Linear Algebra and its Applications 730 (2026) 35–50

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Matrices over finite fields of odd characteristic as sums of diagonalizable and square-zero matrices

Peter Danchev [a,1,3], Esther García [b,*,2,3], Miguel Gómez Lozano [c,2,3]

[a] *Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria*
[b] *Departamento de Matemática Aplicada, Ciencia e Ingeniería de Materiales y Tecnología Electrónica, Universidad Rey Juan Carlos, 28933 Móstoles (Madrid), Spain*
[c] *Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 Málaga, Spain*

A R T I C L E I N F O

Article history:
Received 7 July 2025
Received in revised form 30 September 2025
Accepted 3 October 2025
Available online 9 October 2025
Submitted by C. de Seguins Pazzis

MSC:
15A15
15A21
15A83

Keywords:
Characteristic polynomial
Diagonalizable matrix
Square-zero matrix
Invariant factors

A B S T R A C T

Let $\mathbb{F}$ be a finite field of odd characteristic. When $|\mathbb{F}| \geq 5$, we prove that every matrix A admits a decomposition into $D + M$, where D is diagonalizable and $M^2 = 0$. For $\mathbb{F} = \mathbb{F}_3$, we show that such a decomposition is possible for non-derogatory matrices of order at least 5, and more generally, for matrices whose first invariant factor is not a non-zero trace irreducible polynomial of degree 3; we also establish that matrices consisting of direct sums of companion matrices, all of them associated to the same irreducible polynomial of non-zero trace and degree 3 over $\mathbb{F}_3$, never admit such a decomposition.

These results completely settle the question posed by Breaz (2018) [3] asking if it is true that, for big enough positive integers $n \geq 3$, all matrices A over a field of odd cardinality q admit decompositions of the form $E + M$ with $E^q = E$ and $M^2 = 0$: specifically, the answer is *yes* for $q \geq 5$, but however

* Corresponding author.
E-mail addresses: danchev@math.bas.bg (P. Danchev), esther.garcia@urjc.es (E. García), miggl@uma.es (M. Gómez Lozano).

1 The first author was partially supported by the BIDEB 2221 of TÜBİTAK.
2 The second and third authors were partially supported by Ayuda Puente 2023, URJC, and MTM2017-84194-P (AEI/FEDER, UE).
3 The three authors were partially supported by the Junta de Andalucía FQM264.

https://doi.org/10.1016/j.laa.2025.10.002

there are counterexamples for $q = 3$ and each order $n = 3k$, whenever $k \geq 1$.

1. Backgrounds

All matrices considered in the present paper are square in the sense that they have the same number of columns and rows; this number is often called the *order* of the matrix, sometimes just known as its *size* as well. Now, for completeness of the exposition and the reader's convenience, we recollect some fundamental material. A matrix is said to be *non-derogative* (or, in other words, *non-derogatory*) if its minimal polynomial coincides with the characteristic polynomial of the matrix; in such case, the matrix is similar to the companion matrix of its characteristic polynomial. A matrix X is said to be *diagonalizable* (or, in other terms, *non-defective*) if it is similar to a diagonal matrix, i.e., there exists an invertible matrix U such that the matrix $U^{-1}XU$ is diagonal. Recall also that a *nilpotent* matrix Y is the one for which $Y^k = 0$ for some integer $k > 0$. In the case where $k = 2$, such a matrix is just called *square-zero* for short. Besides, a matrix Z is called t*-potent* for some integer $t \geq 2$, provided $Z^t = Z$. The case when $t = 2$, such a matrix is just termed *idempotent*. Obviously, for any non-zero matrix, the notions of being nilpotent and potent are independent of each other.

In [4] it is shown that every element in the matrix ring $\mathbb{M}_n(\mathbb{F})$ is nil-clean (recall that an element is nil-clean if it can be decomposed as the sum of an idempotent and a nilpotent) if, and only if, $\mathbb{F}$ is the field of two elements (that is, $\mathbb{F} \cong \mathbb{F}_2$). Moreover, the order of nilpotency of the nilpotent part in the nil-clean decomposition of each element of $\mathbb{M}_n(\mathbb{F}_2)$ is always bounded by 4 (for more concrete details, we refer to the works [12,13] and [11]).

On the other hand, a deeper examination of the nil-cleanness of companion matrices in the ring $\mathbb{M}_n(\mathbb{F})$ is done in [5], where the authors prove that every companion matrix in $\mathbb{M}_n(\mathbb{F})$ is nil-clean if, and only if, $\mathbb{F}$ coincides with its prime subfield $\mathbb{F}_p$ and $p < n$.

Furthermore, in [3] Breaz studies those matrices over finite fields of odd characteristic that are a sum of a potent matrix and a nilpotent matrix, proving that each matrix over a field of odd cardinality q is a sum of a q-potent matrix and a nilpotent matrix of order less than or equal to 3 (see [6] too, as well as [1], [2], [8], [9] and [10] for a more detailed information). He also asks in [3, Remark 7] whether for matrices of an enough big order over a finite field of odd cardinality q there will always exist a decomposition into a q-potent matrix plus a square-zero matrix. Notice that all diagonalizable matrices over a finite field of cardinality q are q-potent and, conversely, if $D^q = D \in \mathbb{M}_n(\mathbb{F}_q)$, then D is diagonalizable since its minimal polynomial divides the separable polynomial $x^q - x$.

In [7] we initiated the study of those matrices which are a sum of a diagonalizable matrix and a square-zero matrix. The main two results of [7], which we will considerably extend in this article, are the following ones:

Lemma. *[7, Lemma 3.1] Let $\mathbb{F}$ be a field, let $n \geq 3$ and let $A \in \mathbb{M}_n(\mathbb{F})$ be a non-derogative matrix with trace c. Then:*

- *If $c = 0$ and $|\mathbb{F}| \geq n$, A admits a decomposition into $D + M$, where D is diagonalizable and $M^2 = 0$.*
- *If $c \neq 0$ and $|\mathbb{F}| \geq n + 1$, A admits a decomposition into $D + M$, where D is diagonalizable and $M^2 = 0$.*

Theorem. *[7, Theorem 3.6] Given any field $\mathbb{F}$, all matrices in $\mathbb{M}_2(\mathbb{F})$ admit a decomposition into $D + M$, where D is a diagonalizable matrix and M is a matrix such that $M^2 = 0$.*

Let $n \geq 3$ and let $\mathbb{F}$ be a field with $|\mathbb{F}| \geq n+1$. Then, every matrix $A \in \mathbb{M}_n(\mathbb{F})$ admits a decomposition into $D + M$, where D is a diagonalizable matrix and M is a matrix such that $M^2 = 0$. In particular, square matrices over infinite fields always admit such decompositions.

In this paper, we succeed to establish that any non-derogatory matrix of size at least 5 over a field of odd characteristic can be written as the sum of a diagonalizable matrix and a square-zero matrix (see, for more account, Proposition 2.1 and Proposition 3.1).

With this result in hand, together with the conclusions of [7] cited above, we can show that for all fields $\mathbb{F}$ of odd characteristic and cardinality at least 5, every matrix $A \in \mathbb{M}_n(\mathbb{F})$ can be expressed as $A = D + M$, where D is a diagonalizable matrix and M is a square-zero matrix; in particular, $D^q = D$ and so A can be expressed as the sum of q-potent matrix and a square-zero matrix. Moreover, if $\mathbb{F} = \mathbb{F}_3$ and the smallest invariant factor of a matrix $A \in \mathbb{M}_n(\mathbb{F})$ is not a non-zero trace irreducible polynomial of degree 3, then there exists a square-zero matrix $M \in \mathbb{M}_n(\mathbb{F})$ and a diagonalizable matrix D such that $A = D + M$.

We also show that, for each $n = 3k$, where $k \geq 1$, there exist matrices of order n over the field $\mathbb{F}_3$ that cannot be expressed as $D + M$ with $D^3 = D$ and $M^2 = 0$.

We end our work with a plan for a further research treatment of this type of decompositions of matrices over fields of characteristic 2, by raising two relevant questions.

2. Principal results

We begin with a technical result.

Proposition 2.1. *Let $\mathbb{F}$ be a field of odd characteristic, and let $A \in \mathbb{M}_n(\mathbb{F})$ be a non-derogative matrix of order $n \geq 5$. The following statements hold:*

(1) *If $Trace(A) = u_{n-1} \neq 0$, then there exists a square-zero matrix M such that $A + M$ is diagonalizable with 3 different eigenvalues $\{0, u_{n-1}, -u_{n-1}\}$.*

(2) *If Trace$(A) = 0$ and n is odd, then there exists a square-zero matrix M such that $A + M$ is diagonalizable with 3 different eigenvalues $\{0, 1, -1\}$.*
(3) *If Trace$(A) = 0$, n is even and $\mathbb{F} \neq \mathbb{F}_3$, then there exists a square-zero matrix M such that $A + M$ is diagonalizable with at most 4 different eigenvalues.*

In particular, every non-derogatory matrix of order at least 5 over a field of odd cardinality $q \geq 5$ can be written as the sum of a diagonalizable matrix and a square-zero matrix.

Proof. We can suppose, without loss of generality, that A has the following form:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & u_0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & u_1 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & u_2 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & u_3 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & u_4 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 & 0 & u_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & u_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & u_{n-1} \end{pmatrix}.$$

Along this proof, for smoothness of the presentation, all the matrices will be identified with endomorphisms of the vector space $\mathbb{F}^n$ with respect to the canonical basis $\{e_1, \ldots, e_n\}$.

(1.a) Let us study the case when Trace$(A) = u_{n-1} \neq 0$ and n is odd: let us consider $M = (m_{ij})$, where $m_{k+1,k} = -1$ and $m_{k-1,k} = u_{n-1}^2$ for $k = 3, 5, \ldots, n-2$, $m_{1,n} = -u_0$, $m_{k,n} = -u_{k-1} - u_{n-1}u_k$ for $k = 2, 4, \ldots, n-3$, $m_{n-1,n} = -u_{n-2}$ and the rest equal to 0, i.e.,

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -u_0 \\ 0 & 0 & u_{n-1}^2 & 0 & 0 & \cdots & 0 & 0 & -u_1 - u_{n-1}u_2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & u_{n-1}^2 & \cdots & 0 & 0 & -u_3 - u_{n-1}u_4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 & -u_5 - u_{n-1}u_6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & -u_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Thus, we calculate that $M^2 = 0$ and

$$A+M=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & u_{n-1}^2 & 0 & 0 & \cdots & 0 & 0 & -u_{n-1}u_2 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & u_2 \\ 0 & 0 & 0 & 0 & u_{n-1}^2 & \cdots & 0 & 0 & -u_{n-1}u_4 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & u_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -u_{n-1}u_{n-3} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & u_{n-3} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & u_{n-1} \end{pmatrix}.$$

With respect to the new basis

$$B'=\{e_1,e_2,\ldots,\,e_{n-2},u_{n-1}e_{n-1}+\sum_{i=1}^{(n-3)/2}u_{2i}e_{2i},\,u_{n-1}e_n+\sum_{i=1}^{(n-3)/2}u_{2i}e_{2i+1}\},$$

we have that

$$(A+M)_{B'}=\left(\begin{array}{ccc|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & u_{n-1}^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & u_{n-1}^2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & u_{n-1} \end{array}\right),$$

which consists of the direct sum of a block of the form $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & u_{n-1}^2 \\ 0 & 1 & 0 \end{pmatrix}$ that satisfies the polynomial $x^3-u_{n-1}^2x$, a block of the form $\begin{pmatrix} 0 & 0 \\ 1 & u_{n-1} \end{pmatrix}$ that satisfies the polynomial $x^2-u_{n-1}x$ and, if $n\geq 7$, blocks of the form $\begin{pmatrix} 0 & u_{n-1}^2 \\ 1 & 0 \end{pmatrix}$ that satisfy the polynomial $x^2-u_{n-1}^2$.

Therefore, the matrix $A+M$ satisfies the equation $(x-u_{n-1})(x+u_{n-1})x=0$, and so it is diagonalizable with eigenvalues $\{0,u_{n-1},-u_{n-1}\}$.

(1.b) Let us study the case when $\mathrm{Trace}(A)=u_{n-1}\neq 0$ and n is even: let us consider $M=(m_{ij})$, where $m_{k+1,k}=-1$ and $m_{k-1,k}=u_{n-1}^2$ for $k=2,4,\ldots,n-2$, $m_{k,n}=-u_{k-1}-u_{n-1}u_k$ for $k=1,3,\ldots,n-3$, $m_{n-1,n}=-u_{n-2}$ and the rest equal to 0, i.e.,

$$M=\begin{pmatrix}
0 & u_{n-1}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & -u_0-u_{n-1}u_1\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & -1 & 0 & u_{n-1}^2 & 0 & \cdots & 0 & 0 & -u_2-u_{n-1}u_3\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 & -u_4-u_{n-1}u_5\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \cdots & u_{n-1}^2 & 0 & -u_{n-4}-u_{n-1}u_{n-3}\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & -u_{n-2}\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}.$$

Thus, we compute that $M^2=0$ and

$$A+M=\begin{pmatrix}
0 & u_{n-1}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -u_{n-1}u_1\\
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & u_1\\
0 & 0 & 0 & u_{n-1}^2 & 0 & \cdots & 0 & 0 & 0 & -u_{n-1}u_3\\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & u_3\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -u_{n-1}u_5\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & u_{n-1}^2 & 0 & -u_{n-1}u_{n-3}\\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & u_{n-3}\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & u_{n-1}
\end{pmatrix}.$$

With respect to the basis

$$B'=\{e_1,e_2,\ldots,e_{n-2},\, u_{n-1}e_{n-1}+\sum_{i=1}^{(n-2)/2} u_{2i-1}e_{2i-1},\, u_{n-1}e_n+\sum_{i=1}^{(n-2)/2} u_{2i-1}e_{2i}\},$$

we deduce that

$$(A+M)_{B'}=\left(\begin{array}{cc|cc|cc|cc|cc}
0 & u_{n-1}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & u_{n-1}^2 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\
\hline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & u_{n-1}^2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & u_{n-1}
\end{array}\right),$$

which is the direct sum of blocks of the form $\begin{pmatrix} 0 & u_{n-1}^2 \\ 1 & 0 \end{pmatrix}$ that satisfies $x^2 - u_{n-1}^2$ and a block of the form $\begin{pmatrix} 0 & 0 \\ 1 & u_{n-1} \end{pmatrix}$ that satisfies $x^2 - u_{n-1}x$.

Therefore, the matrix $A + M$ satisfies the equation $(x - u_{n-1})(x + u_{n-1})x = 0$, and so it is diagonalizable with eigenvalues $\{0, u_{n-1}, -u_{n-1}\}$.

(2) Let us study the case when Trace$(A) = 0$ and n odd: let us consider $M = (m_{ij})$, where $m_{k+1,k} = -1$ and $m_{k-1,k} = 1$ for $k = 2, 4, \dots, n-1$ and the rest equal to 0, i.e.,

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix}.$$

Thus, we find that $M^2 = 0$ and

$$A + M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & u_0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & u_1 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & u_2 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & u_3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & u_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & u_{n-3} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & u_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

With respect to the new basis

$$B' = \{e_1, e_2, \dots, e_{n-1}, e_n - \sum_{i=1}^{(n-1)/2} u_{2i-2}e_{2i} - \sum_{i=1}^{(n-1)/2} u_{2i-1}e_{2i-1}\},$$

we derive that

$$(A+M)_{B'} = \left(\begin{array}{cc|cc|ccc|cc} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right),$$

which is the direct sum of blocks of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ that satisfy $x^2 - 1$ and a zero block of order one. Therefore, $A + M$ satisfies the equation $(x-1)(x+1)x = 0$, and so it is diagonalizable with eigenvalues $\{0, 1, -1\}$.

(3) Finally, let us study the case when $\mathrm{Trace}(A) = 0$ and n is even with $\mathbb{F} \neq \mathbb{Z}_3$: Since $|\mathbb{F}| \geq 5$, let us consider $0 \neq \alpha \in \mathbb{F}$ such that $\alpha^2 \neq 1$, and also let us consider $M = (m_{ij})$, where $m_{k+1,k} = -1$ and $m_{k-1,k} = 1$ for $k = 2, 4, \ldots, n-2$, $m_{k,n} = -u_{k-1}$ for $k = 1, 3, \ldots, n-3$, $m_{n-1,n} = \alpha^2 - u_{n-2}$ and the rest equal to 0, i.e.,

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -u_0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & -u_2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & -u_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & -u_{n-4} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \alpha^2 - u_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, we have that $M^2 = 0$ and

$$A + M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & u_1 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & u_3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & u_{n-3} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \alpha^2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}.$$

With respect to the basis

$$B' = \{e_1, e_2, \ldots, e_{n-2},\, e_{n-1} + \sum_{i=1}^{(n-2)/2} \frac{u_{2i-1}}{\alpha^2-1} e_{2i-1},\, e_n + \sum_{i=1}^{(n-2)/2} \frac{u_{2i-1}}{\alpha^2-1} e_{2i}\},$$

we obtain that

$$(A+M)_{B'} = \left(\begin{array}{cc|cc|cc|cc|cc} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \alpha^2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{array}\right),$$

which consists of the direct sum of blocks of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ that satisfies $x^2 - 1$ and a block of the form $\begin{pmatrix} 0 & \alpha^2 \\ 1 & 0 \end{pmatrix}$ that satisfies $x^2 - \alpha^2$.

Therefore, $A + M$ satisfies the equation $(x - \alpha)(x + \alpha)(x - 1)(x + 1) = 0$, and so it is diagonalizable with eigenvalues $\{1, -1, \alpha, -\alpha\}$, completing the arguments after all. □

It was shown in [7, §1] that matrices of order 2 can always be decomposed into a diagonalizable matrix and a square-zero matrix. This result, together with Proposition 2.1, lead to the following theorem, which solves positively the question posed by Breaz in [3, Remark 7] for fields of odd cardinality at least 5.

Theorem 2.2. *Let $\mathbb{F}$ be a finite field of odd cardinality q with $q \geq 5$, and let $n \in \mathbb{N}$. Then, every matrix $A \in \mathbb{M}_n(\mathbb{F})$ can be expressed as $A = D + M$, where D is a diagonalizable matrix and M is a square-zero matrix. In particular, $D^q = D$, so A can be expressed as the sum of a q-potent matrix and a square-zero matrix.*

Proof. Let $A \in \mathbb{M}_n(\mathbb{F})$, and suppose that $f_1(x)| \ldots |f_k(x)$ are its invariant factors; in view of the rational canonical decomposition, A is similar to the direct sum of the companion matrices of $f_1(x), \ldots, f_k(x)$. We now differ two possibilities as follows:

– The companion matrices of polynomials of degree less than or equal to 4 can be decomposed as the sum of a diagonalizable matrix and a square-zero matrix in accordance with the arguments of [7]: those of order 1 are trivially decomposed using the zero matrix;

those of order 2, 3 and 4 can always be decomposed thanks to [7, Theorem 3.6], because the cardinality of the field q is at least 5 by hypothesis.
– The companion matrices of polynomials of degree at least 5 can be decomposed as the sum of a diagonalizable matrix and a square-zero matrix according to Proposition 2.1.

This terminates all arguments necessary to establish the wanted decomposition. □

3. Matrices over the field $\mathbb{F}_3$

In the previous section we have completely solved the problem of decomposing matrices into diagonalizable and square-zero for fields of odd cardinality at least five. In this section, let us consider matrices over $\mathbb{F}_3$. Notice that in Proposition 2.1 we have not included non-derogative matrices of even order and zero trace over $\mathbb{F}_3$. So, let us study them now in what follows.

Proposition 3.1. *Let $A \in \mathbb{M}_n(\mathbb{F}_3)$ be a non-derogative matrix of even order n and zero trace. Then, there exists a square-zero matrix M such that $A + M$ is diagonalizable.*

Proof. If the order n of A is even and not divisible by 3, we can consider the matrix $A-I$, which has non-zero trace. So, Proposition 2.1(1) applies to get a square-zero matrix M such that $A - I + M = D$ is diagonalizable, whence $A + M = D + I$ is diagonalizable as well.

Assume from now on that n is even and a multiple of 3, i.e., $n = 6k$ for $k \in \mathbb{N}$. Let us suppose that

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & u_0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & u_1 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & u_2 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & u_3 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & u_4 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 & 0 & u_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & u_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

and consider $M = (m_{ij})$, where $m_{k+1,k} = -1$ and $m_{k-1,k} = 1$ for $k = 2, 4, \ldots, n-4$, $m_{n-1,n-1} = m_{n-2,n-1} = 1$, $m_{n-1,n-2} = m_{n-2,n-2} = -1$, $m_{k,n} = -u_{k-1} - u_k$ for $k = 1, 3, \ldots, n-5$, $m_{n-3,n} = u_{n-4}$, $m_{n-2,n} = m_{n-1,n} = u_{n-2}$ and the rest equal to 0, i.e.,

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -u_0 - u_1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -u_2 - u_3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -u_4 - u_5 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & -u_{n-6} - u_{n-5} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 & -u_{n-4} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 1 & -u_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 1 & -u_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, we have that $M^2 = 0$ and, if we consider $A + M$, we obtain

$$A + M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -u_1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & u_1 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -u_3 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & u_3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -u_5 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 & 0 & 0 & 0 & 0 & u_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & -u_{n-5} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & u_{n-5} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & 1 & -u_{n-3} - u_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

With respect to the basis

$$\begin{aligned} B' = \{ & e_1, e_2, \ldots, e_{n-2}, \\ & e_{n-1} + \sum_{i=1}^{(n-4)/2} -u_{2i-1}e_{2i} + (u_{n-3} - u_{n-2})e_{n-3} + \frac{(u_{n-3} - u_{n-2} + 1)}{2} e_{n-2}, \\ & e_n + \sum_{i=1}^{(n-4)/2} u_{2i-1}(e_{2i} - e_{2i-1}) + (u_{n-2} - u_{n-3})e_{n-3} \}, \end{aligned}$$

we get

$$
(A+M)_{B'} = \left(\begin{array}{cc|cc|cc|c|cc|cc|cc}
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \vdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right),
$$

which consists of the direct sum of blocks of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ that satisfy $x^2 - 1$, a block of the form $\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$ that satisfies $x^2 + x$, and a block of the form $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ that satisfies $x^2 - x$.

Therefore, $A + M$ satisfies the equation$(x-1)(x+1)x = 0$, and so it is diagonalizable with eigenvalues $\{0, 1, -1\}$, as desired. □

Remark 3.2. It is a well-known fact by [3, Example 6] that the companion matrices of non-zero trace irreducible polynomials of degree 3 cannot be decomposed into a sum of diagonalizable and zero-square matrices. In regard to the previous results, these are the only possible counterexamples for non-derogatory matrices as the next consequence demonstrates.

Corollary 3.3. *A non-derogatory matrix over $\mathbb{F}_3$ can be decomposed as the sum of a diagonalizable matrix and a square-zero matrix if, and only if, it is not similar to the companion matrix of a degree 3 irreducible polynomial with non-zero trace.*

Proof. For matrices of order 1, 2 and 4, the claim follows directly from [7, Theorem 3.6, Proposition 5.4]. Non-derogative matrices of order 3 and zero trace can be decomposed in this way by [7, Lemma 3.1]. For non-derogative matrices of order at least 5, the decomposition follows with the aid of Propositions 2.1(1)(2) and 3.1. □

We can extend this result to general matrices over $\mathbb{F}_3$ unless we cannot elude the counterexample mentioned in Remark 3.2.

Corollary 3.4. *Let $A \in \mathbb{M}_n(\mathbb{F}_3)$ and suppose that the first invariant factor of A is not a degree 3 irreducible polynomial of non-zero trace. Then, there exists a square-zero matrix M and a diagonalizable matrix D such that $A = D + M$.*

Proof. Suppose that $f_1(x)|\dots|f_k(x)$ are the invariant factors of A; by hypothesis, $f_1(x)$ is not a degree 3 irreducible polynomial of non-zero trace. This condition implies that non of the invariant factors are degree 3 irreducible polynomials of non-zero trace, so their associated companion matrices can all of them be decomposed into a diagonalizable matrix and square-zero matrix by Corollary 3.3. □

In the following proposition, we show that direct sums of companion matrices, all of them associated to the same irreducible degree 3 polynomial with non-zero trace, cannot be decomposed as the sum of a diagonalizable matrix and a square-zero matrix, providing counterexamples to the decomposition into diagonalizable and square-zero for matrices of each order $n = 3k$ whenever $k \geq 1$. To that target, let us first start with a technical lemma.

Lemma 3.5. *Let R be a unital ring of characteristic 3, let $x, m \in R$, and suppose that $x^3 - x^2 - x - 1 = 0$, $m^2 = 0$ and $(x+m)^3 = x + m$. Then, $(x^2+1)m = 0$.*

Proof. (1) Let us prove that $mx^2m = -mxmxm$: in fact, one calculates that

$$\begin{aligned}0 &= m\left((x+m)^3 - (x+m)\right)m = mx^3m + mxmxm - mxm \\ &= m(x^2+x+1)m + mxmxm - mxm \\ &= mx^2m + mxmxm.\end{aligned}$$

(2) Let us prove that $mxmxmxm = mxm + mxmxm$: in fact, one computes that

$$\begin{aligned}0 &= m\left((x+m)^4 - (x+m)^2\right)m \\ &= mx^4m + mx^2mxm + mxmx^2m - mx^2m \\ &= m(1+2x+2x^2)m - 2mxmxmxm - mx^2m \\ &= 2mxm - mxmxm + mxmxmxm.\end{aligned}$$

(3) Let us prove that $mxmxm = -mxm$: in fact, one has that

$$\begin{aligned}0 &= m\left((x+m)^5 - (x+m)\right)m \\ &= mx^5m + mx^3mxm + mx^2mx^2m \\ &+ mxmx^3m + mxmxmxm - mxm \\ &= m(4x^2+2)m + m(x^2+x+1)mxm + mx^2mx^2m \\ &+ mxm(x^2+x+1)m + mxm + mxmxm - mxm\end{aligned}$$

$$
\begin{aligned}
&= -mxmxm - mxmxmxm + mxmxm + mxmxmxmxm \\
&- mxmxmxm + mxmxm + mxmxm \\
&= -2mxm + mxmxm + mxmxmxm \\
&= -mxm + 2mxmxm.
\end{aligned}
$$

(4) Let us prove that $mxm = 0 = mx^2m$: in fact, one finds that

$$mxmxmxm =_{(2)} mxm + mxmxm =_{(3)} mxm - mxm = 0,$$

so

$$mxmxm =_{(3)} -mxmxmxm = 0$$

and, combining (3) with (1), we get

$$mxm = 0 = -mx^2m.$$

(5) Let us prove that $(x^2+1)m = 0$: in fact, one discovers that

$$
\begin{aligned}
0 &= \left((x+m)^3 - (x+m)\right) m = x^3m + xmxm + mx^2m - xm \\
&= (x^2+x+1)m - xm = x^2m + m = (x^2+1)m,
\end{aligned}
$$

thus finishing our argumentation. □

We are now prepared to establish the promised proposition.

Proposition 3.6. *Let $p(x) \in \mathbb{F}_3[x]$ be an irreducible polynomial of degree 3 and non-zero trace, and let $A \in \mathbb{M}_n(\mathbb{F}_3)$ be a matrix which is similar to a direct sum of companion matrices all of them associated to $p(x)$. Then, there does not exist a square-zero matrix M such that $A + M$ is diagonalizable.*

Proof. We first observe that over $\mathbb{F}_3$ there are six irreducible polynomials of degree 3 and non-zero trace. Let us suppose that $A \in \mathbb{M}_n(\mathbb{F}_3)$ is similar to a direct sum of companion matrices all of them associated to the polynomial $x^3 - x^2 - x - 1 \in \mathbb{F}_3[x]$, and let us show that for such A there does *not exist* a square-zero matrix $M \in \mathbb{M}_n(\mathbb{F}_3)$ such that $A + M$ is diagonalizable. Once we have shown that fact, it is direct to see that there does not exist such M for the matrices $A + \mathrm{Id}$, $A - \mathrm{Id}$, $2A$, $2A + \mathrm{Id}$, $2A - \mathrm{Id}$, which are similar to direct sums of companion matrices, all of them associated to the other five irreducible polynomials of degree 3 and non-zero trace over $\mathbb{F}_3$.

Suppose now that there exists a square-zero matrix $M \in \mathbb{M}_n(\mathbb{F}_3)$ such that $D = A+M$ is diagonalizable. Then, we have that $A^3 = A^2 + A + \mathrm{Id}$, $M^2 = 0$ and $(A+M)^3 = A+M$ (because every diagonal matrix over $\mathbb{F}_3$ satisfies the equation $x^3 - x = 0$). Looking at

Lemma 3.5, one detects that $(A^2 + \mathrm{Id})M = 0$, and since $A^2 + \mathrm{Id}$ is invertible with inverse $A^2 - A + \mathrm{Id}$, we conclude that $M = 0$, which is an obvious contradiction as A is not diagonalizable knowing that $A^3 \neq A$. □

Remark 3.7. Thanks to Proposition 3.6, we can answer in the negative the question posed by Breaz in [3, Remark 7]; namely, for the field $\mathbb{F}_3$ it is *not* true that for big enough n every matrix of order n can be decomposed into $D + M$ with $D^3 = D$ and $M^2 = 0$.

4. Further works

In this paper, we have dealt with matrices over fields of odd cardinality. It, thereby, remains to examine the analogous problems for fields of characteristic two. To that end, we know from [4] the related result that every matrix in $\mathbb{M}_n(\mathbb{F}_2)$ is a sum of an idempotent matrix and a nilpotent matrix. Nevertheless, Šter constructed in [12, answer to Question 4.1] two companion matrices of order 4 over $\mathbb{F}_2$ that cannot be expressed as $D + M$, where $D^2 = D$ (equivalently, D is diagonalizable) and $M^2 = 0$. These counterexamples were extended by Shitov in [11], showing that the direct sum of an odd number of such matrices cannot be expressed as the sum $D + M$ of an idempotent and a nilpotent of order 2.

Therefore, one needs to study matrices over fields of characteristic two and cardinality at least four via the following two relevant queries.

Questions. Is it true that each matrix in $\mathbb{M}_n(\mathbb{F}_{2^i})$, $i \geq 2$, is a sum of a diagonalizable matrix and a square-zero matrix? If not, is that decomposition valid at least for non-derogative matrices?

Funding

The first-named author (Peter V. Danchev) was supported in part by the project BIDEB 2221 of TÜBÍTAK; the second-named and third-named authors (Esther García and Miguel Gómez Lozano) were partially supported by the projects Ayuda Puente 2023, URJC, and MTM2017-84194-P (AEI/FEDER, UE).

The all three authors were partially supported by the project Junta de Andalucía FQM264.

Declaration of competing interest

There is no competing interest.

Acknowledgements

The authors express their sincere thanks to the anonymous expert referee for his/her careful reading of the manuscript and the insightful comments and suggestions made on it. The authors are also very thankful to the handling editor, Prof. Clement de Seguins Pazzis, for his professional editorial management.

Data availability

No data was used for the research described in the article.

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