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Ergodic theorems for Cesàro bounded operators in L^1 ☆

Francisco J. Martín-Reyes *, Elena de la Rosa

Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain

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We consider a positive invertible Lamperti operator T with positive inverse. Our result concerns the averages $A_{k,n}$, $0 \leq k, n \in \mathbb{Z}$, and the ergodic maximal operator

$$M_T f = \sup_{k,n\geq 0} |A_{k,n} f| = \sup_{k,n\geq 0} \left| \frac{1}{k+n+1} \sum_{j=-k}^{n} T^j f \right|,$$

the ergodic Hilbert transform defined by $H_T f(x) = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j}(T^j f(x) - T^{-j} f(x))$ and the ergodic power functions defined by $P_{r,T} f(x) = \left(\sum_{n=0}^{\infty} |A_{n+1,0} f(x) - A_{n,0} f(x)|^r + |A_{0,n+1} f(x) - A_{0,n} f(x)|^r\right)^{1/r}$, for $1 < r < +\infty$. Several authors proved that if the averages are uniformly bounded in L^p, $1 < p < \infty$, then these operators are bounded in L^p [23,24,27,28,21]. Regarding the case $p = 1$, Gillespie and Torrea [12] showed that this condition is not sufficient to assure M_T is of weak type $(1, 1)$. We provide two sufficient conditions that recall the assumptions in the Dunford-Schwartz theorem: if the averages are uniformly bounded in L^1 and in L^∞ then M_T, H_T and $P_{r,T}$ apply L^1 into weak-L^1 and the corresponding sequences of functions in L^1 converge a.e. and in measure. Furthermore, we reach the same conclusions by assuming a weaker condition: we replace the uniform boundedness of the averages $A_{n,n}$ in L^∞ by the assumption that, for a fixed $p \in (1, \infty)$, the averages associated with a modified operator T_p, related with T, are uniformly bounded in L^p. We end the paper showing examples of nontrivial operators satisfying the assumptions of the two main results.

1. Introduction

We start establishing our setting. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $\mathcal{M}(\mu)$ be the space of measurable functions $f : (X, \mathcal{M}) \to \mathbb{R}$. As usual, we identify functions which are equal almost everywhere.

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* Corresponding author.
E-mail addresses: martin_reyes@uma.es (F.J. Martín-Reyes), elena.rosa@uma.es (E. de la Rosa).

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By $L^p := L^p(\mu)$, $1 \le p < \infty$, we denote the set of measurable functions f such that $\int_X |f|^p \, d\mu < \infty$. For $f \in L^p$, we write $||f||_p = ||f||_{L^p(d\mu)} = \left(\int_X |f|^p \, d\mu\right)^{1/p}$. As usual, for $p = \infty$,

$$||f||_\infty = \inf \left\{ M \ge 0 : \mu(\{x \in X : |f(x)| > M\}) = 0 \right\}$$

and $L^\infty := L^\infty(\mu)$ is the set of measurable functions f such that $||f||_\infty < \infty$.

Let $T : \mathcal{M}(\mu) \to \mathcal{M}(\mu)$ (or alternatively $T : L^p \to L^p$) a linear operator. The classical ergodic theorems refer to the averages

$$A_n f = \frac{1}{n+1} \sum_{j=0}^{n} T^j f, \tag{1.1}$$

and the ergodic maximal operator

$$M_T^+ f = \sup_{n \ge 0} |A_n f|. \tag{1.2}$$

Roughly speaking, Dunford-Schwartz ergodic theorem [8,9] establishes that if T is a contraction in L^1 and a contraction in L^∞ then the following statements hold:

(1) M_T^+ applies L^1 into weak-L^1, more precisely,

$$\mu(\{x : M_T^+ f(x) > \lambda\}) \le \frac{1}{\lambda} ||f||_1$$

for all functions $f \in L^1$ and all $\lambda > 0$.

(2) The sequence of averages $A_n f$ converges a.e. for all $f \in L^1$.

Akcoglu's theorem [2] says that if $1 < p < \infty$ and T is a positive contraction on L^p then

(1) $||M_T^+ f||_p \le \frac{p}{p-1} ||f||_p$ for all $f \in L^p$.

(2) The sequence of averages $A_n f$ converges a.e. and in the norm of L^p for all $f \in L^p$.

(We recall that in this setting an operator T is positive if $Tf \ge 0$ a.e. for all $f \ge 0$ a.e.). The proof of Akcoglu's theorem follows from the particular case of positive isometries which was previously proved by A. Ionescu-Tulcea [14]. The proof of Ionescu-Tulcea's result in Krengel's book [17] follows the lines of the proofs by Kan [16] and de la Torre [31]. It is based on the following key fact: if $1 < p < \infty$ and T is a positive linear isometry on L^p then T is a Lamperti operator or, in other words, T separates supports ($fg = 0$ a.e. $\Rightarrow TfTg = 0$ a.e.).

As we have noticed, Lamperti operators are a very important case; we point out that these operators have a very special structure [16,18] that we resume in Section 2.

It was proved in [24] a kind of generalization of Akcoglu's theorem (see also [29]). On the one hand, more restrictive assumptions are considered: the authors work with positive invertible Lamperti operators with positive inverse. On the other hand, the authors treat with a more general assumption: they do not assume that T is a positive contraction but the averages are uniformly bounded in L^p, $1 < p < \infty$, that is, there exists a positive constant C such that

$$\sup_n ||A_n f||_p \le C ||f||_p$$

for all $f \in L^p$. Under these assumptions, it is proved that the maximal operator M_T^+ is bounded in L^p and the sequence of averages $A_n f$ converges a.e. and in L^p for all $f \in L^p$.

In what follows, we shall work in the setting of [24] (or [29]). Therefore, T will be a positive invertible Lamperti operator with positive inverse (see the definitions and properties in Section 2). Since T is invertible, it is natural to consider, for $0 \le k, n \in \mathbb{Z}$, the averages

$$A_{k,n} f = \frac{1}{k+n+1} \sum_{j=-k}^{n} T^j f, \tag{1.3}$$

and the ergodic maximal operator

$$M_T f = \sup_{k,n \ge 0} |A_{k,n} f|. \tag{1.4}$$

In fact, earlier results are obtained under the assumption

$$\sup_n ||A_{n,n} f||_p \le C||f||_p \quad (f \in L^p)$$

not only for the ergodic maximal operator M_T [23,24] but also for the ergodic Hilbert transform [27] and the ergodic power function [21,28]. We recall the definitions of these operators. The ergodic Hilbert transform is defined as

$$H_T f(x) = \lim_{n \to \infty} H_n f(x),$$

where

$$H_n f(x) = {\sum_{j=-n}^{n}}' \frac{1}{j} T^j f(x) := \sum_{j=1}^{n} \frac{1}{j} \left(T^j f(x) - T^{-j} f(x) \right),$$

and the maximal ergodic Hilbert transform is

$$H_T^* f(x) = \sup_{n \ge 0} |H_n f(x)|.$$

This operator was introduced by Cotlar [4] for operators induced by measure preserving transformations.

For $1 < r < +\infty$ the ergodic power functions are defined by

$$P_{r,T}^+ f(x) = \left(\sum_{n=0}^{\infty} |A_{0,n+1} f(x) - A_{0,n} f(x)|^r \right)^{1/r}, \quad \text{and}$$

$$P_{r,T} f(x) = \left(\sum_{n=0}^{\infty} |A_{n+1,0} f(x) - A_{n,0} f(x)|^r + |A_{0,n+1} f(x) - A_{0,n} f(x)|^r \right)^{1/r}.$$

$P_{r,T}^+$ was introduced by Roger L. Jones [15]. We collect these results in the next theorem (see also [29] for the ergodic maximal operator and the ergodic Hilbert transform).

Theorem A *([23,24,27,28,21]). Let $1 < p, r < +\infty$. Let T be a positive invertible Lamperti operator with positive inverse. Assume that the averages are uniformly bounded in L^p, i.e., there exists $C > 0$ such that*

$$\sup_n ||A_{n,n} f||_p \le C||f||_p \quad \textit{for all } f \in L^p.$$

Then

(a) *The operators M_T, H_T^*, H_T and $P_{r,T}$ are bounded in L^p.*
(b) *The sequences $A_{n,n}f$ and H_nf and the series $P_{r,T}f$ converge a.e. and in L^p for all $f \in L^p$.*

It could be reasonable to think that if the averages are uniformly bounded in L^1 then the maximal operator M_T is of weak type $(1,1)$. However, it was noticed in [12, Example 2.11] that this is not the case in general. It was left open the problem of finding an additional condition in such a way that this condition together with the uniform boundedness of the averages in L^1 implies that the ergodic maximal operator is of weak type $(1,1)$. A cursory discussion of the problem studied in this paper for the continuous version of the ergodic maximal operator M_T^+ appears in [22]. We show in this paper that if the averages are uniformly bounded in L^1 and in L^∞ then M_T is of weak type $(1,1)$ (which resembles Dunford-Schwartz ergodic theorem) but, not only that, we also show that under the same assumptions we have that H_T^*, H_T and $P_{r,T}$ are of weak type $(1,1)$ and the sequences $A_{n,n}f$, H_nf converge a.e. and in measure for all $f \in L^1$. In fact, we are able to prove that the same conclusions hold under weaker assumptions, by changing the assumption in L^∞ by the uniform boundedness in some L^p, $1 < p < +\infty$, of the averages associated to an operator T_p which is a modification of the operator T (see Theorem 3.4). This modified operator T_p is the operator T in the limit case $p = \infty$ and the natural isometry in L^1 when $p = 1$. Notice that these weaker assumptions imply that there exists $s \in (1,\infty)$ such that the averages $A_{n,n}$ of the operator T are uniformly bounded in L^p for all $p \in [1,s]$ while the stronger assumptions imply that the same is true for $s = +\infty$.

The paper is organized in the following way: Section 2 is devoted to establish in a more precise way the setting of the paper; in particular we resume the structure and properties of Lamperti operators. Section 3 contains the main results and the proofs of the results are in the following sections. The last section is devoted to provide examples of operators which satisfy the assumptions of the theorems.

1.1. Notations

Throughout the paper, we will use some standard notations that we detail in what follows. If $E \subset \mathbb{R}$ is a Lebesgue-measurable set then $|E|$ stands for the Lebesgue measure of the set E. In the same way, if A is a subset of integer numbers then $\#A$ denotes the cardinal or the counting measure of A. When we work in the real line and $U : \mathbb{R} \to \mathbb{R}$ is a nonnegative measurable function then $L^p(U)$ is the Lebesgue space $L^p(\mu)$, where μ is the measure with density U with respect to Lebesgue measure. Also, $\ell^p(u)$ is the Lebesgue space $L^p(\mu)$ where μ is the measure with density u with respect to counting measure on $\mathbb{Z}$.

It is said that an operator $\mathcal{L}$ applies $L^p(\mu)$ into $L^p(\mu)$ if there exists a positive constant C such that

$$||\mathcal{L}f||_p \leq C||f||_p \text{ for all functions } f \in L^p(\mu).$$

The space weak-$L^p(\mu)$, denoted by $L^{p,\infty}(\mu)$, is the space of functions f such that

$$||f||_{p.\infty} := \sup_{\lambda>0} \lambda \left(\mu(\{x : |f(x)| > \lambda\})\right)^{\frac{1}{p}} < \infty.$$

Finally, it is said that the operator $\mathcal{L}$ applies $L^p(\mu)$ into weak-$L^p(\mu)$ if there exists $C > 0$ such that

$$||\mathcal{L}f||_{p,\infty} \leq C||f||_p \text{ for all functions } f \in L^p(\mu).$$

2. Lamperti operators

In this section we state the setting of our paper (which is the same as in [29] and we follow the notations introduced in [3]). A Lamperti operator on $\mathcal{M}(\mu)$ is a map $T : \mathcal{M}(\mu) \to \mathcal{M}(\mu)$ of the form

$$Tf(x) = h(x)\Phi f(x), \tag{2.1}$$

where $h \in \mathcal{M}(\mu)$ and $\Phi : \mathcal{M}(\mu) \longrightarrow \mathcal{M}(\mu)$ is linear and multiplicative, that is,

(1) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$
(2) $\Phi(fg) = \Phi(f)\Phi(g)$

Throughout the paper we always assume that T is positive and invertible with positive inverse. It follows that $0 < h(x) < \infty$ a.e. and Φ is invertible and positive. Other properties are $\Phi 1 = 1$, $\Phi(|f|^r) = |\Phi(f)|^r$ for positive r and the following ones (see e.g. [16] and [18]):

(1) There exists a sequence of measurable functions h_j such that

$$T^j f = h_j \Phi^j f \tag{2.2}$$

where $h_1 = h$, $h_0 = 1$ and $h_{j+k} = h_j\, \Phi^j h_k$, for any $j, k \in \mathbb{Z}$.
(2) By the Radon-Nikodym theorem, for every $j \in \mathbb{Z}$ there exists a positive function $J_j \in \mathcal{M}(\mu)$ such that if $f \geq 0$ then

$$\int_X J_j\, \Phi^j f \, d\mu = \int_X f \, d\mu \qquad \text{and} \qquad J_{j+k} = J_j\, \Phi^j J_k, \text{ for any } j, k \in \mathbb{Z}. \tag{2.3}$$

The function J_1 will be simply denoted by J.

3. Statement of the main results

A Cesàro bounded operator in L^p is a linear operator such that the averages are uniformly bounded in L^p, that is, $\sup_{n\in\mathbb{N}} \|A_{0,n}\|_{L^p} < \infty$. Assume that T is invertible. T and its inverse T^{-1} are Cesàro bounded operators in L^p if and only if $\sup_{n\in\mathbb{N}} \|A_{n,n}\|_{L^p} < \infty$ (the averages $A_{n,n}$ are uniformly bounded operators in L^p). The next theorem establishes the weak type $(1,1)$ inequality for the ergodic maximal operator, the maximal ergodic Hilbert transform and the ergodic power operator and the a.e. convergence of the corresponding sequences of functions in L^1, assuming that T and T^{-1} are Cesàro bounded operators in L^1 and in L^∞.

Theorem 3.1. *Let $Tf(x) = h(x)\Phi f(x)$ be a positive invertible Lamperti operator. Let $1 < r < \infty$. If the averages $A_{n,n}$ are uniformly bounded operators in L^1 and in L^∞ then the following statements hold:*

(i) *The maximal operator M_T applies L^1 into weak-L^1, i.e., there exists $C > 0$ such that*

$$\mu(\{x \in X : M_T f(x) > \lambda\}) \leq \frac{C}{\lambda} \int_X |f| \, d\mu \quad (f \in L^1, \lambda > 0)$$

and the sequences of averages $A_{n,n}f$, $A_{0,n}f$ and $A_{n,0}f$ converge a.e. and in measure for all $f \in L^1$.
(ii) *The maximal ergodic Hilbert transform H_T^* applies L^1 into weak-L^1 and the sequence $H_n f$ converges a.e. and in measure for all $f \in L^1$; therefore, the ergodic Hilbert transform $H_T f$ exists in the a.e. sense and belongs to weak-L^1 for all $f \in L^1$.*

(iii) *The ergodic power operator $P_{r,T}$ applies L^1 into weak-L^1 and the series*

$$P_{r,T}f(x) = \left(\sum_{n=0}^{\infty} |A_{n+1,0}f(x) - A_{n,0}f(x)|^r + |A_{0,n+1}f(x) - A_{0,n}f(x)|^r \right)^{1/r}$$

converges a.e. and in measure for all $f \in L^1$.

Remark 3.2. Observe that we can take symmetric means in the maximal definition, and Theorem 3.1 (i) holds. Indeed, if we define

$$M_T^c f(x) = \sup_{n\geq 0} |A_{n,n}f(x)|,$$

it is clear that $M_T^c f \leq M_T f \leq 2M_T^c f$, so M_T applies L^1 into weak-L^1 if and only if M_T^c applies L^1 into weak-L^1.

Remark 3.3. R. Sato proved [26] that if T is a positive operator such that the averages A_n (see the definition in (1.1)) are uniformly bounded in L^1 and in L^∞ then the sequence $A_n f$ converges a.e. for all $f \in L^\infty$. Earlier, Derriennic and Lin [7] proved that there exist a positive linear operator T on L^1 of a finite measure space, with $\sup_{n\in\mathbb{N}} ||T^n||_{L^1} < \infty$ and $T1 = 1$, and a function f in L^1 such that $A_n f$ does not converge a.e. Therefore, for this operator T, the averages A_n are uniformly bounded in L^1 and in L^∞, the sequence $A_n f$ converges a.e. for all functions f in a dense class (L^∞) in L^1 but the ergodic maximal operator $M_T^+ f = \sup_{n\geq 0} |A_n f|$ is not of weak type $(1, 1)$ because there exists a function in L^1 such that $A_n f$ does not converge a.e.

The assumption on L^∞ can be weakened by assuming the uniform boundedness of the averages of a modified operator T_p related to T. This result is stated in the following theorem.

Theorem 3.4. *Let $Tf(x) = h(x)\Phi f(x)$ be a positive invertible Lamperti operator and let J be the function defined in (2.3). Let $1 < p < \infty$ and let T_p be the positive invertible Lamperti operator $T_p f(x) = \left(\frac{J(x)}{h(x)}\right)^{\frac{1}{p}} Tf(x)$. If the averages $A_{n,n}$ are uniformly bounded operators in L^1 and the averages A_{n,n,T_p} defined by*

$$A_{n,n,T_p} f = \frac{1}{2n+1} \sum_{j=-n}^{n} T_p^j f \tag{3.1}$$

are uniformly bounded operators in L^p, $1 < p < \infty$, then the following statements hold:

(i) *The maximal operator M_T, the maximal ergodic Hilbert transform H_T^* and the ergodic power operator $P_{r,T}$, $1 < r < \infty$ apply L^1 into weak-L^1.*
(ii) *The sequences $A_{n,n}f$, $A_{0,n}f$, $A_{n,0}f$ and $H_n f$ and the series $P_{r,T}f$, $1 < r < \infty$, converge a.e. and in measure for all $f \in L^1$.*

As it is shown in Remark 3.6, the assumptions in this theorem are weaker than the ones in Theorem 3.1. We notice that the assumptions in Theorem 3.1 imply that the averages are uniformly bounded in L^p for all $p \in [1, +\infty]$ while the assumptions in Theorem 3.4 imply that there exists $s \in (1, +\infty)$ such that the averages are uniformly bounded in L^p for all $p \in [1, s]$ (see the proof of Theorem 3.4 (ii) at the end on Section 7).

In order to prove Theorem 3.1 and Theorem 3.4, a key result is the following lemma which characterizes the positive invertible Lamperti operators such that T and T^{-1} are Cesàro bounded in L^p, $1 \leq p \leq \infty$. Before stating it, we introduce some notations: if $a : \mathbb{Z} \to \mathbb{R}$ is a sequence of real numbers, the Hardy-Littlewood maximal function of a is defined by

$$ma(j) = \sup_{k,n \geq 0} \left| \frac{1}{k+n+1} \sum_{i=-k}^{n} a(j+i) \right|. \tag{3.2}$$

It is clear that $m = M_T$ where $Ta(j) = a(j+1)$. It is said that a nonnegative function $u : \mathbb{Z} \to \mathbb{R}$ satisfies the discrete Muckenhoupt $A_1(\mathbb{Z})$ condition if there exists $C > 0$ such that

$$mu(n) \leq Cu(n) \text{ for all } n \in \mathbb{Z}, \tag{3.3}$$

and the minimum of all these positive constants is denoted by $[u]_{A_1(\mathbb{Z})}$. We also say that u belongs to $A_1(\mathbb{Z})$. We recall that the discrete Hardy-Littlewood maximal operator m applies the weighted space $\ell^1(u)$ into weak-$\ell^1(u)$ if and only if $u \in A_1(\mathbb{Z})$. It is said that u satisfies the $A_p(\mathbb{Z})$ condition or u belongs to $A_p(\mathbb{Z})$, $1 < p < +\infty$, if there exists $C > 0$ such that

$$\left(\sum_{j \in I} u(j) \right) \left(\sum_{j \in I} u^{-1/(p-1)}(j) \right)^{p-1} \leq C(\#(I))^p \tag{3.4}$$

for all bounded intervals I. The minimum of all these positive constants is denoted by $[u]_{A_p(\mathbb{Z})}$. It is said that u satisfies $A_\infty(\mathbb{Z})$, belongs to $A_\infty(\mathbb{Z})$ or simply $u \in A_\infty(\mathbb{Z})$, if there exists $p \in [1, \infty)$ such that $u \in A_p(\mathbb{Z})$ (we also write $A_\infty(\mathbb{Z}) = \bigcup_{p \in [1,\infty)} A_p(\mathbb{Z})$). We recall that m applies the weighted space $\ell^p(u)$ into $\ell^p(u)$ if and only if $u \in A_p(\mathbb{Z})$. These discrete results can be found in [13].

Lemma 3.5. *Let $Tf(x) = h(x)\Phi f(x)$ be a positive invertible Lamperti operator and let J_j and h_j be the functions defined in (2.3) and (2.2), respectively. Let $1 < p, p' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and let T_p be the positive invertible Lamperti operator $T_p f(x) = \left(\frac{J(x)}{h(x)}\right)^{\frac{1}{p}} Tf(x)$.*

(i) *The averages $A_{n,n}$ are uniformly bounded operators in L^1 if and only if for almost every $x \in X$, the functions $q_x : \mathbb{Z} \to \mathbb{R}$, $q_x(j) = J_j(x) h_j^{-1}(x)$, belong to the discrete Muckenhoupt $A_1(\mathbb{Z})$ class with a uniform constant, that is, there exists $C > 0$ such that for a.e. $x \in X$*

$$mq_x(j) \leq Cq_x(j) \quad \textit{for all } j \in \mathbb{Z}. \tag{3.5}$$

More precisely, the constant $C = \sup_n ||A_{n,n}||_1$ satisfies the above inequality.

(ii) *The averages $A_{n,n}$ are uniformly bounded operators in L^∞ if and only if for almost every $x \in X$, the functions $h_x : \mathbb{Z} \to \mathbb{R}$, $h_x(j) = h_j(x)$, belong to the discrete Muckenhoupt $A_1(\mathbb{Z})$ class with a uniform constant, that is, there exists $C > 0$ such that for a.e. x*

$$mh_x(j) \leq Ch_x(j) \quad \textit{for all } j \in \mathbb{Z}\ . \tag{3.6}$$

More precisely, the constant $C = \sup_n ||A_{n,n}||_\infty$ satisfies the above inequality.

(iii) *([24,23]) The averages $A_{n,n}$ are uniformly bounded operators in L^p if and only if for almost every $x \in X$, the functions $q_x : \mathbb{Z} \to \mathbb{R}$, $q_x(j) = J_j(x) h_j^{-p}(x)$ belong to the discrete Muckenhoupt $A_p(\mathbb{Z})$ class with a uniform constant, that is, there exists $C > 0$ such that for a.e. $x \in X$*

$$\left(\sum_{j=n}^{m} q_x(j)\right)\left(\sum_{j=n}^{m} q_x^{-1/(p-1)}(j)\right)^{p-1} \leq C(n+m+1)^p$$

for all integers n and m with $n \leq m$.

(iv) *The averages A_{n,n,T_p} defined in (3.1) are uniformly bounded operators in L^p if and only if for almost every $x \in X$, the functions $h_x : \mathbb{Z} \to \mathbb{R}$, $h_x(j) = h_j(x)$ belong to the discrete Muckenhoupt $A_{p'}(\mathbb{Z})$ class with a uniform constant.*

Remark 3.6. It is known that the class $A_1(\mathbb{Z})$ is included in $A_{p'}(\mathbb{Z})$, $1 < p' < \infty$. Then it follows that the assumptions in Theorem 3.1 are stronger than the ones in Theorem 3.4.

Proof of Lemma 3.5. (iii) is proved in [24,23], and (iv) is an immediate consequence of (iii) since $T_p f(x) = h(x)^{\frac{1}{p'}} J(x)^{\frac{1}{p}} \Phi f(x)$.

Proof of (i). (3.5) holds if and only if for a.e. x

$$\sup_{n\geq 0} \frac{1}{2n+1} \sum_{i=-n}^{n} J_{j+i}(x) h_{i+j}^{-1}(x) \leq C J_j(x) h_j^{-1}(x) \quad \text{for all } j \in \mathbb{Z}.$$

By (2.2) and (2.3) the above inequality holds if and only if for a.e. x

$$\sup_{n\geq 0} \frac{1}{2n+1} \sum_{i=-n}^{n} J_i(x) h_i^{-1}(x) \leq C.$$

The last inequality is equivalent to

$$\sup_{n\geq 0} \frac{1}{2n+1} \sum_{i=-n}^{n} \int_X f(x) \frac{J_i(x)}{h_i(x)} d\mu(x) \leq C \int_X f(x) d\mu(x)$$

for all non negative measurable functions. Applying again (2.3), we get

$$\sup_{n\geq 0} \frac{1}{2n+1} \sum_{i=-n}^{n} \int_X \Phi^{-i} f(x) \frac{\Phi^{-i} J_i(x)}{\Phi^{-i} h_i(x)} J_{-i}(x) d\mu(x) \leq C \int_X f(x) d\mu(x).$$

By properties (2.2) and (2.3) we deduce $h_{-i}(x)\Phi^{-i} h_i(x) = 1$ and $J_{-i}(x)\Phi^{-i} J_i(x) = 1$. Therefore, all the inequalities are equivalent to

$$\sup_{n\geq 0} \frac{1}{2n+1} \sum_{i=-n}^{n} \int_X h_{-i}(x) \Phi^{-i} f(x) d\mu(x) \leq C \int_X f(x) d\mu(x).$$

This inequality means that the averages $A_{n,n}$ are uniformly bounded in L^1. This finishes the proof of (i). It follows from the proof that we can take the constant $C = \sup_n ||A_{n,n}||_1$ in (3.5).

Proof of (ii). Assume (3.6) holds. Then there exists $C > 0$ such that for a.e. x and for all $n \geq 0$,

$$\frac{1}{2n+1} \sum_{i=-n}^{n} h_{i+j}(x) \leq C h_j(x) \quad \text{for all } j \in \mathbb{Z}. \tag{3.7}$$

By using (2.2) the last inequality is equivalent to

$$\frac{1}{2n+1}\sum_{i=-n}^{n}\Phi^j h_i(x)\le C \quad \text{a.e.}\, x, \text{ for all } n\ge 0 \text{ and for all } j\in\mathbb{Z},$$

or, equivalently,

$$\frac{1}{2n+1}\sum_{i=-n}^{n} h_i(x)\le C \quad \text{a.e.}\, x, \text{ for all } n\ge 0. \tag{3.8}$$

It follows that for all $n\in\mathbb{N}$

$$\begin{aligned}|A_{n,n}f(x)| &\le \frac{1}{2n+1}\sum_{i=-n}^{n} h_i(x)|\Phi^i f(x)| \\ &\le ||f||_\infty \frac{1}{2n+1}\sum_{i=-n}^{n} h_i(x)\le C||f||_\infty \quad \text{a.e.}\, x.\end{aligned}$$

Therefore,

$$\sup_{n\ge 0}||A_{n,n}f||_\infty \le C||f||_\infty. \tag{3.9}$$

Conversely, assume that (3.9) holds. Applying this inequality to $f(x)=1$ we obtain (3.8) which, as we said above, holds if and only if for almost every $x\in X$, the functions h_x belong to the discrete Muckenhoupt $A_1(\mathbb{Z})$ class with a uniform constant. It follows from the proof that we can take the constant $C=\sup_n ||A_{n,n}||_\infty$ in (3.7). □

4. Proof of Theorem 3.1

The proof follows by transference arguments from some results in the integers. Due to the similarity of the proofs of the statements of Theorem 3.1 (the key steps of the transference argument are essentially the same), we will show the first one in detail and the others more concisely.

4.1. Proof of Theorem 3.1(i)

We start stating the result we need on the integers. This kind of inequalities is sometimes called a mixed weak type $(1,1)$ inequality for the operator m.

Theorem 4.1. *Let $u,v:\mathbb{Z}\to\mathbb{R}$ be non negative measurable functions. If $u,v\in A_1(\mathbb{Z})$ then there exists a constant C such that for all $\lambda>0$ and all functions $a:\mathbb{Z}\to\mathbb{R}$,*

$$\sum_{\{n\in\mathbb{Z}:ma(n)>\lambda v(n)\}} u(n)v(n)\le \frac{C}{\lambda}\sum_{n\in\mathbb{Z}}|a(n)|u(n),$$

where C depends only on $[u]_{A_1(\mathbb{Z})}$ and $[v]_{A_1(\mathbb{Z})}$.

The proof follows from the corresponding result in [30]. For reasons of completeness, we give the proof of this result in Section 5.

Proof of Theorem 3.1(i). We may assume without loss of generality that $f\ge 0$. We establish some notation that we will reuse throughout the paper. Let $\mathcal{L}_T$ be the maximal operator M_T^c and let $\mathcal{L}$ denote the Hardy-Littlewood maximal operator m on the integers. We set $L_n=A_{n,n}$ for any natural number n. Then by definition,

$$\mathcal{L}_T f = \sup_{n\geq 0} L_n f.$$

Let N be a positive integer and let us consider the truncated operator

$$\mathcal{L}_{T,N} f(x) = \sup_{0\leq n\leq N} L_n f(x).$$

Since $\{\mathcal{L}_{T,N} f\}_{N=1}^{\infty}$ is an increasing sequence and $\lim_{N\to\infty} \mathcal{L}_{T,N} f(x) = \mathcal{L}_T f(x)$, by the monotone convergence theorem, it suffices to prove that there exists $C > 0$ independent of N such that

$$\mu(\{x \in X : \mathcal{L}_{T,N} f(x) > \lambda\}) \leq \frac{C}{\lambda} \int_X f d\mu. \tag{4.1}$$

Proof of (4.1). Let $O_{\lambda,N} := \{x \in X : \mathcal{L}_{T,N} f(x) > \lambda\}$. Let K be any positive integer. Then, by (2.3),

$$\mu(O_{\lambda,N}) = \frac{1}{K+1} \sum_{j=0}^{K} \int_X J_j \Phi^j \chi_{O_{\lambda,N}} d\mu = \frac{1}{K+1} \int_X \sum_{j=0}^{K} J_j \Phi^j \chi_{O_{\lambda,N}} d\mu,$$

where χ_A denotes the characteristic function of A. Let $E_0 = \{x : f(x) > \lambda\}$ and if $0 < n \leq N$ let

$$E_n = \{x : L_n f(x) > \lambda\} \cap (X \setminus \cup_{i=1}^{n-1} E_i).$$

It is clear that the sets E_n are disjoint and $O_{\lambda,N} = \cup_{n=0}^{N} E_n$. From the definition of the sets,

$$\lambda \chi_{O_{\lambda,N}} \leq \sum_{n=0}^{N} L_n f \chi_{E_n}.$$

Therefore,

$$\lambda \Phi^j \chi_{O_{\lambda,N}} \leq \sum_{n=0}^{N} \Phi^j (L_n f) \Phi^j \chi_{E_n}.$$

Observe, by applying (2.2), that $\Phi^j(L_n f) = h_j^{-1} T^j (L_n f)$, and a direct computation yields $T^j(L_n f) = L_n(T^j f)$, so it follows that

$$\lambda h_j \Phi^j \chi_{O_{\lambda,N}} \leq \sum_{n=0}^{N} L_n(T^j f) \Phi^j \chi_{E_n}.$$

For fixed $x \in X$, let $p_x : \mathbb{Z} \to \mathbb{R}$, $p_x(i) = T^i f(x)$. Since $0 \leq j \leq K$, a simple observation yields $L_n(T^j f) \leq \mathcal{L}(p_x \chi_{[-N,N+K]})(j)$, where $[-N, N+K] = \{j \in \mathbb{Z} : -N \leq j \leq N+K\}$. Then

$$\lambda h_j \Phi^j \chi_{O_{\lambda,N}} \leq \sum_{n=0}^{N} L_n(T^j f) \Phi^j \chi_{E_n} \leq \mathcal{L}(p_x \chi_{[-N,N+K]})(j) \Phi^j \chi_{O_{\lambda,N}}.$$

Since $\Phi^j \chi_{O_{\lambda,N}} = \chi_{\Phi^j O_{\lambda,N}} \leq 1$ (see [16]), the above inequality yields,

$$\sum_{j=0}^{K} J_j(x)\Phi^j\chi_{O_{\lambda,N}}(x) \leq \sum_{\{j\in\mathbb{Z}:\mathcal{L}(p_x\chi_{[-N,K+N]})(j)>\lambda h_j(x)\}} J_j(x)$$
$$= \sum_{\{j\in\mathbb{Z}:\mathcal{L}(p_x\chi_{[-N,K+N]})(j)>\lambda h_j(x)\}} (J_j(x)h_j^{-1}(x))h_j(x).$$

Since the averages $A_{n,n}$ are uniformly bounded operators in L^1 and L^∞, Lemma 3.5 implies that for almost every $x \in X$, the functions $h_x(j) \in A_1(\mathbb{Z})$ and $q_x(j) = J_j(x)h_j^{-1}(x) \in A_1(\mathbb{Z})$ with uniform constants (independent of x). Then, by applying Theorem 4.1, there exists $C > 0$ such that for a.e. x

$$\sum_{\{j\in\mathbb{Z}:\mathcal{L}(p_x\chi_{[-N,K+N]})(j)>\lambda h_j(x)\}} (J_j(x)h_j^{-1}(x))h_j(x) \leq \frac{C}{\lambda}\sum_{j\in\mathbb{Z}} T^j(f)(x)\chi_{[-N,K+N]}(j)J_j(x)h_j^{-1}(x)$$
$$= \frac{C}{\lambda}\sum_{j=-N}^{N+K} \Phi^j(f)(x)J_j(x).$$

Therefore,

$$\mu(O_{\lambda,N}) = \frac{1}{K+1}\int_X \sum_{j=0}^{K} J_j\Phi^j\chi_{O_{\lambda,N}}d\mu \leq \frac{1}{K+1}\frac{C}{\lambda}\sum_{j=-N}^{N+K}\int_X \Phi^j(f)(x)J_j(x)d\mu(x)$$
$$= \frac{2N+K+1}{K+1}\frac{C}{\lambda}\int_X f(x)d\mu(x),$$

where the last equality is due to (2.3). By taking limit as K goes to infinity, (4.1) holds.

By Marcinkiewizc interpolation theorem the averages $A_{n,n}$, $A_{n,0}$ and $A_{0,n}$ are uniformly bounded operators in L^p, $1 < p < \infty$. By the result in [23,24] (see Theorem A), the sequences of averages $A_{n,n}f$, $A_{0,n}f$ and $A_{n,0}f$ converge a.e. for all $f \in L^p$. Therefore we have that the averages converge a.e. for all functions f in $L^p \cap L^1$ which is a dense set in L^1. Moreover, we have just proved that $\mathcal{L}_T = M_T$ applies L^1 into weak-L^1. Then Banach principle yields that the sequences of averages converge in a.e. for all $f \in L^1$.

Regarding the convergence in measure of the sequences of averages $A_{n,n}f$, $A_{0,n}f$ and $A_{n,0}f$, we proceed in the same way; we only have to observe that it follows from Theorem A that these sequences converge in the L^p norm, therefore in measure, for all $f \in L^p$, $1 < p < \infty$. □

4.2. Proof of Theorem 3.1(ii)

Before prove it, we establish some notation: if $a : \mathbb{Z} \to \mathbb{R}$ is any function then the maximal function Hilbert transform h^* of a is defined as follows:

$$h^*a(i) = \sup_{n\geq 0}\left|\sum_{1\leq|j|\leq n}\frac{a(i+j)}{j}\right| = \sup_{n\geq 0}\left|\sum_{j=1}^{n}\frac{1}{j}\left(a(i+j)-a(i-j)\right)\right|. \tag{4.2}$$

We point out that $h^* = H_T^*$, where $Ta(i) = a(i+1)$.

The proof follows by transference arguments from the next result in the integers.

Theorem 4.2. *Let u, $v : \mathbb{Z} \to \mathbb{R}$ be non negative measurable functions. If $u, v \in A_1(\mathbb{Z})$ then there exists a constant $C > 0$ such that for all $\lambda > 0$ and all functions $a : \mathbb{Z} \to \mathbb{R}$,*

$$\sum_{\{n\in\mathbb{Z}:h^*a(n)>\lambda v(n)\}} u(n)v(n) \leq \frac{C}{\lambda}\sum_{n\in\mathbb{Z}}|a(n)|u(n),$$

where C depends only of $[u]_{A_1(\mathbb{Z})}$ and $[v]_{A_1(\mathbb{Z})}$.

This theorem follows from the results in [5]. We provide a proof in Section 5.

Proof of Theorem 3.1(ii). The inequality

$$\mu(\{x\in X : H_T^* f(x)>\lambda\}) \leq \frac{C}{\lambda}\int\limits_X |f|\,d\mu \quad (f\in L^1,\lambda>0) \tag{4.3}$$

is proved by following the same argument of the proof of Theorem 3.1(i), by taking the maximal operator $\mathcal{L}_T = H_T^*$, the operator on the integers $\mathcal{L} = h^*$ and the partial sums $L_n = |H_n|$, and by applying Theorem 4.2 instead of Theorem 4.1.

Now we prove the results about the convergence of H_n. By Marcinkiewizc interpolation theorem, the averages $A_{n,n}$ are uniformly bounded operators in L^p, $1<p<\infty$. Moreover, Theorem A yields that the sequence $H_n f$ converges a.e. and in L^p (and therefore in measure) for all $f\in L^p$ and then it converges a.e. and in measure for all functions $f\in L^p\cap L^1$ which is a dense set in L^1. In addition, we have just proved that H_T^* applies L^1 into weak-L^1, so Banach principle yields that $H_n f$ converges a.e. and in measure for all $f\in L^1$. Therefore the ergodic Hilbert transform $H_T f$ exists in the a.e. sense and belongs to weak-L^1 for all $f\in L^1$. □

4.3. Proof of Theorem 3.1(iii)

For $1<r<\infty$, let $Q_{r,T}$ be the operator defined by the following expression

$$Q_{r,T}(f) = \left(\sum_{k=-\infty}^{\infty}\left(\frac{|T^k f|}{|k|+1}\right)^r\right)^{\frac{1}{r}} = \lim_{n\to\infty} Q_{r,n} f, \tag{4.4}$$

where $Q_{r,n} f = \left(\sum_{k=-n}^{n}\left(\frac{|T^k f|}{|k|+1}\right)^r\right)^{\frac{1}{r}}$. The argument of our proof is based in the following key inequality

$$P_{r,T} f \leq C M_T f + Q_{r,T} f,\ r>1 \tag{4.5}$$

(see [15,28]). Then in order to prove Theorem 3.1(iii) it is enough to show the following result.

Theorem 4.3. *Let $Tf(x) = h(x)\Phi f(x)$ a positive invertible Lamperti operator. Let $1<r<\infty$. If the averages $A_{n,n}$ are uniformly bounded operators in L^1 and in L^∞ then the ergodic operator $Q_{r,T}$ applies L^1 into weak-L^1, i.e. there exists $C>0$ such that*

$$\mu(\{x\in X : Q_{r,T} f(x)>\lambda\}) \leq \frac{C}{\lambda}\int\limits_X |f|\,d\mu \quad (f\in L^1,\lambda>0).$$

When T is the operator defined on functions $a:\mathbb{Z}\to\mathbb{R}$ by $Ta(i) = a(i+1)$, we denote the operator $Q_{r,T}$ simply by Q_r, that is,

$$Q_r(a)(j) = \left(\sum_{k=-\infty}^{\infty}\left(\frac{|a(k+j)|}{|k|+1}\right)^r\right)^{\frac{1}{r}}.$$

The proof of Theorem 4.3 follows by transference arguments from the next result in the integers.

Theorem 4.4. *Let u, $v : \mathbb{Z} \to \mathbb{R}$ be non negative measurable functions. If $u, v \in A_1(\mathbb{Z})$ then there exists a constant $C > 0$ such that for all $\lambda > 0$ and all functions $a : \mathbb{Z} \to \mathbb{R}$,*

$$\sum_{\{n\in\mathbb{Z}: Q_r a(n) > \lambda v(n)\}} u(n)v(n) \leq \frac{C}{\lambda} \sum_{n\in\mathbb{Z}} |a(n)|u(n),$$

where C depends only on $[u]_{A_1(\mathbb{Z})}$ and $[v]_{A_1(\mathbb{Z})}$.

The proof of this result will be given in Section 6. Unlike Theorems 4.1 and 4.2, and as far as we know, this theorem does not follow from any known result in the continuous setting, that is, in Harmonic Analysis. This is the reason why the proof of this result will be more involved and we dedicate the entire Section 6 to it.

Proof of Theorem 4.3. We only have to follow the proof of Theorem 3.1(i) by taking the maximal operator $\mathcal{L}_T = Q_{r,T}$, the operator defined on functions on the integers $\mathcal{L} = Q_r$ and $L_n = Q_{r,n}$. In this case there are some slight differences that can make the proof simpler, for example, it is enough to consider the truncated operator $\mathcal{L}_{T,N} = Q_{r,N}$. Moreover we point out that we apply Theorem 4.4 instead of Theorem 4.1. $\square$

Proof of Theorem 3.1(iii). By inequality (4.5), Theorem 4.3 and Theorem 3.1(i), we obtain that $P_{r,T}$ applies L^1 into weak-L^1. Now, we point out that Marcinkiewizc interpolation theorem yields that the averages $A_{n,n}$ are uniformly bounded operators in L^p, $1 < p < \infty$, so by Theorem A the sequence $P_{r,T}f$ converges in a.e. and in L^p (therefore in measure) for all $f \in L^p$. This fact together with the weak type $(1, 1)$ inequality gives that $P_{r,T}f$ converges a.e and in measure for all $f \in L^1$. $\square$

5. Proof of Theorems 4.1 and 4.2

In order to prove Theorems 4.1, and 4.2 we follow the ideas in [13] which allow to transfer weighted inequalities from the real line to the integers. In the next sections we shall need some notations, definitions and results that we include here. Given a measurable function $f : \mathbb{R} \to \mathbb{R}$, the Hardy-Littlewood maximal function is defined as

$$Mf(x) = \sup_{h,k>0} \frac{1}{h+k} \int_{x-h}^{x+k} |f|.$$

The maximal Hilbert transform is defined as

$$H^*f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy \right|$$

when the integrals make sense. It is said that a nonnegative function $U : \mathbb{R} \to \mathbb{R}$ satisfies the Muckenhoupt A_1 condition if there exists $C > 0$ such that

$$MU(x) \leq CU(x) \text{ for almost every } x \in \mathbb{R},$$

and the minimum of all these positive constants is denoted by $[U]_{A_1}$. We also say that U belongs to A_1 or $U \in A_1$. We recall that M applies the weighted space $L^1(U)$ into weak-$L^1(U)$ if and only if $U \in A_1$ ([25]). The same result holds for the maximal Hilbert transform H.

Moreover it is said that a nonnegative function $U : \mathbb{R} \to \mathbb{R}$ satisfies the Muckenhoupt A_p condition, $1 < p < +\infty$ if there exists $C > 0$ such that

$$\left(\int_I U(x)\,dx\right)\left(\int_I U^{-1/(p-1)}(x)\,dx\right)^{p-1} \le C|I|^p$$

for all bounded intervals I. The minimum of all these positive constants is denoted by $[U]_{A_p}$. We also say that U belongs to A_p or $U \in A_p$.

It is said that U satisfies A_∞, belongs to A_∞ or simply $U \in A_\infty$, if there exists $p \in [1, \infty)$ such that $U \in A_p$ (we also write $A_\infty = \bigcup_{p\in[1,\infty)} A_p$). We recall that M applies the weighted space $L^p(U)$ into $L^p(U)$ if and only if $U \in A_p$ [25]. The same result holds for the maximal Hilbert transform H^*.

Further, a weight $U \in A_p$, $1 \le p < \infty$, satisfies the Reverse Hölder Inequality: there exist $C > 0$ and $\gamma > 1$, that depend only on p and $[U]_{A_p}$ such that

$$\left(\frac{1}{|I|}\int_I U(x)^\gamma\,dx\right)^{\frac{1}{\gamma}} \le C\left(\frac{1}{|I|}\int_I U(x)\,dx\right). \tag{5.1}$$

As a consequence, $U \in A_p$ for some $1 \le p < \infty$ if and only if there exist positive constants C and δ that depend only on p and $[U]_{A_p}$ such that

$$U(F) \le C\left(\frac{|F|}{|I|}\right)^\delta U(I) \tag{5.2}$$

for all bounded intervals I and all measurable subsets $F \subset I$, where, as usual, $U(E)$ stands for $\int_E U(x)\,dx$.

We point out [10] as a general reference for results about Muckenhoupt A_p weights.

In the proofs of Theorems 4.1 and 4.2 we shall need the following result for M and H^* (the result for M can be found in [30] while the corresponding one for the maximal Hilbert transform follows from Theorems 1.7 and 1.3 in [5]).

Theorem B *([30,5]). Let $U, V \in A_1$. Let $\mathcal{L}$ be the Hardy-Littlewood maximal operator M or the maximal Hilbert transform H^*. Then there exists a positive constant C, depending only on $[U]_{A_1}$ and $[V]_{A_1}$, such that*

$$\int_{\{x\in\mathbb{R}:\mathcal{L}f(x)>\lambda V(x)\}} UV \le \frac{C}{\lambda}\int_{\mathbb{R}} |f|U$$

for all $\lambda > 0$ and all measurable functions f.

Proof of Theorem 4.1. We assume without loss of generality that $a(n) \ge 0$, $n \in \mathbb{Z}$. For all $x \in \mathbb{R}$, let $[x]$ be the greatest integer less than or equal to x. Let $U(x) = u([x])$, $V(x) = v([x])$ and $f(x) = a([x])$, $x \in \mathbb{R}$. Let us observe that $u, v \in A_1(\mathbb{Z})$ if and only if $U, V \in A_1$ and the constant $[U]_{A_1}$ and $[V]_{A_1}$ depend on $[u]_{A_1(\mathbb{Z})}$ and $[v]_{A_1(\mathbb{Z})}$, respectively. Moreover, $ma([x]) \le Mf(x)$. Indeed, if $k, m \ge 0$, and $x \in [n, n+1)$

$$\frac{1}{m+k+1}\sum_{j=-k}^{m} a(n+j) = \frac{1}{m+k+1}\sum_{j=-k}^{m}\int_{n+j}^{n+j+1} f(y)dy = \frac{1}{m+k+1}\int_{n-k}^{m+n+1} f(y)dy \le Mf(x).$$

Then Theorem B yields

$$\begin{aligned}
\sum_{\{n\in\mathbb{Z}:ma(n)>\lambda v(n)\}} u(n)v(n) &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} \chi_{\{j\in\mathbb{Z}:ma(j)>\lambda v(j)\}}(n)U(y)V(y)dy \\
&\le \sum_{n\in\mathbb{Z}} \int_n^{n+1} \chi_{\{x\in\mathbb{R}:Mf(x)>\lambda V(x)\}}(y)U(y)V(y)dy \\
&= \int_{\{x\in\mathbb{R}:Mf(x)>\lambda V(x)\}} U(y)V(y)dy \\
&\le \frac{C}{\lambda}\int_{\mathbb{R}} f(x)U(x)dx = \frac{C}{\lambda}\sum_{n\in\mathbb{Z}}\int_n^{n+1} f(x)U(x)dx = \frac{C}{\lambda}\sum_{n\in\mathbb{Z}} a(n)u(n). \quad \square
\end{aligned}$$

Proof of Theorem 4.2. Assume without loss of generality that $a(n) \ge 0$, $n \in \mathbb{Z}$. Let $f = \sum_{i\in\mathbb{Z}} a(i)\chi_{(i-\frac{1}{4},i+\frac{1}{4})}$, $U = \sum_{i\in\mathbb{Z}} u(i)\chi_{(i-\frac{1}{4},i+\frac{1}{4})}$ and $V = \sum_{i\in\mathbb{Z}} v(i)\chi_{(i-\frac{1}{4},i+\frac{1}{4})}$. It is easy to see that U, $V \in A^1$ and the constant $[U]_{A_1}$ and $[V]_{A_1}$ depend on $[u]_{A_1(\mathbb{Z})}$ and $[v]_{A_1(\mathbb{Z})}$, respectively. We also have that for $x \in (i-\frac{1}{4}, i+\frac{1}{4})$ the inequality $h^*a(i) \le 8Mf(x) + 2H^*f(x)$ holds (see [13]). Then Theorem B yields

$$\begin{aligned}
\sum_{\{n\in\mathbb{Z}:h^*a(n)>\lambda v(n)\}} u(n)v(n) &= 2\sum_{n\in\mathbb{Z}} \int_{n-\frac{1}{4}}^{n+\frac{1}{4}} \chi_{\{j\in\mathbb{Z}:h^*a(j)>\lambda v(j)\}}(n)U(y)V(y)dy \\
&\le 2\sum_{n\in\mathbb{Z}} \int_{n-\frac{1}{4}}^{n+\frac{1}{4}} \chi_{\{x\in\mathbb{R}:8Mf(x)+2H^*f(x)>\lambda V(x)\}}(y)U(y)V(y)dy \\
&\le \frac{32C}{\lambda}\int_{\mathbb{R}} |f(x)|U(x)dx + \frac{8C}{\lambda}\int_{\mathbb{R}} |f(x)|U(x)dx = \frac{C'}{\lambda}\sum_{n\in\mathbb{Z}} |a(n)|u(n). \quad \square
\end{aligned}$$

6. Proof of Theorem 4.4

In order to prove Theorem 4.4 it is convenient to introduce the one-sided versions of the operator Q_r, denoted by Q_r^+ and Q_r^-, defined on functions $a : \mathbb{Z} \to \mathbb{R}$, by

$$Q_r^+a(j) = \left(\sum_{k=0}^{\infty}\left(\frac{|a(j+k)|}{k+1}\right)^r\right)^{\frac{1}{r}} \text{ and } Q_r^-a(j) = \left(\sum_{k=0}^{\infty}\left(\frac{|a(j-k)|}{k+1}\right)^r\right)^{\frac{1}{r}};\ r > 1.$$

It is clear that

$$\max\{Q_r^-a, Q_r^+a\} \le Q_ra \le Q_r^-a + Q_r^+a.$$

We shall study the operator Q_r^+ (similar results hold for Q_r^-) and the corresponding theorems for Q_r are obtained putting together the results for Q_r^+ and Q_r^-.

We start with a good-λ inequality for Q_r^+. To state it we introduce the one-sided Hardy-Littlewood maximal operator m^+ on functions $a : \mathbb{Z} \to \mathbb{R}$ defined by

$$m^+a(j) = \sup_{n\geq 0} \frac{1}{n+1}\sum_{k=0}^{n} |a(j+k)|.$$

Theorem 6.1. *Let $a : \mathbb{Z} \to \mathbb{R}$, $a \in \ell^1$. Then there exists $C > 0$ such that for all $\beta > 1$ and all $\gamma < 1$*

$$\#(\{k \in \mathbb{Z} : Q_r^+ a(k) > \beta\lambda,\, m^+a(k) \leq \gamma\lambda\}) \leq \frac{C\gamma}{\beta - 1}\#(\{k \in \mathbb{Z} : Q_r^+ a(k) > \lambda\}).$$

Proof. Let $E_\lambda = \{k \in \mathbb{Z} : Q_r^+ a(k) > \lambda\}$. Since Q_r^+ applies ℓ^1 into weak-ℓ^1 (see [21, Theorem 3.8 with $w = 1$] or [15]) then E_λ is a finite set. Therefore $E_\lambda = \bigcup_{i=1}^{n} I_j$, and I_j are maximal disjoint intervals in $\mathbb{Z}$. Then, it is enough to show that there exists $C > 0$ such that for each $j \in \{1, 2, \ldots, n\}$

$$\#(\{k \in I_j : Q_r^+ a(k) > \beta\lambda,\, m^+a(k) \leq \gamma\lambda\}) \leq \frac{C\gamma}{\beta - 1}\#(I_j). \tag{6.1}$$

Indeed, since $\beta > 1$ and by using (6.1),

$$\begin{aligned}\#(\{k \in \mathbb{Z} : Q_r^+ a(k) > \beta\lambda,\, m^+a(k) \leq \gamma\lambda\}) &= \#(\{k \in E_\lambda : Q_r^+ a(k) > \beta\lambda,\, m^+a(k) \leq \gamma\lambda\}) \\ &= \sum_{j=1}^{n} \#(\{k \in I_j : Q_r^+ a(k) > \beta\lambda,\, m^+a(k) \leq \gamma\lambda\}) \\ &\leq \sum_{j=1}^{n} \frac{C\gamma\#(I_j)}{\beta - 1} = \frac{C\gamma\#(E_\lambda)}{\beta - 1}.\end{aligned}$$

Let us show (6.1) for any I_j. Assume that there exists $k_j \in I_j$ such that $m^+a(k_j) \leq \gamma\lambda$ (otherwise we get $\#(\{k \in I_j : Q_r^+ a(k) > \beta\lambda,\, m^+a(k) \leq \gamma\lambda\}) = 0$ and (6.1) holds). Let k_j be the minimum of $\{k \in I_j : m^+(a)(k) \leq \gamma\lambda\}$ and let $\widetilde{I}_j = \{k \in I_j : k \geq k_j\} = \{k_j, \ldots, k_l\}$, where $k_i = k_j + i - j$. It is obvious that

$$\{k \in I_j : Q_r^+ a(k) > \beta\lambda,\, m^+a(k) \leq \gamma\lambda\} = \{k \in \widetilde{I}_j : Q_r^+ a(k) > \beta\lambda,\, m^+a(k) \leq \gamma\lambda\}.$$

Let us define the functions $a_1, a_2 : \mathbb{Z} \to \mathbb{R}$ as $a_1 = a\chi_{\widetilde{I}_j}$ and $a_2 = a - a_1$. We claim that the next two properties hold.

(i) For each $\beta' > 0$, $\#(\{k \in \widetilde{I}_j : Q_r^+(a_1)(k) > \beta'\lambda\}) \leq \frac{C\gamma}{\beta'}\#(I_j)$.
(ii) For each $k \in \widetilde{I}_j$, $Q_r^+(a_2)(k) \leq \lambda$ holds.

Indeed, by using that Q_r^+ applies ℓ^1 into weak-ℓ^1,

$$\begin{aligned}\#(\{k \in \widetilde{I}_j : Q_r^+(a_1)(k) > \beta'\lambda\}) &\leq \frac{C}{\beta'\lambda}\sum_{k\in\mathbb{Z}} |a_1(k)| = \frac{C\#(\widetilde{I}_j)}{\beta'\lambda\#(\widetilde{I}_j)}\sum_{k\in\widetilde{I}_j} |a(k)| \\ &= \frac{C\#(\widetilde{I}_j)}{\beta'\lambda}\frac{1}{l - j + 1}\sum_{i=0}^{l-j} |a(k_j + i)| \leq \frac{C\#(I_j)}{\beta'\lambda} m^+(a)(k_j) \leq \frac{C\gamma\#(I_j)}{\beta'}.\end{aligned}$$

In order to show (ii), we notice that $Q_r^+(a)(k_l + 1) \leq \lambda$ because $k_l + 1 \notin E_\lambda$ since I_j is a maximal interval of E_λ. Therefore, if $k \in \widetilde{I}_j$,

$$Q_r^+(a_2)(k) = \left[\sum_{j=0}^{\infty} \frac{|a_2(k+j)|^r}{(j+1)^r}\right]^{1/r} = \left[\sum_{j=k_l-k+1}^{\infty} \frac{|a(k+j)|^r}{(j+1)^r}\right]^{1/r} = \left[\sum_{i=0}^{\infty} \frac{|a(k_l+1+i)|^r}{(i+1+k_l+1-k)^r}\right]^{1/r}$$
$$\leq \left[\sum_{i=0}^{\infty} \frac{|a(k_l+1+i)|^r}{(i+1)^r}\right]^{1/r} = Q_r^+(a)(k_l+1) \leq \lambda.$$

Once (i) and (ii) have been established, we apply these properties with $\beta' = \beta - 1$ and we obtain

$$\begin{aligned}
\#(\{k \in I_j : Q_r^+ a(k) > \beta\lambda,\, m^+a(k) \leq \gamma\lambda\}) &\leq \#(\{k \in \widetilde{I}_j : Q_r^+ a(k) > \beta\lambda\}) \\
&\leq \# \left(\{k \in \widetilde{I}_j : Q_r^+(a_1)(k) > \beta'\lambda\} \cup \{k \in \widetilde{I}_j : Q_r^+(a_2)(k) > \lambda\}\right) \\
&= \#(\{k \in \widetilde{I}_j : Q_r^+(a_1)(k) > \beta'\lambda\}) \leq \frac{C\gamma\#(I_j)}{\beta'} = \frac{C\gamma\#(I_j)}{\beta - 1},
\end{aligned}$$

i.e., (6.1) holds and the proof of Theorem 6.1 is complete. □

Lemma 6.2. *Let $w : \mathbb{Z} \to \mathbb{R}$ be a non negative measurable function. Then $w \in A_p(\mathbb{Z})$ for some $1 \leq p < \infty$ if and only if there exist constants $C \geq 1$ and $\delta > 0$, that depend only on $[w]_{A_p(\mathbb{Z})}$ and p, such that*

$$\sum_{n \in E} w(n) \leq C \left(\frac{\#(E)}{b-a+1}\right)^{\delta} \sum_{n=a}^{b} w(n) \tag{6.2}$$

for all integers a and b, $a \leq b$, and each set $E \subset [a,b] = \{n \in \mathbb{Z} : a \leq n \leq b\}$.

Proof. Since $w \in A_p(\mathbb{Z})$, then the function $W : \mathbb{R} \to \mathbb{R}$ defined by $W(x) = w([x])$ belongs to A_p and $[W]_{A_p}$ depends on $[w]_{A_p(\mathbb{Z})}$. Then, by (5.2) there exist positive constants C and δ that depend on $[W]_{A_p}$ and p such that for all bounded intervals I and each measurable subset $F \subset I$

$$W(F) \leq C \left(\frac{|F|}{|I|}\right)^{\delta} W(I). \tag{6.3}$$

By applying the inequality above to $F = \cup_{n \in E}[n, n+1)$ and the interval $I = [a, b+1)$ (notice that $F \subset [a, b+1)$), we obtain (6.2). Conversely, assume that (6.2) holds. It is not difficult to see that (6.3) holds and, arguing as before, $w \in A_p(\mathbb{Z})$ for some $1 \leq p < \infty$. □

Theorem 6.3. *Let $0 < s < \infty$ and $w \in A_q(\mathbb{Z})$, $1 \leq q < \infty$. Then there exists $C > 0$ (that depends only on $[w]_{A_q(\mathbb{Z})}$, q and s) such that*

$$\sum_{k \in \mathbb{Z}} |Q_r(a)(k)|^s w(k) \leq C \sum_{k \in \mathbb{Z}} |ma(k)|^s w(k)$$

for each function $a : \mathbb{Z} \to \mathbb{R}$ such that the sum on the left-hand side is finite.

Proof. We shall prove the above inequality for the operator Q_r^+. The inequality holds also for Q_r^- since $Q_r^- a = \widetilde{Q_r^+ \widetilde{a}}$, where $\widetilde{b}(j) = b(-j)$, and $w \in A_q(\mathbb{Z})$ if and only if $\widetilde{w} \in A_q(\mathbb{Z})$. Then the proof of the theorem follows from $Q_r \leq Q_r^- + Q_r^+$.

Let us show that there exists $A > 0$ and $\alpha > 0$ such that

$$\sum_{\{k \in \mathbb{Z} : Q_r^+ a(k) > 2\lambda,\, ma(k) \leq \gamma\lambda\}} w(k) \leq A\gamma^{\alpha} \sum_{\{k \in \mathbb{Z} : Q_r^+ a(k) > \lambda\}} w(k),\ \gamma < 1. \tag{6.4}$$

Since Q_r^+ applies ℓ^1 into weak-ℓ^1 (see [21] or [15]), then $\#(E_\lambda) = \#(\{k \in \mathbb{Z} : Q_r^+ a(k) > \lambda\}) < \infty$. Therefore $E_\lambda = \bigcup_{i=1}^{n} I_j$, and I_j are maximal and disjoint intervals in $\mathbb{Z}$. Let $O_{\lambda,j} = \{k \in I_j : Q_r^+ a(k) > 2\lambda,\ ma(k) \leq \gamma\lambda\}$, $\gamma < 1$, then Theorem 6.1 yields there exists $B > 0$ such that $\#(O_{\lambda,j}) \leq B\gamma\#(I_j)$, for all $\gamma < 1$, $j \in \{1, 2, \ldots, n\}$. Since $w \in A_q(\mathbb{Z})$, by applying the property (6.2) to sets $O_{\lambda,j} \subset I_j$, $j \in \{1, 2, \ldots, n\}$ there exist $C,\ \delta > 0$ that depend on $[w]_{A_q(\mathbb{Z})}$ and q such that

$$\frac{\sum_{k\in O_{\lambda,j}} w(k)}{\sum_{k\in I_j} w(k)} \leq C\left(\frac{\#(O_{\lambda,j})}{\#(I_j)}\right)^{\delta} \leq CB^{\delta}\gamma^{\delta}, \quad j \in \{1, \ldots, n\}.$$

Then since I_j are disjoint sets,

$$\begin{aligned}\sum_{\{k\in\mathbb{Z}:Q_r^+ a(k)>2\lambda,\, ma(k)\leq\gamma\lambda\}} w(k) &= \sum_{\{k\in E_\lambda:Q_r^+ a(k)>2\lambda,\, ma(k)\leq\gamma\lambda\}} w(k) = \sum_{j=1}^{n}\sum_{k\in O_{\lambda,j}} w(k) \\ &\leq CB^{\delta}\gamma^{\delta} \sum_{\{k\in\mathbb{Z}:Q_r^+ a(k)>\lambda\}} w(k),\end{aligned}$$

so (6.4) holds. Now, by following a classic argument (see for example [10] or [11, Theorem 6.12]), and by taking γ small enough such that $2^s CB^{\delta}\gamma^{\delta} < 1$ we obtain that there exists a constant $C' > 0$ that depends on $[w]_{A_q(\mathbb{Z})}$, q and s such that

$$\sum_{k\in\mathbb{Z}} |Q_r^+(a)(k)|^s w(k) \leq C' \sum_{k\in\mathbb{Z}} |m(a)(k)|^s w(k),$$

for each function $a : \mathbb{Z} \to \mathbb{R}$ such that the sum on the left-hand side is finite. $\square$

Once we have Theorem 6.3 we follow the ideas in [5] to prove Theorem 4.4. More precisely, we will use the next reformulation of [5, Theorem 1.7] that follows from its proof.

Theorem C. *Let $\mathcal{F}$ be a family of pairs of functions, and suppose that for all $0 < s < \infty$ and $w \in A_q$, $1 \leq q < \infty$,*

$$\int_{\mathbb{R}} |f(x)|^s w(x)dx \leq C \int_{\mathbb{R}} |g(x)|^s w(x)dx, \tag{6.5}$$

for each $(f, g) \in \mathcal{F}$ such that the left-hand side is finite, where C depends only on $[w]_{A_q(\mathbb{Z})}$, q and s. Then for all $u \in A_1$ and $v \in A_p$, $1 \leq p < \infty$,

$$\sup_{\lambda>0}\left(\lambda \int_{\{x\in\mathbb{R}:|f(x)|>\lambda v(x)\}} u(x)v(x)dx\right) \leq C \sup_{\lambda>0}\left(\lambda \int_{\{x\in\mathbb{R}:|g(x)|>\lambda v(x)\}} u(x)v(x)dx\right)$$

for each $(f, g) \in \mathcal{F}$ such that the left-hand side is finite, where C depends on $[u]_{A_1}$ and $[v]_{A_p}$ and p.

In fact, we need the next result that follows from the previous theorem.

Theorem 6.4. *Let $\mathcal{F}$ be a family of pairs of functions defined on $\mathbb{Z}$, and suppose that for all $0 < s < \infty$ and $w \in A_q(\mathbb{Z})$, $1 \leq q < \infty$,*

$$\sum_{k\in\mathbb{Z}} |f(k)|^s w(k) \le C \sum_{k\in\mathbb{Z}} |g(k)|^s w(k), \tag{6.6}$$

for each $(f,g) \in \mathcal{F}$ such that the left-hand side is finite, where C depends only on $[w]_{A_q(\mathbb{Z})}$, q and s. Then for all $u \in A_1(\mathbb{Z})$ and $v \in A_p(\mathbb{Z})$, $1 \le p < \infty$

$$\sup_{\lambda>0}\left(\lambda \sum_{\{k\in\mathbb{Z}:|f(k)|>\lambda v(k)\}} u(k)v(k)\right) \le C \sup_{\lambda>0}\left(\lambda \sum_{\{k\in\mathbb{Z}:|g(k)|>\lambda v(k)\}} u(k)v(k)\right)$$

for each $(f,g) \in \mathcal{F}$ such that the left-hand side is finite, where C depends on $[u]_{A_1(\mathbb{Z})}$ and $[v]_{A_p(\mathbb{Z})}$ and p.

Proof. Let us consider the family $\mathcal{F}^*$ of pairs of functions (F,G) defined by $F(x) = f([x])$, $G(x) = g([x])$, $(f,g) \in \mathcal{F}$, and let W be an arbitrary weight, $W \in A_q$, $1 \le q < \infty$. Let us show that the weight $w(k) = \int_k^{k+1} W(x)dx$ belongs to $A_q(\mathbb{Z})$. Observe that

$$1 \le \left(\int_k^{k+1} W(x)dx\right)^{q'-1} \int_k^{k+1} W^{1-q'}(x)dx.$$

Let I be an interval of $\mathbb{Z}$ and $J = \bigcup_{n\in I}[n, n+1)$ the interval in $\mathbb{R}$. Then

$$\begin{aligned}\left(\sum_{k\in I} w(k)\right)^{q'-1}\left(\sum_{k\in I} w^{1-q'}(k)\right) &= \left(\int_J W(x)dx\right)^{q'-1} \sum_{k\in I}\left(\int_k^{k+1} W(x)dx\right)^{1-q'} \\ &\le \left(\int_J W(x)dx\right)^{q'-1}\left(\sum_{k\in I}\int_k^{k+1} W^{1-q'}(x)dx\right) \\ &= \left(\int_J W(x)dx\right)^{q'-1}\left(\int_J W^{1-q'}(x)dx\right) \le [W]_{A_q}^{q'-1}|J|^{q'} = [W]_{A_q}^{q'-1}(\#(I))^{q'}.\end{aligned}$$

Therefore, $w \in A_q(\mathbb{Z})$ and $[w]_{A_q(\mathbb{Z})} \le [W]_{A_q}$. By (6.6), if $F(x) = f([x])$ and $G(x) = g([x])$, $(f,g) \in \mathcal{F}$, we obtain

$$\int_{\mathbb{R}} |F(x)|^s W(x)dx = \sum_{k\in\mathbb{Z}} |f(k)|^s w(k) \le C \sum_{k\in\mathbb{Z}} |g(k)|^s w(k) = C \int_{\mathbb{R}} |G(x)|^s W(x)dx,$$

assuming that the left-hand side is finite, where the constant C depends only on $[W]_{A_q}$, q and s. Thus assumption (6.5) in Theorem C holds for $\mathcal{F}^*$.

Now, as usual, if $u \in A_1(\mathbb{Z})$ and $v \in A_p(\mathbb{Z})$, $1 \le p < \infty$ let us define $U(x) = u([x])$ and $V(x) = v([x])$, so $U \in A_1$ and $V \in A_p$. Then by applying Theorem C, bearing in mind that

$$\sup_{\lambda>0}\left(\lambda \int_{\{x\in\mathbb{R}:|F(x)|>\lambda V(x)\}} U(x)V(x)dx\right) = \sup_{\lambda>0}\left(\lambda \sum_{\{k\in\mathbb{Z}:|f(k)|>\lambda v(k)\}} u(k)v(k)\right),$$

we obtain

$$\sup_{\lambda>0}\left(\lambda \sum_{\{k\in\mathbb{Z}:|f(k)|>\lambda v(k)\}} u(k)v(k)\right) \le C \sup_{\lambda>0}\left(\lambda \sum_{\{k\in\mathbb{Z}:|g(k)|>\lambda v(k)\}} u(k)v(k)\right),$$

where $C>0$ depends on $[u]_{A_1(\mathbb{Z})}$ and $[v]_{A_p(\mathbb{Z})}$ and p. □

Proof of Theorem 4.4. Theorem 6.3 yields that if $w \in A_q(\mathbb{Z})$, $1 \le q < \infty$ and $0 < s < \infty$, there exists $C > 0$ (that depends only on $[w]_{A_q(\mathbb{Z})}$, q and s) such that

$$\sum_{k\in\mathbb{Z}} |Q_r(a)(k)|^s w(k) \le C \sum_{k\in\mathbb{Z}} |ma(k)|^s w(k)$$

so the hypothesis (6.6) of Theorem 6.4 holds. Then for all $u \in A_1(\mathbb{Z})$ and $v \in A_p(\mathbb{Z})$, $1 \le p < \infty$ there exists a constant $C > 0$ that depends on $[u]_{A_1(\mathbb{Z})}$ and $[v]_{A_p(\mathbb{Z})}$ and p such that

$$\sup_{\lambda>0}\left(\lambda \sum_{\{k\in\mathbb{Z}:|Q_r a(k)|>\lambda v(k)\}} u(k)v(k)\right) \le C \sup_{\lambda>0}\left(\lambda \sum_{\{k\in\mathbb{Z}:|ma(k)|>\lambda v(k)\}} u(k)v(k)\right).$$

Therefore the result follows as a consequence of Theorem 4.1. □

7. Proof of Theorem 3.4

7.1. Proof of Theorem 3.4 (i)

By Lemma 3.5, for almost every $x \in X$ the functions $h_x : \mathbb{Z} \to \mathbb{R}$, $h_x(j) = h_j(x)$, and $q_x : \mathbb{Z} \to \mathbb{R}$, $q_x(j) = J_j(x)h_j^{-1}(x)$, belong to the discrete Muckenhoupt $A_{p'}(\mathbb{Z})$ and $A_1(\mathbb{Z})$ classes with a uniform constant, respectively. Then, Theorem 3.4 (i) follows in the same way as Theorem 3.1 by using the next theorem instead of Theorems 4.1, 4.2 and 4.4.

Theorem 7.1. *Let $u \in A_1(\mathbb{Z})$ and $v \in A_p(\mathbb{Z})$, $1 \le p < \infty$, and let $1 < r < \infty$. Let $\mathcal{L}$ be the Hardy-Littlewood maximal function m, the maximal Hilbert transform h^* or the operator Q_r. Then there exists a positive constant C, depending only on $[u]_{A_1(\mathbb{Z})}$, $[v]_{A_p(\mathbb{Z})}$ and p, such that for all $\lambda > 0$ and all functions $a : \mathbb{Z} \to \mathbb{R}$,*

$$\sum_{\{n\in\mathbb{Z}:\mathcal{L}a(n)>\lambda v(n)\}} u(n)v(n) \le \frac{C}{\lambda}\sum_{n\in\mathbb{Z}} |a(n)|u(n).$$

The proof of this theorem for m and h^* follows from the continuous case (we omit the details). The weighted inequality for M was proved in [19, Theorem 1.2] and the corresponding weak-type inequality for H^* is proved by applying [19, Corollary 1.7]. For the sake of completeness, we collect the results obtained in [19] in the next theorem (we give an equivalent statement).

Theorem D. *[19, Theorem 1.2, Corollary 1.7] Let M be the Hardy-Littlewood maximal operator on $\mathbb{R}^n$ and let $u \in A_1$ and $v \in A_p$, $1 \le p < \infty$. Then there is a finite constant c depending on $[u]_{A_1}$ and $[v]_{A_p}$ such that*

$$\left\|\frac{M(fv)}{v}\right\|_{L^{1,\infty}(uv)} \le c\|f\|_{L^{1,\infty}(uv)}.$$

The same result holds for the maximal Hilbert transform H^*.

Finally, the proof of Theorem 7.1 for Q_r is a consequence of Theorem 6.4 and Theorem 7.1 for the operator m.

7.2. Proof of Theorem 3.4 (ii)

We need the following result that concerns a weights property that is interesting by itself.

Lemma 7.2. *Let $U \in A_1$, $V \in A_p$, $1 < p < \infty$. Let $1 < p' < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then there exists $s \in (1, p')$ such that $UV^{1-s} \in A_s$, and $[UV^{1-s}]_{A_s}$ depends only on p, $[V]_{A_p}$ and $[U]_{A_1}$.*

Proof. Since $U \in A_1$, then U satisfies the Reverse Hölder inequality (5.1) for some $\gamma > 1$, $C > 0$ which depend only on $[U]_{A_1}$. Let $1 < \gamma' < \infty$ such that $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$ and set $s = 1 + \frac{p'-1}{\gamma'} \in (1, p')$. Let us see $UV^{1-s} \in A_s$. By Hölder inequality,

$$\left(\frac{1}{|I|}\int_I UV^{1-s}\right)^{\frac{1}{s}} \left(\frac{1}{|I|}\int_I U^{1-s'}V\right)^{\frac{1}{s'}} \leq \left(\frac{1}{|I|}\int_I U^{\gamma}\right)^{\frac{1}{s\gamma}} \left(\frac{1}{|I|}\int_I V^{1-p'}\right)^{\frac{1}{\gamma' s}} \left(\frac{1}{|I|}\int_I U^{1-s'}V\right)^{\frac{1}{s'}},$$

and by applying the Reverse Hölder inequality and $U \in A_1$, there exists $C_1 > 0$ that depends only on $[U]_{A_1}$ such that

$$\begin{aligned}
&\left(\frac{1}{|I|}\int_I UV^{1-s}\right)^{\frac{1}{s}} \left(\frac{1}{|I|}\int_I U^{1-s'}V\right)^{\frac{1}{s'}} \\
&\leq C_1 \left(\frac{1}{|I|}\int_I U\right)^{\frac{1}{s}} \left(\frac{1}{|I|}\int_I V^{1-p'}\right)^{\frac{1}{\gamma' s}} \left(\frac{1}{|I|}\int_I U\right)^{\frac{1-s'}{s'}} \left(\frac{1}{|I|}\int_I V\right)^{\frac{1}{s'}} \\
&= C_1 \left(\left(\frac{1}{|I|}\int_I V^{1-p'}\right)^{\frac{1}{p'}} \left(\frac{1}{|I|}\int_I V\right)^{\frac{1}{p}}\right)^{\frac{p}{s'}} \leq C_2,
\end{aligned}$$

where $C_2 > 0$ depends only on p, $[V]_{A_p}$ and $[U]_{A_1}$. □

In fact, the same property holds in the discrete setting, by following the same argument:

Corollary 7.3. *Let $U \in A_1(\mathbb{Z})$, $V \in A_p(\mathbb{Z})$, $1 < p < \infty$. Let $1 < p' < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then there exists $s \in (1, p')$ such that $UV^{1-s} \in A_s(\mathbb{Z})$, and $[UV^{1-s}]_{A_s(\mathbb{Z})}$ depends only on p, $[V]_{A_p(\mathbb{Z})}$ and $[U]_{A_1(\mathbb{Z})}$.*

Proof of Theorem 3.4 (ii). By Lemma 3.5, for almost every $x \in X$ the functions h_x, $q_x : \mathbb{Z} \to \mathbb{R}$, $h_x(j) = h_j(x)$ and $q_x(j) = J_j(x)h_j^{-1}(x)$, belong to the discrete Muckenhoupt $A_{p'}(\mathbb{Z})$ and $A_1(\mathbb{Z})$ classes with a uniform constant, respectively. Then, by Corollary 7.3, there exists $s \in (1, p)$ such that $h_j^{-s}(x)J_j(x) \in A_s(\mathbb{Z})$ with a uniform constant for almost every $x \in X$. Therefore Lemma 3.5 yields that the averages $A_{n,n}$ are uniformly bounded operators in L^s. The rest of the proof repeats that of Theorem 3.1, by using the corresponding weak-type inequality (1,1) just proved in Theorem 3.4 (i). □

8. Examples

We recall that if T is a positive Cesàro bounded operator on a finite dimensional space then T is necessarily power bounded, i.e. $\sup_{n\in\mathbb{N}} ||T^n|| < \infty$. In [7] there is an example of a positive, Cesàro bounded operator in L^1 which is not power bounded. We can also find in [20] examples of operators $Tf = f \circ \tau$ associated to an invertible ergodic transformation τ, such that T and T^{-1} are Cesàro bounded in some L^p, but they are neither power bounded in L^p nor Cesàro bounded in L^q where $1 \leq q < p < \infty$ are fixed. Examples for averages associated to semiflows can be found in [1].

In this section we provide examples of Lamperti operators which satisfy the assumptions in Theorems 3.1 and 3.4 in a non trivial way. As in [20] and in [1], our method relies heavily on the so called Rubio de Francia algorithm which is an extraordinary tool to obtain powerful results in factorization and extrapolation Theory of weights (see [6] for instance). We would like to highlight that the examples in the present paper require to show an operator (or two operators) which is (are) Cesàro bounded in two different L^p-spaces simultaneously.

We start providing examples for Theorem 3.1. The operators are associated to an ergodic transformation which preserves some measure. In the second result of this section we consider an ergodic transformation without invariant measure.

Proposition 8.1. *Let γ be a finite measure and let $\tau : X \to X$ be an invertible ergodic transformation preserving the measure γ. There exist measurable positive functions h and w such that the equivalent measure $d\mu = wd\gamma$ is finite and the operator $Tf = h(f \circ \tau)$ satisfies the following statements:*

i) *T and T^{-1} are Cesáro bounded in $L^1(\mu)$.*
ii) *T and T^{-1} are Cesáro bounded in $L^\infty(\mu)$.*
iii) $\sup_{n\in\mathbb{Z}} ||T^n||_{L^1(\mu)} = \infty$.
iv) $\sup_{n\in\mathbb{Z}} ||T^n||_{L^\infty(\mu)} = \infty$.

Proof. Let M_τ denote the ergodic maximal function

$$M_\tau f(x) = \sup_{n\geq 0} \left| \frac{1}{2n+1} \sum_{k=-n}^{n} f(\tau^k x) \right|. \tag{8.1}$$

Then M_τ is bounded in $L^2(\gamma)$, $M_\tau : L^2(\gamma) \to L^2(\gamma)$ and $\|M_\tau f\|_{L^2(\gamma)} \leq K\|f\|_{L^2(\gamma)}$. Let us consider two positive functions $g_1, g_2 > 0$ such that $g_1, g_2 \in L^2(\gamma) \setminus L^\infty(\gamma)$. Let

$$U = \sum_{i=0}^{\infty} \frac{M_\tau^i g_1}{(2K)^i} \qquad V = \sum_{i=0}^{\infty} \frac{M_\tau^i g_2}{(2K)^i},$$

where M_τ^i denotes the iteration of the maximal operator ($M_\tau^0 f = f$). Then, $U, V \in L^2(\gamma)$, $g_1 \leq U$, $g_2 \leq V$, $M_\tau U \leq 2KU$ and $M_\tau V \leq 2KV$. The last two inequalities yield that the weights $\{\frac{U(\tau^i x)}{U(x)}\}_i$, $\{\frac{V(\tau^i x)}{V(x)}\}_i$, as functions on the integers, satisfy $A_1(\mathbb{Z})$ for almost every $x \in X$ with uniform constants. Define $h(x) = \frac{U(\tau x)}{U(x)}$ and $w = UV$. Then $w \in L^1(\gamma)$ and μ is a finite measure. Let $Tf = h(f \circ \tau)$.

By construction, $h_j(x) = \frac{U(\tau^j x)}{U(x)} \in A_1(\mathbb{Z})$ with uniform constant; then Lemma 3.5 yields ii) holds. It is clear that the function J with respect the measure μ has the expression $J(x) = \frac{w(\tau x)}{w(x)}$, then for a.e. x, $J_j(x)h_j^{-1}(x) = \frac{V(\tau^j x)}{V(x)} \in A_1(\mathbb{Z})$ with uniform constant, so Lemma 3.5 yields i) holds.

Let us show that iii) holds. If this is not the case then there exists a positive constant C such that for all $f \geq 0$ and for all j,

$$\int h_j(x) f(\tau^j x) w(x) d\gamma \leq C \int f(\tau^j x) w(\tau^j x) d\gamma.$$

Then $h_j(x)w(x) \leq Cw(\tau^j x)$ for almost every $x \in X$ and $j \in \mathbb{Z}$ which implies $V(x) \leq CV(\tau^j x)$ for almost every $x \in X$ and for all $j \in \mathbb{Z}$. Then

$$V(x) \leq C\frac{1}{N+1}\sum_{j=0}^{N} V(\tau^j x) \text{ for almost every } x \in X \text{ and all } N \in \mathbb{N}.$$

Therefore by taking limit as N goes to infinity, bearing in mind $V \in L^2(\gamma) \subset L^1(\gamma)$, we obtain $V(x) \leq \frac{C}{\gamma(X)} \int_X V d\gamma < \infty$ for a.e. x. But this is a contradiction since $V \geq g_2$ and $g_2 \notin L^\infty(\gamma)$.

Finally, let us prove that iv) holds. If this is not the case then there exists a positive constant C such that $\|h_j\|_{L^\infty} \leq C$, and therefore $U(\tau^j x) \leq CU(x)$ for almost every $x \in X$, for all $j \in \mathbb{Z}$, or, equivalently, $U(x) \leq CU(\tau^j x)$ for almost every $x \in X$, for all $j \in \mathbb{Z}$. Now by following the argument above, we obtain that U is bounded which is a contradiction since $U \geq g_1$ and $g_1 \notin L^\infty(\gamma)$. $\square$

The second result studies the same type of examples but assuming that there is no an equivalent measure $d\mu = wd\gamma$ such that τ preserves the measure μ.

Proposition 8.2. *Let γ be a finite measure and let $\tau : X \to X$ be a nonsingular invertible ergodic transformation (τ is non singular means that $\gamma(E) = 0 \iff \gamma(\tau E) = 0$). Assume that there is no an equivalent measure $d\mu = wd\gamma$ such that τ preserves the measure μ. Then there exist measurable positive functions h and w such that the equivalent measure $d\mu = wd\gamma$ is finite and the operator $Tf = h(f \circ \tau)$ satisfies i)-iv) in Proposition 8.1.*

Proof. Let Q be the natural isometry in $L^2(\gamma)$ defined by $Qf = J^{\frac{1}{2}}(f \circ \tau)$, where the function J is defined in (2.3). Let M_Q be the maximal operator defined in (1.4). Then M_Q is bounded in $L^2(\gamma)$ and $\|M_Q f\|_{L^2(\gamma)} \leq K\|f\|_{L^2(\gamma)}$. Let us consider a positive function $g \in L^2(\gamma) \setminus L^\infty(\gamma)$. Let

$$G = \sum_{i=0}^{\infty} \frac{M_Q^i g}{(2K)^i}.$$

Then $G \in L^2(\gamma)$, $g \leq G$ and $M_Q G \leq 2KG$. The last inequality implies that for a.e. x the weights, as functions on the integers, $\{J_j^{\frac{1}{2}}(x)\frac{G(\tau^j x)}{G(x)}\}_j$ belong to $A_1(\mathbb{Z})$ with uniform constant. Define $h(x) = \frac{J(x)^{\frac{1}{2}} G(\tau x)}{G(x)}$, $Tf = h(f \circ \tau)$ and $w = G^2$. Then μ is a finite measure.

By construction, for a.e. x, $h_j(x) = \frac{J_j(x)^{\frac{1}{2}} G(\tau^j x)}{G(x)} \in A_1(\mathbb{Z})$ with uniform constant; then Lemma 3.5 yields ii) holds.

The function $\widetilde{J}$ with respect to the measure μ has the expression $\widetilde{J}(x) = \frac{w(\tau x)}{w(x)} J(x)$, then $\widetilde{J}_j(x)h_j^{-1}(x) = \frac{J_j(x)^{\frac{1}{2}} G(\tau^j x)}{G(x)} \in A_1(\mathbb{Z})$ with uniform constant, so Lemma 3.5 yields i) holds.

Let us show that iii) holds. If this is not the case, then there exists a constant $C > 0$ such that for all $f \geq 0$, and for all j,

$$\int h_j(x) f(\tau^j x) w(x) d\gamma \leq C \int f(\tau^j x) w(\tau^j x) J_j(x) d\gamma,$$

then $h_j(x)w(x) \leq Cw(\tau^j x)J_j(x)$ for almost every $x \in X$ and $j \in \mathbb{Z}$, and that implies

$$G(x) \leq CG(\tau^j x)J_j^{\frac{1}{2}}(x) = CQ^j(G)(x) \text{ for almost every } x \in X, \text{ for all } j \in \mathbb{Z}.$$

Then for all $N \in \mathbb{N}$

$$G(x) \leq C\frac{1}{N+1}\sum_{j=0}^{N} Q^j(G)(x). \tag{8.2}$$

Ionescu Tulcea's theorem [14] yields the sequence of averages in (8.2) converges a.e. and in norm of $L^2(\gamma)$. We denote by H the limit as N goes to infinity. Moreover $QH = H$, and that is $J^{\frac{1}{2}}(x)H(\tau x) = H(x)$. Then the measure given by $d\nu = H^2 d\gamma$ is a finite measure and τ preserve the measure ν. This contradicts the assumption and iii) holds.

Let us show that iv) holds. If this is not the case, then there exists a constant $C > 0$ such that $\|h_j\|_{L^\infty} \le C$ for all $j \in \mathbb{Z}$ and then

$$\frac{J_j(x)^{\frac{1}{2}} G(\tau^j x)}{G(x)} \le C \text{ for almost every } x \in X, \text{ for all } j \in \mathbb{Z}$$

and by using (2.3) this implies

$$G(x) \le C \frac{G(\tau^{-j} x)}{J_j(\tau^{-j} x)^{\frac{1}{2}}} = CG(\tau^{-j} x) J_{-j}^{\frac{1}{2}}(x) \text{ for almost every } x \in X, \text{ for all } j \in \mathbb{Z},$$

so

$$G(x) \le CG(\tau^j x) J_j^{\frac{1}{2}}(x) = CQ^j(G)(x) \text{ for almost every } x \in X, \text{ for all } j \in \mathbb{Z}.$$

Now by following the argument above, we obtain a contradiction so iv) holds, and this finishes the proof of the second example. □

Finally, we are going to show there exist some non trivial Lamperti operators and some measures satisfying the hypothesis of Theorem 3.4.

Proposition 8.3. *Let $1 < p < \infty$ and let γ be a finite measure. Let $\tau : X \to X$ be an invertible ergodic transformation preserving the measure γ. There exist measurable positive functions h and w such that the equivalent measure $d\mu = w d\gamma$ is finite and the operator $Tf = h(f \circ \tau)$ satisfies the following statements:*

i) *T and T^{-1} are Cesáro bounded in $L^1(\mu)$.*
ii) *T_p and T_p^{-1} are Cesáro bounded in $L^p(\mu)$.*
iii) $\sup_{n \in \mathbb{Z}} ||T^n||_{L^1(\mu)} = \infty$.
iv) *For $1 < p < q < \infty$ fixed, T_q and T_q^{-1} are not Cesàro bounded in $L^q(\mu)$.*
v) $\sup_{n \in \mathbb{Z}} ||T_p^n||_{L^p(\mu)} = \infty$.

Proof. Let $1 < p < q < \infty$ and $1 < q' < p' < \infty$ such that $p + p' = pp'$ and $q + q' = qq'$ and let M_τ be the ergodic maximal function defined in (8.1). Then for fixed $1 < r < \frac{p'-1}{q'-1}$, M_τ is bounded in $L^r(\gamma)$, $M_\tau : L^r(\gamma) \to L^r(\gamma)$ and $\|M_\tau f\|_{L^r(\gamma)} \le K\|f\|_{L^r(\gamma)}$. Let us consider a positive function $g \ge 1$ such that $g \in L^r(\gamma) \setminus L^{\frac{p'-1}{q'-1}}(\gamma)$ (in particular, g is not bounded). Let

$$U = \sum_{i=0}^{\infty} \frac{M_\tau^i g}{(2K)^i}.$$

Then, $U \in L^r(\gamma)$, $g \le U$ and $M_\tau U \le 2KU$. The last inequality yields that the weights $\{\frac{U(\tau^i x)}{U(x)}\}_i$ as functions on the integers, satisfy $A_1(\mathbb{Z})$ for almost every $x \in X$ with uniform constants. Define $h(x) = \frac{U(\tau x)^{1-p'}}{U(x)^{1-p'}}$ and $w = U^{2-p'}$. Then μ is finite (if $p' \ge 2$, it follows from $U \ge g \ge 1$ and if $p' \le 2$ it follows from Hölder inequality).

By construction, $h_j(x) = \frac{U^{1-p'}(\tau^j x)}{U^{1-p'}(x)} \in A_{p'}(\mathbb{Z})$ with uniform constant, then Lemma 3.5 yields ii) holds. Moreover, it is easy to see that the function J with respect the measure μ has the expression $J(x) = \frac{w(\tau x)}{w(x)}$, then for a.e. $x \in X$, $J_j(x)h_j^{-1}(x) = \frac{U(\tau^j x)}{U(x)} \in A_1(\mathbb{Z})$ with uniform constant, so Lemma 3.5 yields i) holds.

Let us show that iii) holds. If this is not the case then there exists $C > 0$ such that $h_j(x)w(x) \leq Cw(\tau^j x)$ for almost every $x \in X$ and all $j \in \mathbb{Z}$, and that implies $U(x) \leq CU(\tau^j x)$ for almost every $x \in X$ and for all $j \in \mathbb{Z}$. This inequality yields that U is bounded (see the proof of Proposition 8.1 iii)) but this is a contradiction because $U \geq g$ and $g \notin L^\infty$, so iii) holds.

In order to show iv) we are going to prove that $h_j \notin A_{q'}(\mathbb{Z})$ with uniform constant for almost every $x \in X$. If this is not the case then there exists $C > 0$ such that

$$\left(\frac{1}{N+1}\sum_{i=0}^{N} U^{1-p'}(\tau^i x)\right)^{\frac{1}{q'}} \left(\frac{1}{N+1}\sum_{i=0}^{N} U^{(1-p')(1-q)}(\tau^i x)\right)^{\frac{1}{q}} \leq C,$$

for all $N \in \mathbb{N}$ and almost every $x \in X$. This implies $U^{(1-p')(1-q)} = U^{\frac{p'-1}{q'-1}} \in L^1(\gamma)$, and that is $U \in L^{\frac{p'-1}{q'-1}}(\gamma)$. However $U \geq g$ and $g \notin L^{\frac{p-1}{q-1}}(\gamma)$ and we obtain a contradiction. Then Lemma 3.5 yields iv).

Finally let us show v). If this is not the case, that is, if T_p is power bounded in $L^p(\mu)$ then there exists $C > 0$ such that

$$\int \left(h_j(x)f(\tau^j x)\frac{J_j^{\frac{1}{p}}(x)}{h_j(x)^{\frac{1}{p}}}\right)^p w(x)d\gamma \leq C\int f^p(\tau^j x)w(\tau^j x)d\gamma$$

for every $f \geq 0$ and every $j \in \mathbb{Z}$, and this implies $h_j^{p-1}(x)J_j(x)w(x) \leq Cw(\tau^j x)$ for almost every $x \in X$ and for all $j \in \mathbb{Z}$. Then $U(x) \leq CU(\tau^j x)$ for almost every $x \in X$ and for all $j \in \mathbb{Z}$. Now by following the previous argument shown in Proposition 8.1 iii), we obtain that U is bounded but this is a contradiction because $U \geq g$ and $g \notin L^\infty(\gamma)$, so v) holds. This finishes the proof. □

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