

Class number formulas for certain bicyclic biquadratic number fields as finite sums involving Jacobi elliptic functions

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Abstract

We give formulas for the class numbers of bicyclic biquadratic number fields containing an imaginary quadratic field of class number one. The class number is expressed as a finite sum in terms of the basic Jacobi elliptic functions, playing a similar role as the trigonometric sine in Dirichlet classical class number formula.

1 Introduction

In his memoir “Recherches sur les formes quadratiques à coefficients et à indéterminées complexes. Première partie”, Dirichlet [3] announced a generalization of his class number formula for real quadratic fields to biquadratic fields containing $\mathbb{Q}(i)$, by replacing the circular trigonometric functions with certain elliptic trigonometric functions. He showed the class number as a double infinite sum, and explained afterwards that the development of the formula, as a finite sum in terms of lemniscate trigonometric functions, consisted of three essential steps: a Gauss sum, the sum of a trigonometric Euler series and the summation of a series by means of a formula due to Jacobi and Abel, which leads to the appearance of elliptic functions. He added that the details of the development and subsequent study of the formula would be reserved to the second part of that

*Work partially supported by P08-FQM-03894, FSE and FEDER.

†Work partially supported through the project *Arithmetic of algebraic varieties: theory and computation*, Ministerio de Ciencia e Innovación (MICINN), reference MTM 2009-13-060-C02-01.

memoir, but it never appeared. Later, Nazimow [11] published the corresponding formula. In [5] we analyzed the Dirichlet-Nazimow formula and updated its proof in the context of Class Field Theory.

In the present paper, we obtain similar formulas to the classical Dirichlet class number formula for real quadratic fields, replacing the role of the field of rationals \mathbb{Q} with any imaginary quadratic field F of class number one, distinct from $\mathbb{Q}(i)$. The class number is then expressed as a finite sum in terms of the basic Jacobi elliptic functions sine amplitude, cosine amplitude and delta amplitude, playing a similar role to the trigonometric sine in Dirichlet classical formula.

In Section 2 we introduce the necessary tools for the proof of the formulas. In Section 3 we state the main theorem, with all the formulas, and prove the most intricate case. The remaining cases follow along the line of the given proof, with minor changes as we shall indicate.

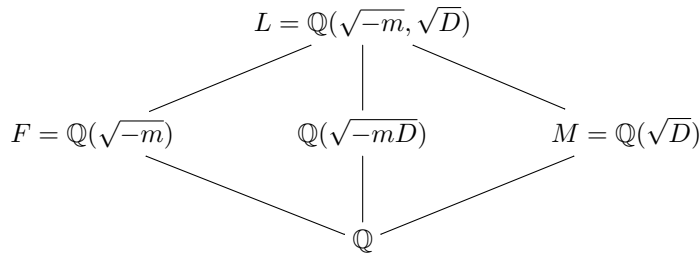
2 Preliminaries

We shall denote by $F = \mathbb{Q}(\sqrt{-m})$, m a squarefree positive integer, an imaginary quadratic field of class number one. The class number one problem for quadratic imaginary fields was discussed by Heegner (1952), using modular forms and modular equations, but his work was not initially accepted. With later work of Stark and Birch (1967) the position was clarified and Heegner's work understood. By a completely different method, Baker (1966) gave a proof based on his theorem on linear forms in logarithms of algebraic numbers. The complete list of imaginary quadratic fields with class number one is

$$\begin{aligned} &\mathbb{Q}(i), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-7}), \mathbb{Q}(\sqrt{-11}), \\ &\mathbb{Q}(\sqrt{-19}), \mathbb{Q}(\sqrt{-43}), \mathbb{Q}(\sqrt{-67}), \mathbb{Q}(\sqrt{-163}). \end{aligned}$$

Since we have already developed the case $\mathbb{Q}(i)$ in [5], we will consider only the cases $m = 2, 3, 7, 11, 19, 43, 67, 163$. We shall denote by \mathcal{O} the ring of integers of F .

Let $M = \mathbb{Q}(\sqrt{D})$ be a quadratic field with squarefree $D \in \mathbb{Z}$, $(D, m) = 1$. The composite field $L = FM$ is a bicyclic biquadratic number field and it is an abelian extension of \mathbb{Q} with Galois group isomorphic to the Klein group V_4 .



We shall focus on the abelian extension L/F and its primitive character

$$\chi: (\mathcal{O}/\mathfrak{m})^* \rightarrow \{\pm 1\}$$

of conductor \mathfrak{m} associated with L/F by Class Field Theory.

2.1 Primitive character associated to the extension

As in the classical case, the proof of the formulas consists in evaluating a certain L -function at $s = 1$ into two different ways, in such a way that one obtains the desired formula. Specifically, we shall consider the function

$$L(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}$$

where \mathfrak{a} runs through the (non-zero) ideals of \mathcal{O} and χ denotes the primitive character modulo \mathfrak{m} associated with the quadratic extension L/F viewed as character of the ray class group $I_F(\mathfrak{m})/P_1(\mathfrak{m}) \simeq \text{Gal}(F\mathfrak{m}/F)$. As usual, we extend χ to all ideals by declaring $\chi(\mathfrak{a}) = 0$ for those ideals \mathfrak{a} such that $(\mathfrak{a}, \mathfrak{m}) \neq 1$. Since \mathcal{O} is a principal ideal domain and χ is trivial on the group of units \mathcal{O}^* , we can write $\chi(\mathfrak{a}) = \chi(k)$ where $\mathfrak{a} = (k)$ for some $k \in \mathcal{O}$. The sum $L(s, \chi)$ is defined for $\text{Re}(s) > 1$ and, by analytic continuation, it can be extended to a meromorphic function on the whole complex plane which will be also denoted $L(s, \chi)$.

The proof of the following lemma is a routine exercise and it will be omitted.

Lemma 2.1

1. The relative discriminant of the extension L/F is $\mathcal{D}_{L/F} = (d)$, with

$$d = \begin{cases} |D| & \text{if } D \equiv 1 \pmod{4}, \\ 4|D| & \text{otherwise,} \end{cases}$$

for $m \neq 2$, and

$$d = \begin{cases} |D| & \text{if } D \equiv 1 \pmod{4}, \\ 2|D| & \text{otherwise,} \end{cases}$$

for $m = 2$.

2. The conductor of L/F is $\mathfrak{m} = (d)$ and the multiplicative character χ extended to the ring \mathcal{O} is given by

$$\chi(k) = \left(\frac{L/F}{(k)} \right) = \left(\frac{D}{|k|^2} \right)$$

if $\gcd(k, 2) = 1$, whereas

$$\chi(\pi) = \begin{cases} 1 & \text{if } D \equiv 1 \pmod{8}, \\ -1 & \text{if } D \equiv 5 \pmod{8}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\pi = \sqrt{-2}$ when $m = 2$ and $\pi = \frac{1 \pm \sqrt{-7}}{2}$ when $m = 7$; for $m \neq 2, 7$, then

$$\chi(2) = \begin{cases} 1 & \text{if } D \equiv 1 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$$

Here the symbols are those of Artin and Kronecker respectively, and $|k|$ denotes the norm of k in F/\mathbb{Q} .

When $m \neq 7$, there is a unique even prime ideal $J = (\pi)$ in \mathcal{O} , so that any nonzero ideal \mathfrak{a} of \mathcal{O} can be written as $\mathfrak{a} = J^n I$ for some odd ideal I , and we obtain the equality

$$L(1, \chi) = \left(\sum_{n=0}^{\infty} \frac{\chi(J)^n}{N(J)^n} \right) \sum_I \frac{\chi(I)}{N(I)}$$

where I runs through the odd ideals of \mathcal{O} . Denoting $c = \sum_{n=0}^{\infty} \frac{\chi(J)^n}{N(J)^n}$ we have

$$c = \begin{cases} 4/3 & \text{if } D \equiv 1 \pmod{4}, \\ 1 & \text{otherwise,} \end{cases}$$

for $m \neq 2, 7$. For $m = 2$, then

$$c = \begin{cases} 2 & \text{if } D \equiv 1 \pmod{8}, \\ 2/3 & \text{if } D \equiv 5 \pmod{8}, \\ 1 & \text{otherwise.} \end{cases}$$

For $m = 7$, we also define c by the equality

$$L(1, \chi) = c \sum_{I \text{ odd}} \frac{\chi(I)}{N(I)},$$

and it turns out that

$$c = \begin{cases} 4 & \text{if } D \equiv 1 \pmod{8}, \\ 4/9 & \text{if } D \equiv 5 \pmod{8}, \\ 1 & \text{otherwise.} \end{cases}$$

2.2 Class number and fundamental unit

It is immediate to check that we have the equality

$$\zeta_L(s) = \zeta_F(s) \cdot L(s, \chi),$$

involving the Dedekind zeta functions of L and F , respectively. Multiplying both sides of the equation by $s - 1$ and taking limit when $s \rightarrow 1$, the general formula for the residue of Dedekind zeta functions at the pole $s = 1$ provides the value

$$L(1, \chi) = \begin{cases} \frac{2\pi h \log |u|}{d\sqrt{2}} & \text{if } m = 2, \\ \frac{4\pi h \log |u|}{d\sqrt{m}} & \text{otherwise,} \end{cases} \quad (1)$$

where h is the class number of L and u is a fundamental unit of the ring of integers of L with $|u| > 1$. Here we have used the value of the discriminants $\Delta_{L/\mathbb{Q}}$, $\Delta_{F/\mathbb{Q}}$ and take into account the number of roots of unity in F and in L . Now we shall recalculate $L(1, \chi)$ in a sequence of steps to achieve another expression for the value that will approach to the right-hand side of the pursued formula. For it, we need some other ingredients.

2.3 Gauss sums

Along with the character χ of conductor \mathfrak{m} we shall need its associated Gauss sum. Noting that the different of K is the ideal (δ) with

$$\delta = \begin{cases} 2\sqrt{-2} & \text{if } m = 2, \\ \sqrt{-m} & \text{otherwise,} \end{cases} \quad (2)$$

for each $\alpha \in 1/\delta\mathfrak{m}^{-1}$, one defines the Gauss sum

$$\tau_\alpha(\chi) = \sum_{k \in (\mathcal{O}/\mathfrak{m})^*} \chi(k) \exp(2\pi i \operatorname{tr}(\alpha k)),$$

where tr denotes the trace linear map on F/\mathbb{Q} . The sum is well-defined and if we consider $\beta \in \mathcal{O}$ with $(\beta, \mathfrak{m}) = 1$, then it holds

$$\tau_{\alpha\beta}(\chi) = \overline{\chi(\beta)} \tau_\alpha(\chi).$$

Moreover, χ being primitive, if $(\delta\alpha\mathfrak{m}, \mathfrak{m}) = 1$ then one has $|\tau_\alpha(\chi)| = N(\mathfrak{m})^{1/2}$. These general properties can be found, for example, in [7], [10], [12]. In our case we shall consider $\alpha = 1/(\delta d)$ and since the character χ is quadratic, we can say even more. Indeed, viewing χ as a Hecke character of F and taking into account the functional equation of $L(s, \chi)$, we obtain the exact value $\tau_\alpha(\chi) = d$. The employed argument goes back to Hecke and only applies to quadratic extensions. The reader can consult, for example, the Appendix in Shimura's text [14] or Rohrlich's notes [13].

2.4 Jacobi elliptic functions

We shall express the class number of the bicyclic biquadratic number field L as a finite sum in terms of the three basic Jacobi elliptic functions, which are denoted by $\operatorname{sn}(z, k)$, $\operatorname{cn}(z, k)$ and $\operatorname{dn}(z, k)$, where k is known as the elliptic modulus [8]. They arise from the inversion of the elliptic integral of the first kind,

$$z = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

and $\phi = \operatorname{am}(z, k)$ is the Jacobi amplitude. One defines

$$\begin{aligned} \operatorname{sn}(z, k) &= \sin(\phi) = \sin(\operatorname{am}(z, k)), \\ \operatorname{cn}(z, k) &= \cos(\phi) = \cos(\operatorname{am}(z, k)), \\ \operatorname{dn}(z, k) &= \sqrt{1 - k^2 \operatorname{sn}^2(z, k)}. \end{aligned}$$

These are doubly periodic functions with

$$\begin{aligned} \operatorname{sn}(z + 2mK + 2niK', k) &= (-1)^m \operatorname{sn}(z, k), \\ \operatorname{cn}(z + 2mK + 2niK', k) &= (-1)^{m+n} \operatorname{cn}(z, k), \\ \operatorname{dn}(z + 2mK + 2niK', k) &= (-1)^n \operatorname{dn}(z, k), \end{aligned}$$

where K is the quarter period, defined by

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

and K' is the quarter period corresponding to the complementary modulus $k' = \sqrt{1 - k^2}$, i.e.

$$K' = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}}.$$

The nome [1] q is defined in terms of k by $q = e^{-\frac{\pi K'}{K}}$ and it determines k by

$$k = \frac{\theta_2^2(0, q)}{\theta_3^2(0, q)}$$

where θ_2 and θ_3 are Jacobi theta functions.

In the course of the proof of the formulas, we will use several properties of these elliptic functions which can be found in [8] and [4]. For the general identities we will use the notation q, k, k', K, K' . We set the nome $q_m = e^{-\pi/\sqrt{m}}$, whose corresponding modulus and quarter period are denoted by k_m and K_m respectively. The nome corresponding to the complementary modulus k'_m will be denoted $q'_m = e^{-\pi/\sqrt{m}}$. Notice that $K'_m = \sqrt{m}K_m$.

m	K_m	k_m	Irred(k_m, x)
2	1.645568395	0.414213562	$x^2 + 2x - 1$
3	1.598142002	0.258819044	$16x^4 - 16x^2 + 1$
7	1.572339753	0.062622912	$256x^4 - 256x^2 + 1$
11	1.570983856	0.021849722	$4096x^{12} - 12288x^{10} + 13056x^8$ $- 5632x^6 + 2864x^4 - 2096x^2 + 1$
19	1.570803423	0.004250780	$4096x^{12} - 12288x^{10} + 13056x^8$ $- 5632x^6 + 56112x^4 - 55344x^2 + 1$
43	1.570796334	0.000134478	$4096x^{12} - 12288x^{10} + 13056x^8$ $- 5632x^6 + 55296816x^4$ $- 55296048x^2 + 1$
67	1.570796327	0.000010425	$4096x^{12} - 12288x^{10} + 13056x^8$ $- 5632x^6 + 9199872816x^4$ $- 9199872048x^2 + 1$
163	1.570796327	0.000000007	$4096x^{12} - 12288x^{10} + 13056x^8$ $- 5632x^6 + 16408588290048816x^4$ $- 16408588290048048x^2 + 1$

3 The formulas

As above, we consider an imaginary quadratic field $F = \mathbb{Q}(\sqrt{-m})$ of class number one, a quadratic field $M = \mathbb{Q}(\sqrt{D})$, with D a squarefree integer, $(D, m) = 1$, and the composite field $L = \mathbb{Q}(\sqrt{-m}, \sqrt{D})$. Let us call d the relative discriminant of the extension $L|F$, its conductor $\mathfrak{m} = (d)$ and let χ be the primitive character modulo \mathfrak{m} . Let h denote the class number of L , u a fundamental unit of L with $|u| > 1$ and c the constant determined in section 2.1. Finally, k_m and K_m denote respectively the elliptic modulus and the quarter period corresponding to the nome $q_m = e^{-\pi\sqrt{m}}$.

Now we present the main result of the paper but, before, some comments on the proof are in order. First, the case $F = \mathbb{Q}(i)$ is proved in [5]. For the sake of readability, we shall start proving the fourth case in the statement. The proof of the remaining cases, once generators for the odd ideals in the ring of integers are determined, essentially reproduces a part of the given proof as we shall indicate.

Theorem 3.1 *With the preceding notations, we have:*

- If $F = \mathbb{Q}(i)$ then

$$h = \frac{-1}{4t \log |u|} \sum_{\substack{r,s=0 \\ (d,r^2+s^2)=1}}^{d-1} \left(\frac{D}{r^2 + s^2} \right) \log \left| \operatorname{sn} \left(\frac{\omega}{d} (r + is), i \right) \right|$$

where $t = 2 - \chi(1 + i)$ and $\omega = 2.622057\dots$, where 2ω is the arc-length of the lemniscate of Bernoulli given by the equation $(x^2 + y^2)^2 = x^2 - y^2$.

- If $F = \mathbb{Q}(\sqrt{-2})$ then

$$h = \frac{-c}{4 \log |u|} \sum_{\substack{r,s=0 \\ (d,r^2+2s^2)=1}}^{d-1} \left(\frac{D}{r^2 + 2s^2} \right) \log \left| \operatorname{sn} \left(\frac{2K_2}{d} (r + s\sqrt{-2}), k_2 \right) \right|.$$

- If $F = \mathbb{Q}(\sqrt{-m})$, $m = 3$ or 7 , then

$$h = \frac{c}{8 \log |u|} \sum_{\substack{r,s=0 \\ (d,r^2+rs+\lambda s^2)=1}}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \log \left| \frac{\operatorname{dn}(w, k_m)}{\operatorname{sn}(w, k_m)} \right|$$

where $\lambda = \frac{m+1}{4}$ and $w = \frac{2K_m}{d}(2r + s + s\sqrt{-m})$.

- If $F = \mathbb{Q}(\sqrt{-m})$, $m = 11, 19, 43, 67$ or 163 , then

$$h = \frac{c}{8 \log |u|} \sum_{\substack{r,s=0 \\ (d,r^2+rs+\lambda s^2)=1}}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \log \left| \frac{(1 + \operatorname{cn})^2 \operatorname{dn}}{\operatorname{sn}^3}(w, k_m) \right|$$

where $\lambda = \frac{m+1}{4}$ and $w = \frac{2K_m}{d}(2r + s + s\sqrt{-m})$.

Assume we are in the fourth case, that is $m \geq 11$. Then, each odd ideal of \mathcal{O} is generated by a unique element of the type either $a + b\sqrt{-m}$ with $a, b \in \mathbb{Z}$ of distinct parity and the odd one positive, or $\frac{a+b\sqrt{-m}}{2}$ with $a, b \in \mathbb{Z}$ both odd and $a > 0$. Therefore, one has

$$\sum_{I \text{ odd}} \frac{\chi(I)}{N(I)} = S_1 + S_2 + 4S_3$$

where

$$\begin{aligned} S_1 &= \sum_{\substack{a=1 \\ a \text{ odd}}}^{\infty} \sum_{\substack{b=-\infty \\ b \text{ even}}}^{\infty} \frac{\chi(a + b\sqrt{-m})}{a^2 + mb^2}, \\ S_2 &= \sum_{\substack{b=1 \\ b \text{ odd}}}^{\infty} \sum_{\substack{a=-\infty \\ a \text{ even}}}^{\infty} \frac{\chi(a + b\sqrt{-m})}{a^2 + mb^2}, \\ S_3 &= \sum_{\substack{a=1 \\ a \text{ odd}}}^{\infty} \sum_{\substack{b=-\infty \\ b \text{ odd}}}^{\infty} \frac{\chi\left(\frac{a+b\sqrt{-m}}{2}\right)}{a^2 + mb^2}. \end{aligned}$$

In the following we shall sum up the three double series separately.

Addition of S_1 : First step. We consider the Gauss sum $\tau_\alpha(\chi)$ associated with the character χ and the element $\alpha = 1/(d\sqrt{-m})$. We know that $\tau_\alpha(\chi) = d$. Then

$$\begin{aligned}\chi(a + b\sqrt{-m}) &= \frac{1}{d} \tau_{(a+b\sqrt{-m})/(d\sqrt{-m})}(\chi) \\ &= \frac{1}{d} \sum_{r,s=0}^{d-1} \chi\left(r + s \frac{1 + \sqrt{-m}}{2}\right) e^{\frac{2\pi i((2r+s)b+ay)}{d}} \\ &= \frac{1}{d} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2}\right) \cos(2\pi(2r+s)b/d) \cos(2\pi sa/d)\end{aligned}$$

and therefore one has

$$S_1 = \frac{1}{d} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2}\right) \sum_{\substack{a=1 \\ a \text{ odd}}}^{\infty} \cos(2\pi sa/d) \sum_{\substack{b=-\infty \\ b \text{ even}}}^{\infty} \frac{\cos(2\pi(2r+s)b/d)}{a^2 + mb^2}.$$

We turn now to the series $T_1 = \sum_{\substack{b=-\infty \\ b \text{ even}}}^{\infty} \frac{\cos(2\pi(2r+s)b/d)}{a^2 + mb^2}$.

Addition of S_1 : Second step. Using the identity

$$\frac{\pi \exp(x\xi) + \exp(-x\xi)}{x \exp(x\pi) - \exp(-x\pi)} = \frac{1}{x^2} + 2 \sum_{n=1}^{\infty} (-1)^n \frac{\cos \xi n}{x^2 + n^2} \quad (3)$$

that holds for all real numbers x and ξ with $-\pi \leq \xi \leq \pi$ (see [15]), we get

$$T_1 = \frac{\pi}{a\sqrt{m}} \frac{q_m^{a/2}}{1 - q_m^a} \cosh(a\xi/(2\sqrt{m}))$$

where $\xi = \pi(4|\tilde{z}|/d - 1)$ and $\tilde{z} \equiv 2r + s \pmod{d}$ with $|\tilde{z}| \leq d/2$. In this way we can write

$$S_1 = \frac{\pi}{d\sqrt{m}} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2}\right) \sum_{\substack{a=1 \\ a \text{ odd}}}^{\infty} \frac{1}{a} \frac{q_m^{a/2}}{1 - q_m^a} \cosh(a\xi/(2\sqrt{m})) \cos(2\pi as/d).$$

We turn now to the series $U_1 = \sum_{\substack{a=1 \\ a \text{ odd}}}^{\infty} \frac{1}{a} \frac{q_m^{a/2}}{1 - q_m^a} \cosh(a\xi/(2\sqrt{m})) \cos(2\pi as/d)$.

Addition of S_1 : Third step. Letting $a = 2n + 1$, we can rewrite

$$U_1 = \sum_{n=0}^{\infty} \frac{\sqrt{q_m^{2n+1}}}{(2n+1)(1 - q_m^{2n+1})} \cosh((2n+1)\xi/(2\sqrt{m})) \cos(2\pi(2n+1)s/d).$$

Using $\cosh(x) = \cos(ix)$, the addition formulas for the cosine function, and letting $A = \frac{\pi}{2d\sqrt{m}} (4s\sqrt{m} + i(4|\tilde{z}| - d))$, and B the complex conjugate of A , we obtain

$$U_1 = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\sqrt{q_m^{2n+1}}}{(2n+1)(1-q_m^{2n+1})} (\cos(2n+1)A + \cos(2n+1)B).$$

Jacobi's formula (10) on page 156 of [8] gives the following identity between circular and elliptic trigonometric functions:

$$\log \sqrt{\frac{1 + k \operatorname{sn}(\frac{2Kx}{\pi}, k)}{1 - k \operatorname{sn}(\frac{2Kx}{\pi}, k)}} = 4 \sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{q^{2n+1}}}{(2n+1)(1-q^{2n+1})} \sin(2n+1)x. \quad (4)$$

This allows us to rewrite

$$U_1 = \frac{1}{8} \left(\log \sqrt{\frac{1 + k'_m \operatorname{sn}(\frac{2K'_m x}{\pi}, k'_m)}{1 - k'_m \operatorname{sn}(\frac{2K'_m x}{\pi}, k'_m)}} + \log \sqrt{\frac{1 + k'_m \operatorname{sn}(\frac{2K'_m y}{\pi}, k'_m)}{1 - k'_m \operatorname{sn}(\frac{2K'_m y}{\pi}, k'_m)}} \right),$$

with $x = A + \pi/2$, and $y = B + \pi/2$ its complex conjugate.

Addition of S_1 : Fourth step. Taking into account the identities

$$\operatorname{sn}(v + K \pm iK') = \frac{\operatorname{dn}(v, k)}{k \operatorname{cn}(v, k)} \quad (5)$$

$$\frac{\operatorname{dn}(v, k)}{\operatorname{cn}(v, k)} = \operatorname{dn}(iv, k') \quad (6)$$

and since $\operatorname{sn}(\bar{v}, k) = \overline{\operatorname{sn}(v, k)}$ when $k \in \mathbb{R}$, we get

$$S_1 = \frac{\pi}{16d\sqrt{m}} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \log \left| \frac{1 + \operatorname{dn}(2w, k_m)}{1 - \operatorname{dn}(2w, k_m)} \right|^2,$$

where $w = \frac{2K_m}{d}(2r + s + s\sqrt{-m})$. From the duplication formulas of dn , we get

$$S_1 = \frac{\pi}{8d\sqrt{m}} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \log \left| \frac{\operatorname{dn}(w, k_m)}{\operatorname{sn}(w, k_m) \operatorname{cn}(w, k_m)} \right|^2. \quad (7)$$

Addition of S_2 . Following Step 1, Step 2 (this time q_m , rather than q'_m , appears directly) and Step 3 as before, we obtain:

$$S_2 = \frac{\pi}{8d\sqrt{m}} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \log \left| \frac{\operatorname{cn}(w, k) \operatorname{dn}(w, k)}{\operatorname{sn}(w, k)} \right|^2. \quad (8)$$

Addition of S_3 . Considering the Gauss sum as in the the preceding additions, one has

$$S_3 = \frac{1}{d} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \sum_{\substack{a=1 \\ a \text{ odd}}}^{\infty} \sum_{\substack{b=-\infty \\ b \text{ odd}}}^{\infty} \frac{\cos(\pi sa/d) \cos(\pi(2r+s)b/d)}{a^2 + mb^2}.$$

We write now

$$S_3 = S_{31} - S_{32}$$

where

$$S_{31} = \frac{1}{d} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \sum_{\substack{a=1 \\ a \text{ odd}}}^{\infty} \sum_{b=-\infty}^{\infty} \frac{\cos(\pi sa/d) \cos(\pi(2r+s)b/d)}{a^2 + mb^2},$$

$$S_{32} = \frac{1}{d} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \sum_{\substack{a=1 \\ a \text{ odd}}}^{\infty} \sum_{\substack{b=-\infty \\ b \text{ even}}}^{\infty} \frac{\cos(\pi sa/d) \cos(\pi(2r+s)b/d)}{a^2 + mb^2}.$$

Due to the similarity between S_{32} and S_1 , one readily obtains

$$S_{32} = \frac{\pi}{8d\sqrt{m}} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \log \left| \frac{\operatorname{dn}(w/2, k)}{\operatorname{sn}(w/2, k) \operatorname{cn}(w/2, k)} \right|^2.$$

The addition of S_{31} follows similar steps, with some differences.

Addition of S_{31} : First step. Using identity (3) we obtain

$$S_{31} = \frac{2\pi}{d\sqrt{m}} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \sum_{\substack{a=1 \\ a \text{ odd}}}^{\infty} \frac{1}{a} \frac{q_m^a}{1 - q_m^{2a}} \cosh(a\xi/\sqrt{m}) \cos(\pi sa/d)$$

where $\xi = \pi \left(\frac{2}{d}(r + \frac{s}{2}) - 1 \right)$. Consider

$$U_{31} = \sum_{\substack{a=1 \\ a \text{ odd}}}^{\infty} \frac{1}{a} \frac{q_m^a}{1 - q_m^{2a}} \cosh(a\xi/\sqrt{m}) \cos(\pi sa/d)$$

$$= \sum_{n=0}^{\infty} \frac{q_m^{2n+1}}{(2n+1)(1 - q_m^{2(2n+1)})} \cosh((2n+1)\xi/(\sqrt{m})) \cos(\pi(2n+1)s/d).$$

Using $\cosh(x) = \cos(ix)$, the addition formulas for the cosine function, and letting $A = \frac{\pi}{d\sqrt{m}} (s\sqrt{m} + i(2r+s-d))$, and B the complex conjugate of A , we obtain

$$U_{31} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{q_m^{2n+1}}{(2n+1)(1 - q_m^{2(2n+1)})} (\cos(2n+1)A + \cos(2n+1)B).$$

Addition of S_{31} : Second step. Jacobi's formula (8) on page 155 of [8] gives the following identity between circular and elliptic trigonometric functions:

$$\log \operatorname{dn}\left(\frac{2Kx}{\pi}, k\right) = \log \sqrt{k'} + 4 \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(2n+1)(1-q^{2(2n+1)})} \cos(2(2n+1)x). \quad (9)$$

Then, one has

$$S_{31} = \frac{\pi}{4d\sqrt{m}} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \log \left| \operatorname{dn} \left(\frac{K_m}{d} (s\sqrt{m} + i(2r+s)) - K_m i, k'_m \right) \right|^2.$$

Using the identities

$$\operatorname{dn}(v + iK', k) = -i \cot \operatorname{am}(v, k) \quad (10)$$

$$\cot \operatorname{am}(v, k) = \frac{1}{i \operatorname{sn}(v/i, k')} \quad (11)$$

we have

$$S_{31} = \frac{\pi}{4d\sqrt{m}} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \log \left| \frac{1}{\operatorname{sn}(w/2, k_m)} \right|^2. \quad (12)$$

To get the formula for S_3 in terms of w , we use the duplication formulas in [8] to obtain

$$\frac{\operatorname{cn}^2(u)}{\operatorname{sn}^2(u) \operatorname{dn}^2(u)} = \frac{1 + \operatorname{cn}(2u)}{1 - \operatorname{cn}(2u)}$$

so that

$$4S_3 = \frac{\pi}{8d\sqrt{m}} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \log \left| \frac{1 + \operatorname{cn}(w, k)}{1 - \operatorname{cn}(w, k)} \right|^4$$

and therefore

$$\sum_{I \text{ odd}} \frac{\chi(I)}{\mathbb{N}(I)} = \frac{\pi}{2d\sqrt{m}} \sum_{\substack{r,s=0 \\ (d,r^2+rs+\lambda s^2)=1}}^{d-1} \left(\frac{D}{r^2 + rs + \lambda s^2} \right) \log \left| \frac{(1 + \operatorname{cn})^2 \operatorname{dn}}{\operatorname{sn}^3} (w, k_m) \right|,$$

what finishes the proof of the fourth case.

As for the third case, that is $m = 3$ or 7 , each odd ideal of \mathcal{O} is generated by a unique element of the type $a + b\sqrt{-m}$ with $a, b \in \mathbb{Z}$ of distinct parity and the odd one positive. Therefore one has

$$\sum_{I \text{ odd}} \frac{\chi(I)}{\mathbb{N}(I)} = S_1 + S_2,$$

where S_1 and S_2 are defined as above. The formula follows from the addition of their expressions in (7) and (8).

In the second case, $m = 2$, each odd ideal is generated by a unique element of the form $a + b\sqrt{-m}$, with $a, b \in \mathbb{Z}$, a odd and positive. Therefore

$$\sum_{I \text{ odd}} \frac{\chi(I)}{N(I)} = \sum_{\substack{a=1 \\ a \text{ odd}}}^{\infty} \sum_{b=-\infty}^{\infty} \frac{\chi(a + b\sqrt{-m})}{a^2 + mb^2}.$$

Considering the Gauss sum as above, one has

$$\sum_{I \text{ odd}} \frac{\chi(I)}{N(I)} = \frac{1}{d} \sum_{r,s=0}^{d-1} \left(\frac{D}{r^2 + ms^2} \right) \sum_{\substack{a=1 \\ a \text{ odd}}}^{\infty} \sum_{b=-\infty}^{\infty} \frac{\cos(\pi 2sa/d) \cos(\pi 2rb/d)}{a^2 + mb^2},$$

and this addition follows the same steps as S_{31} before. The final formula follows from the corresponding expression to (12), with $w = 2r + 2s\sqrt{-m}$.

4 Final remark

It is worth mentioning that there exists a classical product formula for the class number due to Hasse. In our setting, and except for the particular case $L = \mathbb{Q}(\sqrt{-2}, \sqrt{-1})$, the formula reads

$$h = \frac{Q}{2} \cdot h(\mathbb{Q}(\sqrt{-m})) \cdot h(\mathbb{Q}(\sqrt{D})) \cdot h(\mathbb{Q}(\sqrt{-mD})),$$

where as above h is the class number of $L = \mathbb{Q}(\sqrt{-m}, \sqrt{D})$, and Q is the so-called unit index; we refer to [6], [2], and [9] for the details. The following table contains numerical data concerning the evaluation of the class number h by the two methods, either the product class formula or by using elliptic functions. The computations have been carried out with Magma and Maple, respectively, and they have served as a double-check of the given formulas.

$-m$	D	h_{prod}	h_{ell}
-2	3	2	2.0000
-2	-5	2	2.0000
-2	21	2	2.0000
-3	41	1	0.9999
-3	-5	2	1.9999
-3	-7	1	1.0000
-7	6	2	1.9999
-7	23	8	8.0000
-7	15	8	8.0000
-11	5	2	2.0000
-19	10	4	3.9999
-43	-6	2	1.9999
-67	5	9	9.0000
-163	17	13	12.9999

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