Infinity structures and higher products in rational homotopy theory

Ph. D. Dissertation

José Manuel Moreno Fernández

Directed by:
Dr. Urtzi Buijs Martín and Dr. Aniceto Murillo Mas

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AUTOR: José Manuel Moreno Fernández

http://orcid.org/0000-0002-0956-8407

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Urtzi Buijs Martín, Aniceto Murillo Más
Dpto. de Álgebra, Geometría y Topología
Universidad de Málaga
Ap. 59, 29080
Málaga, Spain

Por la presente, Urtzi Buijs Martín y Aniceto Murillo Mas, como directores de la tesis doctoral, y tutor en el segundo caso, de D. José Manuel Moreno Fernández,

AUTORIZAMOS la presentación de la misma para que pueda ser defendida como Tesis Doctoral según la legislación vigente.

Además,

INFORMAMOS que el trabajo mencionado es totalmente original y que ha dado lugar a la siguiente publicación que lo avala:

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Hacemos constar que ningún contenido ni resultado de la anterior publicación ha sido utilizado en ninguna otra tesis ni trabajo de investigación. Para que así conste y surta los efectos oportunos, firmamos el presente escrito en Málaga a 22 de diciembre de 2017.

Urtzi Buijs Martín

Aniceto Murillo Mas

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Contents

INTRODUCTION

Rational homotopy theory classically studies the torsion free phenomena in the homotopy category of topological spaces and continuous maps. Its success is mainly due to the existence of relatively simple algebraic models that faithfully capture this non-torsion homotopical information.

Infinity structures are algebraic gadgets in which some axioms hold up to a hierarchy of coherent homotopies. We will work particularly with two of the most important of these, namely, A_∞ and L_∞ algebras. The former can be thought of as a differential graded algebra (DGA, henceforth, or CDGA if it is commutative) where the associativity law holds up to a homotopy which is determined by a 3-product whose associativity up to homotopy is again given by a 4-product, and so on. The latter can be seen as a differential graded Lie algebra (DGL, henceforth) where similarly, the Jacobi identity holds up to a homotopy which is determined by a 3-product, and so on.

In this work, we use infinity structures to shed light on classical matters of rational homotopy theory (and beyond, as will be explained below). When attempting to classify (or, more modestly, just to distinguish) rational homotopy types, one is naturally led to consider secondary operations in homotopy or cohomology. Among these, we focus on the higher Whitehead and Massey products, complementing the usual Whitehead product in homotopy and cup product in cohomology, respectively [26, 60]. These fundamental homotopical invariants are at the very heart of the theory, but their manipulation can be difficult at times. Infinity structures also classify rational homotopy types [31] and are sometimes more amenable to computations or better adapted to the problem at hand than are the secondary operations.

The main achievement of this thesis is the description of the precise relationship between the higher arity operations of an infinity structure governing a given rational homotopy type and the higher products in homotopy and cohomology of it. We use the developed theory to recover and generalize a classical result in rational homotopy theory [3, Thm. 5.4], and to give some applications in the context of (co)formality. Furthermore, some of the main results on the entanglement between the higher arity operations and the higher order secondary operations still hold when the infinity structure is not necessarily modeling a rational space, making it possible to apply these results in other contexts. More precisely, some of the theorems that we prove are valid over fields of characteristic $p > 0$ and/or for not necessarily finite type or bounded (upper or below) complexes.

We begin with a more accurate description of the results of the thesis (see the corresponding section for complete details). The first chapter is a compendium of necessary background, and we properly start the original work in Chapter 2. We focus first on the homotopy side of things, and ignore the distinction between a map and its homotopy class hereafter. Higher order Whitehead products are homotopy invariant sets introduced in [55] and are constructed as follows. Let $S^{n_1},\dots,S^{n_k}$ be simply connected spheres, denote by $W = S^{n_1}\vee\cdots\vee S^{n_k}$ and $T = T(S^{n_1},\dots,S^{n_k})$ their wedge and fat wedge, respectively. There is a non trivial attaching map $\omega\colon S^{N-1}\to T$, with $N = n_1+\cdots+n_k$, for which

$$S^{n_1}\times\cdots\times S^{n_k} = T\cup_w e^N.$$

For $k\geq 2$ given homotopy classes $x_j\in\pi_{n_j}(X)$, consider the map these induce on the wedge $f\colon W\to X$ and define the *kth order Whitehead product set* $[x_1,\dots,x_k]\subseteq\pi_{N-1}(X)$ as the (possibly empty) set

$$\{\widetilde{f}\circ w\mid \widetilde{f}\colon T\to X \text{ is an extension of } f\},$$

as depicted by the following diagram:

$$\begin{tikzcd} & W \arrow[r, "f"] \arrow[d, hook] & X \\ S^{N-1} \arrow[r, "w"] & T \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

Observe that for $k=2$, one recovers the classical two-fold Whitehead product. This construction is elegantly captured by Quillen's DGL models for rational homotopy theory. As expertly explained in [63, Chap. V], given $k\geq 2$ homology classes $x_j\in H_*(L)$ of a DGL L model of X, define the *kth order Whitehead bracket set* $[x_1,\dots,x_k]\subseteq H_*(L)$ as the (possibly empty) set of homology classes arising from the DGL maps closing the diagram below, arising from translating the previous topological diagram into the DGL setting:

$$\begin{tikzcd} & \mathbb{L}(u_1,\dots,u_k) \arrow[r, "f"] \arrow[d, hook] & L \\ \mathbb{L}(w) \arrow[r, "w"] & \mathbb{L}(U) \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

This translation identifies topological and algebraic higher Whitehead products. Recall that, given any DGL L there is a structure of minimal L_∞ algebra on $H = H_*(L)$, unique up to L_∞ isomorphism, for which L and H are quasi-isomorphic L_∞ algebras. This structure can be inherited from L by exhibiting H as a homotopy retract of it, but there is no canonical way of doing so. Hence, different choices give rise to different (although L_∞ quasi-isomorphic) L_∞ structures on H. The rational homotopy type of a simply connected space X is governed by the Quillen model, which is in turn determined by many possibly different L_∞ structures on H.

One of the main results of Quillen's rational homotopy theory lets us identify

$$H = H_*(L)\cong\pi_*(\Omega X)\otimes\mathbb{Q}\cong\pi_{*-1}(X)\otimes\mathbb{Q}.$$

Roughly speaking, our aim in Chapter 2 (and in some sense, in the whole thesis, as will become clear by the end of this introduction) is to detect and recover (whenever defined) higher Whitehead products of order k of X via the operation of arity-k of an inherited L_∞ structure $\{\ell_n\}$ on H. The first and most general result, which is essential for many others, reads as follows.

Theorem 1 (Thm. 2.8) *Let $x_1,\dots x_k\in H$ and assume that $[x_1,\dots,x_k]$ is non empty. Then, for any homotopy retract of L, and for any $x\in[x_1,\dots,x_k]$,*

$$\varepsilon\ell_k(x_1,\dots,x_k) = x+\Gamma,\quad \Gamma=\sum_{j=1}^{k-1}\operatorname{Im}\ell_j,$$

where $\varepsilon = (-1)^{\sum_{i=1}^{k-1}(k-i)|x_i|}$. *In particular, if* $\ell_j = 0$ *for* $j \le k-1$, *then up to a sign,* $\ell_k(x_1,...,x_k) \in [x_1,...,x_k]$.

An immediate but interesting consequence of the result above is

Corollary 2 (Cor. 2.9) *If for some homotopy retract of L onto H, the induced higher brackets vanish up to arity* $k-1 \ge 2$, *then for any* $x_1,...,x_k \in H$, *the set* $[x_1,...,x_k]$ *is non empty, and moreover, it consists of the single homology class*

$$[x_1,...,x_k] = \{x = \varepsilon \ell_k(x_1,...,x_k)\}.$$

We say that an L_∞ structure *recovers higher Whitehead products* if $\pm\ell_k(x_1,...,x_k) \in [x_1,...,x_k]$. Theorem 1 suggests that it might not always be the case that ℓ_k recovers Whitehead products. In fact, it does not necessarily happen. Hence, we are led to define *adapted homotopy retracts* to a given $x \in [x_1,..,x_k]$ (see Def. 2.11). We prove then

Theorem 3 (Thm. 2.12) *Let* $x \in [x_1,...,x_k]$. *Then, for any homotopy retract of L adapted to x,*

$$\ell_k(x_1,...,x_k) = x.$$

We show (see thms. 2.13, 2.19 and 2.20, respectively) the existence of explicit spaces X and L_∞ structures on $\pi_*(\Omega X)\otimes\mathbb{Q}$ governing the rational homotopy type of X for which

(1) the evaluation $\ell_4(x_1,x_2,x_3,x_4)$ does not recover any Whitehead product. In fact, it happens that $\mathrm{Im}(\ell_4)\cap[x_1,x_2,x_3,x_4] = \emptyset$, even though $[x_1,x_2,x_3,x_4]$ is non empty and ℓ_4 is non trivial,

(2) for any choice $x \in [x_1,x_2,x_3]$ there exists an L_∞ structure $\{\ell_n\}$ on $\pi_*(\Omega X)\otimes\mathbb{Q}$ for which $\ell_3(x_1,x_2,x_3) = x$, and

(3) for many elements $x \in [x_1,x_2,x_3]$ there exists some L_∞ structure $\{\ell_n\}$ with $\ell_3(x_1,x_2,x_3) = x$, while for many others $x' \in [x_1,x_2,x_3]$ there exists no L_∞ structure $\{\ell'_n\}$ with $\ell'_3(x_1,x_2,x_3) = x'$.

The most general result ensuring the recovery of kth order Whitehead products via the operation ℓ_k is the following.

Theorem 4 (Thm. 2.15) *Let L be a DGL such that, on H,* $\ell_i = 0$ *for* $i \le k-2$ *with* $k \ge 3$. *If* $[x_1,...,x_k] \ne \emptyset$, *then*

$$\ell_k(x_1,...,x_k) \in [x_1,...,x_k].$$

Two important and useful consequences are collected as

Corollary 5 (Cor. 2.17, 2.18)

(1) If $x_1,x_2,x_3 \in H$ *are such that* $[x_1,x_2,x_3]$ *is non empty, then for any homotopy retract,*

$$\ell_3(x_1,x_2,x_3) \in [x_1,x_2,x_3].$$

(2) If $x_1,x_2,x_3,x_4 \in H$ *are such that* $[x_1,x_2,x_3,x_4]$ *is non empty, and H is abelian as graded Lie algebra, then for any homotopy retract,*

$$\ell_3(x_1,x_2,x_3,x_4) \in [x_1,x_2,x_3,x_4].$$

The results obtained up to this point are the core of Chapter 2, and provide a good framework in which to give applications. We turn to this task next.

First, we go to Sullivan models and we recall P. Andrews and M. Arkowitz's classical and celebrated result on higher Whitehead products [3, Thm. 5.4]. This result shows how the kth order Whitehead products of a simply connected space with finite type rational homology are captured by the kth homogeneous component of the differential of its Sullivan minimal model. This is Theorem 2.23 in the thesis, but we do not reproduce it here because it requires quite a bit of notation to set up. We give a careful exposition of the topic in Section 2.4, and then explain the precise relationship between L_∞ and Sullivan algebras (which is not original to this thesis). We translate [3, Thm. 5.4] to the L_∞ context. The result reads as follows, where H is the L_∞ structure equivalent to the Sullivan algebra $(\Lambda V, d)$ of a simply connected space X and $\langle\ ;\ \rangle$ is the pairing recalled in Section 2.4.

Theorem 6 (Thm. 2.29) *Let $x_j \in H$, $1 \le j \le k$, be such that $[x_1,\dots,x_k]$ is defined. Let $v \in V^{N-1}$ be such that $dv \in \Lambda^{\ge k} V$. Then, for every $x \in [x_1,\dots,x_k]$,*

$$\langle v; sx\rangle = \varepsilon\langle v; s\ell_k(x_1,\dots,x_k)\rangle.$$

We show how to recover P. Andrews and M. Arkowitz's Theorem 5.4 from our results, and furthermore how to extend it, in the presence of an adapted homotopy retract, avoiding the hypothesis that $dv \in \Lambda^{\ge k} V$.

Second, we turn to formality and coformality, a very important topic in several fields (see the introduction of Chapter 3 for a brief summary). Recall that either a DGA or a DGL L is *formal* if it has the same homotopy type, or it is weakly equivalent to its (co)homology H. In other words, if there exists a zig-zag of quasi-isomorphisms,

$$L \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} H.$$

Recall also that a connected space X is *formal* if $H^*(X;\mathbb{Q})$ is a CDGA model of X. Whenever X is a simply connected CW-complex of finite type, this is equivalent to saying that the rational homotopy type of X is characterized by its rational cohomology. We say that X is *intrinsically formal* when there exists exactly one rational homotopy type with cohomology algebra $H^*(X;\mathbb{Q})$. On the other hand, a simply connected space X is *coformal* if $\pi_*(\Omega X)\otimes\mathbb{Q}$ is a DGL model of X. In other words, the rational homotopy type of X is characterized by its rational homotopy Lie algebra. We say that X is *intrinsically coformal* when there exists exactly one rational homotopy type with rational homotopy Lie algebra $\pi_*(\Omega X)\otimes\mathbb{Q}$.

We explain how infinity structures are related to the concept of formality, and do some work whose final aim is to show how higher order Whitehead brackets can be useful in discarding formality of a completely arbitrary DGL, giving the following two criteria:

Theorem 7 (Thm. 3.8) *Let L be a DGL and let $x_1,\dots,x_k \in H = H_*(L)$ be such that $[x_1,\dots,x_k]$ is non empty. Denote by $[,\dots,]^H$ the higher order Whitehead products in H. Then, L is not formal if one of the following conditions hold:*

(1) $0 \notin [x_1,\dots,x_k]$.

(2) The sets $[x_1,\dots,x_k]$ and $[x_1,\dots,x_k]^H$ are not bijective.

We show how and when to use the *zero criterion* and the *cardinality criterion* above in several examples. Finally, we characterize the intrinsic coformality of certain spaces, which we summarize together here next. Observe that item (2) below is the Eckmann-Hilton dual of Baues' theorem ([5], [26, Thm. 1.5]).

Theorem 8 (Thms. 3.12,3.14)

(1) The product of k simply connected odd dimensional spheres $S^{n_1}\times\cdots\times S^{n_k}$ is intrinsically coformal if and only if $k\le 4$ or $k\ge 5$ and $n_i\neq n_{j_1}+\cdots+n_{j_r}-1$ for every i and subset $\{n_{j_1},\dots,n_{j_r}\}\subseteq\{n_1,\dots,n_k\}$ with $r\ge 4$ even.

(2) An arbitrary product of simply connected even dimensional Eilenberg-Mac Lane spaces is intrinsically coformal.

The thesis then studies the Eckmann-Hilton dual problem, turning to the cohomology side of matters. We refer the reader to Chapter 4 to find all details in the sketchy explanations that follow. We will fix a (not necessarily commutative) DGA A and denote by H its cohomology. Hereafter, the cohomology class of a cocyle z is denoted by $[z]$, and $\overline{z}=(-1)^{|z|+1}z$.

We assume the reader of this summary is familiar with the definition of higher Massey products (see Section 4.1 to recall it). In order to fix notation, we denote the triple Massey product set of the cohomology classes $x_1,x_2,x_3\in H$ by $\langle x_1,x_2,x_3\rangle$, and that (whenever $x_1x_2=x_2x_3=0$) it consists of the set of all cohomology classes represented by a cocycle of the form

$$\overline{a}_{01}a_{13}+\overline{a}_{02}a_{23},$$

where a_{01},a_{12},a_{23} are cocycles representing x_1,x_2,x_3 respectively, and a_{02},a_{13} are such that $da_{02}=\overline{a}_{01}a_{12}$ and $da_{13}=\overline{a}_{12}a_{23}$. If the condition $x_1x_2=x_2x_3=0$ is not satisfied, $\langle x_1,x_2,x_3\rangle$ is the empty set. Each set $\{a_{ij}\}$ is a *defining system* for $\langle x_1,x_2,x_3\rangle$, and different choices of defining systems give rise to possibly different cohomology classes in the set $\langle x_1,x_2,x_3\rangle$. Higher order Massey products $\langle x_1,\dots,x_n\rangle$ of n cohomology classes $x_1,\dots,x_n$ are inductively defined in an analogous manner, depending on the condition that certain lower order Massey products are defined and contain the zero class.

Given any DGA A, there is a structure of minimal A_∞ algebra on H, unique up to A_∞ isomorphism, for which A and H are quasi-isomorphic A_∞ algebras. This structure can be inherited from A by exhibiting H as a homotopy retract of it, but there is no canonical way of doing so. Hence, different choices give rise to different (although A_∞ quasi-isomorphic) A_∞ structures on H. Fixed a Massey product set $\langle x_1,\dots,x_n\rangle$, we will say that the A_∞ structure $\{m_n\}$ *recovers Massey products* if $\pm m_n(x_1,\dots,x_n)\in\langle x_1,\dots,x_n\rangle$.

We wish to underline here a quite important point. What we do in this final chapter of the thesis applies to a more general context than that of strict rational homotopy theory, because *our theorems hold for fields of characteristic* $p>0$. Nevertheless, we are motivated by the following well known but remarkable fact in characteristic zero: the rational homotopy type of a nilpotent space with finite type rational cohomology X is governed by its Sullivan minimal model M_X, which is in turn determined by many possibly different commutative A_∞ structures on its cohomology algebra H. One of Sullivan's main theorems states that

$$H=H^*(M_X)\cong H^*(X;\mathbb{Q}).$$

In the same spirit as in the beginning of the thesis, our main motivating goal is to detect and recover (whenever defined) higher Massey products of order k of X via the operation of arity k of an inherited A_∞ structure on H. To do so, we prove the Eckmann-Hilton dual of all the results in Chapter 2, but with some extra results.

We start by defining *adapted retracts* to Massey products (Def. 4.3), and prove the Eckmann-Hilton dual of Theorem 3:

Theorem 9 (Thm. 4.4) *Let* $x \in \langle x_1, ..., x_n \rangle$. *Then, for any homotopy retract adapted to* x,

$$m_n(x_1, ..., x_n) = (-1)^{\varepsilon} x,$$

where $\varepsilon = 1 + |x_{n-1}| + |x_{n-3}| + \cdots$.

We show (see examples 4.5 and 4.9, respectively) the existence of explicit spaces X and A_∞ structures on $H^*(X;\mathbb{Q})$ governing the rational homotopy type of X for which,

(1) the evaluation $m_4(x_1, x_2, x_3, x_4)$ does not recover Massey products. That is, $\pm m_4(x_1, x_2, x_3, x_4) \notin \langle x_1, x_2, x_3, x_4 \rangle$, even though $\langle x_1, x_2, x_3, x_4 \rangle$ is non empty and m_4 is not trivial, and

(2) there are countably many different elements in $\langle x_1, x_2, x_3 \rangle$, but for every A_∞ structure $\{m_n\}$ induced on H, we have that $m_3(x_1, x_2, x_3) = x$ for the same cohomology class $x \in \langle x_1, x_2, x_3 \rangle$.

We emphasize here an important point of this thesis. It has been widely accepted by the mathematical community that Theorem 9 above holds in complete generality, that is, without any assumptions on the homotopy retract. This was stated in Theorem 3.1 of the very interesting and remarkable reference [42], but unfortunately the proof contains a gap. Item (1) above is a clear counterexample. We devote Section 4.2 to explain what fails in the mentioned proof. We amend the statement of [42, Thm 3.1], obtaining the following result.

Theorem 10 (Thm. 4.11) *Let* A *be a DGA and assume* $\langle x_1, ..., x_n \rangle$ *is defined,* $n \geq 3$. *Then, for any homotopy retract of* A *such that the elements*

$$\left\{a_{ij} := K\lambda_{j-i+1}(x_i, ..., x_j) \mid 2 < j - i < n - 1\right\}$$

assemble into a defining system,

$$\varepsilon\, m_n(x_1, ..., x_n) \in \langle x_1, ..., x_n \rangle.$$

Provided that the Massey product set $\langle x_1, ..., x_n \rangle$ is non empty, we may use this result to construct a particular homotopy retract such that the assumption in this result holds and therefore, the nth multiplication $m_n(x_1, \ldots, x_n)$ in the corresponding A_∞ algebra structure on H recovers a Massey product.

Theorem 11 (Thm. 4.12) *If* $\langle x_1, ..., x_n \rangle \neq \emptyset$, *then there exists an* A_∞ *structure on* H *such that, up to a sign,*

$$m_n(x_1, ..., x_n) \in \langle x_1, ..., x_n \rangle\,.$$

We extend Theorem 9 to the more general following result which complements it.

Theorem 12 (Thm. 4.7) *If, given a homotopy retract onto* H, *the set* $\left\{b_{ij}\right\}$ *given by*

$$b_{ij} = K d a_{ij} \text{ for } j - i \geq 2, \quad \text{and} \quad b_{i-1,i} = a_{i-1,i} \text{ for } i = 1, ..., n$$

is a defining system for some Massey product element $x \in \langle x_1, ..., x_n \rangle$, *then* $m_n(x_1, ..., x_n) = \varepsilon x$.

The following result is a useful, easy to check criterion to decide if it is possible to recover a particular Massey product element.

Proposition 13 *(Prop. 4.10)* If there exists a defining system $\{a_{ij}\}$ for $x \in \langle x_1, ..., x_k \rangle$ such that $\left\{d a_{ij}\right\}_{j-i\geq 2}$ is a linearly independent set, then there exists an A_∞ structure on H such that $m_k(x_1, ..., x_k) = \pm\, x$.

We turn to Section 4.3, which closes the thesis. It consists of the statement and proof of the results that follow, which are the precise Eckmann-Hilton duals to most of the results proven for higher Whitehead products.

Theorem 14 (Thm. 4.13) *If* $\langle x_1,\dots,x_n\rangle \neq \emptyset$*, then, for any homotopy retract, and for any* $x \in \langle x_1,\dots,x_n\rangle$*,*

$$\varepsilon m_n(x_1,\dots,x_n) = x + \Gamma, \qquad \Gamma \in \sum_{j=1}^{n-1} \operatorname{Im}\left(m_j\right),$$

where $\varepsilon = (-1)^{\sum_{j=1}^{n-1}(n-j)|x_j|}$*. In particular,* $m_j = 0$ *for* $j \leq n-1$ *implies that* $\varepsilon m_n(x_1,\dots,x_n) \in \langle x_1,\dots,x_n\rangle$*.*

Corollary 15 (Cor. 4.14) *Let* A *be a DGA such that for some homotopy retract of* A *into* H*, the induced higher multiplications satisfy* $m_1 = \dots = m_{k-1} = 0$*, with* $k \geq 2$*. Then, for any* $x_1,\dots,x_k \in H$*, one has that* $\langle x_1,\dots,x_k\rangle$ *is defined, and moreover, it consists of a single cohomology class:*

$$\langle x_1,\dots,x_k\rangle = \{x\} = \{\varepsilon m_k(x_1,\dots,x_k)\},$$

where $x = \varepsilon m_k(x_1,\dots,x_k)$*, with* ε *as in Proposition 4.13.*

Theorem 16 (Thm. 4.16) *If for some homotopy retract of* A *onto* H*, the induced higher multiplications* m_n *vanish up to* m_{k-2}*, with* $k \geq 3$*, and* $\langle x_1,\dots,x_k\rangle \neq \emptyset$*, then*

$$\varepsilon m_k(x_1,\dots,x_k) \in \langle x_1,\dots,x_k\rangle,$$

with ε *as in Proposition 4.13.*

The following are two important and useful corollaries.

Corollary 17 (Cor. 4.18, 4.19)

(1) If $x_1, x_2, x_3 \in H$ *are such that* $\langle x_1, x_2, x_3\rangle$ *is non empty, then for any homotopy retract,*

$$m_3(x_1,x_2,x_3) \in \langle x_1,x_2,x_3\rangle.$$

(2) If $x_1,x_2,x_3,x_4 \in H$ *are such that* $\langle x_1,x_2,x_3,x_4\rangle$ *is non empty, and* H *has trivial product, then for any homotopy retract,*

$$m_3(x_1,x_2,x_3,x_4) \in \langle x_1,x_2,x_3,x_4\rangle.$$

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CHAPTER 1

Background

In this chapter we summarize all the notation and standard results on which this work stands.

1.1 Notation and conventions

A *space* is a pointed topological space of the homotopy type of a CW-complex, and a map between spaces will always be continuous and preserve the base point. Notationally, we do not distinguish a map from its homotopy class. The base field for all constructions is $\mathbb{Q}$ unless otherwise specified. Gradings are taken over the integers $\mathbb{Z}$. We write $A \in \mathscr{C}$ to indicate that A is an object of the category $\mathscr{C}$.

We will make use of the *Koszul sign convention*: in any algebraic formula, whenever two graded objects x and y are permuted, the sign $(-1)^{|x||y|}$ appears.

The *suspension* of a DG module (M,d) is the DG module (sM,d), where $(sM)_i = M_{i-1}$, and whose differential is $d(sm) = -sd(m)$. Its *desuspension* is $(s^{-1}M,d)$, where $\left(s^{-1}M\right)_i = M_{i+1}$, and whose differential is $d(s^{-1}m) = -s^{-1}d(m)$.

The permutation group of n elements will be denoted by S_n. A permutation $\sigma \in S_{p+q}$ is said to be a (p,q)*-shuffle* if

$$\sigma(1) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \cdots < \sigma(p+q).$$

We will denote by $S(p,q)$ those permutations of S_{p+q} which are (p,q)-shuffles, and by $\widetilde{S}(p,q)$ those (p,q)-shuffles σ verifying $\sigma(1) = 1$.

1.2 Infinity structures

An L_∞ *algebra* is a graded vector space $L = \{L_n\}_{n\in\mathbb{Z}}$ together with skew-symmetric linear maps $\ell_k : L^{\otimes k} \to L$ of degree $k-2$, for $k \geq 1$, satisfying the *generalized Jacobi identities* for every $n \geq 1$:

$$\sum_{i+j=n+1} \sum_{\sigma\in S(i,n-i)} \varepsilon(\sigma)\,\mathrm{sgn}(\sigma)(-1)^{i(j-1)} \ell_j \left(\ell_i \left(x_{\sigma(1)},\dots,x_{\sigma(i)}\right), x_{\sigma(i+1)},\dots,x_{\sigma(n)}\right) = 0.$$

Here, $\varepsilon(\sigma)$ and $\mathrm{sgn}(\sigma)$ stand for the Koszul sign and the signature associated to σ, respectively. A *differential graded Lie algebra*, DGL henceforth, is an L_∞ algebra L for which $\ell_k = 0$ for all $k \geq 3$. In this case $\ell_1 = \partial$ and $\ell_2 = [\ ,\ ]$ are the *differential* and the *Lie bracket*, respectively.

An L_∞ algebra is *minimal* if $\ell_1 = 0$, *abelian* if $\ell_k = 0$ for every $k \geq 2$, and *reduced* if $L_n = 0$ for every $n \leq 0$. The *homology* of an L_∞ algebra is defined as the homology of the chain complex (L, ℓ_1).

An L_∞ *morphism* $f : L \to L'$ is a family of skew-symmetric linear maps $\left\{f^{(n)} : L^{\otimes n} \to L'\right\}$ of degree $n-1$ such that the following equation is satisfied for every $n \geq 1$:

$$\begin{aligned}
&\sum_{p=1}^{n} \sum_{\sigma \in S(p,n-p)} \varepsilon(\sigma)\operatorname{sgn}(\sigma)(-1)^{p(n-p)} f^{(n+1-p)}\left(\ell_p\left(x_{\sigma(1)},\dots,x_{\sigma(p)}\right), x_{\sigma(p+1)},\dots,x_{\sigma(n)}\right) = \\
&\sum_{k\geq 1} \sum_{\substack{\tau \in S(i_1,\dots,i_k) \\ i_1+\dots+i_k=n}} \varepsilon(\tau)\operatorname{sgn}(\tau)(-1)^{\eta}\ell'_k\left(f^{(i_1)}\left(x_{\tau(1)},\dots,x_{\tau(i_1)}\right),\dots,f^{(i_k)}\left(x_{\tau(n-i_k+1)},\dots,x_{\tau(n)}\right)\right).
\end{aligned} \tag{1.1}$$

Here, $\eta = \sum_{j=1}^{n}(n-j)(i_j-1) + \sum_{j=2}^{n}(i_j-1)\sum_{l=1}^{i_j-1}|x_{\tau(l)}|$, although these signs are not relevant nor useful for our purposes because we will reinterpret L_∞ morphisms in a more compact form (Thm. 1.4). Such an L_∞ morphism is said to be an L_∞ *quasi-isomorphism* if $f_1 : (L,\ell_1) \to \left(L',\ell'_1\right)$ is a quasi-isomorphism of chain complexes, and it is said to be an L_∞ *isomorphism* if there exist an inverse L_∞ morphism for f. A quasi-isomorphism of minimal L_∞ algebras is an isomorphism ([34, Thm. 4.6]).

An A_∞ *algebra* is a graded vector space $A = \left\{A^n\right\}_{n\in\mathbb{Z}}$ together with linear maps $m_k : A^{\otimes k} \to A$ of degree $2-k$, for $k \geq 1$, satisfying the *Stasheff identities* for every $i \geq 1$:

$$\sum_{k=1}^{i}\sum_{n=0}^{i-k}(-1)^{k+n+kn} m_{i-k+1}\left(\mathrm{id}^{\otimes n} \otimes m_k \otimes \mathrm{id}^{\otimes i-k-n}\right) = 0.$$

An A_∞ algebra A is *commutative*, or it is a C_∞ *algebra* if, for each $k \geq 2$, the k-th multiplication m_k vanishes on the *shuffle products*, that is, $m_k \nu_k = 0$, with

$$\nu_k \colon A^{\otimes k} \to A^{\otimes k}, \quad \nu_k(a_1 \otimes \cdots \otimes a_k) = \sum_{i=1}^{k-1} \sum_{\sigma \in S(i,k-i)} \operatorname{sgn}(\sigma)\varepsilon(\sigma)\, a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)},$$

where $S(i,k-i)$ denotes the set of $(i,k-i)$-shuffles.

A *differential graded algebra*, DGA henceforth (CDGA if it is commutative), is an A_∞ algebra for which $m_k = 0$ for all $k \geq 3$. In this case $m_1 = d$ and $m_2 = m$ are the *differential* and *multiplication*, respectively.

Let V be a graded vector space. Define the *free algebra*, or *tensor algebra*, on V as

$$TV := \bigoplus_{n\geq 0} T^n V = \bigoplus_{n\geq 0} V^{\otimes n} = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots,$$

where the product is given by concatenation:

$$(v_1 \otimes \cdots \otimes v_k)(v_{k+1} \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_n.$$

Define the *free commutative algebra*, or *symmetric algebra*, on V as $\Lambda V := TV/I$, where I is the ideal generated by elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x$, for all $x, y \in V$. Its elements are equivalence classes $v_1 \wedge \ldots \wedge v_n$ (sometimes also denoted by juxtaposition, $v_1 \cdots v_n$) where

$$v_1 \wedge \ldots \wedge v_n = \varepsilon(\sigma) v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(n)} \quad \forall\, \sigma \in S_n.$$

A *morphism* of A_∞ algebras $f: A \to B$ is a family of linear maps $f_n: A^{\otimes n} \to B$ of degree $1-n$ such that the following equation holds for every $n \geq 1$:

$$\sum_{\substack{n=r+s+t\\ s\geq 1\\ r,t\geq 0}} (-1)^{r+st} f_{r+1+t}\left(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}\right) = \sum_{\substack{1\leq r\leq n\\ n=i_1+\cdots+i_r}} (-1)^{\alpha} m'_r\left(f_{i_1}\otimes\cdots\otimes f_{i_r}\right) \tag{1.2}$$

where $\alpha = \sum_{k=1}^{r-1} k(i_{r-k}-1)$.

An A_∞ algebra is *minimal* if $m_1 = 0$. The *cohomology* of an A_∞ algebra is defined as the cohomology of the cochain complex (A, m_1). An A_∞ morphism $f: A \to B$ is said to be an A_∞ *quasi-isomorphism* if $f_1 : (A, m_1) \to \left(B, m'_1\right)$ is a quasi-isomorphism of cochain complexes, and it is said to be an A_∞ *isomorphism* if there exists an inverse A_∞ morphism for f. A quasi-isomorphism of minimal A_∞ algebras is an isomorphism ([37]).

A *strictly unital* A_∞ algebra is an A_∞ algebra A endowed with an element $1 \in A^0$ such that $m_1(1) = 0$, $m_2(1, a) = a = m_2(a, 1)$ for all $a \in A$ and such that, for all $k \geq 3$, and all $a_1, \ldots, a_k$ in A, the product $m_k(a_1, \ldots, a_k)$ vanishes if any $a_j = 1$. An A_∞ morphism between strictly unital A_∞ algebras is *strictly unital* if we have $f_1(1_A) = 1_B$ and if for all $k \geq 2$ and all $a_1, \ldots, a_k$, the element $f_k(a_1, \ldots, a_k)$ vanishes if some $a_j = 1_A$. If A is a strictly unital A_∞ algebra, then there exists a canonical strict (hence strictly unital) morphism $\eta : \mathbb{K} \to A$ with $1_{\mathbb{K}} \mapsto 1_A$. It is *augmented* if it is moreover provided with a strictly unital morphism $\varepsilon : A \to \mathbb{K}$ such that $\varepsilon \circ \eta = \mathrm{id}_{\mathbb{K}}$. A *morphism of augmented A_∞ algebras* is a strictly unital morphism $f : A \to B$ such that $\varepsilon_B \circ f = \varepsilon_A$. The *augmentation ideal* is defined as $\bar{A} = \mathrm{Ker}(\varepsilon)$.

Let V be a graded vector space. The tensor algebra TV is endowed with a graded Lie algebra structure under the *commutator brackets*, defined on homogeneous elements $x, y \in TV$ as:

$$[x, y] = x \otimes y - (-1)^{|x||y|} y \otimes x.$$

Define the *free graded Lie algebra on V*, and denote it by $\mathbb{L}(V)$, as the graded Lie subalgebra of TV generated by V. Observe that $\mathbb{L}(V)$ is *filtered by bracket length*,

$$\mathbb{L}(V) = \bigoplus_{n\geq 1} \mathbb{L}^n(V),$$

where $\mathbb{L}^n(V) = \mathbb{L}(V) \cap T^n V$ consists of those elements with Lie bracket length equal to n. A *free DGL* is a DGL whose underlying graded Lie algebra is of the form $\mathbb{L}(V)$ for some graded vector space V. In particular, the differential ∂ decomposes as a sum of derivations

$$\partial = \sum_{n\geq 1} \partial_n,$$

where the sum squares to zero. Each ∂_n is determined by its restriction $\partial_n| : V \to \mathbb{L}^n(V)$. Conversely, any differential ∂ on $\mathbb{L}(V)$ is determined by a family of degree -1 linear maps ∂_n as above. We call ∂_1 the *linear part* of ∂, which turns (V, ∂_1) into a chain complex, and ∂_2 the *quadratic part* of ∂.

An A_∞ *coalgebra* is a graded vector space $C = \{C_n\}_{n\in\mathbb{Z}}$ together with linear maps $\Delta_k : C \to C^{\otimes k}$ of degree $k-2$, for $k \geq 1$, such that for all $i \geq 1$,

$$\sum_{k=1}^{i}\sum_{n=0}^{i-k} (-1)^{k+n+kn}\left(\mathrm{id}^{\otimes i-k-n} \otimes \Delta_k \otimes \mathrm{id}^n\right)\Delta_{i-k+1} = 0.$$

An A_∞ coalgebra C is *cocommutative*, or it is a C_∞ *coalgebra* if, for each $k \geq 2$, the *unshuffle products* vanish on the image of the k-th comultiplication Δ_k, that is, $\tau \circ \Delta_k = 0$ with

$$\tau : C^{\otimes k} \to C^{\otimes k}, \quad \tau(c_1 \otimes \cdots \otimes c_k) = \sum_{i=1}^{k} \sum_{\sigma\in S(i,k-i)} \varepsilon_\sigma\, c_{\sigma^{-1}(1)} \otimes \cdots \otimes c_{\sigma^{-1}(k)}.$$

A *differential graded coalgebra*, DGC henceforth (CDGC if it is cocommutative), is an A_∞ coalgebra for which $\Delta_k = 0$ for all $k \geq 3$. In this case $\Delta_1 = \delta$ and $\Delta_2 = \Delta$ are the *codifferential* and *comultiplication* respectively. A 1*-connected* DGC is of the form $C = \mathbb{Q} \oplus \{C_n\}_{n\geq 2}$.

Let V be a graded vector space. Define the *tensor coalgebra* on V as the graded coalgebra having the same underlying graded vector space as the tensor algebra $TV = \bigoplus_{n\geq 0} T^n V = \bigoplus_{n\geq 0} V^{\otimes n}$, where $1 \in V^0 = \mathbb{K}$ is the counit, $\Delta(1) = 1 \otimes 1$, and the coproduct on the rest of elements is given by the following formula:

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=0}^{n} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n).$$

Define the *cofree cocommutative coalgebra* or *symmetric coalgebra* on V as having underlying graded vector space ΛV, where $\Delta(1) = 1 \otimes 1$ and

$$\Delta(v_1 \wedge \ldots \wedge v_n) = \sum_{i=0}^{n} \sum_{\sigma \in S(i,n-i)} \varepsilon(\sigma)(v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \wedge \ldots \wedge v_{\sigma(n)}).$$

It will be clear from the context if we are treating TV or ΛV as an algebra or as a coalgebra. In both cases, we use the notation T^+V and Λ^+V for the *reduced tensor algebra* and the *reduced symmetric algebra*, respectively. That is,

$$T^+V = \bigoplus_{n\geq 1} T^n V \qquad \text{and} \qquad \Lambda^+ V = \bigoplus_{n\geq 1} \Lambda^n V.$$

Here, $\Lambda^n V = \operatorname{span}\{v_1 \wedge \ldots \wedge v_n\}$ is the subspace of ΛV generated by the elements of *word length* n, for any $n \geq 1$.

A morphism of A_∞ coalgebras $f : C \to D$ is a family of linear maps $f_n : C \to D^{\otimes n}$ of degree $n-1$ such that the following equation holds for every $n \geq 1$:

$$\sum_{\substack{n=r+s+t\\ s\geq 1\\ r,t\geq 0}} (-1)^{2st-rs-r-2t} \left(\mathrm{id}^{\otimes r} \otimes \Delta_s \otimes \mathrm{id}^{\otimes t}\right) f_{r+t+1} = \sum_{\substack{1\leq r\leq n\\ n=i_1+\cdots+i_r}} (-1)^{\beta} \left(f_{i_1} \otimes \cdots \otimes f_{i_r}\right) \Delta'_r, \tag{1.3}$$

where $\beta = \sum_{k=1}^{r} (2r-k-2)(i_k-1)$.

A *strictly counital A_∞ coalgebra* is an A_∞ coalgebra C endowed with an element $1 \in C_0$ such that we have $\Delta_1(1) = 0$, $\Delta_2(1) = 1 \otimes 1$ and such that, for all $i > 2$, the coproduct $\Delta_i(1)$ vanishes. If C and D are strictly counital A_∞ coalgebras, a morphism of A_∞ coalgebras $f : C \to D$ is *strictly counital* if we have $f_1(1_C) = 1_D$ and if for all $k \geq 2$, the element $f_k(1, \ldots, 1)$ vanishes. Each strictly counital A_∞ coalgebra is canonically endowed with a strict (hence strictly counital) morphism $\eta : C \to \mathbb{K}$ mapping 1_C to $1_{\mathbb{K}}$. It is *coaugmented* if it is moreover endowed with a strictly counital morphism $\varepsilon : \mathbb{K} \to C$ such that $(\mathrm{id}_C \otimes \varepsilon)\Delta = (\varepsilon \otimes \mathrm{id}_C)\Delta = \mathrm{id}_C$. A *morphism of coaugmented A_∞ coalgebras* is a strictly counital morphism $f : C \to D$ such that $\varepsilon_D \circ f = \varepsilon_C$. The kernel of the coaugmentation is denoted by $\bar{C} = \operatorname{Ker}(\varepsilon)$.

An A_∞ coalgebra is *minimal* if $\Delta_1 = 0$. The *homology* of an A_∞ coalgebra is defined as the homology of the chain complex (C, Δ_1).

A morphism of A_∞ coalgebras $f : C \to D$ is said to be an A_∞ *quasi-isomorphism* if $f_1 : (C, \Delta_1) \to \left(D, \Delta'_1\right)$ is a quasi-isomorphism of chain complexes. It is said to be an A_∞ *isomorphism* if there exists an inverse A_∞ morphism for f. A quasi-isomorphism of minimal A_∞ coalgebras is an isomorphism.

1.3 The homotopy transfer theorem

Let (M,d) and $\left(N,d'\right)$ be (co)chain complexes. We say that N is a *homotopy retract* of M if there exist chain maps i,q and a chain homotopy K

$$\begin{tikzcd} K \arrow[loop left] & (M,d) \arrow[r, shift left, "q"] & \left(N,d'\right) \arrow[l, shift left, "i"] \end{tikzcd} \tag{1.4}$$

satisfying the following two conditions:

1. (Deformation retraction) $\quad qi=\mathrm{id}_N \quad$ and $\quad \mathrm{id}_M - iq = dK+Kd, \quad$ and
2. (Annihilation conditions) $\quad K^2 = Ki = qK = 0.$

Here, $|q|=|i|=0$, and $|K|=1$ or -1, if we are working with chain or cochain complexes, respectively. A homotopy retract like above is denoted by (M,N,i,q,K).

Remark 1.1 If, given a diagram as in (1.4), the annihilation conditions $Ki=qK=0$ are not satisfied, then the new homotopy $G=(Kd+dK)K(Kd+dK)$ does satisfy them. If $K^2\neq 0$, then the new homotopy $G'=KdK$ does satisfy it. Hence, combining both constructions, one can always obtain a homotopy satisfying the annihilation conditions.

Proposition 1.2 Let (M,d) be a (co)chain complex and $H=(H(M,d),0)$. There is a bijective correspondence between homotopy retracts of M of the form (M,H,i,q,K) and decompositions of M of the form

$$M = A\oplus dA\oplus C,$$

where A is a complement of $\operatorname{Ker} d$ (and thus $d\colon A\xrightarrow{\cong} dA$) and $C\cong H$.

Proof: Let $M=A\oplus dA\oplus C$ be such a decomposition. Define $i\colon H\cong C\hookrightarrow M$, $q\colon M\twoheadrightarrow C\cong H$ and $K(A)=K(C)=0$, $K\colon \partial A\xrightarrow{\cong} A$. It is easy to check that (M,H,i,q,K) is a homotopy retract.

Conversely, let (M,H,i,q,K) be a homotopy retract of M. Then, one has:

$$dKd = d. \tag{1.5}$$

Indeed, $dKd = d(\mathrm{id}_M - iq - dK) = d\,\mathrm{id}_M = d$. We define

$$A = KdM$$

and check that this is a complement of $\operatorname{Ker} d$: If $x\in A\cap \operatorname{Ker} d$, then $dx=0$ and $x=Kda$. Hence, by formula (1.5), $da=dKda=dx=0$ and therefore $x=Kda=0$. On the other hand, any $x\in M$ can be written as $x=Kdx+(x-Kdx)$ where $Kdx\in KdM$ and $x-Kdx\in \operatorname{Ker} d$. To finish, define $C=\operatorname{Im} i$ and another trivial computation shows that $M=A\oplus dA\oplus C$. $\square$

If (M,N,i,q,K) is a homotopy retract, and M is endowed with an L_∞, A_∞ or C_∞ structure (of algebra, or coalgebra), then it is possible to induce an infinity quasi-isomorphic structure on N of the same type. Moreover, there are explicit formulas for the transferred structures, and for the involved maps. This is a particular instance of the so-called *homotopy transfer theorem* [20, 35, 36, 40, 49], also known as the *homological perturbation lemma* [24, 25, 28, 33, 7]. We state this result in the cases in which M has a DGL, (C)DGA or (C)DGC structure, as these are the relevant for this thesis. For the next result, the annihilation conditions on a homotopy retract are not required. In what follows, for any $k\geq 2$, $\mathscr{P}\mathscr{T}_k$ denotes the set of isomorphism classes of planar rooted binary trees of k leaves, while $\mathscr{T}_k$ consists of isomorphism classes of (non planar) rooted binary trees with k leaves.

Theorem 1.3 *Let (M,d) be a (co)chain complex, and assume that (M,H,i,q,K) is a homotopy retract of M onto its (co)homology. Then:*

(1) If $(M,d)=(L,\partial)$ is a DGL, then there exists an L_∞ structure $\{\ell_k\}$ on H, unique up to isomorphism, and L_∞ quasi-isomorphisms

$$(L,\partial) \underset{I}{\overset{Q}{\rightleftarrows}} (H,\{\ell_k\}),$$

such that $I_1=i$ and $Q_1=q$. Moreover, the transferred higher brackets and the components of I are explicitly given by

$$\ell_k=\sum_{T\in\mathscr{T}_k}\frac{\ell_T}{|\mathrm{Aut}(T)|},\qquad I_k=\sum_{T\in\mathscr{T}_k}\frac{\ell'_T}{|\mathrm{Aut}(T)|}. \tag{1.6}$$

(2) If $(M,d)=(A,d)$ is a DGA, then there exists an A_∞ algebra structure $\{m_k\}$ on H, unique up to isomorphism, and A_∞ algebra quasi-isomorphisms

$$(A,d) \underset{I}{\overset{Q}{\rightleftarrows}} (H,\{m_k\}),$$

such that $I_1=i$ and $Q_1=q$. Moreover, the transferred higher products and the components of I are explicitly given by

$$m_k=\sum_{T\in\mathscr{P}\mathscr{T}_k} m_T,\qquad I_k=\sum_{T\in\mathscr{P}\mathscr{T}_k} m'_T. \tag{1.7}$$

If (A,d) is commutative, then $(H,\{m_k\})$ is a C_∞ algebra.

(3) If $(M,d)=(C,\delta)$ is a DGC, then there exists an A_∞ coalgebra structure $\{\Delta_k\}$ on H, unique up to isomorphism, and A_∞ coalgebra quasi-isomorphisms

$$(C,\delta) \underset{I}{\overset{Q}{\rightleftarrows}} (H,\{\Delta_k\}),$$

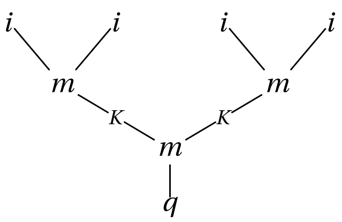
such that $I_1=i$ and $Q_1=q$. Moreover, the transferred higher coproducts and the components of I are explicitly given by

$$\Delta_k=\sum_{T\in\mathscr{P}\mathscr{T}_k} \Delta_T,\qquad I_k=\sum_{T\in\mathscr{P}\mathscr{T}_k} \Delta'_T. \tag{1.8}$$

If (C,δ) is cocommutative, then $(H,\{m_k\})$ is a C_∞ coalgebra.

We describe in each case, the explicit description of the transferred structure in the above theorem.

Let first $M=(A,d,m)$ be a (commutative) DGA (m denotes its multiplication), for each $T\in\mathscr{P}\mathscr{T}_k$, we define a linear map $m_T\colon H^{\otimes k}\to H$ as follows: label the root by q, each internal edge by K, each internal vertex by m, and each leaf by i. Then, m_T is defined as the composition of the different labels moving down from the leaves to the root. For instance, the tree $T\in\mathscr{P}\mathscr{T}_4$

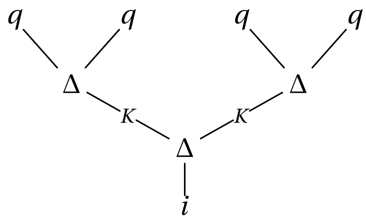


produces $m_T = q\circ m\circ(K\circ m\otimes K\circ m)\circ i^{\otimes 4}\colon H^{\otimes 4}\to H$. The transferred (commutative) A_∞ algebra structure in H provided by Thm. 1.3 is given by $\{m_k\}_{k\geq 1}$, where $m_1 = d$ and, for $k\geq 2$,

$$m_k = \sum_{T\in\mathscr{P}\mathscr{T}_k} m_T.$$

The map m'_T appearing in the description of I is defined as m_T except that we label the root by K instead of q.

Dually, let $M=(C,\delta,\Delta)$ be a (commutative) DGC. For each $T\in\mathscr{P}\mathscr{T}_k$, we define a linear map $\Delta_T\colon V\to V^{\otimes k}$ as follows: label the root by i, each internal edge by K, each internal vertex by Δ, and each leaf by q. Then, Δ_T is defined as the composition of the different labels moving up from the root to the leaves. Now, the same tree above

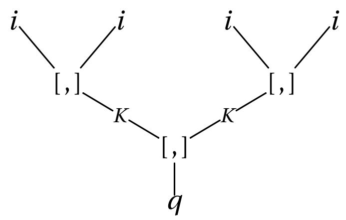


yields the map $\Delta_T = q^{\otimes 4}\circ(\Delta\circ K\otimes\Delta\circ K)\circ\Delta\circ i\colon H\to H^{\otimes 4}$. By Thm. 1.3 the transferred (commutative) A_∞ coalgebra structure in H is given by $\{\Delta_k\}_{k\geq 1}$, where $\Delta_1=\delta$ and, for $k\geq 2$,

$$\Delta_k = \sum_{T\in\mathscr{P}\mathscr{T}_k} \Delta_T.$$

The map Δ'_T appearing in the description of I is defined as Δ_T except that we label the leaves by K instead of q.

The case in which $M=(L,\partial,[\,,\,])$ is a DGL is slightly different. For each T in $\mathscr{T}_k$ define a linear map $\ell_T\colon V^{\otimes k}\longrightarrow V$ as follows: choose a planar embedding of T, label each internal edge by K, each internal vertex by $[\,,\,]$, and each leaf by i. Then, the map $\widetilde{\ell}_T\colon H^{\otimes k}\longrightarrow H$ is defined as the composition of the different labels moving down from the leaves to the root. For instance the planar embedding



of the corresponding $T\in\mathscr{T}_4$ produces the map

$$q\circ[\,,]\circ(K\circ[\,,]\otimes K\circ[\,,])\circ i^{\otimes 4}.$$

Define

$$\ell_T = \widetilde{\ell}_T\circ\mathscr{S}_k$$

where

$$\mathscr{S}_k\colon H^{\otimes k}\to H^{\otimes k},\quad \mathscr{S}_k(a_1\dots a_k) = \sum_{\sigma\in S_k}\varepsilon(\sigma)\,\mathrm{sgn}(\sigma)\,a_{\sigma(1)}\dots a_{\sigma(k)},$$

is the symmetrization map, in which $\mathrm{sgn}(\sigma)$ denotes, as usual, the signature of the permutation and $\varepsilon(\sigma)$ is the sign given by the Koszul convention. The map ℓ_T is independent of the chosen planar embedding and, by Thm. 1.3, the transferred L_∞-algebra structure in H is given by $\{\ell_k\}_{k\geq 1}$, where $\ell_1 = \partial$ and, for $k \geq 2$,

$$\ell_k = \sum_{T \in \mathcal{T}_k} \frac{\ell_T}{|\operatorname{Aut} T|}$$

where Aut T is the automorphism group of the tree T. The map ℓ'_T appearing in the description of I is defined as ℓ_T except that we label the root by K instead of q.

The uniqueness properties in Thm. 1.3 follow from the fact that different homotopy retracts of M produce quasi-isomorphic infinity structures on H. Since all of them are minimal, these are isomorphic. Another invariant of transferred infinity structures on H, in fact on isomorphism classes of minimal infinity algebras, is the least k for which the operation ℓ_k, m_k or Δ_k is non trivial, together with the operation itself.

Either, via an induction using formula (1.6) above, or simply referring to classical formulae on homological perturbation theory, see for instance [29, Thm. 2.7] (cf. [20] or [43, Thm. 6.1]), the components of I and the higher brackets on generators are given recursively by,

$$\begin{aligned} I_n(x_1,\dots,x_n) &= \sum_{j=1}^{n-1} \sum_{\sigma\in\widetilde{S}(j,n-j)} \varepsilon(\sigma) K\left[I_j\left(x_{\sigma(1)},\dots,x_{\sigma(j)}\right), I_{n-j}\left(x_{\sigma(j+1)},\dots,x_{\sigma(n)}\right)\right], \\ \ell_n(x_1,\dots,x_n) &= \sum_{j=1}^{n-1} \sum_{\sigma\in\widetilde{S}(j,n-j)} \varepsilon(\sigma) q\left[I_j\left(x_{\sigma(1)},\dots,x_{\sigma(j)}\right), I_{n-j}\left(x_{\sigma(j+1)},\dots,x_{\sigma(n)}\right)\right]. \end{aligned} \tag{1.9}$$

Here, $\varepsilon(\sigma)$ is the given by Koszul sign convention, times a term $(-1)^{|x_{i_{\sigma(1)}}|+\dots+|x_{i_{\sigma(j)}}|+j-1}$, and $\widetilde{S}(j,n-j)$ are the shuffle permutations such that $\sigma(1)=1$.

For the transfer of A_∞ algebras (see for instance [32, 49]), the components of I and the higher multiplications can also be given recursively as follows. Formally, set $K\lambda_1 := -i$, and define $\lambda_n \colon H^{\otimes n} \to A$, $n \geq 2$, recursively by

$$\lambda_n = m\left(\sum_{s=1}^{n-1} (-1)^{s+1} K\lambda_s \otimes K\lambda_{n-s}\right). \tag{1.10}$$

Then,

$$m_n = q\circ\lambda_n \qquad \text{and} \qquad I_n = K\circ\lambda_n \qquad \text{for all } n\geq 2.$$

A straightforward dualization of the fact above provides recursive formulas for the transfer of A_∞ coalgebra structures. The components of I and the higher comultiplications are given recursively as follows. Formally, set $\mu_1 K := -i$, and define $\mu_n \colon H \to C^{\otimes n}$, $n \geq 2$, recursively by

$$\mu_n = \left(\sum_{s=1}^{n-1} (-1)^{s+1} \mu_{n-s} K \otimes \mu_s K\right)\Delta.$$

Then,

$$\Delta_n = \mu_n \circ q \qquad \text{and} \qquad I_n = \mu_n\circ K \qquad \text{for all } n\geq 2.$$

1.4 The infinity bar, cobar and Quillen constructions

We recall how to interpret L_∞ algebras, A_∞ algebras and A_∞ coalgebras as differentials or codifferentials in certain associated algebraic structures. We state these correspondences as theorems and give precise references for their proofs, which are also sketched here for the reader not familiar with this particular subject.

Theorem 1.4 [38] *(1) L_∞ structures on a graded vector space L are in bijective correspondence with codifferentials on the cofree cocommutative coalgebra ΛsL.*

(2) L_∞ morphisms from L to L' are in bijective correspondence with CDGC morphisms $(\Lambda sL,\delta)\to(\Lambda sL',\delta')$, δ and δ' being the codifferentials determining the L_∞ structures.

Proof: (1) Indeed, a codifferential δ on ΛsL is determined by a degree -1 linear map $\Lambda^+ sL\to sL$ which is written as the sum of linear maps $h_k\colon \Lambda^k sL\to sL$, $k\geq 1$. In fact, δ is written as the sum of coderivations,

$$\delta=\sum_{k\geq 1}\delta_k,\quad \delta_k\colon \Lambda sL\to\Lambda sL, \tag{1.11}$$

each of which is the extension as a coderivation of the corresponding h_k:

$$\delta_k\left(sx_1\wedge\ldots\wedge sx_p\right)=\sum_{i_1<\cdots<i_k}\varepsilon\, h_k\left(sx_{i_1}\wedge\ldots\wedge sx_{i_k}\right)\wedge sx_1\wedge\ldots\widehat{sx}_{i_1}\ldots\widehat{sx}_{i_k}\ldots\wedge sx_p. \tag{1.12}$$

Observe, that each δ_k decreases word length by $k-1$, that is, $\delta_k(\Lambda^p sL)\subset\Lambda^{p-k+1}sL$ for any p.

Then, the operators $\{\ell_k\}_{k\geq 1}$ on L and the maps $\{h_k\}_{k\geq 1}$ (and hence δ) uniquely determine each other as follows:

$$\begin{aligned}\ell_k&=s^{-1}\circ h_k\circ s^{\otimes k}\colon L^{\otimes^k}\to L,\\ h_k&=(-1)^{\frac{k(k-1)}{2}}s\circ\ell_k\circ\left(s^{-1}\right)^{\otimes k}\colon \Lambda^k sL\to sL.\end{aligned} \tag{1.13}$$

(2) Observe that a CDGC morphism

$$f\colon(\Lambda sL,\delta)\longrightarrow(\Lambda sL',\delta')$$

is determined by $\pi f\colon\Lambda sL\to sL'$ (π denotes the projection onto the indecomposables) which can be written as $\sum_{k\geq 1}(\pi f)^{(k)}$, where $(\pi f)^{(k)}\colon\Lambda^k sL\to sL'$. Note that the collection of linear maps $\left\{(\pi f)^{(k)}\right\}_{k\geq 1}$ is in one-to-one correspondence with a system $\left\{f^{(k)}\right\}_{k\geq 1}$ of skew-symmetric maps $f^{(k)}\colon L^{\otimes^k}\to L'$ of degree $1-k$ satisfying equations (1.1). Indeed, each $f^{(k)}$ and $(\pi f)^{(k)}$ determines the other by:

$$\begin{aligned}f^{(k)}&=s^{-1}\circ(\pi f)^{(k)}\circ s^{\otimes k},\\ (\pi f)^{(k)}&=(-1)^{\frac{k(k-1)}{2}}s\circ f^{(k)}\circ\left(s^{-1}\right)^{\otimes k}.\end{aligned}$$

□

Remark 1.5 Observe that, whenever L is a DGL with differential ∂, then the corresponding CDGC $(\Lambda sL,\delta)$ given by (1) of Theorem 1.4 is precisely the classical construction of [57] or [19, Chap. 22]. That is, $\delta=\delta_1+\delta_2$ where these are produced as before by,

$$h_1(sx)=-s\partial x,\quad h_2(sx\wedge sy)=-(-1)^{|x|}s[x,y].$$

Thus, from now on, given a general L_∞ algebra L we denote by $\mathscr{C}(L)=(\Lambda sL,\delta)$ the corresponding CDGC and call it the *(Quillen) chains on L.*

Theorem 1.6 *(1) A_∞ coalgebra structures on a graded vector space C are in bijective correspondence with differentials on the complete tensor algebra $\widehat{T}\left(s^{-1}C\right)$.*

(2) A_∞ coalgebra morphisms from C to D are in bijective correspondence with DGA morphisms $\left(\widehat{T}\left(s^{-1}C\right), d\right) \to \left(\widehat{T}\left(s^{-1}D\right), d'\right)$, d and d' being the differentials determining the A_∞ coalgebra structures.

Proof: (1) Recall that the complete tensor algebra on a graded vector space V is the graded associative algebra,

$$\widehat{T}(V) = \varprojlim_k T(V)/T^{\geq k}(V) \cong \Pi_k T^k(V).$$

Thus, an element of $\widehat{T}(V)$ can be regarded as a series $\sum_{k\geq 0} a_k$, $a_k \in T^k(V)$. Note that there is a canonical injection $T(V) \subseteq \widehat{T}(V)$.

Now observe that a differential d on $\widehat{T}\left(s^{-1}C\right)$ is determined by its image on $s^{-1}C$, which is written as a sum $d = \sum_{k\geq 1} d_k$, with $d_k\left(s^{-1}C\right) \subseteq T^k\left(s^{-1}C\right)$, for $k \geq 1$. Then, the operators $\{\Delta_k\}_{k\geq 1}$ and $\{d_k\}_{k\geq 1}$ determine each other via

$$\Delta_k = -s^{\otimes k} \circ d_k \circ s^{-1} \colon C \to C^{\otimes k},$$
$$d_k = -(-1)^{\frac{k(k-1)}{2}} \left(s^{-1}\right)^{\otimes k} \circ \Delta_k \circ s \colon s^{-1}C \to T^k(s^{-1}C).$$

(2) Observe that a DGA morphism

$$f \colon \left(\widehat{T}\left(s^{-1}C\right), d\right) \to \left(\widehat{T}(s^{-1}D), d'\right)$$

is determined by its image on $s^{-1}C$ which can be written as $\sum_{k\geq 1} f^k$, where $f^k \colon s^{-1}C \to T^k(s^{-1}D)$. Note that the collection $\{f^k\}_{k\geq 1}$ is in one-to-one correspondence with a system $\{f_k\}_{k\geq 1}$ of linear maps $f_k \colon C \to D^{\otimes k}$ of degree $1-k$ satisfying the equations (1.3). Indeed, each f^k and f_k determines the other by:

$$f_k = s^{\otimes k} \circ f^k \circ s^{-1} \colon C \to D^{\otimes k},$$
$$f^k = (-1)^{\frac{k(k-1)}{2}} \left(s^{-1}\right)^{\otimes k} \circ f_k \circ s \colon s^{-1}C \to T^k(s^{-1}D).$$

□

Remark 1.7 Observe that if C is a DGC, then the corresponding differential d on $\widehat{T}\left(s^{-1}C\right)$ given by (1) of Theorem 1.6 has only linear and quadratic part, $d = d_1 + d_2$,

$$d_1 s^{-1}c = -s^{-1}dc, \quad d_2 s^{-1}c = \sum_i (-1)^{|a_i|} s^{-1}a_i \otimes s^{-1}b_i,$$

where $\Delta c = \sum_i a_i \otimes b_i$. Hence, d restricts to a differential on $T\left(s^{-1}C\right) \subseteq \widehat{T}\left(s^{-1}C\right)$ and $\left(T\left(s^{-1}C\right), d\right)$ is precisely the *cobar construction of* C. Given a general A_∞ coalgebra C, we abuse the language and also call the associated DGA $\left(\widehat{T}\left(s^{-1}C\right), d\right)$ the *cobar construction of* C.

Given a graded vector space V, the *free complete Lie algebra generated by* V is defined as

$$\widehat{\mathbb{L}}(V) = \varprojlim_k \mathbb{L}(V)/\mathbb{L}^{\geq k}(V) \cong \Pi_k \mathbb{L}^k(V).$$

Thus, an element of $\widehat{\mathbb{L}}(V)$ can be regarded as a series $\sum_{k\geq 1} a_k$, $a_k \in \mathbb{L}^k(V)$. Note that there are canonical injections $\mathbb{L}(V) \subseteq \widehat{\mathbb{L}}(V) \subseteq \widehat{T}(V)$. Then, we have:

Theorem 1.8 [11, Thm. 3.1] *Let $\left(\widehat{T}\left(s^{-1}C\right), d\right)$ be the cobar construction of the cocommutative A_∞ coalgebra C. Then, the differential on any generator $s^{-1}c \in s^{-1}C$ is a Lie polynomial, that is, $ds^{-1}c \in \widehat{\mathbb{L}}(s^{-1}C)$. In particular, the differential d makes $\widehat{\mathbb{L}}(s^{-1}C)$ a DGL.*

Remark 1.9 Under the hypothesis of the result above, we call the DGL $\widehat{\mathbb{L}}(s^{-1}C)$ the *Quillen construction of* C and denote it by $\mathscr{L}(C)$. Observe that, whenever C is a CDGC, then the restriction to the non-complete free Lie algebra $\mathscr{L}(C) = \left(\mathbb{L}(s^{-1}C), \partial_1 + \partial_2\right)$ is the classical Quillen construction where,

$$\partial_1 s^{-1}c = -s^{-1}\delta c, \quad \partial_2 s^{-1}c = -\frac{1}{2}\sum_i (-1)^{|a_i|}\left[s^{-1}a_i, s^{-1}b_i\right],$$

where $\bar{\Delta}c = \sum_i a_i \otimes b_i$.

Theorem 1.10 *(1) A_∞ algebra structures on a graded vector space A are in bijective correspondence with codifferentials on the tensor coalgebra $T(sA)$.*

(2) A_∞ algebra morphisms from A to B are in bijective correspondence with DGC morphisms $(T(sA), \delta) \to \left(T(sB), \delta'\right)$, δ and δ' being the codifferentials determining the A_∞ algebra structures.

Proof: (1) Recall that a codifferential δ on $T(sA)$ is determined by a degree -1 linear map $T^+(sA) \to sA$ which is written as the sum of linear maps $g_k : T^k(sA) \to sA$, $k \geq 1$. In fact, δ is written as the sum of coderivations,

$$\delta = \sum_{k\geq 1} \delta_k, \quad \delta_k\colon T(sA) \to T(sA), \tag{1.14}$$

each of which is the extension as a coderivation of the corresponding g_k,

$$\delta_k\left(sa_1 \otimes \cdots \otimes sa_p\right) = \sum_{i=1}^{p-k} \varepsilon\, sa_1 \otimes \cdots \otimes sa_{i-1} \otimes sg_k(a_i \otimes \cdots \otimes a_{i+k-1}) \otimes sa_{i+k} \otimes \cdots \otimes sa_p.$$

Observe, that each δ_k decreases word length by $k-1$, that is, $\delta_k(T^p(sA)) \subset T^{p-k+1}(sA)$ for any p.

Then, the operators $\{m_k\}_{k\geq 1}$ on A and the maps $\{g_k\}_{k\geq 1}$ (and hence δ) uniquely determine each other as follows:

$$\begin{aligned} m_k &= s^{-1} \circ g_k \circ s^{\otimes k}\colon A^{\otimes^k} \to A, \\ g_k &= (-1)^{\frac{k(k-1)}{2}} s \circ m_k \circ \left(s^{-1}\right)^{\otimes k}\colon T^k(sA) \to sA. \end{aligned} \tag{1.15}$$

(2) Observe that a DGC morphism

$$f\colon (T(sA), \delta) \longrightarrow \left(T(sB), \delta'\right)$$

is determined by $\pi f\colon T(sA) \to sB$ (π denotes the projection onto the indecomposables) which can be written as $\sum_{k\geq 1}(\pi f)^{(k)}$, where $(\pi f)^{(k)}\colon T^k(sA) \to sB$. Note that the collection of linear maps $\{(\pi f)^{(k)}\}_{k\geq 1}$ is in one-to-one correspondence with a system $\left\{f^{(k)}\right\}_{k\geq 1}$ of linear maps $f^{(k)}\colon A^{\otimes^k} \to B$ of degree $1-k$ satisfying equations (1.2). Indeed, each $f^{(k)}$ and $(\pi f)^{(k)}$ determines the other by:

$$\begin{aligned} f^{(k)} &= s^{-1} \circ (\pi f)^{(k)} \circ s^{\otimes k}, \\ (\pi f)^{(k)} &= (-1)^{\frac{k(k-1)}{2}} s \circ f^{(k)} \circ \left(s^{-1}\right)^{\otimes k}. \end{aligned}$$

□

Remark 1.11 Observe that if A is a DGA, then the corresponding codifferential δ on $T(sA)$ given by (1) of Theorem 1.10 has only linear and quadratic parts, $\delta = \delta_1 + \delta_2$, determined by

$$\delta_1 sa = -sda, \quad \delta_2(sa_1 \otimes sa_2) = -(-1)^{|a_1|} s(a_1 a_2).$$

Hence, this is precisely the *bar construction of* A. Given a general A_∞ algebra A, we abuse the language and also call the associated DGC $(T(sA), \delta)$ the *bar construction of* A.

1.5 Rational homotopy theory

Rational homotopy theory, for which the books [18, 19, 63] are excellent references, concerns the study of the torsion free part of homotopy types. There are two classical approaches to the subject. The Sullivan approach [62], completed by Bousfield and Guggenheim [9], is based on the existence of two adjoint functors

$$A_{PL}\colon \mathsf{Top} \rightleftarrows \mathsf{CDGA^c}\colon \langle\cdot\rangle$$

between the categories of connected topological spaces and connected commutative differential graded algebras. Recall that a CDGA A is connected if it is non negatively graded and $A^0 = \mathbb{Q}$. These functors induce equivalences

$$\mathsf{H_o Top}^1_{\mathsf{f},\mathbb{Q}} \rightleftarrows \mathsf{H_o CDGA}^1_{\mathsf{f}}$$

between the homotopy categories of rational simply connected spaces of the homotopy type of CW-complexes of finite type over $\mathbb{Q}$ and that of simply connected CDGA's with finite type cohomology algebra. A connected CDGA A is simply connected if $A^1 = 0$.

A *Sullivan model* of a connected space X is a pair $\big((\Lambda V, d), \varphi\big)$, or simply $(\Lambda V, d)$ by abuse notation, in which:

(1) the free commutative algebra ΛV is generated by a positively graded vector space V which has a well ordered basis $\{v_\alpha\}_{\alpha\in I}$ such that, for each $\alpha \in I$, $dv_\alpha \in \Lambda V_{<\alpha}$. Here $V_{<\alpha}$ denotes the subspace of V generated by the elements of the basis with subscript smaller than α. We call $\{v_\alpha\}$ a *Koszul-Sullivan basis* or *KS-basis.*

(2) the map $\varphi\colon (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$ is a quasi-isomorphism of cochain algebras.

A Sullivan model is called *minimal* if the well ordered basis of V is compatible with the degree, that is, $\beta < \alpha$ if $|v_\beta| < |v_\alpha|$. Whenever ΛV is simply connected, that is, $V = \bigoplus_{p\geq 2} V^p$, a Sullivan model $(\Lambda V, d)$ is minimal if and only if d is decomposable, i.e., $dV \subseteq \Lambda^{\geq 2} V$. Every connected space X admits a Sullivan minimal model which is unique up to isomorphism. Moreover, whenever X is a simply connected space of the homotopy type of a finite type CW-complex, its Sullivan minimal model $(\Lambda V, d)$ characterizes its rational homotopy type via the above equivalence. In particular,

$$H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q})$$

and there is a pairing

$$\langle\,;\,\rangle\colon V \times \pi_*(X) \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$$

which induces an isomorphism

$$\pi_*(X) \otimes \mathbb{Q} \cong V^\sharp.$$

Via this pairing, the *Whitehead product* $[\,,\,]$ on $\pi_*(X) \otimes \mathbb{Q}$ (see next Chapter for a brief review on this classical invariant) is identified with the bracket induced by the quadratic part d_2 of the differential d. Explicitly,

$$\langle v; [\alpha, \beta]\rangle = (-1)^{p+q+1} \langle d_2 v; \alpha, \beta\rangle, \quad \alpha \in \pi_p(X) \otimes \mathbb{Q}, \beta \in \pi_q(X) \otimes \mathbb{Q}, v \in V.$$

Here,

$$\langle\,;\,,\rangle\colon \Lambda^2 V \times \pi_*(X)\otimes\mathbb{Q} \times \pi_*(X)\otimes\mathbb{Q} \longrightarrow \mathbb{Q}$$

is the obvious pairing whose generalization shall be considered and studied more deeply later on, particularly in Section 2.4.

In general we say that a connected CDGA A is a *model* of a connected space X if it is of the homotopy type of, or weakly equivalent to, $A_{PL}(X)$, that is, if A is connected by quasi-isomorphisms to $A_{PL}(X)$. In particular, $H^*(A)\cong H^*(X;\mathbb{Q})$.

On the other hand, the Quillen approach [57] to rational homotopy theory is based on the existence of two functors

$$\lambda\colon \mathsf{Top}^1 \rightleftarrows \mathsf{DGL}_1\colon \langle\cdot\rangle$$

between the categories of simply connected topological spaces and reduced DGL's. Recall that a DGL L is reduced if it is concentrated in positive degrees. These functors induce equivalences

$$\mathsf{H_oTop}^1_{\mathbb{Q}} \rightleftarrows \mathsf{H_oDGL}_1$$

between the homotopy categories of rational simply connected spaces of the homotopy type of CW-complexes and that of reduced DGL's.

A *Quillen model* of a simply connected space X is a pair $\big((\mathbb{L}(U),\partial),\psi\big)$, or simply $(\mathbb{L}(U),\partial)$, in which $\psi\colon (\mathbb{L}(U),\partial)\xrightarrow{\simeq}\lambda(X)$ is a quasi-isomorphism from a reduced free DGL.

A Quillen model is called *minimal* if the differential is decomposable, i.e., $\partial U\subset \mathbb{L}^{\geq 2}(U)$. Every simply connected space X has a minimal Quillen model which is unique up to isomorphism and characterizes its rational homotopy type via the above equivalence.

If $(\mathbb{L}(U),\partial)$ is a Quillen model of X and ∂_1 is the linear part of its differential, then

$$H_*(U,\partial_1)\cong s^{-1}\widetilde{H}_*(X;\mathbb{Q}).$$

On the other hand, there is a graded Lie algebra isomorphism

$$H_*(\mathbb{L}(U),\partial)\cong \pi_*(\Omega X)\otimes\mathbb{Q}$$

where the Lie bracket on the right is given by the Samelson product (see next Chapter for a brief introduction of this classical invariant).

In general, we say that a reduced DGL L is a *model* of a simply connected space X if it is of the homotopy type of, or weakly equivalent to, $\lambda(X)$, that is, if L is connected by quasi-isomorphisms to $\lambda(X)$. In particular, $H_*(L)\cong\pi_*(\Omega X)\otimes\mathbb{Q}$ as graded Lie algebras.

The following result shows how to obtain a DGL model of a simply connected CW-complex from a given cellular decomposition.

Theorem 1.12 [63, III.3. (6)] Let $Y = X\cup_f e^{n+1}$ be the space obtained by attaching an $(n+1)$-cell to a simply connected space X via the map $f: S^n\to X$. Assume that $L=\mathbb{L}(V)$ is a model of X. Let z be a cycle representing the unique homology class in $H_{n-1}(L)$ which is identified via the isomorphism (1) above with the homotopy class $f\in\pi_n(X)\otimes\mathbb{Q}$. Then, the canonical injection

$$\mathbb{L}(V)\hookrightarrow\mathbb{L}(V\oplus\mathbb{Q}a)$$

where $\partial a = z$ is a Lie model of the inclusion $X\hookrightarrow Y$. In particular, $\mathbb{L}(V\oplus\mathbb{Q}a)$ is a model of Y.

We finish by remarking that the development of rational homotopy theory for completely arbitrary spaces (not necessarily connected, nor simply connected or nilpotent) is an active area of research, see for instance [11, 13, 12, 15, 39].

1.6 The Quillen and Eilenberg-Moore spectral sequences

We assume that the reader is acquainted with the basics of spectral sequences. An excellent reference is [48], although Chapter 18 of [19] is sufficient for our purposes.

Our first aim is to define the Quillen spectral sequence, for which we need some coalgebra prerequisites. In this section, C is a coaugmented graded coalgebra, and $\bar{C}$ is the kernel of the coaugmentation. Define the *reduced diagonal* $\bar{\Delta}\colon \bar{C} \to \bar{C} \otimes \bar{C}$ as

$$x \longmapsto \Delta(x) - (1 \otimes x + x \otimes 1).$$

More generally, the *iterated reduced diagonals* are inductively given by $\bar{\Delta}^p : \bar{C} \to \bar{C}^{\otimes(p+1)}$, for $p \geq 0$, as:

$$\begin{cases} \bar{\Delta}^0 = \mathrm{id}_{\bar{C}}, \\ \bar{\Delta}^1 = \bar{\Delta}, \\ \bar{\Delta}^2 = \left(\bar{\Delta} \otimes \mathrm{id}_{\bar{C}}\right) \bar{\Delta}^1 : \bar{C} \to \bar{C}^{\otimes 2}, \\ \vdots \\ \bar{\Delta}^p = \left(\bar{\Delta} \otimes \mathrm{id}_{\bar{C}}^{\otimes(p-1)}\right) \bar{\Delta}^{(p-1)} : \bar{C} \to \bar{C}^{\otimes p+1}. \end{cases}$$

Define the *filtration* $\mathscr{F}$ *of* C *by the primitives* as the ascending filtration

$$F_p C = \mathrm{Ker}\left(\bar{\Delta}^p\right) \text{ for every } p \geq 1.$$

We say that C is *conilpotent* if $\bar{C} = \cup_p F_p C$, that is, if the filtration is exhaustive. Every graded coalgebra of the form $C = \mathbb{K} \oplus C_{>0}$ is conilpotent. In particular, given (L, ∂) a reduced DGL, its Quillen chains $\mathscr{C}(L)$ is conilpotent. If $C = \Lambda V$ is a free cocommutative graded coalgebra, then, $F_p C = \Lambda^{\leq p} V$ for every $p \geq 1$. Finally, $(C, \delta, \mathscr{F})$ is a DG filtered module compatible with the coalgebra structure.

Define the *Quillen spectral sequence associated to an* L_∞ *algebra* L as the coalgebra spectral sequence determined by the DG filtered module $(\mathscr{C}(L), \mathscr{F})$.

Next, we write explicitly the first two pages of this spectral sequence.

$$\left(E^0, d^0\right) = (\mathscr{G}(\Lambda sL), \mathscr{G}(\delta)) \cong (\Lambda sL, \delta_1)$$

and

$$\left(E^1, d^1\right) \cong (H(\Lambda sL, \delta_1), H(\delta_2)) \cong \left(\Lambda sH, \bar{\delta}_2\right).$$

We check it for the sake of completeness. For any $p, q \geq 0$:

$$E^0_{p,q} = \mathscr{G}_{p,q}(\Lambda sL) = \left(F_p / F_{p-1}\right)_{p+q} = \left(\frac{\Lambda^{\leq p} sL}{\Lambda^{\leq p-1} sL}\right)_{p+q} \cong \left(\Lambda^p sL\right)_{p+q}$$

$$= \left\langle sx_1 \wedge \ldots \wedge sx_p \in \Lambda^p sL \;\middle|\; \sum_{i=1}^{p} |x_i| = q \right\rangle.$$

Summing over all p, q gives the bigraded module $E^0 \cong \Lambda sL$. On the other hand, the differential d^0, of bidegree $(0, -1)$, is induced by the restriction $\delta : Z^0_{p,q} \to Z^0_{p,q-1}$, that is:

$$d^0\left(sx_1 \wedge \ldots \wedge sx_p + \Lambda^{\leq p-1} sL\right) = \delta(sx_1 \wedge \ldots \wedge sx_p) + \Lambda^{\leq p-1} sL = \delta_1(sx_1 \wedge \ldots \wedge sx_p) + \Lambda^{\leq p-1} sL,$$

where we used in the last step that δ_k decreases word length by $k-1$, and therefore the terms $\delta_k(sx_1 \wedge \ldots \wedge sx_p)$ are absorbed in the quotient when $k \geq 2$. Summarizing, there is an isomorphism of differential bigraded modules

$$\left(E^0, d^0\right) \cong (\Lambda sL, \delta_1).$$

We compute now the E^1 page. We use Künneth's theorem ([57, Prop. 2.1]) in the first isomorphism.

$$E^1 = H_*\left(E^0, d^0\right) = H_*\left(\Lambda sL, \delta_1\right) \cong \Lambda H_*\left(sL, \delta_1\right) \cong \Lambda sH_*(L, \ell_1) = \Lambda sH,$$

where $H = H_*(L, \ell_1)$. The differential d^1, of bidegree $(-1,0)$, is induced by the restriction $\delta : Z^1_{p,q} \to Z^1_{p-1,q}$. Note that we may write the following as a direct sum,

$$Z^0_{p-1} + D^0_p = \Lambda^{\leq p-1} sL + \delta\left(\Lambda^{\leq p} sL\right) = \Lambda^{\leq p-1} sL \oplus \delta_1\left(\Lambda^p sL\right),$$

so that:

$$\begin{aligned} d^1\left(sx_1 \wedge \ldots \wedge sx_p + Z^0_{p-1} + D^0_p\right) &= \delta(sx_1 \wedge \ldots \wedge sx_p) \quad + \Lambda^{\leq p-1} sL \oplus \delta_1\left(\Lambda^p sL\right) \\ &= \sum_{k \geq 1} \delta_k\left(sx_1 \wedge \ldots \wedge sx_p\right) \quad + \Lambda^{\leq p-1} sL \oplus \delta_1\left(\Lambda^p sL\right) \\ &= \delta_2\left(sx_1 \wedge \ldots \wedge sx_p\right) \quad + \Lambda^{\leq p-1} sL \oplus \delta_1\left(\Lambda^p sL\right), \end{aligned}$$

where $\delta_1(sx_1 \wedge \ldots \wedge sx_p)$ and $\delta_k(sx_1 \wedge \ldots \wedge sx_p)$ for $k \geq 3$ are absorbed in the quotient by $\delta_1\left(\Lambda^p sL\right)$ and by $\Lambda^{\leq p-1} sL$, respectively (using again that δ_k reduces word length by $k-1$). Therefore,

$$\left(E^1, d^1\right) \cong \left(\Lambda sH, \bar{\delta}_2\right).$$

By [57, Thm. 6.9], if X is a simply connected space, then the spectral sequence of a Quillen model for X converges to $H_*(X;\mathbb{Q})$ as a graded coalgebra.

A morphism of coaugmented DGC's filtered by the primitives $f : (C, \delta, \mathscr{F}) \to (C', \delta', \mathscr{F}')$ induces a morphism of Quillen spectral sequences $E^*(f) : E^*(C) \to E^*(C')$. In particular, a DGL or an L_∞ morphism induces a morphism of the correponding Quillen spectral sequences, in view of Thm. 1.4.

Now we turn to the Eilenberg-Moore spectral sequence. For this, assume that A is an augmented A_∞ algebra. Define the exhaustive ascending filtration on its bar construction given by $\mathscr{F} = \left\{F_p BA\right\}_{p \geq 0}$, where $F_0 BA = \mathbb{K}$, and for $p \geq 1$,

$$F_p BA = \sum_{k=0}^{p} \left(s\bar{A}\right)^{\otimes k} = T^{\leq k}\left(s\bar{A}\right).$$

With the notation above, $(BA, \delta, \mathscr{F})$ is a differential graded filtered module compatible with the coalgebra structure. We adopt the convention that BA is bigraded so that $x \in T^p\left(s\bar{A}\right)$ has filtration degree $-p$. Define the *Eilenberg-Moore spectral sequence associated to* A as the coalgebra spectral sequence determined by the DG filtered module $(BA, \delta, \mathscr{F})$.

Its first two pages can be determined by mimicking what was done for the Quillen spectral sequence. More precisely, these are

$$\left(E^0, d^0\right) = \left(\mathscr{G}(T\left(s\bar{A}\right), \mathscr{G}(\delta)\right) \cong \left(T\left(s\bar{A}\right), \delta_1\right)$$

and

$$\left(E^1,d^1\right)\cong\left(H\left(T\left(s\bar{A}\right),\delta_1\right),H(\delta_2)\right)\cong\left(T\left(s\bar{H}\right),\bar{\delta}_2\right).$$

We have, for any $p,q\geq 0$:

$$E^0_{p,q}=\mathscr{G}_{p,q}\left(T\left(s\bar{A}\right)\right)=\left(F_p/F_{p-1}\right)_{p+q}=\left(\frac{T^{\leq p}\left(s\bar{A}\right)}{T^{\leq p-1}\left(s\bar{A}\right)}\right)_{p+q}\cong\left(T^p\left(s\bar{A}\right)\right)_{p+q}$$

$$=\left\langle sa_1\otimes\cdots\otimes sa_p\in T^p\left(s\bar{A}\right)\ \mid\ \sum_{i=1}^{p}|a_i|=q\right\rangle.$$

Summing over all p,q gives the bigraded module $E^0\cong T\left(s\bar{A}\right)$. The differential d^0, of bidegree $(0,-1)$, is induced by the restriction $\delta: Z^0_{p,q}\to Z^0_{p,q-1}$, that is:

$$d^0\left(sa_1\otimes\cdots\otimes sa_p+T^{\leq p-1}\left(s\bar{A}\right)\right)=\delta(sa_1\otimes\cdots\otimes sa_p)+T^{\leq p-1}\left(s\bar{A}\right)=\delta_1(sa_1\otimes\cdots\otimes sa_p)+T^{\leq p-1}\left(s\bar{A}\right),$$

where we used in the last equality that δ_k decreases word length by $k-1$, and therefore the terms $\delta_k(sa_1\otimes\cdots\otimes sa_p)$ are absorbed in the quotient when $k\geq 2$. Summarizing, there is an isomorphism of differential bigraded modules

$$\left(E^0,d^0\right)\cong\left(T\left(s\bar{A}\right),\delta_1\right).$$

We compute now the E^1 page, using Künneth's theorem ([57, Prop. 2.1]),

$$E^1=H_*\left(E^0,d^0\right)=H_*\left(T\left(s\bar{A}\right),\delta_1\right)\cong TH_*\left(s\bar{A},\delta_1\right)\cong Ts\bar{H}_*(A,m_1)=Ts\bar{H},$$

where $H=H_*(A,m_1)$. The differential d^1, of bidegree $(-1,0)$, is induced by the restriction δ : $Z^1_{p,q}\to Z^1_{p-1,q}$. We write the following as a direct sum,

$$Z^0_{p-1}+D^0_p=T^{\leq p-1}\left(s\bar{A}\right)+\delta\left(T^{\leq p}\left(s\bar{A}\right)\right)=T^{\leq p-1}\left(s\bar{A}\right)\oplus\delta_1\left(T^p\left(s\bar{A}\right)\right),$$

so that:

$$\begin{aligned}d^1\left(sa_1\otimes\cdots\otimes sa_p+Z^0_{p-1}+D^0_p\right)&=\delta(sa_1\otimes\cdots\otimes sa_p)\ \ +T^{\leq p-1}\left(s\bar{A}\right)\oplus\delta_1\left(T^p\left(s\bar{A}\right)\right)\\&=\sum_{k\geq 1}\delta_k\left(sa_1\otimes\cdots\otimes sa_p\right)\ \ +T^{\leq p-1}\left(s\bar{A}\right)\oplus\delta_1\left(T^p\left(s\bar{A}\right)\right)\\&=\delta_2\left(sa_1\otimes\cdots\otimes sa_p\right)\ \ +T^{\leq p-1}\left(s\bar{A}\right)\oplus\delta_1\left(T^p\left(s\bar{A}\right)\right),\end{aligned}$$

where $\delta_1(sa_1\otimes\cdots\otimes sa_p)$ and $\delta_k(sa_1\otimes\cdots\otimes sa_p)$ for $k\geq 3$ are absorbed in the quotient by $\delta_1\left(T^p\left(s\bar{A}\right)\right)$ and by $T^{\leq p-1}\left(s\bar{A}\right)$, respectively (noting again that δ_k reduces word length by $k-1$). Therefore,

$$\left(E^1,d^1\right)\cong\left(Ts\bar{H},\bar{\delta}_2\right).$$

If X is a simply connected CW-complex with finite type rational homology, then the Eilenberg-Moore spectral sequence of a CDGA model of X converges to $H^*(\Omega X;\mathbb{Q})$ as a graded algebra ([63, III.6.(3)] or [26, §7]).

CHAPTER 2

Higher Whitehead products and L_∞ structures

Let X be a path connected space. The rational homotopy groups of ΩX enjoy a graded Lie algebra structure which corresponds to the rational Whitehead products on $\pi_*(X)$. Assume X simply connected, and let L be a model of X, so that $H_*(L) \cong \pi_*(\Omega X) \otimes \mathbb{Q}$ as graded Lie algebras. The homotopy transfer theorem 1.3 (1) allows us to endow these groups $\pi_*(\Omega X) \otimes \mathbb{Q}$ with a minimal L_∞ structure, retaining the rational homotopy type of X. Here, the L_∞ structure is given by operations ℓ_n which take n homotopy classes of $\pi_*(\Omega X) \otimes \mathbb{Q} \cong \pi_{*-1}(X) \otimes \mathbb{Q}$ as input, to produce a new homotopy class. In particular, ℓ_2 is identified with the Whitehead product. But endowing the rational homotopy groups of ΩX with such an L_∞ structure is not the only way of obtaining new homotopy classes out of n of them: one may also produce new homotopy classes by forming, when possible, higher order Whitehead products.

Higher order Whitehead products were introduced by G. Porter in [55], and are homotopy invariant sets arising from a natural extension problem (see also [54]). Since the appearance of these higher products, they have been used mainly to understand and distinguish homotopy types of spaces and maps. Recently, higher Whitehead products have been used in [23, 30] to exhibit the homotopy type of certain polyhedral products, and in [59] to describe the attaching maps in a minimal cellular decomposition for Euclidean configuration spaces. The internal structure of the higher Whitehead products is still a research topic, and has been successfully applied to understand the relationship among certain homotopy classes, in particular for spheres and projective spaces [21, 22]. Rationally, higher Whitehead products from the point of view of Quillen were first studied in [1, 2], and were given its definitive exposition in [63]. From Sullivan's perspective, these were studied in [3]. Rational higher Whitehead products also provide useful numerical invariants [10, §4].

In this Chapter, our aim is to understand and make precise the entanglement between L_∞ structures and higher Whitehead products on $\pi_*(\Omega X) \otimes \mathbb{Q}$.

2.1 Whitehead products

Let X be a path connected space. The *Whitehead product* is the operation $[\ ,\ ] : \pi_p(X) \times \pi_q(X) \to \pi_{p+q-1}(X)$ defined by

$$[f, g] = (f \vee g) \circ w,$$

where $w: S^{p+q-1} \to S^p \vee S^q$ is the universal Whitehead element. Recall that this element w is the homotopy class of the attaching map gluing a $(p+q)$-cell to $S^p \vee S^q$, so that

$$S^p \times S^q = \left(S^p \vee S^q\right) \cup_w e^{p+q}.$$

Consider now the long exact sequence in homotopy of the (Moore) path space fibration,

$$\Omega X \to PX \to X.$$

As PX is contractible in the based category, the boundary operator induces a sequence of isomorphisms,

$$\partial : \pi_{*+1}(X) \cong \pi_*(\Omega X).$$

It is a theorem of Milnor and Moore [50] that, over the rationals, the operation obtained on the graded vector space $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$ by transferring the Whitehead product via the boundary operator ∂ equips it with a graded Lie algebra structure. It is called the *rational homotopy Lie algebra* of X, and it is a fundamental rational homotopy invariant of X. The reader can find much more information in several chapters of [19].

2.2 Higher Whitehead products

Recall that, given spaces $X_1, \dots, X_k$ with $k \geq 2$, their *fat wedge* $T(X_1, \dots, X_k)$ is the subspace of the product $X_1 \times \cdots \times X_k$ formed by all k-tuples in which at least one coordinate is the base point. In particular, given spheres $S^{n_1}, \dots, S^{n_k}$, denote the product $S^{n_1} \times \cdots \times S^{n_k}$ and the fat wedge $T(S^{n_1}, \dots, S^{n_k})$ by P and T, respectively.

Let $N = n_1 + \cdots + n_k$, and let $a_i \in H_{n_i}(S^{n_i}; \mathbb{Z}) \cong \mathbb{Z}$ be a generator for each i. Then, the homology cross product $a_1 \times \cdots \times a_k \in H_N(P; \mathbb{Z}) \cong \mathbb{Z}$ is a generator. Denote by $j: P \to (P, T)$ the inclusion, and let $\partial : \pi_N(P, T) \to \pi_{N-1}(T)$ be the boundary morphism in the homotopy sequence of the pair (P, T). The Hurewicz homomorphism $h: \pi_N(P, T) \to H_N(P, T)$ turns out to be an isomorphism, because the pair (P, T) is $(N-1)$-connected. The *universal Whitehead element* of order k and type $n_1, \dots, n_k$ is the non trivial homotopy class $w \in \pi_{N-1}(T)$ defined by

$$w = \partial h^{-1} H_*(j)(a_1 \times \cdots \times a_k).$$

The type of a universal Whitehead element is usually omitted, it being clear from the context.

Let $W = S^{n_1} \vee \ldots \vee S^{n_k}$, and let X be a path connected space. For $x_i \in \pi_{n_i}(X)$, $1 \leq i \leq k$, denote by $f = (x_1, \dots, x_k): W \to X$ the induced map.

Definition 2.1 Let X be a path connected space, and let $x_i \in \pi_{n_i}(X)$, with $1 \leq i \leq k$ and $k \geq 2$. The *kth order Whitehead product* is the (possibly empty) set

$$[x_1, \dots, x_k] = \left\{\widetilde{f} \circ w \mid \widetilde{f} : T \to X \text{ is an extension of } f\right\} \subseteq \pi_{N-1}(X),$$

as depicted in the diagram below:

$$\begin{tikzcd} & W \arrow[r, "f"] \arrow[d, hook] & X \\ S^{N-1} \arrow[r, "w"] & T \arrow[ur, dotted, "\widetilde{f}"'] & \end{tikzcd} \tag{2.1}$$

Our next aim is to read off the higher rational Whitehead products from a Lie model of a simply connected space. To do so, we recall the models involved in diagram (2.1).

Theorem 2.2 [63, Chap. V]. *The minimal models of the wedge, product and fat wedge of the simply connected spheres $S^{n_1},\dots,S^{n_k}$ are given by:*

(1) (Wedge) $\mathbb{L}(u_1,\dots,u_k)$, *where* $|u_i| = n_i - 1$, *with trivial differential.*

(2) (Product) $\mathbb{L}\big(\widetilde{U}\big)$, *where*

$$\widetilde{U} = \big\langle u_{i_1\dots i_s} \mid 1 \le i_1 < \cdots < i_s \le k, \quad 1 \le s \le k \big\rangle, \quad \text{with} \quad |u_{i_1\dots i_s}| = n_{i_1} + \cdots + n_{i_s} - 1,$$

and whose differential is determined by

$$\partial u_{i_1\dots i_s} = \sum_{p=1}^{s-1} \sum_{\sigma\in\widetilde{S}(p,s-p)} \varepsilon(\sigma) \left[u_{i_{\sigma(1)}\dots i_{\sigma(p)}}, u_{i_{\sigma(p+1)}\dots i_{\sigma(s)}} \right]. \tag{2.2}$$

Here, $\widetilde{S}(p,s-p)$ *are those shuffles fixing* 1, *and* $\varepsilon(\sigma)$ *is given by the Koszul sign times* $(-1)^{|u_{i_{\sigma(1)}\cdots i_{\sigma(p)}}|}$.

(3) (Fat wedge) $\mathbb{L}(U)$, *where* U *is a complement of* $u_{1\dots k}$ *in* $\widetilde{U}$, *and the same differential as above. That is,*

$$U = \big\langle u_{i_1\dots i_s} \mid 1 \le i_1 < \cdots < i_s \le k, \quad 1 \le s < k \big\rangle, \quad \text{with} \quad |u_{i_1\dots i_s}| = n_{i_1} + \cdots + n_{i_s} - 1.$$

Although the generator $u_{1\dots k}$ does not belong to the fat wedge model, the expression defining its boundary, $\partial u_{1\dots k}$, *does* live in it. We denote by $w = \partial u_{1\dots k}$ this non trivial cycle of $\mathbb{L}(U)$,

$$w = \partial u_{1\dots k} = \sum_{p=1}^{k-1} \sum_{\sigma\in\widetilde{S}(p,k-p)} \varepsilon(\sigma) \left[u_{i_{\sigma(1)}\dots i_{\sigma(p)}}, u_{i_{\sigma(p+1)}\dots i_{\sigma(k)}} \right].$$

We refer to w as the *universal Whitehead element* (of order k). Observe also that the free DGL's $\mathbb{L}(u_1,\dots,u_k), \mathbb{L}(\widetilde{U})$ and $\mathbb{L}(U)$ of Theorem 2.2 are well defined when the generators are allowed to have non positive degrees.

From now on, H will denote the homology $H_*(L)$ of a given DGL, and the homology class of a cycle z will be denoted by $\overline{z}$.

Definition 2.3 Let L be a DGL, and let $x_i \in H_{n_i}$ for $1 \le i \le k$, with $k \ge 2$. Define $\varphi\colon \mathbb{L}(u_1,\dots,u_k) \to L$ by mapping u_i to a representative $\varphi(u_i)$ of x_i. The *kth order Whitehead bracket set* is the (possibly empty) set

$$[x_1,\dots,x_k] = \left\{ \overline{\phi(w)} \mid \phi\colon \mathbb{L}(U) \to L \ \text{ is a DGL extension of } \varphi \right\} \subseteq H_{N+1}(L),$$

as depicted in the diagram below:

$$\begin{tikzcd} \mathbb{L}(u_1,\dots,u_k) \arrow[r, "\varphi"] \arrow[d, hook] & L \\ \mathbb{L}(U) \arrow[ur, dotted, "\phi"'] & \end{tikzcd} \tag{2.3}$$

The Whitehead bracket set does not depend on the chosen representatives.

Remark 2.4 The definition of higher Whitehead brackets is completely valid for not necessarily reduced DGL's, but then we have to forget its topological interpretation. Apart from their obvious algebraic utility (see for instance Section 3.2), it would be highly interesting to give topological meaning to higher Whitehead brackets when elements of negative degree are involved, in the context of rational models for spaces and simplicial sets which are not necessarily connected or simply connected [11, 12, 15].

From all of the above, we immediately deduce:

Theorem 2.5 *There is a bijection between rational Whitehead products in a simply connected space X and rational Whitehead brackets in a DGL model of it.*

The following Lemma will be used.

Lemma 2.6 If the Whitehead bracket set $[x_1,...,x_k]$ is non empty, and $2 \le p \le k-1$, then

$$0 \in \left[x_{i_1},...,x_{i_p}\right] \quad \text{for any} \quad 1 \le i_1 < \cdots < i_p \le k.$$

Proof: With the notation of Theorem 2.2 rename

$$\begin{aligned} &v_1 = u_{i_1},\ \dots,\ v_j = u_{i_j},\\ &\widetilde{V} \subset \widetilde{U}, \quad \widetilde{V} = \left\langle v_{\ell_1\dots\ell_s} \mid 1 \le \ell_1 < \cdots < \ell_s \le j,\ 1 \le s \le j \right\rangle,\\ &V = \text{Complement of } v_{1\dots j} \text{ in } \widetilde{V}. \end{aligned}$$

As $[x_1,\dots,x_k] \neq \emptyset$, choose ϕ as in diagram (2.3) and observe that in its restriction to

$$\begin{tikzcd} \mathbb{L}\left(v_1,...,v_j\right) \arrow[r, "\varphi"] \arrow[d, hook] & L \\ \mathbb{L}(V) \arrow[ur, dotted, "\phi"'] & \end{tikzcd}$$

the element $\phi(\partial v_{1\dots j}) = \partial\phi(v_{1\dots j})$ is a boundary. $\square$

The following classical result of C. Allday will also be used.

Theorem 2.7 ([2, Thm. 4.1]) *If $x_1,...,x_k \in H$ are such that $[x_1,...,x_k] \neq \emptyset$, then*

(1) the element $sx_1 \wedge ... \wedge sx_k$ survives to the E^{k-1} page of the Quillen spectral sequence of L, and

(2) for any $x \in [x_1,...,x_k]$, we have that $\delta^{k-1}\left(\overline{sx_1 \wedge ... \wedge sx_k}\right) = \overline{sx}$,

where $\overline{\bullet}$ denotes the class in the E^{k-1} page of the Quillen spectral sequence (E^*,δ^*) of L.

2.3 Higher Whitehead products and L_∞ structures

In this section DGL's are not assumed to be reduced except in Theorem 2.8 and its corollary. To avoid overloading the statements, whenever we choose an element x of a Whitehead bracket set $[x_1,...,x_k]$, we are assuming the existence of a fixed DGL L and homology classes $x_1,...,x_k \in H$ such that $x \in [x_1,...,x_k]$. In particular, $[x_1,...,x_k]$ is assumed to be non empty.

The most general result relating Whitehead brackets on H and brackets of the transferred L_∞ structure depends on Theorem 2.7.

Theorem 2.8 *If $[x_1,...,x_k] \neq \emptyset$, then, for any homotopy retract of L, and for any $x \in [x_1,...,x_k]$,*

$$\varepsilon\ell_k(x_1,...,x_k) = x + \Gamma, \quad \Gamma = \sum_{j=1}^{k-1} \operatorname{Im}\ell_j,$$

where $\varepsilon = (-1)^{\sum_{i=1}^{k-1}(k-i)|x_i|}$. In particular, if $\ell_j = 0$ for $j \le k-1$, then up to a sign, $\ell_k(x_1,...,x_k) \in [x_1,...,x_k]$.

Proof: Recall that the Quillen spectral sequence of L is defined by filtering the chains $\mathscr{C}(L)$ by the kernel of the reduced diagonals, $F_p = \Lambda^{\leq p} sL$. Consider the DGC quasi-isomorphisms induced by applying chains in Thm. 1.3,

$$\mathscr{C}(L) \underset{I}{\overset{Q}{\rightleftarrows}} (\Lambda sH, \delta) \ ,$$

choose the same filtration on ΛsH, and observe that at the E^1 level the induced morphisms of spectral sequences are both the identity on ΛsH. By comparison, all the terms in both spectral sequences are also isomorphic. Now, translating Thm. 2.7 to the spectral sequence on ΛsH we obtain that if $[x_1, \ldots, x_k]$ is non empty, then the element $sx_1 \wedge \ldots \wedge sx_k$ survives to the $k-1$ page (E^{k-1}, δ^{k-1}). Moreover, given any $x \in [x_1, \ldots, x_k]$, one has

$$\delta^{k-1}\, \overline{sx_1 \wedge \ldots \wedge sx_k}^{\,k-1} = \overline{sx}^{\,k-1}.$$

Here $\overline{(\cdot)}^{\,k-1}$ denotes the class in E^{k-1}. This is to say that there exists $\Phi \in \Lambda^{\leq k-1} sH$ such that

$$\delta\,(sx_1 \wedge \ldots \wedge sx_k + \Phi) = sx. \tag{2.4}$$

Write $\delta = \sum_{i \geq 2} \delta_i$ with each δ_i as in equation (1.11), and decompose $\Phi = \sum_{i=2}^{k-1} \Phi_i$ with $\Phi_i \in \Lambda^i sH$. By a word length argument,

$$\delta_k(sx_1 \wedge \ldots \wedge sx_k) + \sum_{i=2}^{k-1} \delta_i(\Phi_i) = sx.$$

Note also that $\delta_k = h_k$ for elements of word length k, with h_k as in equation (1.12). Therefore,

$$h_k\,(sx_1 \wedge \ldots \wedge sx_k) + \sum_{i=2}^{k-1} h_i(\Phi_i) = sx.$$

To finish, apply to this equation the identity (1.13), which is

$$\ell_i = s^{-1} \circ h_i \circ s^{\otimes i} \quad \text{for any} \quad i \geq 1.$$

In particular, the sign ε appears when writing

$$\ell_k(x_1, \ldots, x_k) = s^{-1} \circ h_k \circ s^{\otimes k}(x_1, \ldots, x_k) = \varepsilon\, s^{-1} h_k(sx_1 \wedge \ldots \wedge sx_k).$$

□

Corollary 2.9 *Let L be a DGL such that for some homotopy retract of L onto H, the induced higher brackets satisfy $\ell_1 = \ldots = \ell_{k-1} = 0$, with $k \geq 2$. Then, for any $x_1, \ldots, x_k \in H$, one has that $[x_1, \ldots, x_k] \neq \emptyset$, and moreover, it consists of a single homology class:*

$$[x_1, \ldots, x_k] = \{x\}\,,$$

where $x = \varepsilon \ell_k\,(x_1, \ldots, x_k)$, and ε is as in Thm. 2.8.

Remark 2.10 As the least k for which the operation ℓ_k is non trivial is an invariant of the L_∞ structure, this result is independent of the chosen retract, and x is well defined.

Proof: First, we prove by induction that

$$\{0\} = [x_{i_1},\dots,x_{i_s}] \neq \emptyset \quad \text{for all} \quad 1 \le i_1 < \dots < i_s \le k \quad \text{and} \quad s \le k-1. \tag{2.5}$$

For $s=2$ is straightforward. Assume that assertion (2.5) holds for all $s<k-1$; we prove it for $s=k-1$. Using the induction hypothesis and Lemma 2.6, $\left[x_{i_1},\dots,x_{i_{k-1}}\right]$ is non empty. Now, given $x \in \left[x_{i_1},\dots,x_{i_{k-1}}\right]$, Thm. 2.8 gives that

$$\ell_{k-1}\left(x_{i_1},\dots,x_{i_{k-1}}\right) = x + \Gamma,$$

where the vanishing of Γ and of ℓ_{k-1} implies that of x, proving assertion (2.5).

To finish, use assertion (2.5) and Lemma 2.6 to obtain that $[x_1,\dots,x_k] \neq \emptyset$. Then, since $\ell_j = 0$ for $j \le k-1$, Thm. 2.8 implies that any $x \in [x_1,\dots,x_k]$ coincides with $\varepsilon \ell_k(x_1,\dots,x_k)$. □

Our next aim is to find kth order Whitehead brackets that are detected precisely by kth brackets of the L_∞ structure.

Recall that, any $x \in [x_1,\dots,x_k]$ is given by $x = \overline{\phi(w)}$, for some DGL morphism $\phi : \mathbb{L}(U) \to L$ as in diagram (2.3). Write

$$U = \langle u_1,\dots,u_k\rangle \oplus V, \quad \text{that is,} \quad V = \left\langle u_{i_1\dots i_s} \in U \mid s \ge 2\right\rangle.$$

On the other hand, by Prop. 1.2, any homotopy retract of L onto H is identified with a decomposition $L = A \oplus \partial A \oplus C$, where $\partial : A \xrightarrow{\cong} \partial A$ and $C \cong H$.

Definition 2.11 With the notation above, a homotopy retract (L,H,i,q,K) of L onto H is *adapted* to $x \in [x_1,\dots,x_k]$ if $\phi(V) \subseteq A$.

Using the fact that K is a homotopy from id_L to iq, one readily sees that, for an adapted retract,

$$K\partial\phi\left(u_{i_1\dots i_s}\right) = \phi\left(u_{i_1\dots i_s}\right) \quad \text{for any generator } u_{i_1\dots i_s} \in V. \tag{2.6}$$

As $A = \mathrm{Ker}(q) \cap \mathrm{Ker}(K)$, being adapted to x is equivalent to $K\phi(V) = q\phi(V) = 0$.

Theorem 2.12 *Let $x \in [x_1,\dots,x_k]$. Then, for any homotopy retract of L adapted to x,*

$$\ell_k(x_1,\dots,x_k) = x.$$

Proof: Let $x \in [x_1,\dots,x_k]$, and let $\phi : \mathbb{L}(U) \to L$ be a DGL morphism such that $\overline{\phi(w)} = x$. Consider in H the L_∞ structure induced by a given homotopy retract which is adapted to x. We first prove by induction on p, with $1 \le p \le k-1$, that

$$I_p\left(x_{i_1},\dots,x_{i_p}\right) = \phi\left(u_{i_1\dots i_p}\right), \quad 1 \le i_1 < \dots < i_p \le k. \tag{2.7}$$

The assertion is trivial for $p=1$. Assume it is satisfied for $p<s\le k-1$, and prove it for $p=s$. In what follows we use, and in this order, the recursive formula for the components of I in equation (1.9), the induction hypothesis, the definition of the differential in $\mathbb{L}(U)$ given in formula (2.2), and the fact that the retract is adapted to x:

$$
\begin{aligned}
I_s\left(x_{i_1},\dots,x_{i_s}\right) &= \sum_{j=1}^{s-1}\sum_{\widetilde{S}(j,s-j)} \varepsilon(\sigma)K\left[I_j\left(x_{i_{\sigma(1)}},\dots,x_{i_{\sigma(j)}}\right), I_{s-j}\left(x_{i_{\sigma(j+1)}},\dots,x_{i_{\sigma(s)}}\right)\right] \\
&= \sum_{j=1}^{s-1}\sum_{\widetilde{S}(j,s-j)} \varepsilon(\sigma)K\left[\phi\left(u_{i_{\sigma(1)}\dots i_{\sigma(j)}}\right), \phi\left(u_{i_{\sigma(j+1)}\dots i_{\sigma(s)}}\right)\right] \\
&= K\phi\left(\partial u_{i_1\dots i_s}\right) = K\partial\phi\left(u_{i_1\dots i_s}\right) = \phi\left(u_{i_1\dots i_s}\right).
\end{aligned}
$$

Now use the recursive formula (1.9) for ℓ_k, and then equation (2.7):

$$
\begin{aligned}
\ell_k\left(x_1,\dots,x_k\right) &= \sum_{j=1}^{s-1}\sum_{\widetilde{S}(j,s-j)} \varepsilon(\sigma)q\left[I_j\left(x_{\sigma(1)},\dots,x_{\sigma(j)}\right), I_{k-j}\left(x_{\sigma(j+1)},\dots,x_{\sigma(s)}\right)\right] \\
&= \sum_{j=1}^{k-1}\sum_{\widetilde{S}(j,k-j)} \varepsilon(\sigma)q\left[\phi\left(u_{\sigma(1)\dots\sigma(j)}\right), \phi\left(u_{\sigma(j+1)\dots\sigma(k)}\right)\right] \\
&= q\phi(w).
\end{aligned}
$$

□

Let $x_1,\dots,x_k$ be homotopy classes of a space X such that the Whitehead product set $[x_1,\dots,x_k]$ is non empty. Endow the rational homotopy Lie algebra L_X of X with a minimal L_∞ structure, making it homotopy equivalent to the Quillen minimal model of X. Then, the evaluation $\ell_k(x_1,\dots,x_k)$ need not recover a Whitehead bracket, as the following result shows.

Theorem 2.13 *Let* $X = T\left(S^3,S^3,S^3,S^3\right)\vee S^6$*, and denote by* $x_i : S^3 \to X$ *the homotopy class of the inclusion of the* 3*-sphere, for* $i = 1,2,3,4$*. Then,*

(1) the rational Whitehead product set $[x_1,x_2,x_3,x_4]$ *is non empty, consisting of a single class, and*

(2) there exists a minimal L_∞ *structure* $\{\ell_n\}$ *on* $\pi_*(\Omega X)\otimes\mathbb{Q}$ *retaining the rational homotopy type of* X *for which* $\operatorname{Im}(\ell_4)\cap[x_1,x_2,x_3,x_4] = \emptyset$*. In particular,* $\ell_4(x_1,x_2,x_3,x_4)\notin[x_1,x_2,x_3,x_4]$*.*

Proof: From the standard cellular decomposition of X, it follows that its Quillen minimal model is given by

$$L = \mathbb{L}\left(v_1,v_2,v_3,v_4,v_{12},v_{13},v_{14},v_{23},v_{24},v_{34},v_{123},v_{124},v_{134},v_{234},z\right),$$

where $|v_i| = 2$ for $i = 1,2,3,4$ and $|z| = 5$, and whose differential is determined by

$$
\begin{aligned}
\partial v_i &= 0, \\
\partial v_{ij} &= [v_i,v_j] \quad \text{for } i<j, \\
\partial v_{ijk} &= [v_i,v_{jk}] - [v_{ij},v_k] - [v_j,v_{ik}] \quad \text{for } i<j<k, \text{ and} \\
\partial z &= 0.
\end{aligned}
$$

(1) We prove that fixing $x_i = \overline{v}_i$ for $1\le i\le 4$, the rational Whitehead bracket set is given by a unique homology class

$$[x_1,x_2,x_3,x_4] = \left\{\overline{\Phi}\mid \Phi = [v_{123},v_4]+[v_{12},v_{34}]-[v_{124},v_3]+[v_1,v_{234}]+[v_{14},v_{23}]-[v_{13},v_{24}]+[v_{134},v_2]\right\}.$$

To prove the claim, let us build all the possible DGL extensions $\phi : \mathbb{L}(U) \to L$ representing Whitehead products as in diagram (2.3). Necessarily, $\phi(u_i) = v_i$ for $1 \le i \le 4$. In the next level, for degree reasons, we can define

$$\phi(u_{ij}) = v_{ij} + \lambda_{ij} z \quad \forall\ i < j, \quad \lambda_{ij} \in \mathbb{Q}.$$

In principle, the following choices are valid:

$$\begin{aligned} \partial\phi(u_{ij}) &= \partial\left(v_{ij} + \lambda_{ij} z\right) = \partial v_{ij} = [v_i, v_j] \\ \phi\partial(u_{ij}) &= \phi([u_i, u_j]) = [\phi(u_i), \phi(u_j)] = [v_i, v_j]. \end{aligned}$$

But then, $\phi(u_{ijk})$ should be compatible with the choices made for $\phi(u_{rs})$. For instance, the boundary $\partial\phi(u_{123})$ should be

$$\begin{aligned} \partial\phi(u_{123}) &= [\phi(u_1), \phi(u_{23})] - [\phi(u_{12}), \phi(u_3)] - [\phi(u_2), \phi(u_{13})] \\ &= [v_1, v_{23}] - [v_{12}, v_3] - [v_2, v_{13}] + \lambda_{23}[v_1, z] - \lambda_{12}[z, v_3] - \lambda_{13}[v_2, z]. \end{aligned}$$

But the expression above is not a boundary unless every $\lambda_{ij} = 0$, because for degree reasons (see the table below) it should come from $\partial\left(\alpha v_{123} + \beta v_{124} + \gamma v_{134} + \delta v_{234}\right)$. The same argument applies to every other $\phi(u_{ijk})$, hence we are forced to set

$$\phi(u_{ij}) = v_{ij} \quad \text{for every } i < j.$$

The possible choices for $\phi(u_{ijk})$ are of the form

$$\phi(u_{ijk}) = v_{ijk} + \partial b_{ijk} + h_{ijk}$$

for some b_{ijk} and non trivial homology classes h_{ijk}. But a straightforward computation (see the table below) shows that $H_8(L) = 0$, and the claim then follows.

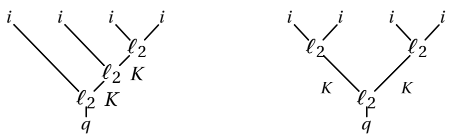
(2) Next, we give an explicit homotopy retract from L onto its homology for which

$$\ell_4(x_1, x_2, x_3, x_4) = \overline{\Phi} + \overline{[z, z]} \notin [x_1, x_2, x_3, x_4].$$

We write next a basis for the decomposition $L = A \oplus \partial A \oplus C$ in a table, up to degree 10. The rest of the decomposition is irrelevant for the question at hand. We have written elements in $(\partial A)_n$ in the order given by differentiating those of A_{n+1}. A dot $\cdot$ indicates that the corresponding subspace is the trivial one, and we omit filling those cells in the table which are not necessary.

deg	A	∂A	C
2	$\cdot$	$\cdot$	v_1, v_2, v_3, v_4
3	$\cdot$	$\cdot$	$\cdot$
4	$\cdot$	$[v_1, v_2], [v_1, v_3], [v_1, v_4],$ $[v_2, v_3], [v_2, v_4], [v_3, v_4]$	$\cdot$
5	$v_{12} + z, v_{13}, v_{14},$ $v_{23}, v_{24}, v_{34} + z$	$\cdot$	z
6	$\cdot$	$[v_1, [v_1, v_2]], [v_1, [v_1, v_3]],$ $\vdots \quad \vdots$ $[v_3, [v_1, v_4]], [v_2, [v_3, v_4]]$	$\cdot$
7	$[v_1, v_{12}], [v_1, v_{13}], [v_1, v_{14}],$ $[v_2, [v_{12}]], [v_2, v_{23}], [v_2, v_{24}],$ $[v_3, v_{13}], [v_3, v_{23}], [v_3, v_{34}],$ $[v_4, v_{14}], [v_4, v_{24}], [v_4, v_{34}],$ $[v_1, v_{23}], [v_2, v_{13}], [v_1, v_{24}],$ $[v_2, v_{14}], [v_1, v_{34}], [v_3, v_{14}],$ $[v_2, v_{34}], [v_3, v_{24}]$	$-[v_{12}, v_3] + [v_1, v_{23}] - [v_2, v_{13}],$ $-[v_{12}, v_4] + [v_1, v_{24}] - [v_2, v_{14}],$ $-[v_{13}, v_4] + [v_1, v_{34}] - [v_3, v_{14}],$ $-[v_{23}, v_4] + [v_2, v_{34}] - [v_3, v_{24}],$	$[z, v_1], [z, v_2],$ $[z, v_3], [z, v_4]$
8	$v_{123}, v_{124}, v_{134}, v_{234}$		$\cdot$
9		$[[v_1, v_2], z], [[v_1, v_3], z], [[v_1, v_4], z],$ $[[v_2, v_3], z], [[v_2, v_4], z], [[v_3, v_4], z]$	
10	$[v_{12}, z], [v_{13}, z], [v_{14}, z],$ $[v_{23}, z], [v_{24}, z], [v_{34}, z]$		$\Phi, [z, z]$

By the explicit formula of Theorem 1.3, the only two trees contributing to ℓ_4 are the following (we denote $\ell_2 = [\ ,\ ]$)



The first tree has 2 automorphisms and contributes to ℓ_4 with a sum over all permutations of $S(1,3)$. The second tree has 8 automorphisms and contributes a sum over all permutations of $S(2,2)$. The evaluation $\ell_4(x_1, x_2, x_3, x_4)$ is obtained then by applying q to the following sum,

$$\frac{1}{2}\sum_{\sigma\in S(1,3)} \varepsilon(\sigma)\left[K\left[K\left[u_{\sigma(1)}, u_{\sigma(2)}\right], u_{\sigma(3)}\right], u_{\sigma(4)}\right] + \frac{1}{8}\sum_{\sigma\in S(2,2)} \varepsilon(\sigma)\left[K\left[u_{\sigma(1)}, u_{\sigma(2)}\right], K\left[u_{\sigma(3)}, u_{\sigma(4)}\right]\right].$$

We first prove that $\ell_4(x_1, x_2, x_3, x_4) \notin [x_1, x_2, x_3, x_4]$. To do so, instead of filling the page with the rather long explicit computation, observe that by (1) it is enough to show that $\ell_4(x_1, x_2, x_3, x_4)$ contains a non trivial term $\overline{[z,z]}$. Indeed, from the identity permutation seen as an element of $S(2,2)$ (hence contributing to the second tree above), $\ell_4(x_1, x_2, x_3, x_4)$ contains the term obtained from applying q to

$$\begin{aligned}[K[u_1, u_2], K[u_3, u_4]] &= [K\partial(u_{12} + z), K\partial(u_{34} + z)] = [v_{12} + z, v_{34} + z] \\ &= [v_{12}, v_{34}] + [v_{12}, z] + [z, v_{34}] + [z, z].\end{aligned}$$

Here, the term $[z,z]$ cannot be obtained from any other permutation in any of the two trees above, because z is an indecomposable element such that $z \notin \operatorname{Im} K$. So, the term $[z,z]$ cannot be canceled in the following final expression,

$$\ell_4(x_1,x_2,x_3,x_4) = q\left(\underbrace{\Phi}_{\in C} + \underbrace{[v_{12},z]+[z,v_{34}]}_{\in A} + \underbrace{[z,z]}_{\in C}\right) = \overline{\Phi} + \overline{[z,z]}.$$

To prove that in fact $\operatorname{Im}(\ell_4) \cap [x_1,x_2,x_3,x_4]$ is empty, just observe that by degree reasons, any element in the intersection should come from the following evaluation, where we plug an arbitrary $\mathbb{Q}$-linear combination of x_1,x_2,x_3,x_4 in each entry of ℓ_4 and then use the antisimmetry of the bracket,

$$\ell_4\left(\sum_{j=1}^{4}\lambda_j^1 x_j,\ \dots,\ \sum_{j=1}^{4}\lambda_j^4 x_j\right) = \left(\sum_{i_1<\cdots i_4}\lambda_{i_1}^1\cdots\lambda_{i_4}^4\right)\ell_4(x_1,x_2,x_3,x_4) \notin [x_1,x_2,x_3,x_4].$$

$\square$

Remarks 2.14 (1) It will be a consequence of Corollary 2.17 to follow that the result above is the best possible in the sense that one will never find a space satisfying the analogous statement for Whitehead products of third order. (2) In view of Theorem 2.13 (2) above, and Theorem 2.8, the way in which L_∞ structures capture higher Whitehead products of order k is not by finding elements z_i such that $\ell_k(z_1,\dots,z_k) \in [x_1,\dots,x_k]$ but by forming the quotient space $H/\sum_{j=2}^{k-1}\operatorname{Im}\ell_j$ and studying its vector space generators.

Having seen that the higher brackets induced on H by some homotopy retract do not always recover higher Whitehead products, we next seek for conditions that do imply it. The following is the most general result in that direction.

Theorem 2.15 *Let L be a DGL such that, on H, $\ell_i = 0$ for $i \le k-2$ with $k \ge 3$. If $[x_1,\dots,x_k] \neq \emptyset$, then*

$$\ell_k(x_1,\dots,x_k) \in [x_1,\dots,x_k].$$

Remark 2.16 As in Remark 2.10, the assumption on the vanishing of the ℓ_i is independent of the chosen retract, hence the result is valid for any of them.

Proof: By Lemma 2.6, we have that

$$0 \in \left[x_{i_1},\dots,x_{i_{k-1}}\right], \quad \text{for any } 1 \le i_1 < \cdots < i_{k-1} \le k.$$

Therefore, we apply Thm. 2.8, taking into account that $\ell_i = 0$ for $i \le k-2$, to deduce:

$$\ell_{k-1}\left(x_{i_1},\dots,x_{i_{k-1}}\right) = 0 \quad \text{for any} \quad 1 \le i_1 < \cdots < i_{k-1} \le k. \tag{2.8}$$

Next, for each $p \le k-1$, let $U_p \subseteq U$ be the following subspace:

$$U_p = \left\langle u_{i_1\dots i_s} \mid 1 \le i_1 < \cdots < i_s \le k,\ s \le p\right\rangle.$$

Clearly, $\mathbb{L}\left(U_p\right)$ is a sub DGL of $\mathbb{L}(U)$ and $\mathbb{L}(U_{k-1}) = \mathbb{L}(U)$. We also denote

$$V_p = \left\langle u_{i_1\dots i_s} \in U_p \mid s \ge 2\right\rangle.$$

Again, $U_p = V_p \oplus \langle u_1,\dots,u_k\rangle$ and $V_{k-1} = V$. Let $L = A\oplus\partial A\oplus C$ be the decomposition equivalent to the chosen arbitrary homotopy retract. By induction on p, with $2 \le p \le k-1$, we will construct a DGL morphism $\phi : \mathbb{L}\left(U_p\right) \to L$ for which $\phi\left(V_p\right) \subseteq A$.

For $p = 2$, as $[x_1,\dots,x_k]$ is nonempty, let $\psi : \mathbb{L}(U) \to L$ be such that $\overline{\psi(w)} \in [x_1,\dots,x_k]$ as in diagram (2.3). We define $\phi : \mathbb{L}(U_2) \to L$ by

$$\phi(u_i) = \psi(u_i), \quad i = 1,\dots,k, \quad \phi\left(u_{i_1 i_2}\right) = K\partial\psi\left(u_{i_1 i_2}\right), \quad 1 \le i_1 < i_2 \le k.$$

Obviously, $\phi(V_2) \subseteq A$, and using the trivial identity for any homotopy retract $\partial K\partial = \partial$ we also see that ϕ commutes with the differentials:

$$\partial\phi\left(u_{i_1 i_2}\right) = \partial K\partial\psi\left(u_{i_1 i_2}\right) = \partial\psi\left(u_{i_1 i_2}\right) = \psi\partial\left(u_{i_1 i_2}\right) = \phi\partial\left(u_{i_1 i_2}\right).$$

Assume the assertion true for $p \le k-2$. Hence, there exists a DGL morphism

$$\phi : \mathbb{L}(U_{k-2}) \to L$$

for which $\phi(V_{k-2}) \subseteq A$. In particular, we have that

$$K\partial\phi\left(u_{i_1\dots i_s}\right) = \phi\left(u_{i_1\dots i_s}\right) \quad \text{for any generator} \quad u_{i_1\dots i_s} \in V_{k-2},$$

which is equation (2.6) for ϕ. Then, the same argument as in the proof of Theorem 2.12 proves that for any $p \le k-1$,

$$\ell_p\left(x_{i_1},\dots,x_{i_p}\right) = q\phi(\partial u_{i_1\dots i_p}), \qquad 1 \le i_1 < \cdots < i_p \le k. \tag{2.9}$$

In particular,

$$\ell_{k-1}\left(x_{i_1},\dots,x_{i_{k-1}}\right) = q\phi\left(\partial u_{i_1\dots i_{k-1}}\right) \quad \text{for any } 1 \le i_1 < \cdots < i_{k-1} \le k.$$

By equation (2.8) we conclude that

$$q\phi\left(\partial u_{i_1\dots i_{k-1}}\right) = 0.$$

Hence, $\phi\left(\partial u_{i_1\dots i_{k-1}}\right) \in \partial A$. Define

$$\phi\left(u_{i_1\dots i_{k-1}}\right) = K\phi\left(\partial u_{i_1\dots i_{k-1}}\right).$$

Obviously $\phi(V_{k-1}) = \phi(V) \subseteq A$, and we see that ϕ commutes with differentials:

$$\partial\phi\left(u_{i_1\dots i_{k-1}}\right) = \partial K\phi\left(\partial u_{i_1\dots i_{k-1}}\right) = \left(\mathrm{id} - iq - K\partial\right)\phi\left(\partial u_{i_1\dots i_{k-1}}\right) = \phi\partial\left(u_{i_1\dots i_{k-1}}\right).$$

Therefore, $\overline{\phi(w)}$ is an element in $[x_1,\dots,x_k]$ to which we can apply Theorem 2.12, and the proof is finished. $\square$

Corollary 2.17 *Let $x_1, x_2, x_3 \in H$ be such that $[x_1, x_2, x_3] \neq \emptyset$. Then, for any homotopy retract,*

$$\ell_3(x_1, x_2, x_3) \in [x_1, x_2, x_3].$$

Corollary 2.18 *Let L be a DGL such that H is abelian. If $[x_1, x_2, x_3, x_4] \neq \emptyset$, then, for any homotopy retract,*

$$\ell_4(x_1, x_2, x_3, x_4) \in [x_1, x_2, x_3, x_4].$$

Let $x_1,\dots,x_k$ be homotopy classes of a space X such that the Whitehead product set $[x_1,\dots,x_k]$ is non empty, and consists in several elements. In some instances, it is possible to endow the rational homotopy Lie algebra L_X of X with several L_∞ structures exhausting the Whitehead product set. That is: for each class in the set $[x_1,\dots,x_k]$, one may choose a minimal L_∞ structure on L_X recovering the chosen class, as in the following result.

Theorem 2.19 *Let $X=\mathbb{C}P^2\vee S^4$. Then:*

(1) there exists a triple rational Whitehead product set $[x_1,x_2,x_3]$ in X with countably many different classes, and

(2) for any choice $x\in[x_1,x_2,x_3]$, one may endow the rational homotopy Lie algebra L_X with a minimal L_∞ structure which makes it homotopy equivalent to the Quillen minimal model of X, and with the property that $\ell_3(x_1,x_2,x_3)=x$.

Proof: From the standard cellular decomposition, it follows that the Quillen minimal model of X is given by $L=\mathbb{L}(u,v,z)$, with $|u|=1,|z|=3$ and the differential is determined by $\partial v=\frac{1}{2}[u,u]$, and $\partial z=0$.

Choose $x_1=x_2=x_3=\overline{u}$. We next compute $[x_1,x_2,x_3]$. To do so, we will find all DGL extensions for the diagram

$$\begin{tikzcd} \mathbb{L}(u_1,u_2,u_3) \arrow[r, "\phi", "\simeq"'] \arrow[d, hook] & L \\ \mathbb{L}(U) \arrow[ur, dotted, "\widetilde{\phi}"'] & \end{tikzcd}$$

where $\phi(u_i)=u$ for every $i=1,2,3$, and $\mathbb{L}(U)$ is the Quillen minimal model of the fat wedge $T\left(S^3,S^3,S^3\right)$. All possible extensions $\widetilde{\phi}$ are of the form

$$\widetilde{\phi}(u_{ij})=2v+\lambda_{ij}z\quad\text{for any }\lambda_{ij}\in\mathbb{Q},\quad i<j.$$

Each choice $\left\{\lambda_{ij}\right\}\subseteq\mathbb{Q}$ gives rise to a different extension $\widetilde{\phi}$, and the set $[x_1,x_2,x_3]$ is then given by all homology classes whose representatives are of the form

$$\begin{aligned}\widetilde{\phi}(w)&=\left[\widetilde{\phi}(u_{12}),u\right]+\left[\widetilde{\phi}(u_{13}),u\right]+\left[u,\widetilde{\phi}(u_{23})\right]\\&=6[v,u]+(\lambda_{12}+\lambda_{13}+\lambda_{23})[z,u],\end{aligned}$$

where $\lambda_{ij}\in\mathbb{Q}$. It is easy to check (see the table below) that $(\partial L)_4=0$, hence each representative above corresponds to a unique homology class in $[x_1,x_2,x_3]$, proving (1).

Next, we choose $x\in[x_1,x_2,x_3]$, and prove (2). Assume that x is represented by

$$6[v,u]+(\lambda_{12}+\lambda_{13}+\lambda_{23})[z,u]$$

for some fixed λ_{ij}. Define $\alpha=\frac{1}{3}(\lambda_{12}+\lambda_{13}+\lambda_{23})$ and consider any L_∞ structure induced on H which is equivalent to a decomposition $L=A\oplus\partial A\oplus H$ whose basis coincides up to degree 6 with the one given in the table below. The rest of the basis is irrelevant for the question at hand. We have written elements in $(\partial A)_n$ in the order given by differentiating those of A_{n+1}. A dot $\cdot$ indicates that the corresponding subspace is the trivial one, and we omit filling those cells which are not needed.

degree	A	∂A	C
1	$\cdot$	$\cdot$	u
2	$\cdot$	$[u,u]$	$\cdot$
3	$2v+\alpha z$	$\cdot$	z
4	$\cdot$	$\cdot$	$[u,v],[u,z]$
5	$\cdot$	$\frac{1}{2}[[u,u],z]$	$[u,[u,v]]$
6	$[v,z]$		

The associated ternary bracket is:

$$-\ell_3(x_1,x_2,x_3) = 3q[K[u,u],u] = 3q[2v+\alpha z,u] = q(6[v,u]+3\alpha[z,u]) = x.$$

□

One may come up with situations like the one above for fourth and higher order products. But in general this is not the case, as one may also find spaces in which some $x \in [x_1,x_2,x_3]$ is *not recovered by any* L_∞ *structure.*

Theorem 2.20 *There exists a simply connected rational space* X*, and homotopy classes* $x_1,x_2,x_3 \in \pi_*(X)$ *such that:*

(1) the triple Whitehead product set $[x_1,x_2,x_3]$ *has countably many different classes, and*

(2) there exist choices $x \in [x_1,x_2,x_3]$ *for which there does not exist a transferred minimal* L_∞ *structure on the rational homotopy Lie algebra* L_X *such that* $\ell_3(x_1,x_2,x_3) = x$*.*

Proof: Let $\mathbb{L}(V)$ be the Quillen minimal model of $T\left(S^3,S^3,S^3\right)$, so that

$$V = \langle v_1,v_2,v_3,v_{12},v_{13},v_{23}\rangle, \quad |v_i| = 2\ \forall i \quad \text{and} \quad \partial v_{ij} = [v_i,v_j] \quad \forall i<j.$$

Take the quotient of this DGL by the differential ideal generated by $\{[v_1,v_2],[v_1,v_3]\}$, and denote it by (L,∂). The realization $\langle L\rangle$ of this DGL will be our space X. All elements in the quotient are denoted as in the original DGL without confusion. We give next a decomposition $L = A\oplus\partial A\oplus C$, up to degree 7, to help with the computations.

degree	A	∂A	C
2	$\cdot$	$\cdot$	v_1,v_2,v_3
3	$\cdot$	$\cdot$	$\cdot$
4	$\cdot$	$[v_2,v_3]$	$\cdot$
5	v_{23}	$\cdot$	v_{12},v_{13}
6	$\cdot$	$[v_2,[v_2,v_3]],[v_3,[v_2,v_3]]$	
7	$[v_2,v_{23}],[v_3,v_{23}]$	$\cdot$	$[v_1,v_{13}],[v_3,v_{13}],[v_1,v_{23}],$ $[v_1,v_{12}],[v_2,v_{12}],[v_2,v_{13}],$ $[v_3,v_{12}]$

Choose x_i as the homology class of v_i for $i=1,2,3$, and compute $[x_1,x_2,x_3]$. For it, consider the diagram

$$\begin{tikzcd}
\mathbb{L}(u_1,u_2,u_3) \arrow[r, "\phi"] \arrow[d, hook] & L \\
\mathbb{L}(U) \arrow[ur, dotted] &
\end{tikzcd}$$

where $\phi(u_i)=v_i$ for $i=1,2,3$, and $\mathbb{L}(U)$ is the Quillen minimal model of the corresponding fat wedge. All possible extensions $\widetilde{\phi}$ are of the form:

$$\begin{aligned}
\widetilde{\phi}(u_{12}) &= a_1 v_{12} + a_2 v_{13} \\
\widetilde{\phi}(u_{13}) &= a_3 v_{12} + a_4 v_{13} \qquad\qquad \text{for any choices } a_i \in \mathbb{Q}. \\
\widetilde{\phi}(u_{23}) &= a_5 v_{12} + a_6 v_{13} + v_{23}
\end{aligned}$$

The set $[x_1,x_2,x_3]$ is then given by all homology classes whose representatives are of the form:

$$\begin{aligned}
\widetilde{\phi}(w) &= -\left[\widetilde{\phi}(u_{12}), v_3\right] + \left[v_1, \widetilde{\phi}(u_{23})\right] - \left[v_2, \widetilde{\phi}(u_{13})\right] \\
&= -a_1[v_{12},v_3] - a_2[v_{13},v_3] - a_3[v_2,v_{12}] - a_4[v_2,v_{13}] + a_5[v_1,v_{12}] + a_6[v_1,v_{13}] + [v_1,v_{23}].
\end{aligned}$$

We see from the table that $(\partial L)_7 = 0$. This implies that each representative above gives a different homology class in $[x_1,x_2,x_3]$, hence statement (1) is proven.
We claim now that for many choices of $x \in [x_1,x_2,x_3]$, there does not exist a homotopy retract of L onto H such that $\ell_3(x_1,x_2,x_3) = x$. For example, choose $a_1 = 1$ and all other $a_i = 0$. Then,

$$z = -[v_{12}, v_3] + [v_1, v_{23}]$$

is a representative of a non trivial element $x \in [x_1,x_2,x_3]$ for which there does not exist one such retract. To see this, observe that for any homotopy retract (L,i,q,K),

$$\begin{aligned}
\ell_3(x_1,x_2,x_3) &= q\left(-\left[K\underbrace{[v_1,v_2]}_{0}, v_3\right] + [v_1, K[v_2,v_3]] + \left[K\underbrace{[v_1,v_3]}_{0}, v_2\right]\right) \\
&= q\left(\left[v_1, v_{23} + \alpha v_{12} + \beta v_{13}\right]\right) = q\ell_3'(x_1,x_2,x_3).
\end{aligned}$$

For $\ell_3(x_1,x_2,x_3)$ to be equal to x, it is necessary and sufficient that $\ell_3'(x_1,x_2,x_3) - z \in \operatorname{Ker}(q) = A \oplus \partial A$. But this is impossible, because

$$\ell_3'(x_1,x_2,x_3) - z = \alpha[v_1,v_{12}] + \beta[v_1,v_{13}] + [v_{12},v_3]$$

is a non trivial cycle for any election of α and β, and so we are done. One may go a step forward, and characterize which elements in $[x_1,x_2,x_3]$ are recovered by some ℓ_3. To do so, it suffices to write down the condition $\ell_3'(x_1,x_2,x_3) - \widetilde{\phi}(w) \in \operatorname{Ker}(q)$ and check if it is satisfied. □

The situation above can be manipulated to obtain similar examples for products of fourth and above order.

2.4 Higher Whitehead products and Sullivan L_∞ algebras

The detection of higher Whitehead products in Sullivan models was studied by P. Andrews and M. Arkowitz in [3]. In that work, the authors very explicitly determined how to read off the (ordinary, as well as) higher Whitehead products from the differential of a minimal Sullivan model, for simply connected rational spaces. We first state the main result of [3] introducing all the necessary material. Then, we recall (see [15, 8]) that, under certain assumptions which we fix for the rest of the section, L_∞ algebras correspond uniquely to Sullivan algebras. Finally, we achieve the main goal of this section by showing how some results of Section 2.3 generalize and/or complement the main result of [3].

All spaces are assumed to be simply connected rational CW-complexes of finite type.

2.4.1 Higher Whitehead products in Sullivan models

Here, we briefly recall the most relevant result of [3] which, in general terms, reads off the rational higher Whitehead products from the differential d of the Sullivan minimal model of a space X. For it, we need first some linear algebra.

Let $n_1,\dots,n_r$ be fixed positive integers, and denote by $M_r(\mathbb{Q})$ the square matrices of size r over the rationals. Define the map

$$\widetilde{\rho}\colon M_r(\mathbb{Q}) \longrightarrow \mathbb{Q}, \qquad \widetilde{\rho}(A) = \sum_{\sigma\in S_r} \varepsilon_\sigma\, a_{1\sigma(1)}\cdots a_{r\sigma(r)},$$

where $A=(a_{ij})$, and ε_σ is the sign arising from associating to each element a_{ij} a generator w_j of degree n_j in the free graded commutative algebra $\Lambda\{w_1,\dots,w_r\}$ and writing $w_1\cdots w_r = \varepsilon_\sigma\, w_{\sigma(1)}\cdots w_{\sigma(r)}$. More precisely, ε_σ is the parity of

$$\sum_{i=1}^{r-1}\ \sum_{\substack{1\le j<\sigma^{-1}(i)\\ \sigma(j)>i}} n_i n_{\sigma(j)},$$

and it is 1 whenever the above sum is empty.

Remark 2.21 In other words, if one considers $\mathbb{Q}$-linear combinations

$$y_1 = a_{11}w_1+\cdots+a_{r1}w_r, \quad \dots, \quad y_r = a_{r1}w_1+\cdots+a_{rr}w_r$$

in the commutative graded algebra $\Lambda\{w_1,\dots,w_r\}$, and one forms the matrix $A=(a_{ij})$, then $\widetilde{\rho}(A)$ is simply the coefficient of the term $w_1\cdots w_r$ in the product $y_1\cdots y_r$, with the correct Koszul sign. In particular, if the integers $n_1,\dots,n_r$ are all odd, i.e., $w_1,\dots,w_r$ are oddly graded, then $\widetilde{\rho}$ is simply the determinant. Hence, in general, we may think of ρ as a "graded determinant". In particular, observe that, if a row or a column of some matrix A is the zero vector then $\widetilde{\rho}(A)=0$. However, in general, $\widetilde{\rho}$ does not vanish on matrices whose rows (or columns) are linearly dependent.

Let $(\Lambda V,d)$ be the minimal Sullivan model of the simply connected complex X, fix a KS-basis $\{v_i\}_{i\ge1}$ of V and homotopy classes $x_j\in\pi_{n_j}(X)$, for $1\le j\le r$, and define a map

$$\rho=\rho_{\{v_i\}\{x_1,\dots,x_r\}}\colon \Lambda^{\ge r}V\longrightarrow\mathbb{Q}$$

as follows: for a given $\Phi\in\Lambda^{\ge r}V$, write

$$\Phi=\sum_{i_1\le\cdots\le i_r}\lambda_{i_1\dots i_r}\, v_{i_1}\cdots v_{i_r}+\beta,$$

where $\lambda_{i_1\dots i_r}\in\mathbb{Q}$ and $\beta\in\Lambda^{>r}V$. For each $i_1\le\cdots\le i_r$ let $A_{i_1\dots i_r}\in M_r(\mathbb{Q})$ be the matrix whose entries are given by $a_{pq}=\langle v_{i_p};x_q\rangle$, where $\langle\,;\,\rangle$ is the usual pairing (see Section 1.5). Then,

$$\rho(\Phi)=\sum_{i_1\le\cdots\le i_r}\lambda_{i_1\dots i_r}\,\widetilde{\rho}\big(A_{i_1\dots i_r}\big).$$

Remark 2.22 Whenever the homotopy classes $x_1,\dots,x_r$ are linearly independent, $\rho(\Phi)$ has the following interpretation: identify $x_1,\dots,x_r$ with vectors $w_1,\dots,w_r$ of V through the usual pairing, and extend this to a new basis of V which in turn produces a new basis of ΛV. Write

$$\Phi = \lambda\, w_1 \cdots w_r + \Gamma$$

as a linear combination of this basis. Then, it follows by Remark 2.21 that $\rho(\Phi)$ is precisely λ.

We are ready to recall how the differential of a Sullivan minimal model captures higher Whitehead products. Denote $N = n_1 + \cdots + n_r$.

Theorem 2.23 [3, Thm. 5.4] *Let $(\Lambda V, d)$ be the Sullivan minimal model of the simply connected complex X. Fix a KS-basis $\{v_i\}$ of V and homotopy classes $x_j \in \pi_{n_j}(X)$, $1 \le j \le r$, such that $[x_1,\dots,x_r]$ is defined. Let $\Phi \in (\Lambda V)^{N-1}$ such that $d\Phi \in \Lambda^{\ge r} V$. Then, for every $x \in [x_1,\dots,x_r]$,*

$$\langle \overline{\Phi}; x\rangle = (-1)^{\alpha} \rho(d\Phi),$$

where $\rho = \rho_{\{v_i\}\{x_1,\dots,x_r\}}$, $\alpha = \sum_{i<j} n_i n_j$ and $\overline{\Phi} \in V$ is the linear part of Φ. In particular, if $v \in V^{N-1}$ is such that $dv \in \Lambda^{\ge r} V$,

$$\langle v; x\rangle = (-1)^{\alpha} \rho(dv).$$

Note that the rational number $\rho(dv)$ depends on the chosen basis for V but only of the rth part d_r of the differential d.

As an illustrative example, we keep the notation of this theorem in the following result, which is obvious in view of Remark 2.22:

Corollary 2.24 *Let $v \in V$ be such that, in the chosen homogeneous basis of V,*

$$dv = \sum_{i_1 \le \cdots \le i_r} \lambda_{i_1 \dots i_r} v_{i_1} \cdots v_{i_r} + \beta, \qquad \beta \in \Lambda^{>r} V.$$

Fix $i_1 \le \cdots \le i_r$ and let $x_{i_1},\dots,x_{i_r}$ be the homotopy classes dual to $v_{i_1},\dots,v_{i_r}$ through the usual pairing. Then, for each $x \in [x_{i_1},\dots,x_{i_r}]$,

$$\langle v; x\rangle = (-1)^{\alpha} \lambda_{i_1 \dots i_r}. \qquad \square$$

2.4.2 Sullivan L_∞ algebras

To compare the results in Section 2.3 with those of the above subsection we characterize here those L_∞ algebras which are equivalent to finite type Sullivan algebras, as defined in Section 1.5. Such a characterization is not original to this work, and the reader is referred to [15] and [8] in order to find the most general results and a meticulous study of the subtleties concerning this duality.

Recall that, by Theorem 1.4, an L_∞ structure on a graded vector space L corresponds uniquely with a codifferential δ on the cofree cocommutative coalgebra ΛsL. If, in addition, $L = L_{\ge 0}$ is finite type and non-negatively graded, then sL is concentrated in positive degrees and therefore ΛsL is also finite type and connected.

On the other hand, it is well known that dualization provides a one to one correspondence between finite type CDGA's and CDGC's. Hence, an L_∞ structure on a finite type, non negatively graded vector space L corresponds to the CDGA $\big((\Lambda sL)^\sharp, \delta^\sharp\big)$ dual of the CDGC $(\Lambda sL, \delta)$. Under the same finiteness and bounding hypothesis, a slight generalization of the proof of [19, Lemma 23.1] provides an isomorphism of commutative graded algebras

$$\Lambda (sL)^\sharp \cong (\Lambda sL)^\sharp$$

via the following pairing [19, p. 294], which is essential in what follows: for each $k \geq 1$, denote $V = (sL)^\sharp$ and define

$$\langle\, ;\,\rangle \colon \Lambda^k V \times \Lambda^k sL \longrightarrow \mathbb{Q}, \tag{2.10}$$

$$\langle v_1 \cdots v_k ; sx_k \wedge \ldots \wedge sx_1 \rangle = \sum_{\sigma \in S_k} \varepsilon_\sigma \langle v_{\sigma(1)} ; sx_1 \rangle \cdots \langle v_{\sigma(k)} ; sx_k \rangle,$$

where $v_1 \ldots v_k = \varepsilon_\sigma\, v_{\sigma(1)} \ldots v_{\sigma(k)}$.

Through this isomorphism the differential $\delta^\sharp$ in $(\Lambda sL)^\sharp$ becomes the differential d in $\Lambda(sL)^\sharp = \Lambda V$ whose kth part d_k satisfies

$$\langle d_k v ; sx_1 \wedge \ldots \wedge sx_k \rangle = \varepsilon \langle v ; s\ell_k(x_1, \ldots, x_k) \rangle \tag{2.11}$$

where $v \in (sL)^\sharp$, $x_i \in L$, $i = 1, \ldots, k$ and $\varepsilon = (-1)^{|v| + \sum_{j=1}^{k-1}(k-j)|x_j|}$. Observe that, via this equality, for each $v \in (sL)^\sharp$ and any $k \geq 1$, $d_k v$ is well defined and vanishes for k big enough, due to the finite type and the bounding assumption respectively.

Summarizing,

Proposition 2.25 L_∞ algebras on a finite type, non negatively graded vector space L are in one to one correspondence with differentials d on the free commutative graded algebra ΛV generated by $V = (sL)^\sharp$. Explicitly, for each $k \geq 1$, the kth bracket ℓ_k on L and the kth part d_k determine each other via the formula (2.11) above. □

Moreover, we can easily detect when a given L_∞ algebra on L provides a Sullivan algebra through this correspondence. From now on, and along this section, L_∞ algebras are all non negatively graded and of finite type.

Proposition 2.26 Let L be an L_∞ algebra and let $(\Lambda V, d)$ be the corresponding CDGA. Then, the following are equivalent:

(1) $(\Lambda V, d)$ is a Sullivan algebra.

(2) There exists an ordered homogeneous basis $\{x_i\}_{i \in I}$ of L such that for every $k \geq 1$, the class of $\ell_k\left(x_{i_1}, \ldots, x_{i_k}\right)$ vanishes in the quotient $L/L^{>j}$, where $j = \max\{i_1, \ldots, i_k\}$ and $L^{>j}$ is the linear span of $\left\{x_i \mid i > j\right\}$.

(3) L is "degree-wise nilpotent": in the *lower central series* $L = \Gamma^0 L \supseteq \Gamma^1 L \supseteq \ldots$ of L, where each $\Gamma^k L$ is the subspace of L generated by all possible brackets using at least k elements of L, for every n there exists some k with the property that $(\Gamma^k L)_n = 0$.

Proof: The equivalence between (1) and (2) arises simply from translating the property defining a Sullivan algebra in Section 1.5 to the brackets of L through the formula (2.11). On the other hand, the equivalence between (2) and (3) is precisely [8, Thm. 3.2]. □

Definition 2.27 A *Sullivan L_∞ algebra* is an L_∞ algebra satisfying any of the above equivalent conditions.

Let $(\Lambda V, d)$ be the minimal Sullivan algebra equivalent to the minimal Sullivan L_∞ algebra $(L, \{\ell_k\})$ and fix a KS-basis $\{v_i\}$ of V.

Observe that, fixing elements $x_1, \ldots, x_r \in L$ (not necessarily of the given basis), the map

$$\rho = \rho_{\{v_i\}\{x_1, \ldots, x_r\}} \colon \Lambda^{\geq r} V \to \mathbb{Q}$$

of the past section can be defined using the same procedure: write any element $\Phi\in\Lambda^{\geq r}V$ as

$$\Phi=\sum_{i_1\leq\cdots\leq i_r}\lambda_{i_1\dots i_r}\,v_{i_1}\cdots v_{i_r}+\beta,$$

where $\lambda_{i_1\dots i_r}\in\mathbb{Q}$ and $\beta\in\Lambda^{>r}V$, let $A_{i_1\dots i_r}\in M_r(\mathbb{Q})$ be the matrix whose entries are given by $a_{pq}=\left\langle v_{i_p};sx_q\right\rangle$, and recall that $V=(sL)^\sharp$. Then,

$$\rho(\Phi)=\sum_{i_1\leq\cdots\leq i_r}\lambda_{i_1\dots i_r}\,\widetilde{\rho}\left(A_{i_1\dots i_r}\right).$$

Remark 2.28 In view of the pairing (2.10), for any $v\in V$ and any $sx_1,\dots,sx_k\in sL$, a short computation shows that,

$$\langle d_k v;sx_1\wedge\ldots\wedge sx_k\rangle=\rho(d_k v).$$

with $\rho=\rho_{\{v\}\{x_1,\dots,x_k\}}$.

2.4.3 Extending Andrews and Arkowitz's theorem to L_∞ algebras

Let L be a Lie model of the simply connected, finite type complex X, consider a homotopy retract (L,H,i,q,K) of L and let $(H,\{\ell_k\})$ be the corresponding L_∞ structure on H. Observe that H is a Sullivan L_∞ algebra whose associated Sullivan algebra $(\Lambda V,d)$ is the minimal model of X for which we fix a KS-basis. Then, the translation of Theorem 2.23 in this context reads:

Theorem 2.29 *Let $x_j\in H$, $1\leq j\leq r$, be such that $[x_1,\dots,x_r]$ is defined. Let $v\in V^{N-1}$ be such that $dv\in\Lambda^{\geq r}V$. Then, for every $x\in[x_1,\dots,x_r]$,*

$$\langle v;sx\rangle=\varepsilon\langle v;s\ell_r(x_1,\dots,x_r)\rangle.$$

Proof: Indeed, recall that

$$V=(sH)^\sharp\cong\pi_*(X)\otimes\mathbb{Q}.$$

Hence, Theorem 2.23 states that $\langle v;x\rangle=(-1)^\alpha\rho(dv)$. But, $\rho(dv)=\rho(d_kv)$, and in view of Remark 2.28 and formula (2.11),

$$(-1)^\alpha\rho(dv)=(-1)^\alpha\langle d_kv;sx_1\wedge\ldots\wedge sx_k\rangle=\varepsilon\langle v;s\ell_k(x_1,\dots,x_k)\rangle.$$

□

Remarks 2.30 (i) Observe that Theorem 2.29, and hence Theorem 2.23, can be easily deduced from our Theorem 2.8. Indeed, via this result, and for any $x\in[x_1,\dots,x_r]$,

$$\varepsilon\ell_r(x_1,\dots,x_r)=x+\sum_{j=2}^{r-1}\ell_j\left(\Phi_j\right),\qquad \Phi_j\in H^{\otimes j}.$$

Hence, for any $v\in V^{N-1}$,

$$\langle v;sx\rangle=\varepsilon\langle v;s\ell_r(x_1,\dots,x_r)\rangle-\sum_{j=2}^{r-1}\left\langle v;s\ell_j\left(\Phi_j\right)\right\rangle.$$

However, if $dv\in\Lambda^{\geq r}V$, then $d_j(v)=0$ for every $j<r$ and therefore, in view of formula (2.11), the second term vanishes. Therefore,

$$\langle v;sx\rangle=\varepsilon\langle v;s\ell_r(x_1,\dots,x_r)\rangle,$$

which is the statement of Theorem 2.29.

(ii) Observe also that Theorem 2.12 is a generalization of Theorem 2.29, and hence of Theorem 2.23, under the presence of an adapted homotopy retract. Indeed, given $x \in [x_1,\dots,x_r]$, Theorem 2.12 asserts that

$$\ell_r(x_1,\dots,x_r) = x,$$

whenever the L_∞ structure on H arises from a homotopy retract adapted to x. That is,

$$\langle v; sx\rangle = \langle v; s\ell_r(x_1,\dots,x_r)\rangle = (-1)^\alpha \rho(dv), \quad \text{for all } v \in V^{N-1},$$

and not only for those v with $dv \in \Lambda^{\geq r} V$.

CHAPTER 3

Higher Whitehead products and formality

Since it was first introduced in [17, §4], the notion of formality has been a key tool in several branches of mathematics (such as mathematical physics, operads, homotopy theory, differential geometry, algebraic geometry and deformation theory). Deep results, just to mention two of them, where the formality of a DGL or of a CDGA plays a fundamental role are Kontsevich's groundbreaking proof of the Quantization Theorem [34] and the existence of a large family of symplectic manifolds with no Kähler structure [16, 17].

In this Section, we study the relationship of formality of DGL's with L_∞ structures and higher Whitehead products. In particular, we provide two criteria for discarding the formality of a (completely arbitrary) DGL, which complements the characterization of formality on [44, §7], and characterize the intrinsic coformality of products of odd dimensional spheres.

As usual, H will stand for the homology of the L_∞ algebra or DGL under consideration, endowed with the zero differential.

3.1 Formality of DGL's

Recall that either a DGA or a DGL L is *formal* if it has the same homotopy type, or it is weakly equivalent to its (co)homology H. In other words, if there exists a zig-zag of quasi-isomorphisms,

$$L \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} H.$$

A connected space X is *formal* if the CDGA $A_{PL}(X)$ is formal, that is, if $H^*(X;\mathbb{Q})$ is a CDGA model of X. Whenever X is a simply connected CW-complex of finite type, this is equivalent to say that the rational homotopy type of X is characterized by its rational cohomology. On the other hand, a simply connected space X is *coformal* if the DGL $\lambda(X)$ is formal, that is, if $\pi_*(\Omega X)\otimes\mathbb{Q}$ is a DGL model of X. In other words, the rational homotopy type of X is characterized by its rational homotopy Lie algebra.

We now extend to any DGL a well known fact on rational homotopy theory for reduced DGL's with finite type homology (see the discussion below).

Proposition 3.1 A differential graded Lie algebra L is formal if and only if it is L_∞ quasi-isomorphic to a minimal L_∞ algebra L' for which $\ell_n = 0$ for all $n \geq 3$.

Note that necessarily $L' = H(L)$.

Proof: It is well known, see for instance [34, Theorem 4.6] or [40, Section 10.4], that if f is a quasi-isomorphism of L_∞ algebras from M to N, then there exists another L_∞ quasi-isomorphism g from N to M such that

$$H(f_1)\colon H(M,\ell_1) \rightleftarrows H(N,\ell_1)\colon H(g_1)$$

are inverses of each other. Hence, being quasi-isomorphic is a well defined concept for a pair of L_∞ algebras. In particular, if both M and N are DGL's, a zig-zag

$$M \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} N$$

of DGL quasi-isomorphism exists if and only if M and N are quasi-isomorphic as L_∞ algebras, that is, there are L_∞ quasi-isomorphisms $f\colon M \rightleftarrows N\colon g$ as before.

If L is formal, then the above discussion implies that there are L_∞ quasi-isomorphisms

$$L \underset{I}{\overset{Q}{\rightleftarrows}} H.$$

The converse follows from the fact that an L_∞ quasi-isomorphism between two DGL's is equivalent to a zig-zag of DLG quasi-isomorphisms between these two [40, Thm. 11.4.14]. □

Remark 3.2 As in the proposition above, let L be a reduced DGL of finite type L_∞ quasi-isomorphic to a minimal L_∞ algebra with vanishing ℓ_k, for $k \geq 3$. In other words, by Theorem 1.4, we have a CDGC quasi-isomorphism

$$(\Lambda sL,\delta) \xrightarrow{\simeq} (\Lambda sH,\delta).$$

Dualizing, we obtain a CDGA quasi-isomorphism,

$$(\Lambda(sH)^\sharp,d) \xrightarrow{\simeq} \mathscr{C}^*(L)$$

where, on the right, we have the classical cochains on L, dual of its chains (see Remark 1.5) and on the left, we use the isomorphism $(\Lambda sH,\delta)^\sharp \cong (\Lambda(sH)^\sharp,d)$ [19, Lemma 23.1], explicitly given in Section 2.4.2. Also, we recalled in Section 2.4.2 that, for each $k \geq 2$, the bracket ℓ_k on H (which defines δ and vanishes for $k \geq 3$) is identified with the kth part d_k of the differential d by the formula (2.11). In particular d is decomposable and just quadratic, that is, $(\Lambda(sH)^\sharp,d)$ is a Sullivan minimal model with quadratic differential.

Thus, Proposition 3.1 implies the well known fact that a reduced DGL L of finite type is formal if and only if the Sullivan minimal model on the cochains of L admits a quadratic differential [63, II.7(6)].

In particular, let X be a simply connected CW-complex of finite type and choose L a finite type DGL model of X. The above arguments translates to the classical fact: X is coformal if and only if there is a purely quadratic differential on its Sullivan minimal model [63, II.7(6)].

The following example shows, from the L_∞ point of view, this known fact: in order to detect non-formality of a DGL L (respec. non-coformality of a space X), it is *not* sufficient to find a non-vanishing bracket of order greater or equal than 3 in a particular minimal L_∞ algebra quasi-isomorphic to L (respec. to a Lie model of X).

Example 3.3 Consider the formal and coformal space $S^2 \vee S^4$ whose Quillen minimal model is the DGL $(\mathbb{L}(a,b),0)$, where $|a| = 1$ and $|b| = 3$. Consider the homotopy retract in which both i and q are the identity on $\mathbb{L}(a,b)$ and K is the zero chain homotopy except for $K[a,a] = b$. Recall that, although this homotopy retract does not satisfy the annihilation conditions, Theorem 1.3 applies. For this retract, the transferred quasi-isomorphic L_∞ structure is such that

$$\ell_3(a,a,a) = q\,[3K\,[i(a),i(a)]\,,i(a)] = 3[b,a] \neq 0.$$

3.2 Formality and higher Whitehead products

We shall give here two criteria in terms of higher order Whitehead products to detect non-formality of DGL's. As we do not impose any restriction on the considered DGL, the following is needed.

Recall (see for instance [27, §2]) that the category of DGL's admits a model category structure [56] in which fibrations are surjective morphisms, weak equivalences are quasi-isomorphisms and cofibrations are morphisms satisfying the left lifting property with respect to trivial fibrations.

Consider $\mathbb{L}(U)$, the free DGL of Definition 2.3. Even in the non-reduced case, that is, when $u_1,\dots,u_n$ are allowed to be of arbitrary degrees, the proof of the well known *Lifting Lemma* [63, Theorem II.5. (13)] to surjective quasi-isomorphisms works in this case to prove:

Lemma 3.4 The DGL $\mathbb{L}(U)$ is cofibrant. That is, given a trivial fibration, i.e., a surjective quasi-isomorphism $L \overset{\simeq}{\twoheadrightarrow} L'$, any morphism $\eta\colon \mathbb{L}(U)\to L'$ lifts to a DGL morphism $\theta\colon \mathbb{L}(U)\to L$:

$$\begin{tikzcd} & L \arrow[d, "\varphi", "\simeq"'] \\ \mathbb{L}(U) \arrow[r, "\eta"'] \arrow[ur, dashed, "\theta"] & L'. \end{tikzcd}$$

In particular, a well known general fact on model categories shows the following:

Lemma 3.5 Given a (non necessarily surjective) DGL quasi-isomorphism $\varphi\colon L \xrightarrow{\simeq} L'$, composition with φ induces a bijection between the sets of homotopy classes,

$$\varphi_\sharp\colon \big[\mathbb{L}(U), L\big] \xrightarrow{\cong} \big[\mathbb{L}(U), L'\big].$$

Recall that in this model structure, a *path object* for a given DGL L is the DGL $L\otimes\Lambda(t,dt)$ in which $|t|=0$, together with morphisms

$$L\otimes\Lambda(t,dt) \underset{\varepsilon_1}{\overset{\varepsilon_0}{\rightrightarrows}} L,$$

defined by the identity on L and $\varepsilon_0(t)=0$, $\varepsilon_1(t)=1$. Then, two morphisms $f,g\colon \mathbb{L}(U)\to L$ are homotopic if there is a morphism $\Psi\colon \mathbb{L}(U)\to L\otimes\Lambda(t,dt)$ such that $\varepsilon_0\Psi=f$ and $\varepsilon_1\Psi=g$. In particular, as $\Lambda(t,dt)$ is acyclic, two homotopic morphisms induce the same morphism on homology.

Proposition 3.6 Let $\varphi: L \xrightarrow{\simeq} L'$ be a DGL quasi-isomorphism and let $x_1,\dots,x_k\in H_*(L)$, $k\geq 2$. Then, denoting $H_*(\varphi)=\varphi_*$, we have:

(1) $0\in[x_1,\dots,x_k]$ if and only if $0\in\big[\varphi_*(x_1),\dots,\varphi_*(x_k)\big]$.

(2) The cardinality of the higher order Whitehead product sets agree:

$$\#[x_1,\dots,x_k] = \#\big[\varphi_*(x_1),\dots,\varphi_*(x_k)\big].$$

Proof: First, observe that $[x_1,\ldots,x_k]$ is non empty if and only if $[\varphi_*(x_1),\ldots,\varphi_*(x_k)]$ is also non empty. The result is now a straightforward consequence of lemmas 3.4 and 3.5. Indeed, given $\overline{\phi(w)}\in[x_1,...,x_k]$, the fact that φ_* is an isomorphism implies that $\overline{\varphi\circ\phi(w)}\in\left[\varphi_*(x_1),\ldots,\varphi_*(x_k)\right]$ for a unique homology class. And conversely, given $\overline{\phi'(w)}\in\left[\varphi_*(x_1),\ldots,\varphi_*(x_k)\right]$, there exists a unique (up to homotopy) DGL morphism $\theta:\mathbb{L}(U)\to L$ such that $\varphi\circ\theta\simeq\phi'$, which in particular implies that $\overline{\theta(w)}\in[x_1,...,x_k]$ is unique with the property that $\overline{\varphi\circ\theta(w)}=\overline{\phi'(w)}$. □

Lemma 3.7 Let H be a DGL with zero differential and let $x_1,\ldots,x_k\in H$. If $[x_1,\ldots,x_k]\neq\emptyset$, then $0\in[x_1,\ldots,x_k]$.

Proof: Consider the diagram

$$\begin{tikzcd}
\mathbb{L}(u_1,...,u_k) \arrow[r, "\varphi"] \arrow[d, hook] & L \\
\mathbb{L}(U) \arrow[ur, dashed, "\phi"'] &
\end{tikzcd}$$

where $\varphi(u_i)$ is a representative of each x_i, and define

$$\phi\left(u_{i_1\ldots i_s}\right)=0\quad\forall\ 1\le i_1<\ldots<i_s\le k,\quad 2\le s\le k-1.$$

Then, $\partial\phi\left(u_{i_1\ldots i_s}\right)=\partial 0=0$, and

$$\begin{aligned}
\phi\partial\left(u_{i_1\ldots i_s}\right)&=\phi\left(\sum_{p=1}^{s-1}\sum_{\tilde{S}(p,s-p)}\varepsilon(\sigma)\left[u_{i_{\sigma(1)}\ldots i_{\sigma(p)}},u_{i_{\sigma(p+1)}\ldots i_{\sigma(s)}}\right]\right)\\
&=\sum_{p=1}^{s-1}\sum_{\tilde{S}(p,s-p)}\varepsilon(\sigma)\left[\phi\left(u_{i_{\sigma(1)}\ldots i_{\sigma(p)}}\right),\phi\left(u_{i_{\sigma(p+1)}\ldots i_{\sigma(s)}}\right)\right]=0,
\end{aligned}$$

where we used that each bracket in the last sum contains at least one zero factor. As $\phi(w)=0$, the result follows. □

Then the following is an immediate corollary from Proposition 3.6 and Lemma 3.7:

Theorem 3.8 *Let L be a DGL and let $x_1,\ldots,x_k\in H=H_*(L)$ be such that $[x_1,\ldots,x_k]$ is non empty. Denote by $[,\ldots,]^H$ the higher order Whitehead products in H. Then, L is not formal if one of the following conditions hold:*

(1) $0\notin[x_1,\ldots,x_k]$.

(2) The sets $[x_1,\ldots,x_k]$ and $[x_1,\ldots,x_k]^H$ are not bijective.

3.3 Examples

In the following results we will see that both, the *zero element criterion* and the *cardinality criterion* of Theorem 3.8, are independent and useful in order to detect non-formal DGL's, or equivalently, non-coformal spaces. We start with the zero element criterion.

Example 3.9 Although well known to be non-coformal [63, p.91], probably the simplest space to which one can apply the zero element criterion might be $\mathbb{C}P^2$: recall that the Quillen minimal model of $\mathbb{C}P^2$ is given by $L=\mathbb{L}(u,v)$, with $|u|=1,|v|=3$ and $\partial v=\frac{1}{2}[u,u]$. The following table displays a basis of L up to degree 5.

u	$\cdot$	v	$[u,v]$	$[u,[u,v]]$
1	2	3	4	5

The resulting homology is easily seen to be as in the following table.

$\overline{u}$	$\cdot$	$\cdot$	$\overline{[u,v]}$	$\cdot$
1	2	3	4	5

The triple Whitehead bracket $[\overline{u},\overline{u},\overline{u}]$ consist of those homology classes arising from the extensions in the following diagram, where $\phi(u_1)=\phi(u_2)=\phi(u_3)=u$, and $\mathbb{L}(U)$ is the fat wedge model.

$$\begin{tikzcd} \mathbb{L}(u_1,u_2,u_3) \arrow[r, "\phi"] \arrow[d, hook] & L \\ \mathbb{L}(U) \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

Here, the extensions are determined by the elements $\varphi(u_{12}),\varphi(u_{13})$ and $\varphi(u_{23})$. With the help of the two tables above, it is easy to check that there is a unique possible extension given by $\varphi(u_{ij})=2v$ for every $i<j$. Hence, there is a unique non trivial homology class in $[\overline{u},\overline{u},\overline{u}]$, represented by $\varphi(w)=6[u,v]$. Therefore, $\mathbb{C}P^2$ is not coformal. $\square$

Example 3.10 Let $X=T\left(S^{n_1},\dots,S^{n_k}\right)$ be the fat wedge of $k\geq 3$ simply connected spheres. Then:

(1) Its rational homotopy Lie algebra is isomorphic to the free product of an abelian Lie algebra with a free Lie algebra on w,

$$\pi_*(\Omega X)\otimes\mathbb{Q}\cong\langle u_1,\dots,u_k\rangle\coprod\mathbb{L}(w),$$

where $|u_i|=n_i-1$ and $|w|=n_1+\cdots+n_k-2$.

(2) X is not coformal, and this is detected precisely by a non trivial Whitehead product w of order k.

Proof: Statement (1) can be found in [63, V.2. (6)]. Recall from Thm. 2.2 (3) that the Quillen minimal model of X is given by $\mathbb{L}(U)$ where

$$U=\left\langle u_{i_1\dots i_s}\right\rangle,\quad 1\leq i_1<\cdots<i_s\leq k,\quad 1\leq s\leq k-1,\quad |u_{i_1\dots i_s}|=n_{i_1}+\dots+n_{i_s}-1,$$

and whose differential is determined by

$$\partial u_{i_1\dots i_s}=\sum_{p=1}^{s-1}\sum_{\sigma\in\widetilde{S}(p,s-p)}\varepsilon(\sigma)\left[u_{i_{\sigma(1)}\dots i_{\sigma(p)}},u_{i_{\sigma(p+1)}\dots i_{\sigma(s)}}\right]. \tag{3.1}$$

For any choice

$$1\leq i_1<\cdots<i_s\leq k,\quad \text{with}\quad 3\leq s\leq k-2,$$

we claim that

$$\left[u_{i_1},\dots,u_{i_s}\right]=\{0\}.$$

Indeed, the Whitehead bracket set above arises from the solutions to the following extension problem, in which $\mathbb{L}(V)$ models the corresponding fat wedge $T(S^{n_{i_1}},\dots,S^{n_{i_s}})$:

$$\begin{tikzcd}
\mathbb{L}(v_1,\dots,v_s) \arrow[r, "\phi"] \arrow[d, hook] & \mathbb{L}(U) \\
\mathbb{L}(V) \arrow[ur, dashed, "\varphi"'] &
\end{tikzcd}$$

Here, $\phi(v_j) = \varphi(v_j) = u_j$ for $1 \le j \le s$. The possible choices for the rest of generators are

$$\varphi\left(v_{r_1\dots r_m}\right) = u_{i_{r_1}\dots i_{r_m}} \quad \text{for all} \quad 1 \le r_1 < \cdots < r_m \le k, \quad m \le s-2,$$

and

$$\varphi\left(v_{i_1\dots i_{s-1}}\right) = u_{i_1\dots i_{s-1}} + z_{i_{r_1}\dots i_{r_{s-1}}}, \quad \text{for any cycle} \quad z_{i_{r_1}\dots i_{r_{s-1}}} \in L_{n_{i_1}+\dots+n_{i_s}+1}.$$

To see this, observe that setting $\varphi\left(v_{r_1\dots r_m}\right) = u_{i_{r_1}\dots i_{r_m}} + z_{i_{r_1}\dots i_{r_m}}$ for some cycle $z_{i_{r_1}\dots i_{r_m}}$ breaks the compatibility of φ with the differentials when $m \ne s-1$,

$$\begin{aligned}
\partial\varphi\left(v_{r_1\dots r_m}\right) &= \partial u_{i_{r_1}\dots i_{r_m}} \\
&= \sum_{p=1}^{m-1} \sum_{\sigma\in\widetilde{S}(p,m-p)} \varepsilon(\sigma)\left[u_{i_{r_{\sigma(1)}}\dots i_{r_{\sigma(p)}}}, u_{i_{r_{\sigma(p+1)}}\dots i_{r_{\sigma(s)}}}\right],
\end{aligned}$$

meanwhile

$$\begin{aligned}
\varphi\partial\left(v_{r_1\dots r_m}\right) &= \sum_{p=1}^{m-1} \sum_{\sigma\in\widetilde{S}(p,m-p)} \varepsilon(\sigma)\left[\varphi\left(u_{i_{r_{\sigma(1)}}\dots i_{r_{\sigma(p)}}}\right), \varphi\left(u_{i_{r_{\sigma(p+1)}}\dots i_{r_{\sigma(s)}}}\right)\right] \\
&= \sum_{p=1}^{m-1} \sum_{\sigma\in\widetilde{S}(p,m-p)} \varepsilon(\sigma)\left[u_{i_{r_{\sigma(1)}}\dots i_{r_{\sigma(p)}}}, u_{i_{r_{\sigma(p+1)}}\dots i_{r_{\sigma(s)}}}\right] \\
&\quad + \left(\quad \text{summands of brackets involving } z_\alpha\text{'s} \quad\right).
\end{aligned}$$

When $m = s-1$, the differentials and the map φ only involve lower order terms, which are compatible.

The upshot of the discussion above is that:

- every Whitehead bracket set $[u_{i_1},\dots,u_{i_s}]$ with $s \le k-1$ is non empty, and consists in a single class. By Lemma 2.6 (or directly because $u_{i_1\dots i_s} \in U$ for $s \le s-1$), it is the zero class. Hence,
- the top Whitehead bracket set $[u_1,\dots,u_k]$ is non empty, and consists in the unique non trivial homology class represented by

$$\sum_{p=1}^{k-1} \sum_{\sigma\in\widetilde{S}(p,k-p)} \varepsilon(\sigma)\left[u_{i_{\sigma(1)}\dots i_{\sigma(p)}}, u_{i_{\sigma(p+1)}\dots i_{\sigma(k)}}\right].$$

In particular, $0 \notin [u_1,\dots,u_k]$, so Thm. 3.8 (1) applies to prove that X is not coformal. $\square$

There are instances where the zero criterion does not apply, and one may use then the cardinality criterion. The following result is one such instance.

Example 3.11 Let $X = \left(\left(S^3\times S^3\times S^3\times S^3\right)\vee S^6\right)\cup\left\{e^9_a, e^9_b, e^9_c\right\}$, where the 9 cells e^9_a, e^9_b and e^9_c are attached by the following Whitehead products, respectively:

$$\left[\mathrm{id}_{S^6}, \mathrm{id}_{S^3_2}\right], \quad \left[\mathrm{id}_{S^6}, \mathrm{id}_{S^3_3}\right] \quad \text{and} \quad \left[\mathrm{id}_{S^6}, \mathrm{id}_{S^3_4}\right].$$

Here, the subindex i of $\mathrm{id}_{S^3_i}$ indicates that this map is the composition of a generator of $\pi_3\left(S^3\right) \cong \mathbb{Z}$ with the inclusion as the ith copy inside the product $S^3\times S^3\times S^3\times S^3$. The space X is not coformal.

Proof: In view of the cellular decomposition, the Quillen minimal model of X is given by

$$\mathbb{L}\,(v_1, v_2, v_3, v_4, v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}, v_{123}, v_{124}, v_{134}, v_{234}, v_{1234}, z, a, b, c)\,,$$

where

$$|v_{i_1\dots i_s}| = 3s-1 \quad \text{for every} \quad 1 \le i_1 < \cdots < i_s \le 4 \quad \text{and} \quad 1 \le s \le 3, \quad \text{and}$$
$$|z| = 5, \qquad |a| = |b| = |c| = 8.$$

The differential is given on the elements $v_{i_1\dots i_s}$ by equation (3.1) for every $1 \le i_1 < \cdots < i_s \le 4$, and

$$\partial z = 0, \quad \partial a = [z, v_2], \quad \partial b = [z, v_3], \quad \text{and} \quad \partial c = [z, v_4].$$

We claim that $[v_1, v_2, v_3, v_4] = \{0\}$. Indeed, proceeding as in Example 3.10, any extension ϕ in the corresponding diagram is of the form

$$\begin{aligned} \phi(u_i) &= v_i && \text{for } i = 1,2,3,4,\\ \phi(u_{ij}) &= v_{ij} && \text{for } 1 \le i < i \le 4,\\ \phi(u_{ijk}) &= v_{ijk} + c_{ijk} && \text{for } 1 \le i < j < k \le 4. \end{aligned}$$

Here, c_{ijk} can be any degree 8 cycle. But as $H_8\,(L) = 0$, each c_{ijk} is necessarily a boundary ∂b_{ijk}, of which there are many. The four-fold Whitehead product is then given by all those homology classes represented by

$$\begin{aligned} \phi(w) = {} & [v_{123}, v_4] - [v_{124}, v_3] + [v_{12}, v_{34}] + [v_{14}, v_{23}] + [v_1, v_{234}] - [v_{13}, v_{24}] + [v_{134}, v_2]\\ & + \partial\,([b_{123}, v_4] - [b_{124}, v_3] + [v_1, b_{234}] + [b_{134}, v_2])\\ = {} & \partial v_{1234} + \partial c, \end{aligned}$$

and the claim is proven. Hence, the zero criterion does not apply to prove the non coformality of X, at least for the election of cycles we have done. But we claim it is not coformal, and that we may use the chosen cycles to prove it. We show next that the corresponding Whitehead bracket set in homology has many non trivial homology classes. In fact, we prove that

$$[\bar v_1, \bar v_2, \bar v_3, \bar v_4]' = \left\{\alpha\,[\bar z, \bar z] \mid \alpha \in \mathbb{Q}\right\},$$

and then an application of Thm. 3.8 (2) will finish the proof. Note that the possible choices for the extensions (now onto the homology) are given by:

$$\begin{aligned} \phi(u_{ij}) &= \lambda_{ij}\bar z \quad && \text{for any } \lambda_{ij} \in \mathbb{Q}, \quad 1 \le i < j \le 4,\\ \phi(u_{ijk}) &= 0 && \text{for every} \quad 1 \le i < j < k \le 4. \end{aligned}$$

This is because there is a unique non trivial homology class in H_5, represented by z, and the differential commutes under this choice,

$$\begin{aligned} \partial\phi(u_{ij}) &= 0\\ \phi\partial(u_{ij}) &= \phi\big([u_i, u_j]\big) = \overline{[v_i, v_j]} = \overline{\partial v_{ij}} = 0. \end{aligned}$$

On the other hand, there are no non trivial cycles in L_8, and therefore 0 is the only election for $\phi(u_{ijk})$. Therefore, $\left[\overline{v}_1, \overline{v}_2, \overline{v}_3, \overline{v}_4\right]'$ consist of all those homology classes represented by

$$\begin{aligned} \phi(w) &= [\lambda_{12}\bar z, \lambda_{34}\bar z] + [\lambda_{14}\bar z, \lambda_{23}\bar z] + [\lambda_{13}\bar z, \lambda_{24}\bar z]\\ &= (\lambda_{12}\lambda_{34} + \lambda_{14}\lambda_{23} + \lambda_{13}\lambda_{24})\,[\bar z, \bar z]\,. \end{aligned}$$

These are all non trivial unless the elections of λ_{ij} make the coefficient in the line above vanish. This shows that

$$\#\,[\bar v_1, \bar v_2, \bar v_3, \bar v_4]' \quad \neq \quad \#\,[\bar v_1, \bar v_2, \bar v_3, \bar v_4]\,,$$

and so X is not coformal. $\square$

3.4 Intrinsic (co)formality

In this brief Section, we characterize the intrinsic (co)formality of some spaces combining L_∞ structures with higher Whitehead products.

Recall that a connected space X is *intrinsically formal* if any connected space whose rational cohomology algebra is isomorphic to $H^*(X;\mathbb{Q})$ has the same rational homotopy type as X. In other words, if there is a unique rational homotopy type whose rational cohomology algebra is isomorphic to $H^*(X;\mathbb{Q})$. Dually, a simply connected space X is *intrinsically coformal* if any simply connected space whose rational homotopy Lie algebra is isomorphic to $\pi_*(\Omega X)\otimes\mathbb{Q}$ has the same rational homotopy type as X. In other words, if there is a unique rational homotopy type whose rational homotopy Lie algebra is isomorphic to $\pi_*(\Omega X)\otimes\mathbb{Q}$.

It is well known (see [53], or [26, Lemma 1.6]) that the wedge $W = S^{n_1}\vee\cdots\vee S^{n_k}$ of k simply connected spheres is formal, and it is a theorem of Baues ([5], see also [26, Thm. 1.5]) that W is intrinsically formal if every n_i is odd. Dually, ([53]), the product $P = S^{n_1}\times\cdots\times S^{n_k}$ is coformal for any k simply connected spheres. We prove the following.

Theorem 3.12 *The product of k simply connected odd dimensional spheres $S^{n_1}\times\cdots\times S^{n_k}$ is intrinsically coformal if and only if*

(1) $k\le 4$, or

(2) $k\ge 5$ and

$$n_i \neq n_{j_1}+\cdots+n_{j_r}-1 \text{ for every } i \text{ and subset } \{n_{j_1},\dots,n_{j_r}\}\subseteq\{n_1,\dots,n_k\}, \text{ where } r\ge 4 \text{ is even.}$$

Proof: Denote $P = S^{n_1}\times\cdots\times S^{n_k}$. Assume that L is an L_∞ algebra whose underlying graded vector space is generated by homogeneous elements $\langle x_1,\dots,x_k\rangle$ where each $|x_i| = n_i-1$, and that $\ell_1=\ell_2=0$. Hence, $L\cong\pi_*(\Omega P)\otimes\mathbb{Q}$ as graded Lie algebras. Every odd dimensional bracket ℓ_{2n+1} vanishes, because

$$|\ell_{2n+1}(x_{i_1},\dots,x_{i_{2n+1}})| = \underbrace{|x_{i_1}|+\cdots+|x_{i_{2n+1}}|}_{\text{even}}+\underbrace{2n+1-2}_{odd}\in L_{\text{odd}}=0.$$

For $k\le 2$ spheres, the result is straightforward. For $k=3$ (resp. $k=4$), the fact that even dimensional brackets ℓ_{2n} vanish whenever two arguments are linearly dependent, and that $\dim L=3$ (resp. $\dim L=4$) imply that the whole L_∞ structure is trivial. Hence, the homotopy type represented by L is precisely P, which is therefore intrinsically coformal. [1]

Let $k\ge 5$. If $n_i = n_{j_1}+\cdots+n_{j_r}-1$, endow L with the L_∞ structure all of whose brackets vanish except for

$$\ell_r(x_{j_1},\dots,x_{j_r}) = x_i.$$

By Corollary 2.9, the homotopy type represented by L carries the non trivial rth order Whitehead product $[x_{j_1},\dots,x_{j_r}]=\{x_i\}$, hence it is not coformal. This prevents P from being intrinsically coformal, as we have found a different homotopy type.

If on the other hand, $n_i\neq n_{j_1}+\cdots+n_{j_r}-1$ for any choice of i, r even and j_k, then every bracket ℓ_n vanishes. Indeed, assume that for some even integer r (necessarily $4\le r\le k$) and for some $z_i\in L$ we have that $\ell_r(z_1,\dots,z_r)=z_{r+1}\neq 0$. Then, the elements $z_1,\dots,z_r$ are linearly

[1] The homotopy type represented by L is to be understood to be that of $\mathscr{L}\mathscr{C}(L)=\mathbb{L}(s^{-1}\Lambda^+ sL)$.

independent, so after a change of basis we may assume that $\{z_1,\dots,z_r\}=\{x_{j_1},\dots,x_{j_r}\}\subseteq\{x_1,\dots,x_k\}$. But then, for some index i,

$$|\ell_r(z_1,\dots,z_r)|=(n_{j_1}-1)+\cdots+(n_{j_r}-1)+r-2=n_i-1,$$

a contradiction. $\square$

Remark 3.13 When P is not intrinsically coformal, we have proven the existence of a *finite* complex with the same homotopy Lie algebra. This is because L is concentrated in even degrees, hence $\mathscr{C}(L)=\Lambda sL$ has graded vector space of generators sL of odd degree, forcing $\mathscr{C}(L)$ to be finite dimensional. So, $\mathscr{L}\mathscr{C}(L)=\mathbb{L}(s^{-1}\Lambda^+ sL)$ is of finite type. The classical approach (i.e., not using infinity structures) for building a different homotopy type X with the rational homotopy Lie algebra of P would very likely involve attaching infinitely many cells to abelianize $\pi_*(\Omega X)\otimes\mathbb{Q}$.

Baues' theorem can be proven as a straightforward consequence of a degree argument for A_∞ structures (see for instance [31, Thm. 11]). For even dimensional spheres, one can proceed as in the proof of Theorem 3.12, applying the results of Chapter 4 on higher Massey products to give a statement characterizing those wedges of even dimensional spheres which are intrinsically formal. But, the lack of antisymmetry for the m_n's gives a weaker statement, namely that the intrinsic formality will hold if and only if the degree of the generators involved imply that the whole A_∞ structure vanishes.

We conclude from all of the above that the Eckmann-Hilton dual to Baues' theorem is not Theorem 3.12, but as follows. Its proof is a straightforward observation that the whole L_∞ structure vanishes by a degree argument. Observe that Quillen's theory does not require finite type hypothesis.

Theorem 3.14 *An arbitrary product of simply connected even dimensional Eilenberg-Mac Lane spaces $\prod_{i\in I}K(\mathbb{Q},n_i)$ is intrinsically coformal.*

CHAPTER 4

Massey products and A_∞ structures

Higher order Massey products were introduced in [45] and generalized in [47]. These are of fundamental importance not only in the study of DGA's per se, but they also exhibit a remarkable behavior in those geometrical contexts where a DGA plays a role. Classical instances of this fact are the detection of linking numbers of knots [46], or the obstructions to formality of Kähler manifolds [17]. Recently, Massey products have been useful useful in a wide range of applications: in homotopy theory, the authors of [41] use them to prove that in general, the homotopy type of a manifold M does not determine that of its configuration space in $k \geq 2$ points, $F_k(M)$. In group cohomology, the authors of [58] show that 2-groups of maximal nilpotency class are determined by their mod 2 group cohomology algebra and iterated Massey products. In number theory, these represent obstructions for solving certain Galois embedding problems [51, 52]. Structural properties of the Massey products are still being discovered, and might be applied to symplectic and algebraic geometry [4, 64]. There are more applications of Massey products to other fields, but our aim here is just to comment how versatile these are.

In this Chapter, we carefully relate the higher Massey products in the cohomology of a DGA with the possible A_∞ structures induced on it via homotopy transfer techniques (Theorem 1.3). This relationship has been previously studied in [61, 42, 40]. Our approach consists in proving the Eckmann-Hilton dual of all the results in Chapter 2. By doing so, we were puzzled by the fact that the Eckmann-Hilton dual of our Theorem 2.12 was proved in the interesting paper [42, Thm. 3.1] in full generality. That is, without imposing any assumption on the considered homotopy retract. After a deep reading we found a gap in the proof of Theorem 3.1 of op. cit. and arrived at the conclusion that in the A_∞ setting it is also necessary to work with retracts which are adapted to the considered higher Massey product. All of this is explicitly developed with full details in this Chapter.

All algebraic structures are consider over a field $\mathbb{K}$, not necessarily of characteristic zero,[1] and A_∞ algebras will be *connected*, that is, concentrated in non negative degrees with $A^0 = \mathbb{K}$.

[1]Most of the results are valid for $\mathbb{K}$ any associative unital ring, as long as H^i is a free $\mathbb{K}$-module, see [32].

4.1 Massey products and A_∞ structures

The cohomology class of a (homogeneous) cocycle $x \in A$ will be denoted by $[x]$, its degree as usual by $|x|$, and we write $\overline{a} = (-1)^{|a|+1} a$. The cohomology $H^*(A,d)$ will be denoted simply by H.

Define the *Massey product* of two cohomology classes $x_1, x_2 \in H$ as the usual product $x_1 x_2$. We give the definition of third order Massey products in order to see the pattern which generalizes to Massey products of any order.

Definition 4.1 Let $x_1, x_2, x_3 \in H$ be such that $x_1 x_2 = x_2 x_3 = 0$. A *defining system* (for the triple Massey product) is a set $\{a_{ij}\}_{0 \leq i < j \leq 3, 1 \leq j-i \leq 2} \subseteq A$ defined as follows.

- For $i = 1,2,3$ choose a cocycle $a_{i-1,i}$ representative of x_i. These are $\{a_{01}, a_{12}, a_{23}\}$.
- For $0 \leq i < j \leq 3$ and $j - i = 2$, choose $a_{ij} \in A$ with the property that $d(a_{ij}) = \overline{a}_{i,i+1} a_{i+1,j}$. These are $\{a_{02}, a_{13}\}$, with differential given by:

$$d(a_{02}) = \overline{a}_{01} a_{12} \quad \text{and} \quad d(a_{13}) = \overline{a}_{12} a_{23}.$$

  The existence of such elements is guaranteed by the condition $x_1 x_2 = x_2 x_3 = 0$.

The *triple Massey product* is defined as the set

$$\langle x_1, x_2, x_3 \rangle = \left\{ [\overline{a}_{01} a_{13} + \overline{a}_{02} a_{23}] \mid \{a_{ij}\} \text{ is a defining system} \right\} \subseteq H^{s-1},$$

where $s = |x_1| + |x_2| + |x_3|$. If the condition $x_1 x_2 = x_2 x_3 = 0$ is not satisfied, define the triple Massey product $\langle x_1, x_2, x_3 \rangle$ as the empty set.

It is also traditional to define

$$\mathrm{In}(x_1, x_2, x_3) = x_1 H^{|x_2|+|x_3|-1} + H^{|x_1|+|x_2|-1} x_3$$

as the *indeterminacy subgroup*, and see triple Massey products as elements in the quotient $H^{s-1} / \mathrm{In}(x_1, x_2, x_3)$, but we do not work under this point of view here.

The definition of higher Massey products is inductively given as follows.

Definition 4.2 Let $x_1, \ldots, x_n \in H$ be such that for $1 \leq i < j \leq n$ and $j - i \leq n-2$, $\langle x_i, \ldots, x_j \rangle_M$ exists and contains the zero class. A *defining system* (for the nth order Massey product) is a set $\{a_{ij}\}_{0 \leq i < j \leq n, 1 \leq j-i \leq n-1} \subseteq A$ defined as follows.

- For $i = 1, \ldots, n$ choose a cocycle $a_{i-1,i}$ representative of x_i.
- For $0 \leq i < j \leq n$ and $2 \leq j - i \leq n-1$, choose $a_{ij} \in A$ with the property that

$$d(a_{ij}) = \sum_{0 \leq i < k < j \leq n} \overline{a}_{ik} a_{kj}.$$

  Their existence follows from the condition imposed on $\langle x_i, \ldots, x_j \rangle_M$.

The *nth order Massey product* is defined as the set

$$\langle x_1, \ldots, x_n \rangle = \left\{ \left[\sum_{0 \leq i < k < j \leq n} \overline{a}_{ik} a_{kj} \right] \mid \{a_{ij}\} \text{ is a defining system} \right\}.$$

Observe that $\langle x_1, \ldots, x_n \rangle \subseteq H^{s+2-n}$ where $s = \sum_{i=1}^n |x_i|$. If the condition on $\langle x_i, \ldots, x_j \rangle$ is not satisfied, define $\langle x_1, \ldots, x_n \rangle$ as the empty set.

Whenever we choose an element $x \in \langle x_1, \dots, x_n \rangle$, we are implicitly assuming that we have chosen a DGA, as well as cohomology classes $x_1, \dots, x_n \in H$ such that $x \in \langle x_1, \dots, x_n \rangle \neq \emptyset$. When talking about an A_∞ structure on H we are understanding that it has been fixed on H via Theorem 1.3. Given a Massey product set $\langle x_1, \dots, x_n \rangle$, we will say that the A_∞ structure $\{m_n\}$ *recovers Massey products* if $\pm m_n(x_1, \dots, x_n) \in \langle x_1, \dots, x_n \rangle$. Given a Massey product element $x \in \langle x_1, \dots, x_n \rangle$, we will say that the A_∞ structure $\{m_n\}$ *recovers* x if $\pm m_n(x_1, \dots, x_n) = x$.

Recall from Proposition 1.2 that homotopy retracts (A, i, q, K) correspond to decompositions

$$A = B \oplus dB \oplus C,$$

where B is a complement of $\operatorname{Ker} d$ (and thus $d\colon B \xrightarrow{\cong} dB$) and $C \cong H$. The next definition is essential in what follows.

Definition 4.3 Let $x \in \langle x_1, \dots, x_n \rangle$. A retract (A, i, q, K) is *adapted to* x if there exists a defining system $\{a_{ij}\}$ for x such that $\{a_{ij}\}_{j-i\geq 2} \subseteq B$ with B as above, and moreover that $i(x_j) = a_{j-1,j}$ for $j = 1, \dots, n$.

The following result gives a sufficient condition for higher order multiplications to recover Massey products.

Theorem 4.4 *Let $x \in \langle x_1, \dots, x_n \rangle$. Then, for any homotopy retract adapted to x,*

$$m_n(x_1, \dots, x_n) = (-1)^\varepsilon x,$$

where $\varepsilon = 1 + |x_{n-1}| + |x_{n-3}| + \cdots$.

Proof: Let $x \in \langle x_1, \dots, x_n \rangle$ and let $\{a_{ij}\}$ be a defining system for which (A, i, q, K) is an adapted retract to x. Consider the map

$$\lambda_n\colon H^{\otimes n} \longrightarrow A, \quad n \geq 2,$$

defined recursively in formula (1.10) of Section 1.3, by setting formally $K\lambda_1 = -i$ and for $n \geq 2$,

$$\lambda_n = m\left(\sum_{s=1}^{n-1} (-1)^{s+1} K\lambda_s \otimes K\lambda_{n-s}\right).$$

Recall also that, for $n \geq 2$, this map is a section of the nth multiplication m_n of the A_∞ structure induced on H by the given homotopy retract. That is,

$$m_n = q \circ \lambda_n.$$

First, we prove by induction on s, for $2 \leq s \leq n-1$, the following two equalities:

$$\begin{aligned} K\lambda_s(x_1, \dots, x_s) &= (-1)^{b_s} a_{0s}, \\ K\lambda_{n-s}(x_{s+1}, \dots, x_n) &= (-1)^{b^{n-s}} a_{sn}. \end{aligned} \tag{4.1}$$

Here, $b_s = |x_{s-1}| + |x_{s-3}| + \cdots + 1$ and $b^{n-s} = |x_{n-1}| + |x_{n-3}| + \cdots + 1$. For $s = 2$, it is straightforward. Assume equations (4.1) are true for every $p \leq s-1$. We prove it for $p = s$. Now,

$$
\begin{aligned}
K\lambda_s(x_1,\dots,x_s) &= K\lambda_2\left(\sum_{i=1}^{s-1}(-1)^{i+1}K\lambda_i\otimes K\lambda_{s-i}\right)(x_1,\dots,x_s)\\
&= K\lambda_2\left(\sum_{i=1}^{s-1}(-1)^{i+1+(|x_1|+\cdots+|x_i|)(i-s+1)}K\lambda_i(x_1,\dots,x_i)\otimes K\lambda_{s-i}(x_{i+1},\dots,x_s)\right).
\end{aligned}
$$

Apply the induction hypothesis and note that, recursively,

$$
1+|a_{0k}| = |x_1|+\cdots+|x_k|-(k-2) \qquad \text{for every } k=1,\dots,i.
$$

Then,

$$
\begin{aligned}
&= K\lambda_2\left(\sum_{i=1}^{s-1}(-1)^{i+1+(|x_1|+\cdots+|x_i|)(i-s+1)+b_i+b^{s-i}}\,a_{0i}\otimes a_{is}\right)\\
&= K\lambda_2\left(\sum_{i=1}^{s-1}(-1)^{i+1+(|x_1|+\cdots+|x_i|)(i-s+1)+b_i+b^{s-i}+1+|a_{0k}|}\,\bar{a}_{0i}\otimes a_{is}\right)\\
&= (-1)^{b_s}K\left(\sum_{0<i<s}\bar{a}_{0i}a_{is}\right) = (-1)^{b_s}Kd(a_{0s}) = (-1)^{b_s}a_{0s}.
\end{aligned}
$$

This proves formula (4.1) for $K\lambda_s$. For $K\lambda_{n-s}$, the proof is analogous. Proven equations (4.1), we compute λ_n:

$$
\begin{aligned}
\lambda_n(x_1,\dots,x_n) &= \lambda_2\left(\sum_{s=1}^{n-1}(-1)^{s+1}K\lambda_s\otimes K\lambda_{n-s}\right)(x_1,\dots,x_n)\\
&= \lambda_2\left(\sum_{s=1}^{n-1}(-1)^{s+1+(\sum_{i=1}^{s}|x_i|)(s-n+1)}K\lambda_s(x_1,\dots,x_s)\otimes k\lambda_{n-s}(x_{s+1},\dots,x_n)\right)\\
&= \sum_{s=1}^{n-1}(-1)^{b}\bar{a}_{0s}a_{sn},
\end{aligned}
$$

where $b=|x_{n-1}|+|x_{n-3}|+\cdots+1$. This sum defines the Massey element x, up to a sign, hence the result is proven. □

The next example shows that if a retract is not adapted, then the associated higher multiplication might *not* recover a Massey product.

Example 4.5 Let $(\Lambda V,d)$ be the CDGA over $\mathbb{Q}$ in which

$$
V=\mathrm{Span}\{\underbrace{a_{01},a_{12},a_{23},a_{34},}_{\text{degree }3}\ \underbrace{a_{02},a_{13},a_{24},z_1,z_2,}_{\text{degree }5}\ \underbrace{a_{03},a_{14}}_{\text{degree }7}\},
$$

where $a_{i-1,i}$ and z_1,z_2 are cocycles, and for the rest of the elements, $da_{ij}=\sum_{i<k<j}a_{ik}a_{kj}$.

This is a very simple DGA: it is a finite dimensional elliptic Sullivan algebra with purely quadratic differential. This shows that examples in which higher operations do not recover Massey products are not bizarre.

It is straightforward to check that, fixing the cohomology classes $x_i = [a_{i-1,i}]$ for $i = 1,2,3,4$, there is a unique defining system $\{a_{ij}\}$ (given by the obvious choices) which gives rise to a unique non trivial Massey product $[x] = [a_{01}a_{14} + a_{02}a_{24} + a_{03}a_{34}]$. That is,

$$\langle x_1, x_2, x_3, x_4\rangle = \big\{[x] \mid x = a_{01}a_{14} + a_{02}a_{24} + a_{03}a_{34}\big\} \subseteq H^{10}, \quad \text{and} \quad [x] \neq [0].$$

It is also easy to see that the cohomology in degree 10 admits the basis $\{[x], [z_1 z_2]\}$. Fix the decomposition of A given in the following table, in which elements appearing in $(dB)^s$ come from differentiating the elements of B^{s-1} in the order written, and a dot $\cdot$ indicates that the corresponding subspace is the trivial one. The decomposition above degree 10 is irrelevant for our purposes.

degree	B	dB	H
3	$\cdot$	$\cdot$	$a_{01}, a_{12}, a_{23}, a_{34}$
4	$\cdot$	$\cdot$	$\cdot$
5	$a_{02}+z_1, a_{13},$ $a_{24}+z_2$	$\cdot$	z_1, z_2
6	$\cdot$	$a_{01}a_{12}, a_{12}a_{23},$ $a_{23}a_{34}$	$a_{01}a_{23}, a_{01}a_{34}, a_{12}a_{34}$
7	a_{03}, a_{14}	$\cdot$	$\cdot$
8	$a_{01}a_{13}, a_{01}a_{24},$ $a_{02}a_{34}, a_{12}a_{24}$	$a_{01}a_{13}+a_{02}a_{23},$ $a_{12}a_{24}+a_{13}a_{34}$	$a_{01}a_{23}, a_{02}a_{12}, a_{12}a_{13},$ $a_{13}a_{23}, a_{23}a_{24}, a_{23}a_{34},$ $a_{01}z_1, a_{12}z_1, a_{23}z_1, a_{34}z_1,$ $a_{01}z_2, a_{12}z_2, a_{23}z_2, a_{34}z_2$
9	$\cdot$	$a_{01}a_{13}a_{23}, a_{01}a_{23}a_{34},$ $a_{01}a_{12}a_{34}, a_{12}a_{23}a_{34}$	$\cdot$
10		$\cdot$	$x = a_{01}a_{14}+a_{02}a_{24}+a_{03}a_{34}$ z_1z_2

For the homotopy retract associated to this decomposition, one has that

$$m_4(x_1,x_2,x_3,x_4) = -q(x - a_{02}a_{24} - a_{02}z_2 - z_1a_{24} - z_1z_2) = -[x] - [z_1z_2],$$

and this cohomology class is *not* a Massey product by the discussion above. □

Remarks 4.6 (1) Basic facts on rational homotopy theory [19] show that the CDGA $(\Lambda V, d)$ of the example above is the minimal model of a simply connected elliptic complex X which is of the form

$$X = S^5 \times S^5 \times Y$$

where Y lies as the total space of a fibration of this sort

$$(S^5)^{\times 3} \times (S^7)^{\times 2} \longrightarrow Y \longrightarrow (S^3)^{\times 4}.$$

Hence, the example above shows that in $H^*(X;\mathbb{Q})$ the set $\langle x_1,x_2,x_3,x_4\rangle$ reduces to the single element x which is not recovered by the A_∞ structure induced by the given decomposition.
(2) The reason why the counterexample above concerns $4th$ order products, and not $3rd$, is that, as we prove in Corollary 4.18 below, m_3 *always recovers Massey products*. That is, for *any* induced A_∞ structure on H, whenever $\langle x_1,x_2,x_3\rangle \neq \emptyset$, it happens that $m_3(x_1,x_2,x_3) \in \langle x_1,x_2,x_3\rangle$, up to a sign.

Heuristically analyzing the proof of Theorem 4.4, we see that if a retract is not adapted to a given Massey product element, perhaps the corresponding projections onto B, $\{Kda_{ij}\}_{j-i\geq 2}$ *do* give rise to a defining system of some (possibly different) Massey product element. If so, the higher operation does recover some Massey product, and Theorem 4.4 may be upgraded to include the following statement, whose proof is obvious.

Theorem 4.7 *If, given a homotopy retract onto H, the set $\{b_{ij}\}$ given by*

$$b_{ij} = Kda_{ij} \text{ for } j-i \geq 2, \quad \text{and} \quad b_{i-1,i} = a_{i-1,i} \text{ for } i = 1,\dots,n$$

is a defining system for some Massey product element $x \in \langle x_1,\dots,x_n\rangle$, then $m_n(x_1,\dots,x_n) = \varepsilon x$. The sign ε is as in Theorem 4.4.

In any case, the combination of theorems 4.4 and 4.7 *does not* characterize the recovery of Massey products, as the following example corroborates.

Example 4.8 (A non adapted retract whose projections do not define a Massey product, but $m_4(x_1,x_2,x_3,x_4) \in \langle x_1,x_2,x_3,x_4\rangle$). Let $(\Lambda V, d)$ be the CDGA over $\mathbb{Q}$ in which

$$V = \operatorname{Span}\{\underbrace{a_{01},a_{12},a_{23},a_{34},}_{\text{degree }3}\ \underbrace{a_{02},a_{13},a_{24},z,}_{\text{degree }5}\ \underbrace{a_{03},a_{14}}_{\text{degree }7}\},$$

with $a_{i-1,i}$ and z cocycles, and for the rest of the elements $da_{ij} = \sum_{i<k<j} a_{ik}a_{kj}$.
It is a straightforward check that, fixing $x_i = [a_{i-1,i}]$ for $i = 1,2,3,4$, there is a unique defining system $\{a_{ij}\}$ (given by the obvious choices) which yields a unique non trivial Massey product $[x] = [a_{01}a_{14} + a_{02}a_{24} + a_{03}a_{34}]$. That is,

$$\langle x_1,x_2,x_3,x_4\rangle = \{[x] \mid x = a_{01}a_{14} + a_{02}a_{24} + a_{03}a_{34}\} \subseteq H^{10}, \quad \text{and} \quad [x] \neq [0].$$

Note also that there are more cohomology classes in degree 10 apart from $[x]$. Fix the obvious decomposition $A = B \oplus dB \oplus H$, and substitute a_{02} by $a_{02}+z$ in the degree 5 basis:

degree	B	dB	H
5	$a_{02}+z$		z
6		$a_{01}a_{12}$	

That is, we have chosen that $K(a_{01}a_{12}) = a_{02}+z$, and have not changed anything else. Consider the set $\{l_{ij}\}$, where $l_{ij} = Kd(a_{ij})$. Then, $l_{ij} = a_{ij}$ for every ij except for $l_{02} = a_{02}+z$. Therefore $\{l_{ij}\}$ is not a defining system for a Massey product, and a straightforward computation shows that

$$m_4(x_1,x_2,x_3,x_4) = -q(x + za_{24}) = -q(x) \in \langle x_1,x_2,x_3,x_4\rangle. \qquad \square$$

It is easy to come up with examples of Massey product sets containing many different cohomology classes in such a way that for any choice $x \in \langle x_1,\dots,x_n\rangle$, one may endow H with an A_∞ algebra structure such that $m_n(x_1,\dots,x_n) = x$, all these infinity structures being A_∞ quasi-isomorphic to the original DGA. But in general, it is not always possible to make such choices, as the following example corroborates.

Example 4.9 Let $(\Lambda V, d)$ be the CDGA over $\mathbb{Q}$ where

$$V = \operatorname{Span}\{\underbrace{a_{01}, a_{12}, a_{23}}_{\text{degree } 3}, \underbrace{a_{02}, a_{13}}_{\text{degree } 5}\}$$

and

$$da_{01} = da_{12} = da_{23} = 0, \quad da_{02} = a_{01}a_{12}, \quad da_{13} = a_{12}a_{23}.$$

Let J be the differential ideal generated by $\{a_{01}a_{12}, a_{12}a_{23}\}$, and consider (A, d) the quotient of ΛV by J, with the induced differential. We denote the elements of the quotient algebra as in the original without confusion.

Fixed $x_1 = [a_{01}], x_2 = [a_{12}]$ and $x_3 = [a_{23}]$, the possible defining systems $\{b_{ij}\}$ for the triple Massey product $\langle x_1, x_2, x_3\rangle$ are of the following form, where $\alpha_k, \beta_k \in \mathbb{Q}$:

$$b_{01} = a_{01}, \quad b_{12} = a_{12}, \quad b_{23} = a_{23},$$
$$b_{02} = \alpha_1 a_{02} + \alpha_2 a_{13} \quad \text{and} \quad b_{13} = \beta_1 a_{02} + \beta_2 a_{13}.$$

This implies that the triple Massey product set is

$$\langle x_1, x_2, x_3\rangle = \big\{\alpha_1[a_{01}a_{02}] + \alpha_2[a_{01}a_{13}] + \beta_1[a_{02}a_{23}] + \beta_2[a_{13}a_{23}] \mid \alpha_k, \beta_k \in \mathbb{Q}\big\}.$$

The zero class belongs to the set, but there are infinitely many other non trivial Massey product elements. It is a straightforward computation now to check that *for any* homotopy retract, one has that $m_3(x_1, x_2, x_3) = 0$. Therefore, one never recovers a non trivial Massey product element. $\square$

Sometimes, it is possible to build an A_∞ structure recovering a given $x \in \langle x_1, ..., x_n\rangle$. The following result is an easy to check criterion in this direction.

Proposition 4.10 If there exists a defining system $\{a_{ij}\}$ for $x \in \langle x_1, ..., x_k\rangle$ such that $\big\{da_{ij}\big\}_{j-i\geq 2}$ is a linearly independent set, then there exists an A_∞ structure on H such that $m_k(x_1, ..., x_k) = \pm\, x$.

Proof: Assume that $\{da_{ij}\}_{j-i\geq 2}$ is a linearly independent set. Then, for each $n = j - i \geq 2$, the set $\{a_{ij}\}_{j-i=n}$ is also linearly independent. Fix $\alpha_n = |a_{ij}|$ with $j - i = n \geq 2$. A degree argument forces the graded subspace A^{α_n} to be different from $A^{\alpha_{n'}}$, for any other $n' \neq n$. Hence, by induction on $p = j - i$, we build a decomposition $A = B \oplus \operatorname{Ker} d$ in which, at each step p, we enlarge $\{a_{ij}\}_{j-i=p}$ to a basis of all of B^{α_p}. So, $Kda_{ij} = a_{ij}$ (i.e., we can build an adapted retract to x), and therefore $m_k(x_1, ..., x_k) = \pm\, x$ either by construction, or by Theorem 4.4. $\square$

4.2 Discussion on a result of Lu et al.

In Theorem 3.1 of the very interesting paper [42], the authors show the following: for any cohomology classes $x_1, \ldots, x_n \in H$ of a given DGA A such that $\langle x_1, \ldots, x_n\rangle$ is non empty, and for any A_∞ structure on H induced by a homotopy retract,

$$\varepsilon\, m_n(x_1, \ldots, x_n) \in \langle x_1, \ldots, x_n\rangle,$$

where ε is as in Theorem 4.4.

Unfortunately, as stated, this result is only valid for $n = 3$ which is the first inductive step in its proof, and also, our Corollary 4.18 below. For $n \geq 4$, Example 4.5 is a clear counterexample. The small gap in the proof occurs when assuming that the elements

$$\big\{a_{ij} := K\lambda_{j-i+1}(x_i, ..., x_j) \mid 2 < j - i < n-1\big\}$$

together with suitable representatives $a_{i-1,i}$ of the classes x_i form a defining system. This is not the case, for instance, in the homotopy retract chosen in Example 4.5. Indeed,

$$m_3(x_1, x_2, x_3) = q\big(-a_{01}K(\overline{a}_{12}a_{23}) + K(\overline{a}_{01}a_{12})a_{23}\big) = q\big(\underbrace{-a_{01}a_{13} + a_{02}a_{23}}_{\in dB}\big) \in \langle x_1, x_2, x_3\rangle,$$

and

$$m_3(x_2, x_3, x_4) = q\big(-a_{12}K(\overline{a}_{23}a_{34}) + K(\overline{a}_{12}a_{23})a_{34}\big) = q\big(\underbrace{-a_{12}a_{24} + a_{13}a_{34}}_{\in dB}\big) \in \langle x_2, x_3, x_4\rangle,$$

but,

$$K(-a_{01}a_{13} + a_{02}a_{23}) = a_{03} + z_1 \quad \text{and} \quad K(-a_{12}a_{24} + a_{13}a_{34}) = a_{14} + z_4$$

do not assemble into a defining system.

Nevertheless, with this extra assumption, the inductive proof in [42, Thm. 3.1] works and it shows the following:

Theorem 4.11 *Let A be a DGA and assume $\langle x_1, ..., x_n\rangle \neq \emptyset$, $n \geq 3$. Then, for any homotopy retract of A such that the elements*

$$\big\{a_{ij} := K\lambda_{j-i+1}(x_i, ..., x_j) \mid 2 < j - i < n-1\big\}$$

assemble into a defining system,

$$\varepsilon\, m_n(x_1, ..., x_n) \in \langle x_1, ..., x_n\rangle$$

with ε as in Theorem 4.4. □

Nevertheless, provided that the Massey product set $\langle x_1, ..., x_n\rangle$ is non empty, we may use this result to construct a particular homotopy retract such that the assumption in this result holds and therefore, the nth multiplication $m_n(x_1, \ldots, x_n)$ in the corresponding A_∞ algebra structure on H recovers a Massey product.

Theorem 4.12 *If $\langle x_1, ..., x_n\rangle \neq \emptyset$, then there exists an A_∞ structure on H such that, up to a sign,*

$$m_n(x_1, ..., x_n) \in \langle x_1, ..., x_n\rangle\,.$$

Proof: Fix an A_∞ algebra structure on H. For $n=3$, if a_{01}, a_{12} and a_{23} are cocycle representatives of x_1, x_2 and x_3, respectively, then the elements $a_{02} := K(\bar{a}_{01}a_{12})$ and $a_{13} := K(\bar{a}_{12}a_{23})$ are valid elections. Assume that for every $p \le n-2$ we have found a decomposition $A = B \oplus dB \oplus H$ (possibly different to the fixed one at the beginning) and elements $\{a_{ij}\}_{j-i\le n-2}$ with the property that

$$\begin{aligned} a_{ij} &= K\lambda_{j-i+1}(x_i,\dots,x_j), \quad \text{so that} \\ da_{ij} &= \sum_{0\le i<k<j<n} \bar{a}_{ik}a_{kj} \in dB. \end{aligned}$$

Define the elements

$$a_{0,n-1} := K\lambda_{n-1}(x_1,\dots,x_{n-1}) \quad \text{and} \quad a_{1,n} := K\lambda_{n-1}(x_2,\dots,x_n).$$

Then,

$$\begin{aligned} da_{0,n-1} &= (dK)\lambda_{n-1}(x_1,\dots,x_{n-1}) = (dK)\left(\sum_{s=1}^{n-2} \pm K\lambda_s(x_1,\dots,x_s)K\lambda_{n-1-s}(x_{s+1},\dots,x_{n-1})\right) \\ &= (dK)\underbrace{\left(\sum_{s=1}^{n-2} \bar{a}_{0s}a_{s,n-2}\right)}_{\in dB} = \sum_{s=1}^{n-2} \bar{a}_{0s}a_{s,n-2}, \end{aligned}$$

and similarly,

$$da_{1,n} = \sum_{s=1}^{n-2} \bar{a}_{1s}a_{s,n-1}.$$

Therefore, the hypothesis of Theorem 4.11 hold and the proof is complete. □

4.3 Recovering Massey products

Having seen in the previous sections that higher multiplications do not always recover Massey products, we close this Chapter by giving some extra conditions under which this occurs. The result in this section, as well as theorems 4.4 and 4.7 can be seen as the Eckmann-Hilton dual of those in Section 2.3.

Theorem 4.13 If $\langle x_1,\dots,x_n\rangle \ne \emptyset$, then, for any homotopy retract, and for any $x \in \langle x_1,\dots,x_n\rangle$,

$$\varepsilon m_n(x_1,\dots,x_n) = x + \Gamma, \qquad \Gamma \in \sum_{j=1}^{n-1} \operatorname{Im}(m_j),$$

where $\varepsilon = (-1)^{\sum_{j=1}^{n-1}(n-j)|x_j|}$. In particular, $m_j = 0$ for $j \le n-1$ implies that $\varepsilon m_n(x_1,\dots,x_n) \in \langle x_1,\dots,x_n\rangle$.

Proof: Recall (see Section 1.6) that the Eilenberg-Moore spectral sequence of A is the coalgebra spectral sequence obtained by filtering the bar construction $(T(sA),\delta)$ by the ascending filtration $F_p = (sA)^{\otimes\le p}$. Consider the DGC quasi-isomorphisms of Theorem 1.3,

$$(T(sA),\delta) \underset{I}{\overset{Q}{\rightleftarrows}} (T(sH),\delta)\ ,$$

and choose the same filtration on $T(sH)$. Observe that at the E^1 level the induced morphisms of spectral sequences are both the identity on $T(sH)$. By comparison, all the terms in both spectral sequences are also isomorphic. Now, translating [63, Thm. V.7(6)] to the spectral sequence on ΛsH we obtain that if $\langle x_1,\dots,x_n\rangle$ is non empty, then the element $sx_1\otimes\cdots\otimes sx_n$ survives to the $n-1$ page (E^{n-1},δ^{n-1}). Moreover, given any $x\in\langle x_1,\dots,x_n\rangle$, one has

$$\delta^{n-1}\,\overline{sx_1\otimes\cdots\otimes sx_n}=\overline{sx}.$$

Here $\overline{\cdot}$ denotes the class in E^{n-1}. In other words, there exists $\Phi\in T^{\leq n-1}(sH)$ such that

$$\delta\left(sx_1\otimes\cdots\otimes sx_n+\Phi\right)=sx. \tag{4.2}$$

Write $\delta=\sum_{i\geq 2}\delta_i$ with each δ_i as in equation (1.14), and decompose $\Phi=\sum_{i=2}^{n-1}\Phi_i$ with $\Phi_i\in T^i(sH)$. By a word length argument,

$$\delta_k(sx_1\otimes\cdots\otimes sx_n)+\sum_{i=2}^{n-1}\delta_i(\Phi_i)=sx.$$

Note also that $\delta_n=g_n$ for elements of word length n, with g_n as in equation (1.15). Therefore,

$$g_n\left(sx_1\otimes\cdots\otimes sx_n\right)+\sum_{i=2}^{n-1}g_i(\Phi_i)=sx.$$

To finish, apply the identities (1.15) and write each g_i in terms of the corresponding m_i for all $i=1,\dots,n$. In particular, the sign ε appears when writing

$$m_n(x_1,\dots,x_n)=s^{-1}\circ g_n\circ s^{\otimes n}(x_1,\dots,x_n)=\varepsilon\, s^{-1}g_n(sx_1\otimes\cdots\otimes sx_n).$$

□

Corollary 4.14 *Let A be a DGA such that for some homotopy retract of A into H, the induced higher multiplications satisfy $m_1=\ldots=m_{k-1}=0$, with $k\geq 2$. Then, for any $x_1,\ldots,x_k\in H$, one has that $\langle x_1,\ldots,x_k\rangle$ is defined, and moreover, it consists of a single cohomology class:*

$$\langle x_1,\ldots,x_k\rangle=\{x\},$$

where $x=\varepsilon m_k\left(x_1,\ldots,x_k\right)$, with ε as in Theorem 4.13.

Remark 4.15 Note that the least k for which the nth multiplication m_n is non trivial is an invariant of a given A_∞ structure and therefore, it is independent of the chosen homotopy retract. Hence the result holds for all of them. The same applies for Theorem 4.16.

Proof: By induction on $s=j-i$, we have that

$$\{0\}=\left\langle x_i,\ldots,x_j\right\rangle\neq\emptyset\quad\text{for all}\quad 1\leq i<\cdots<j\leq k,\quad\text{and } j-i\leq k-1.$$

Now given $x\in\langle x_1,\ldots,x_k\rangle$, the result follows from a direct application of Theorem 4.13. □

The following Theorem uses the vanishing of certain higher multiplications to ensure the recovery of Massey products.

Theorem 4.16 *If for some homotopy retract of A onto H, the induced higher multiplications m_n vanish up to m_{k-2}, with $k\geq 3$, and $\langle x_1,\ldots,x_k\rangle\neq\emptyset$, then*

$$\varepsilon m_k\left(x_1,\ldots,x_k\right)\in\langle x_1,\ldots,x_k\rangle,$$

with ε as in Theorem 4.13.

Remark 4.17 Again, as observed in Remark 4.15, this result is general and holds for any homotopy retract.

Proof: Recall that $\langle x_1,\dots,x_k\rangle \neq \emptyset$ implies

$$0 \in \langle x_i,\dots,x_j\rangle, \quad \text{for any} \quad 3 \le j-i \le k-1.$$

Therefore, we apply Theorem 4.13, taking into account that $m_i = 0$ for $i \le k-2$ to deduce,

$$m_{k-1}(x_i,\dots,x_j) = 0 \quad \text{for any} \quad j-i = k-1. \tag{4.3}$$

Let $A = B \oplus dB \oplus H$ be the decomposition equivalent to the chosen homotopy retract. By induction on p, with $2 \le p \le k-1$, we will construct a set of elements $\{a_{ij}\}_{2\le j-i\le p} \subseteq B$ with the property that $d(a_{ij}) = \sum_{i<l<j} \bar{a}_{il}a_{lj}$.

For each i, we denote by x_i' a cocycle representing x_i. Let $p=2$. As $\langle x_1,\dots,x_k\rangle \neq \emptyset$, we can (and do) define $a_{ij} := Kdb_{ij}$, being b_{ij} any choice such that $d(b_{ij}) = x_i'x_j'$. Then, $a_{ij} \in B$ by construction, and the differential behaves as expected.
Assume the assertion true for $p \le k-2$. Then, there exists a family of elements $\{a_{ij}\}_{2\le j-i\le k-2} \subseteq B$ such that $d(a_{ij}) = \sum_{i<l<j} \bar{a}_{il}a_{lj}$. Now, as the defining system we are building is adapted, the same argument as in the proof of Theorem 4.4 proves that

$$m_p(x_i,\dots,x_j) = q\left(\sum_{i<l<j} \bar{a}_{il}a_{lj}\right) \quad \text{for any} \quad 3 \le p = j-i \le k-2.$$

By equation (4.3),

$$q\left(\sum_{i<l<j} \bar{a}_{il}a_{lj}\right) = 0 \quad \text{for} \quad j-i = k-1.$$

Hence, there exists some Ψ with $d\Psi = \sum_{i<l<j} \bar{a}_{il}a_{lj}$. Finally, define

$$a_{ij} := K\left(\sum_{i<l<j} \bar{a}_{il}a_{lj}\right) \quad \text{for} \quad j-i = k-1.$$

This belongs to B and satisfies our claim, proving the result. $\square$

Corollary 4.18 *Let A be a DGA and let $x_1, x_2, x_3 \in H$ be such that $\langle x_1, x_2, x_3\rangle \neq \emptyset$. Then, for any homotopy retract of A onto H,*

$$m_3(x_1,x_2,x_3) \in \langle x_1,x_2,x_3\rangle. \qquad \square$$

Corollary 4.19 *Let A be a DGA such that the product on H is trivial. If $\langle x_1,x_2,x_3,x_4\rangle \neq \emptyset$, then, for any homotopy retract,*

$$m_4(x_1,x_2,x_3,x_4) \in \langle x_1,x_2,x_3,x_4\rangle. \qquad \square$$

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