}w(E^0_{k,\eta})\\ &\quad+\ell C_\ell\sum_{k,\eta}\Phi_p(l(Q^j_{k,\eta}))^p\prod_{i=1}^m||f_i||^p_{L\log L,3Q^j_{k,\eta}}w(E^j_{k,\eta})\\ &\le C_\ell\sum_{k,\eta}\int_{E^0_{k,\eta}}[\mathcal{M}_{\Phi_p}\vec f(x)]^p w(x)\,dx\\ &\quad+\ell C_\ell\sum_{k,\eta}\int_{E^j_{k,\eta}}[\mathcal{M}_{\Phi_p,L\log L}\vec f(x)w(x)]^p\,dx\\ &\le C_\ell\int_{\mathbb{R}^n}[\mathcal{M}_{\Phi_p,L(\log L)^\ell}\vec f(x)]^p w(x)\,dx.\end{aligned}$$

The case $p > 1$ follows from the case $p = 1$ by applying the extrapolation theorem in [CUMP]. $\square$

We shall use the next two results to prove Theorem 3.11. The first one is a corollary of Lemma 4.1 and the second one was proved in [CN].

(4.3) **Corollary:** *Let $\phi \in \mathfrak{D}$ and let v be a weight satisfying the RH_∞ condition. If u is a weight, then there exists a positive constant C_ℓ such that*

$$\begin{aligned}\int_{\mathbb{R}^n} |\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}(x)|\, u(x)\, v(x)\, dx \;&\leq\; C_\ell \int_{\mathbb{R}^n} \mathcal{M}_{\Phi_1,L}\vec{f}(x)\, M_{L(\log L)^\ell}u(x)\, v(x)\, dx \\ &\quad + \ell C_\ell \int_{\mathbb{R}^n} \mathcal{M}_{\Phi_1,L\log L}\vec{f}(x)\, Mu(x)\, v(x)\, dx.\end{aligned}$$

Proof: Since $v \in RH_\infty$ then $v \in A_\infty$. Thus, if $|Q| \leq C|E|$ then $v(Q) \leq Cv(E)$. From Lemma 4.1 with $q = 1$ we have that

$$\begin{aligned}\int_{\mathbb{R}^n} |\mathcal{T}_{b_j^\ell,\phi}(\vec{f})|\; u\; v \;&\leq\; C_\ell \sum_{k,\eta} \Phi_1(l(Q^0_{k,\eta}))||u||_{L(\log L)^\ell,3Q^0_{k,\eta}} \prod_{i=1}^m ||f_i||_{L,3Q^0_{k,\eta}} \sup_{3Q^0_{k,\eta}} v\; |E^0_{k,\eta}| \\ &\quad + \ell C_\ell \sum_{k,\eta} \Phi_1(l(Q^j_{k,\eta}))||u||_{L,3Q^j_{k,\eta}} \prod_{i=1}^m ||f_i||_{L(\log L)^{\delta_{i,j}},3Q^j_{k,\eta}} \sup_{3Q^j_{k,\eta}} v\; |E^j_{k,\eta}| \\ &\leq\; C_\ell \sum_{k,\eta} \Phi_1(l(Q^0_{k,\eta}))||u||_{L(\log L)^\ell,3Q^0_{k,\eta}} \prod_{i=1}^m ||f_i||_{L,3Q^0_{k,\eta}} v(E^0_{k,\ell}) \\ &\quad + \ell C_\ell \sum_{k,\eta} \Phi_1(l(Q^j_{k,\eta}))||u||_{L,3Q^j_{k,\eta}} \prod_{i=1}^m ||f_i||_{L(\log L)^{\delta_{i,j}},3Q^j_{k,\eta}} v(E^j_{k,\eta}) \\ &\leq\; C_\ell \sum_{k,\eta} \int_{E^0_{k,\eta}} \mathcal{M}_{\Phi_1,L}(\vec{f})\; M_{L(\log L)^\ell}(u)\; v \\ &\quad + \ell C_\ell \sum_{k,\eta} \int_{E^j_{k,\eta}} \mathcal{M}_{\Phi_1,L(\log L)}(\vec{f})\; M(u)\; v.\end{aligned}$$

(4.4) **Lemma:** *Let g be a function such that $M(g)$ is finite a.e. and $\alpha > 0$. Then $M(g)^{-\alpha} \in RH_\infty$.*

Proof of theorem 3.11 :

i) We follow similar arguments to those in [CPSS], (see also [P] for the multilinear case). We shall use the duality in $L^p(\mu)$ spaces for $p < 1$: if $f \geq 0$

$$\|f\|_{L^p(\mu)} = \inf\{\int f u^{-1}\, d\mu : \|u^{-1}\|_{L^{p'}(\mu)} = 1\} = \int f u_0^{-1}\, d\mu,$$

for some $u_0 \geq 0$ such that $\|u_0^{-1}\|_{L^{p'}(\mu)} = 1$, with $p' = \frac{p}{p-1} < 0$. This follows from the following reverse Hölder's inequality, which is a consequence of the Hölder's inequality,

$$\int fg\,d\mu \geq \|f\|_{L^p(\mu)}\|g\|_{L^{p'}(\mu)}. \tag{4.5}$$

Given a weight u, we shall use the above results with an absolutely continuous measure μ with density $M_{L(\log L)^\ell}u$. In fact, let g be a nonnegative function such that $\|g^{-1}\|_{L^{p'}(M_{L(\log L)^\ell}u)} = 1$ and

$$\|\mathcal{M}_{\Phi_1,L(\log L)^\ell}\vec{f}\|_{L^p(M_{L(\log L)^\ell}u)} = \int \mathcal{M}_{\Phi_1,L(\log L)^\ell}\vec{f}\,\frac{M_{L(\log L)^\ell}u}{g}.$$

Let $\delta > 0$. By the Lebesgue differentiation theorem we get

$$\|\mathcal{M}_{\Phi_1,L(\log L)^\ell}\vec{f}\|_{L^p(M_{L(\log L)^\ell}u)} \geq \int \mathcal{M}_{\Phi_1,L(\log L)^\ell}\vec{f}\,\frac{M_{L(\log L)^\ell}u}{M_\delta(g)},$$

where $M_\delta(g) = M(g^\delta)^{1/\delta}$. Then applying Lemma 4.4 and Corollary 4.3 to the weight $M_\delta(g)^{-1}$ and the reverse Hölder's inequality (4.5), we obtain that

$$\begin{aligned}(C_\ell + \ell C_\ell)\|\mathcal{M}_{\Phi_1,L(\log L)^\ell}\vec{f}\|_{L^p(M_{L(\log L)^\ell}u)} &\geq C_\ell \int \mathcal{M}_{\Phi_1,L}\vec{f}\,\frac{M_{L(\log L)^\ell}u}{M_\delta(g)} \\ &\quad + \ell C_\ell \int \mathcal{M}_{\Phi_1,L(\log L)^\ell}\vec{f}\,\frac{Mu}{M_\delta(g)} \\ &\geq \int_{\mathbb{R}^n} |\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}(x)|\,\frac{u(x)}{M_\delta(g)}\,dx \\ &\geq \|\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}\|_{L^p(u)}\|M_\delta(g)^{-1}\|_{L^{p'}(u)}.\end{aligned}$$

To finish the proof we shall show that

$$\|M_\delta(g)^{-1}\|_{L^{p'}(u)} \geq \|g^{-1}\|_{L^{p'}(Mu)} = 1.$$

Since $p' < 0$, this is equivalent to prove that

$$\int_{\mathbb{R}^n} M_\delta(g)^{-p'}(x)u(x)\,dx \leq C\int g^{-p'}(x)Mu(x)\,dx.$$

By choosing δ such that $0 < \delta < \frac{p}{1-p}$, we have that $-p'/\delta > 1$ and the above inequality follows from the classical weighted norm inequality of Fefferman-Stein (see [FS]).

ii) It is enough to prove that

$$\int_{\mathbb{R}^n} |\mathcal{T}_{\vec{b}^\ell,\phi}(\vec{f})(x)|u(x)^{1/p}g(x)\,dx \leq C_\ell\left(\int_{\mathbb{R}^n}[\mathcal{M}_{\Phi_1,L(\log L)^\ell}\vec{f}(x)]^p M_{L(\log L)^{[\ell p+p]}}u(x)\,dx\right)^{1/p}\|g\|_{p'}$$

for every function $g \in L^{p'}(\mathbb{R}^n)$. From Lemma 4.1 we get that

$$\begin{aligned}\int_{\mathbb{R}^n} |\mathcal{T}_{b_j^\ell,\phi}(\vec{f})|\ u^{1/p}\ g \ &\leq\ C_\ell \sum_{k,\eta} \Phi_1(l(Q^0_{k,\eta}))||u^{1/p}g||_{L(\log L)^\ell,3Q^0_{k,\eta}} \prod_{i=1}^m ||f_i||_{L^1,3Q^0_{k,\eta}}|E^0_{k,\eta}| \\ &\quad + \ell C_\ell \sum_{k,\eta} \Phi_1(l(Q^j_{k,\eta}))||u^{1/p}g||_{L,3Q^j_{k,\eta}} \prod_{i=1}^m ||f_i||_{L(\log L)^{\delta_{i,j}},3Q^j_{k,\eta}}|E^j_{k,\eta}|.\end{aligned}$$

Let ϵ_0 and ϵ_1 be two positive numbers to be chosen later and consider the Young functions $\mathcal{A}_\ell(t) = t^p(1+\log^+ t)^{\ell p+(p-1)(1+\epsilon_\ell)}$ and $\mathcal{C}_\ell(t) = t^{p'}(1+\log^+ t)^{-(1+\epsilon_\ell)}$. Then, it is easy to check that $\mathcal{A}_\ell^{-1}(t)\mathcal{C}_\ell^{-1}(t) \leq \mathcal{B}_\ell^{-1}(t)$, where $\mathcal{B}_\ell(t) = t(1+\log^+ t)^\ell$, and $\mathcal{C}_\ell \in B_{p'}$. Thus, by the generalized Hölder's inequality

$$M_{L(\log L)^\ell}(u^{1/p}g) \leq M_{\mathcal{A}_\ell}(u^{1/p})M_{\mathcal{C}_\ell}g = \left(M_{\tilde{\mathcal{A}}_\ell}u\right)^{1/p}\ M_{\mathcal{C}_\ell}g$$

where $\tilde{\mathcal{A}}_\ell(t) = t(1+\log^+ t)^{\ell p+(p-1)(1+\epsilon_\ell)}$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} |\mathcal{T}_{b_j^\ell,\phi}(\vec{f})|u^{1/p}g \ &\leq\ C_\ell \int_{\mathbb{R}^n} \mathcal{M}_{\Phi_1,L}\vec{f}(x)M_{\mathcal{A}_\ell}(u^{1/p})(x)M_{\mathcal{C}_\ell}(g)(x)\,dx \\ &\quad + \ell C_\ell \int_{\mathbb{R}^n} \mathcal{M}_{\Phi_1,L\log L}\vec{f}(x)M_{\mathcal{A}_0}(u^{1/p})(x)M_{\mathcal{C}_0}(g)(x)\,dx \\ &\leq\ C_\ell \int_{\mathbb{R}^n} \mathcal{M}_{\Phi_1,L(\log L)^\ell}\vec{f}M_{\mathcal{A}_\ell}(u^{1/p})[M_{\mathcal{C}_0}g + M_{\mathcal{C}_1}g] \\ &\leq\ C_\ell \left(\int_{\mathbb{R}^n} [\mathcal{M}_{\Phi_1,L(\log L)^\ell}\vec{f}M_{\mathcal{A}_\ell}(u^{1/p})]^p\right)^{1/p} \|g\|_{p'} \\ &\leq\ C_\ell \left(\int_{\mathbb{R}^n} [\mathcal{M}_{\Phi_1,L(\log L)^\ell}\vec{f}]^p M_{\tilde{\mathcal{A}}_\ell}(u)\right)^{1/p} \|g\|_{p'}\end{aligned}$$

Thus, by taking $\epsilon_\ell = \frac{[\ell p+p]-\ell p-p+1}{p-1}$ we are done. □

5 Proofs of the weak type results

Proof of theorem 3.13 : Let $\Omega_\lambda = \{x \in \mathbb{R}^n : \mathcal{M}_{\varphi,\mathcal{B}}\vec{f}(x) > \lambda^m\}$. By homogeneity we may assume that $\lambda = 1$. Let K be a compact set in Ω_1. Since K is a compact set and using Vitali's covering lemma we obtain a finite family of disjoint cubes $\{Q_j\}$ for which

$$\varphi(|Q_j|)\prod_{i=1}^m \|f_i\|_{\mathcal{B},Q_j} > 1,$$

or, equivalently

$$\prod_{i=1}^m \|\varphi(|Q_j|)^{1/m} f_i\|_{\mathcal{B},Q_j} > 1$$

and $K \subset U_j 3Q_j$. The proof follows now similar arguments as in the proof of Theorem 4.1 of [PPTT]. We include it for the sake of completeness.

Let $g_i = \varphi(|Q_j|)^{1/m} f_i$ and C_h^m the family of all subset $\sigma = \{\sigma(1), ..., \sigma(h)\}$ of different elements from the index $\{1, ..., m\}$ with $1 \leq h \leq m$. Given $\sigma \in C_h^m$ and a cube Q_i, we say that $i \in B_\sigma$ if $\|g_{\sigma(k)}\|_{\mathcal{B},Q_i} > 1$ for $k = 1, ..., h$ and $\|g_{\sigma(k)}\|_{\mathcal{B},Q_i} \leq 1$ for $k = h+1, ..., m$. For $\sigma \in C_h^m$ and $i \in B_\sigma$ denote

$$\Pi_k = \prod_{j=1}^{k} \|g_{\sigma(j)}\|_{\mathcal{B},Q_i}$$

and $\Pi_0 = 1$. Then $\Pi_k > 1$ for every $1 \leq k \leq m$ and thus

$$1 < \Pi_k = \|g_{\sigma(k)}\|_{\mathcal{B},Q_i} \, \Pi_{k-1} = \|g_{\sigma(k)} \Pi_{k-1}\|_{\mathcal{B},Q_i}$$

or, equivalently

$$\frac{1}{|Q_i|} \int_{Q_i} \mathcal{B}\left(g_{\sigma(k)} \, \Pi_{k-1}\right) > 1. \tag{5.1}$$

In particular,

$$1 < \frac{1}{|Q_i|} \int_{Q_i} \mathcal{B}\left(g_{\sigma(m)} \, \Pi_{m-1}\right) \leq \frac{1}{|Q_i|} \int_{Q_i} \mathcal{B}\left(g_{\sigma(m)}\right) \mathcal{B}\left(\Pi_{m-1}\right). \tag{5.2}$$

Now, by taking into account the equivalence

$$\|g\|_{\mathcal{B},Q} \simeq \inf_{\mu>0} \left\{ \mu + \frac{\mu}{|Q|} \int_Q \mathcal{B}(|g|/\mu) \right\},$$

if $1 \leq j \leq m-h-1$, by (5.1) we get

$$\begin{aligned}
\mathcal{B}^j(\Pi_{m-j}) &= \mathcal{B}^j(\|g_{\sigma(m-j)} \Pi_{m-j-1}\|_{\mathcal{B},Q_i}) \\
&\leq C\mathcal{B}^j \left(1 + \frac{1}{|Q_i|} \int_{Q_i} \mathcal{B}\left(g_{\sigma(m-j)} \, \Pi_{m-j-1}\right)\right) \\
&\leq \frac{C}{|Q_i|} \int_{Q_i} \mathcal{B}^{j+1}\left(g_{\sigma(m-j)}\right) \mathcal{B}^{j+1}\left(\Pi_{m-j-1}\right).
\end{aligned}$$

From (5.2), by iterating the inequality above, we obtain

$$\begin{aligned}
1 &< C \frac{1}{|Q_i|} \int_{Q_i} \mathcal{B}\left(g_{\sigma(m)}\right) \frac{1}{|Q_i|} \int_{Q_i} \mathcal{B}^2\left(g_{\sigma(m-1)}\right) \mathcal{B}^2\left(\Pi_{m-2}\right) \\
&\leq C \left(\prod_{j=0}^{m-h-1} \frac{1}{|Q_i|} \int_{Q_i} \mathcal{B}^{j+1}\left(g_{\sigma(m-j)}\right) \right) \mathcal{B}^{m-h}\left(\Pi_h\right) \\
&\leq C \left(\prod_{j=0}^{m-h-1} \frac{1}{|Q_i|} \int_{Q_i} \mathcal{B}^{j+1}\left(g_{\sigma(m-j)}\right) \right) \left(\prod_{j=1}^{h} \mathcal{B}^{m-h}(\|g_{\sigma(j)}\|_{\mathcal{B},Q_i}) \right).
\end{aligned}$$

since $\mathcal{B}$ is submultiplicative. Thus, since $i \in B_\sigma$, we have $\|g_{\sigma(j)}\|_{\mathcal{B},Q_i} > 1$ for $j = 1, ...h$, and it follows that

$$1 < C \left(\prod_{j=0}^{m-h-1} \frac{1}{|Q_i|} \int_{Q_i} \mathcal{B}^{j+1} \left(g_{\sigma(m-j)} \right) \right) \left(\prod_{j=1}^{h} \frac{1}{|Q_i|} \int_{Q_i} \mathcal{B}^{m-h+1}(g_{\sigma(j)}) \right). \tag{5.3}$$

Now, since for $1 \le h \le m$ and $0 \le j \le m-h-1$ we have that $\mathcal{B}^{j+1}(t) \le \mathcal{B}^{m-h}(t) \le \mathcal{B}^m(t)$ and $\mathcal{B}^{m-h+1}(t) \le \mathcal{B}^m(t)$. Then

$$1 < C \prod_{j=1}^{m} \frac{1}{|Q_i|} \int_{Q_i} \mathcal{B}^m(g_j)$$

or equivalently

$$|Q_i| < C \prod_{j=1}^{m} \left(\int_{Q_i} \mathcal{B}^m(g_j) \right)^{1/m}.$$

Thus, we obtain that

$$|Q_j| \le C \prod_{i=1}^{m} \left(\int_{Q_j} \mathcal{B}^m \left(\varphi(|Q_j|)^{1/m} |f_i| \right) \right)^{1/m} \le C \prod_{i=1}^{m} \left(\mathcal{B}^m \left(\varphi(|Q_j|)^{1/m} \right) \int_{Q_j} \mathcal{B}^m \left(|f_i| \right) \right)^{1/m}$$

since B is submultiplicative.

By applying Hölder's inequality we have that $\frac{u(Q)}{|Q|} \le \prod_{i=1}^m \left(\frac{u_i(Q)}{|Q|} \right)^{1/m}$. Then from the above inequalities we obtain that

$$\begin{aligned}
u(K) &\le C \sum_j \frac{u(3Q_j)}{|3Q_j|} |Q_j| \\
&\le C \sum_j \prod_{i=1}^{m} \left(\frac{1}{|3Q_j|} \int_{3Q_j} u_i \right)^{1/m} \left(\mathcal{B}^m \left(\varphi(|Q_j|)^{1/m} \right) \int_{Q_j} \mathcal{B}^m \left(|f_i| \right) \right)^{1/m} \\
&\le C \prod_{i=1}^{m} \left(\sum_j \mathcal{B}^m \left(\varphi(|Q_j|)^{1/m} \right) \frac{1}{|3Q_j|} \int_{3Q_j} u_i \int_{Q_j} \mathcal{B}^m \left(|f_i| \right) \right)^{1/m} \\
&\le C \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} \mathcal{B}^m \left(|f_i| \right) M_{\psi,L} u_i \right)^{1/m}
\end{aligned}$$

where $\psi = \mathcal{B}^m \circ \varphi^{1/m}$ and the proof concludes. □

In order to prove Theorem 3.14 we need the following lemma.

(5.4) **Lemma:** *Let $q > 1$, $\epsilon > 0$, $\mathcal{B}_\ell(t) = t(1+\log^+ t)^\ell$, $\tilde{\mathcal{A}}_\ell(t) = t(1+\log^+ t)^{\ell q + q - 1 + \epsilon}$ and let w, u be weights. Then there exists a positive constant C such that the inequality*

$$\int_{\mathbb{R}^n} M_{\mathcal{B}_\ell}(f)^{q'} M_{\tilde{\mathcal{A}}_\ell}(u)^{1-q'} w \le \int_{\mathbb{R}^n} |f|^{q'} u^{1-q'} M w \tag{5.5}$$

holds.

Proof: Let $\mathcal{A}_\ell(t)=t^q(1+\log^+ t)^{\ell q+q-1+\epsilon}$ and $\mathcal{C}(t)=t^{q'}(1+\log^+ t)^{-(1+\frac{\epsilon}{q-1})}$. Then, it is easy to check that $\mathcal{A}_\ell^{-1}(t)\mathcal{C}^{-1}(t)\le \mathcal{B}_\ell^{-1}(\mathrm{t})$. Thus, by Hölder's inequality

$$M_{\mathcal{B}_\ell}(gu^{1/q})\le M_{\mathcal{C}}(g)M_{\mathcal{A}_\ell}(u^{1/q})=M_{\mathcal{C}}(g)M_{\tilde{A}_\ell}(u)^{1/q}.$$

Then, since $\mathcal{C}\in B_{q'}$

$$\int_{\mathbb{R}^n} M_{\mathcal{B}_\ell}(gu^{1/q})^{q'}M_{\tilde{A}_\ell}(u)^{1-q'}w \;\le\; C\int_{\mathbb{R}^n} M_{\mathcal{C}}(g)^{q'}w\le C\int_{\mathbb{R}^n}|g|^{q'}Mw,$$

where the last inequality was proved in [Pe2]. Finally, by taking $g=fu^{-1/q}$ we obtain (5.5). □

Proof of theorem 3.14: Let $p>1$ to be chosen later. Thus, since $L^{p,\infty}$ and $L^{p',1}$ are associated spaces, we have that

$$\|\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}\|^{1/(pm)}_{L^{1/m,\infty}(u)}=\||\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}|^{1/(pm)}\|_{L^{p,\infty}(u)}=\sup_{\|g\|_{L^{p',1}(u)}\le 1}\int_{\mathbb{R}^n}|\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}|^{1/(pm)}g\,u.$$

By Theorem 3.11 i) we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n}|\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}|^{1/(pm)}g\,u &\le C_\ell\int_{\mathbb{R}^n}(\mathcal{M}_{\Phi_1,B_\ell}\vec{f})^{1/(pm)}M_{B_\ell}(gu)\\ &= C_\ell\int_{\mathbb{R}^n}(\mathcal{M}_{\Phi_1,B_\ell}\vec{f})^{1/(pm)}\frac{M_{B_\ell}(gu)}{M_{\tilde{A}_\ell}(u)}M_{\tilde{A}_\ell}(u),\end{aligned}$$

where $\tilde{A}_\ell$ is a Young function to be chosen later. Then, by applying Hölder's inequality in Lorentz spaces we obtain that

$$\int_{\mathbb{R}^n}|\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}|^{1/(pm)}g\,u \;\le\; \|(\mathcal{M}_{\Phi_1,B_\ell}\vec{f})^{1/(pm)}\|_{L^{p,\infty}(M_{\tilde{A}_\ell}(u))}\left\|\frac{M_{B_\ell}(gu)}{M_{\tilde{A}_\ell}(u)}\right\|_{L^{p',1}(M_{\tilde{A}_\ell}(u))}.$$

Now the argument follows in similar way to that in [CPSS] (see also [P] for the multilinear fractional integral). If we define $S(f)=\frac{M_{B_\ell}(fu)}{M_{\tilde{A}_\ell}(u)}$,, the result will be done if we prove that $S: L^{p',1}(u)\to L^{p',1}(M_{\tilde{A}_\ell}(u))$.

We shall take $\tilde{A}_\ell$ such that $M_{B_\ell}(u)\le M_{\tilde{A}_\ell}(u)$, then $S: L^\infty(u)\to L^\infty(M_{\tilde{A}_\ell}(u))$. Thus, by the Marcinkiewicz's interpolation theorem in Lorentz spaces due to Hunt (see [BS]), it is enough to prove that $S: L^{(p+\epsilon)'}(u)\to L^{(p+\epsilon)'}(M_{\tilde{A}_\ell}(u))$, which is equivalent to prove the following inequality

$$\int_{\mathbb{R}^n}(M_{B_\ell}(f))^{(p+\epsilon)'}(M_{\tilde{A}_\ell}(u))^{1-(p+\epsilon)'}\le\int_{\mathbb{R}^n}|f|^{(p+\epsilon)'}u^{1-(p+\epsilon)'}. \tag{5.6}$$

By choosing $\tilde{A}_\ell=t(1+\log^+ t)^{\ell p+p-1+(\ell+2)\epsilon}$, $\epsilon>0$, the above result holds from Lemma 5.6 with $q=p+\epsilon$ and $w=1$. Then, we obtain that

$$\|\mathcal{T}_{\vec{b}^\ell,\phi}\vec{f}\|_{L^{1/m,\infty}(u)}\le C\|(\mathcal{M}_{\Phi_1,B_\ell}\vec{f})\|_{L^{1/m,\infty}(M_{\tilde{A}_\ell}(u))}.$$

Thus, taking $p=1+\frac{\delta_\ell-\epsilon(\ell+2)}{\ell+1}$ with $0<\epsilon(\ell+2)<\delta_\ell$ we get the theorem, since $\tilde{A}_\ell(t)=t(1+\log^+ t)^{\ell+\delta_\ell}$.

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