Mediterr. J. Math. (2023) 20:66
https://doi.org/10.1007/s00009-023-02285-6
1660-5446/23/020001-15
published online January 17, 2023

Mediterranean Journal of Mathematics

A Curvature Inequality Characterizing Totally Geodesic Null Hypersurfaces

Benjamín Olea

Abstract. A well-known application of the Raychaudhuri equation shows that, under geodesic completeness, totally geodesic null hypersurfaces are unique which satisfy that the Ricci curvature is nonnegative in the null direction. The proof of this fact is based on a direct analysis of a differential inequality. In this paper, we show, without assuming the geodesic completeness, that an inequality involving the squared null mean curvature and the Ricci curvature in a compact three-dimensional null hypersurface also implies that it is totally geodesic. The proof is completely different from the above, since Riemannanian tools are used in the null hypersurface thanks to the rigging technique.

Mathematics Subject Classification. Primary 53C50; Secondary 53C40.

Keywords. Null hypersurface, rigging technique, Raychaudhuri equation.

1. Introduction

Traditionally, the hypersurfaces of a Lorentzian manifold are classified depending on its causal character in three types: timelike, spacelike or null. Spacelike and timelike hypersurfaces are well studied compared to null hypersurfaces because they inherit a nondegenerate metric from the ambient space. Conversely, a null hypersurface inherits a degenerate metric and its orthogonal direction is tangent to the hypersurface itself, so there is not a well-defined canonical projection. This issue can be partially overcome by different ways [7,17]. One of them is to fix (arbitrarily) a transverse vector field to the null hypersurface which is called a rigging, [11]. From it we can construct several geometric objects as the null second fundamental form and the null mean curvature. Moreover, we can construct a Riemannian metric on the null hypersurface which can be used as an auxiliary tool to prove some results, [3,12–14].

There are several examples of inequalities involving the squared mean curvature of a hypersurface. For example, Willmore proved that for a compact surface S in $\mathbb{R}^3$ it holds

$$\int_S ||\vec{H}||^2 \geq 16\pi,$$

and the equality holds if and only if S is a round sphere, [23]. This was generalized by Chen in [4] to compact submanifold in $\mathbb{R}^n$ and in [5] he showed that for a hypersurface L in a n-dimensional Riemannian manifold of constant curvature c it holds

$$Ric^L(u,u) \leq (n-2)c + \frac{1}{4}||\vec{H}||^2 \quad (1.1)$$

for all unitary vector u tangent to L. Moreover, the equality holds for all unitary vectors if and only if L is totally geodesic or $n=3$ and L is totally umbilical. A similar inequality can be obtained in the case of a spacelike hypersurface of a semi-Riemannian manifold of constant curvature, [6].

On the other hand, it is well-known that under geodesic completeness and the null convergence condition ($Ric(u,u) \geq 0$ for all null vector u) we can get information about the null mean curvature of a null hypersurface.

Proposition 1.1. (see for example [9]) *Let M be a spacetime which obeys the null convergence condition and L a null hypersurface. If the null geodesics of L are future geodesically complete, then the null mean curvature of L is nonpositive.*

Recall that the sign for the null mean curvature used in this paper is the opposite than in [9]. Moreover, its sign depends on the chosen time orientation. As a corollary of the above proposition, if the null geodesics of L are also past complete, then the null mean curvature is zero, which jointly with the null convergence condition implies that L is totally geodesic.

As pointed out by Galloway, proposition 1.1 is the most rudimentary form of Hawking's black hole area theorem, since it says that cross sections of L do not decrease in area as one moves towards the future. Clearly, the completeness of the null geodesics is necessarily to obtain the conclusion of proposition 1.1. Moreover, it does not hold even if there is a complete vector field in L which integral curves are null pregeodesics, as it is shown in example 3.6.

In this paper we generalize proposition 1.1 to the case of a compact null hypersurface L in a four-dimensional Lorentzian manifold without assuming the completeness of the null geodesics of L (recall that compactness does not implies geodesic completeness in this context, see example 3.12). Our main result is theorem 3.11, where an inequality between the Ricci curvature and the square of the null mean curvature similar to the inequality (1.1) is assumed instead of the null convergence condition. The proof of proposition 1.1 is a quite easy argument based on the analysis of a differential inequality (see proposition 3.2). However, the proof of theorem 3.11 uses as main tool the rigging technique introduced in [11], which we will briefly review in the next section.

2. Preliminaries

Let (M, g) be a n-dimensional $(n \geq 3)$ Lorentzian connected manifold. A hypersurface L is null if the induced metric tensor is degenerate on L. In this case, the radical of L, which is given by $TL \cap TL^{\perp}$, is a one-dimensional null distribution and all other directions in L are spacelike.

Definition 2.1. ([11]) A rigging for L is a vector field ζ defined in a open set containing L such that $\zeta_p \notin T_pL$ for all $p \in L$. If it is only defined over L, then it is called a restricted rigging.

Locally, it always exists a rigging for a null hypersurface. It may not exist globally, but we can ensure its existence under the orientability hypothesis.

Lemma 2.2. *If (M, g) is an orientable Lorentzian manifold and L a null hypersurface, then there is a (restricted) rigging for L if and only if L is orientable.*

On the other hand, it always exists a rigging for any null hypersurface in a time-orientable Lorentzian manifold, since there is not timelike tangent directions to a null hypersurface.

From now on, we always suppose that there is a rigging ζ for any null hypersurface. From it, we can define the rigged vector field ξ as the unique null vector field in L such that $g(\xi, \zeta) = 1$, the null transverse vector field

$$N = \zeta - \frac{1}{2} g(\zeta, \zeta)\xi$$

and the screen distribution $\mathcal{S}_p = \zeta_p^{\perp} \cap T_pL$ for all $p \in L$. Taking into account that $g(\nabla_U \xi, \xi) = 0$ for all $U \in \mathfrak{X}(L)$ and the decompositions $T_pM = T_pL \oplus span(N_p)$ and $T_pL = \mathcal{S}_p \oplus span(\xi_p)$, we can write

$$\nabla_U V = \nabla^L_U V + g(A^*(U), V)N, \tag{2.1}$$

$$\nabla_U \xi = -A^*(U) - \tau(U)\xi \tag{2.2}$$

for all $U, V \in \mathfrak{X}(L)$, where ∇^L is the induced connection, $A^* : TL \to \mathcal{S}$ is an endomorphism field called the shape operator and τ is a one-form. It is easy to check that A^* is self-adjoint and $A^*(\xi) = 0$. This last fact has two important consequences. The first one is that A^* is diagonalizable, since $\mathcal{S}$ is spacelike. The second one is that $\nabla_\xi \xi = \nabla^L_\xi \xi = -\tau(\xi)\xi$ and $\nabla_U \xi = \nabla^L_U \xi$ for all $U \in \mathfrak{X}(L)$.

The null mean curvature of L is given by

$$H_p = trace\,(A^*) = -\sum_{i=1}^{n} g(\nabla_{e_i} \xi, e_i)$$

for all $p \in L$, being $\{e_1, \dots, e_{n-2}\}$ an orthonormal basis of $\mathcal{S}_p$. Since A^* is diagonalizable, it holds the inequality

$$\frac{1}{n-2} H^2 \leq ||A^*||^2. \tag{2.3}$$

If A^* is identically zero, then L is totally geodesic and if A^* is proportional to the identity map, then L is totally umbilical.

The following proposition shows how the geometric objects constructed in a null hypersurface from a rigging are transformed under a rigging change.

Proposition 2.3. *([19]) If $\zeta' = \Phi N + X_0 + a\xi$, with $X_0 \in \Gamma(\mathcal{S})$ and $\Phi, a \in C^\infty(L)$, is another rigging for L, then*

$$g(A^{*\prime}(U), V) = \frac{1}{\Phi} g(A^*(U), V),$$
$$\tau'(U) = \tau(U) + \frac{1}{\Phi} g(A^*(X_0), U) + \frac{1}{\Phi} d\Phi(U)$$

for all $U, V \in \mathfrak{X}(L)$. Moreover, $\xi' = \frac{1}{\Phi}\xi$ and $H' = \frac{1}{\Phi}H$ and in particular

$$\tau'(\xi') = \frac{1}{\Phi}\tau(\xi) + \frac{1}{\Phi^2} d\Phi(\xi).$$

Therefore, although A^* depends on the chosen rigging, being totally geodesic, totally umbilical or having zero null mean curvature do not depend on any choice. Moreover, a change on the sign of the rigging only produces a change on the sign of the rigged vector field and the null mean curvature.

The curvature tensor R^L of the induced connection ∇^L is related to the curvature tensor R of the ambient by the Gauss-Codazzi equations, [7]. For our purpose we recall that

$$R^L_{UV}\xi = R_{UV}\xi \tag{2.4}$$

for all $U, V \in \mathfrak{X}(L)$.

We can also define a Riemannian metric on L, called the rigged metric, as

$$\widetilde{g} = g + \omega \otimes \omega, \tag{2.5}$$

where $\omega = i^*(\alpha)$, $i : L \to M$ is the canonical inclusion and α is the metrically equivalent one form to ζ. In this manner, once we have fixed a rigging for a null hypersurface, two connections appear: the induced connection ∇^L and the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$, which is called the rigged connection. The difference tensor $D^L(X, Y) = \nabla^L_X Y - \widetilde{\nabla}_X Y$ is a symmetric tensor.

Proposition 2.4. ([11]) *Given $U, V, W \in \mathfrak{X}(L)$, it holds*

$$g(D^L(U, V), W) = -\frac{1}{2}\left(\omega(W)(L_\xi \widetilde{g})(U, V) + \omega(U) d\omega(V, W) + \omega(V) d\omega(U, W)\right) - g(A^*(U), V)\omega(W).$$

where L_ξ is the Lie derivative along ξ.

Remark 2.5. If $E \in \mathfrak{X}(M)$ is a timelike and unitary vector field, then we can construct the Riemannian metric on M given by

$$h = g + 2g(E, \cdot)g(E, \cdot). \tag{2.6}$$

This is a classical construction which has been used in some situations. For example, in [8] it is used to prove, under suitable conditions, the existence of periodic timelike geodesics in a compact Lorentzian manifold and in [22] to show that a compact Lorentzian manifold furnished with a timelike Killing vector field is geodesically complete, although the Riemannian metric (2.6) is not explicitly written in this last paper.

On the other hand, in [18], to prove the geodesic completeness of certain Lorentzian manifolds, the authors construct a Riemannian metric h on the whole Lorentzian manifold from two null vector fields $Z, V \in \mathfrak{X}(M)$ such that $g(Z, V) = 1$. For this, they define h by the conditions

$$\begin{aligned} h(X,Y) &= g(X,Y), \\ h(V,V) &= h(Z,Z) = 1, \\ h(V,X) &= h(Z,X) = h(Z,V) = 0 \end{aligned}$$

for all $X, Y \in V^{\perp} \cap Z^{\perp}$. This construction is equivalento to (2.6) taking $E = \frac{1}{\sqrt{2}}(Z - V)$.

If we have a null hypersurface L in (M, g), then it inherits a Riemannian metric from the Riemannian manifold (M, h). However, this construction seems to be too rigid since we need a timelike and unitary vector field whereas to construct the rigged metric (2.5) we only need a transverse vector field to the null hypersurface. Another fact supporting this is that if L is totally umbilical in (M, g), then it is not totally umbilical in (M, h) even if E is parallel, [20, Proposition 6.1]. Nevertheless, some characterizations of totally geodesic null hypersurfaces can be proven using the Riemannian metric (2.6), [20, Theorem 6.5].

As an immediate consequence of proposition 2.4, the difference tensor D^L holds the following properties.

Corollary 2.6. *Given $X, Y, Z \in \Gamma(\mathcal{S})$ and $U \in \mathfrak{X}(L)$ it holds*

1. $\widetilde{g}(D^L(X,\xi),Y) = -\widetilde{g}(D^L(Y,\xi),X)$. *In particular* $\widetilde{g}(D^L(X,\xi),X) = 0$.
2. $\widetilde{g}(D^L(X,Y),Z) = 0$
3. $\widetilde{g}(D^L(U,\xi),\xi) = -\tau(U)$.

If we take $W = \xi$ and $U = X, V = Y \in \Gamma(\mathcal{S})$ in proposition 2.4, then we get

$$(L_{\xi}\widetilde{g})(X,Y) = -2g(A^*(X),Y) \tag{2.7}$$

for all $X, Y \in \Gamma(\mathcal{S})$. In particular, we have $\widetilde{g}(\widetilde{\nabla}_X \xi, X) = -g(A^*(X), X)$ for all $X \in \Gamma(\mathcal{S})$ and

$$H = -\widetilde{\operatorname{div}}\xi. \tag{2.8}$$

The rigged metric $\widetilde{g}$ allows us to use Riemannian tools in the study of null hypersurfaces. One of them is the so-called Bochner technique, which has its starting point at the general equation

$$Ric(Z,Z) = \operatorname{div}\nabla_Z Z - Z(\operatorname{div} Z) - trace\,(S^2), \tag{2.9}$$

where S is the endomorphism given by $S(U) = \nabla_U Z$. This equation holds for any vector field Z in a semi-Riemannian manifold. Moreover, since

$$\begin{aligned} trace\,(S) &= \operatorname{div} Z, \\ \operatorname{div}\left((\operatorname{div} Z)\, Z\right) &= Z\left(\operatorname{div} Z\right) + \left(\operatorname{div} Z\right)^2, \end{aligned}$$

if the manifold is compact and orientable, then from equation (2.9) we get

$$\int Ric(Z,Z) = \int trace\,(S)^2 - trace\,(S^2). \tag{2.10}$$

On the other hand, the Raychaudhuri equation states that for a null hypersurface it holds (see for example [9,10])

$$Ric(\xi,\xi) = \tau(\xi)H + \xi(H) - ||A^*||^2. \tag{2.11}$$

This equation can be derived from (2.9). Indeed, fixed $p \in L$ we extend ξ to get a null vector field in a neighbourhood $p \in \theta \subset M$ which we still call ξ. If we take a basis $\{e_1,\ldots,e_{n-2}\}$ of $\mathcal{S}_p$, then we can construct a pseudo-orthonormal basis of T_pM adding ξ_p and N_p. Now we have

$$\begin{aligned}
\operatorname{div}\nabla_\xi\xi &= \sum_{i=1}^{n-2} g(\nabla_{e_i}\nabla_\xi\xi, e_i) + g(\nabla_\xi\nabla_\xi\xi, N) + g(\nabla_N\nabla_\xi\xi,\xi)\\
&= -\sum_{i=1}^{n-2}\tau(\xi)g(\nabla_{e_i}\xi, e_i) - \xi(\tau(\xi)) + \tau(\xi)^2 - g(\nabla_\xi\xi,\nabla_N\xi)\\
&= \tau(\xi)H - \xi(\tau(\xi)) + \tau(\xi)^2 + \tau(\xi)g(\xi,\nabla_N\xi)\\
&= \tau(\xi)H - \xi(\tau(\xi)) + \tau(\xi)^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\operatorname{div}\xi &= \sum_{i=1}^{n-2} g(\nabla_{e_i}\xi, e_i) + g(\nabla_\xi\xi, N) + g(\nabla_N\xi,\xi)\\
&= -H - \tau(\xi).\\
trace\,(S^2) &= \sum_{i=1}^{n-2} g(S^2(e_i), e_i) + g(S^2(\xi), N) + g(S^2(N),\xi)\\
&= ||A^*||^2 - \tau(\xi)g(\nabla_\xi\xi, N) = ||A^*||^2 + \tau(\xi)^2.
\end{aligned}$$

Therefore, taking $Z=\xi$ in equation (2.9) we obtain equation (2.11).

3. Main Results

An immediate but important consequence of equation (2.11) is stated in the following well-known proposition, which we will use later.

Proposition 3.1. *Let L be a null hypersurface in a Lorentzian manifold.*

1. *If L is totally geodesic, then $Ric(\xi,\xi)=0$.*
2. *If $Ric(\xi,\xi)\geq 0$ and L has zero null mean curvature, then it is totally geodesic.*

We also need a slight different formulation of proposition 1.1. We include the proof to make the paper self-contained.

Proposition 3.2. *Let L be a null hypersurface in a time-oriented Lorentzian manifold and ζ a rigging for it such that ξ is future pointing and $cH^2 \leq Ric(\xi,\xi)$ for some constant $c > -\frac{1}{n-2}$.*

1. *If the null geodesics of L are future geodesically complete, then H is nonpositive.*
2. *If the null geodesics of L are future and past geodesically complete, then L is totally geodesic.*

Proof. Take $p \in L$ and $\gamma : [0, \infty) \to L$ a future pointing null geodesic with $\gamma(0) = p$. Using [14, lemma 3.2], there is a locally defined null section, which we still call ξ, such that $\xi_{\gamma(t)} = \gamma'(t)$. Observe that $\tau(\xi)_{\gamma(t)} = 0$ since γ is a geodesic. Moreover, from proposition 2.3 it still holds $cH^2 \leq Ric(\xi, \xi)$, although we have changed the null section ξ. Using the inequality (2.3) in equation (2.11) we get

$$\left(c + \frac{1}{n-2}\right) H^2_{\gamma(t)} \leq \frac{d}{dt} H_\gamma(t).$$

If $H_p > 0$, then the above inequality implies that $H_{\gamma(t)}$ is not defined for all $t > 0$, which is a contradiction. If the null geodesics of L are also past complete, then $H = 0$ and proposition 3.1 ensures that L is totally geodesic. □

Corollary 3.3. *Let L be a null hypersurface in a time-oriented Lorentzian manifold and ζ a rigging for it such that $\tau(\xi) = 0$ and $cH^2 \leq Ric(\xi, \xi)$ for some constant $c > -\frac{1}{n-2}$. If ξ is a complete vector field, then L is totally geodesic.*

We can drop the geodesic completeness assumption in proposition 3.2 if we suppose that the null hypersurface is compact.

Theorem 3.4. *Let L be a compact orientable null hypersurface in a time-oriented Lorentzian manifold and ζ a rigging for it such that $cH^2 \leq Ric(\xi, \xi)$ for some $c > -\frac{1}{n-2}$. If $\tau(\xi)$ is constant, then L is totally geodesic.*

Proof. From proposition 2.3, if we change the sign of the rigging, then the hypotheses of the theorem still hold. Therefore, we can suppose that ξ is future pointing.

The vector field ξ is complete since L is compact. If $\tau(\xi) = 0$, then we can apply corollary 3.3. Suppose for example that $\tau(\xi) < 0$. If $\alpha : \mathbb{R} \to L$ is an integral curve of ξ, then we can check that $\gamma(t) = \alpha\left(-\frac{1}{\tau(\xi)} \ln(-\tau(\xi)t + 1)\right)$ is a null geodesic, which is defined for $\frac{1}{\tau(\xi)} < t$. Therefore, from proposition 3.2 we get that $H \leq 0$. From equation (2.8) we have $\int_L H d\widetilde{g} = 0$, so $H = 0$ and proposition 3.1 ensures that L is totally geodesic. □

As is pointed out in the proof of proposition 3.2, the condition $cH^2 \leq Ric(\xi, \xi)$ does not depend on the chosen rigging. However, from proposition 2.3 we see that the condition $\tau(\xi)$ constant does depend on the chosen rigging.

A null hypersurface L is called a Killing horizon if there is a Killing vector field K such that K_p is null and tangent to L for all $p \in L$. It is a remarkable fact that for a Killing horizon in a spacetime which satisfies the null dominant energy condition there is a null section ξ such that $\tau(\xi)$ is constant (see [15] for a proof of this using the rigging technique).

Corollary 3.5. *Let L be a compact orientable null hypersurface in a time-orientable Einstein Lorentzian manifold. If there is a rigging for L such that $\tau(\xi)$ is constant, then L is totally geodesic.*

Theorem 3.4 does not hold if L is not supposed compact. The following is an example of this and a counterexample of corollary 3.3 if ξ is not a complete vector field or $\tau(\xi) \neq 0$.

Example 3.6. Take (F, g_R) a Riemannian manifold and $M = \mathbb{R}^2 \times F$ furnished with the Lorentzian metric given by

$$g = 2e^s dtds + e^{2s} g_R.$$

We can check that $L = \{t_0\} \times \mathbb{R} \times F$ is a null hypersurface for any fixed $t_0 \in \mathbb{R}$. If we consider the rigging $\zeta = e^{-s}\partial t$, then the rigged vector field is $\xi = \partial s$ and the screen distribution is given by $\mathcal{S} = TF$. If $X, Y \in \Gamma(\mathcal{S})$, then

$$g(A^*(X), Y) = -\frac{1}{2}\partial s(g(X, Y)) = -g(X, Y).$$

Therefore $A^*(X) = -X$ for all $X \in \Gamma(\mathcal{S})$ and so L is totally umbilical with null mean curvature $H = 2 - n$. Since $g(\xi, \partial t) = e^s$, from the Koszul formula we get

$$-\tau(\xi)e^s = g(\nabla_\xi \xi, \partial t) = \xi g(\xi, \partial t) = e^s$$

and thus $\tau(\xi) = -1$. Moreover, from equation (2.11) we have $Ric(\xi, \xi) = 0$. Thus, the inequality in theorem 3.4 holds with $c = 0$, but L is not totally geodesic.

If we consider the rigging $\zeta' = e^{-2s}\partial t$, then its rigged vector field is $\xi' = e^s \xi$ and it holds $\tau'(\xi') = 0$, but we can not apply corollary 3.3 because ξ' is not a complete vector field.

Now, we try to generalize theorem 3.4 in some manner such that $\tau(\xi)$ does not need to be constant. From now on, we suppose that L has dimension three and it is oriented. Fix a $\widetilde{g}$-orthonormal local positive basis $\{X, Y, \xi\}$ and write

$$\begin{cases} \widetilde{\nabla}_X \xi &= \sigma_{11} X + \sigma_{12} Y, \\ \widetilde{\nabla}_Y \xi &= \sigma_{21} X + \sigma_{22} Y. \end{cases} \tag{3.1}$$

We can define a function which measures the integrability of $\mathcal{S}$. In fact, $d\omega(X, Y)$ does not depend on the chosen local positive orthonormal basis, so it determines a well-defined global function $\mu \in C^\infty(L)$. If μ vanishes everywhere, then $\mathcal{S}$ is integrable. Observe that μ is locally given by

$$\mu = \sigma_{12} - \sigma_{21}.$$

Lemma 3.7. *In the above situation, it holds*

$$D^L(X, \xi) = -\frac{1}{2}\mu Y - \tau(X)\xi,$$

$$D^L(Y, \xi) = \frac{1}{2}\mu X - \tau(Y)\xi.$$

In particular, $\widetilde{\nabla}_X \xi = -A^*(X) + \frac{1}{2}\mu Y$ *and* $\widetilde{\nabla}_Y \xi = -A^*(Y) - \frac{1}{2}\mu X$.

Proof. From proposition 2.4, we have $\widetilde{g}(D^L(X,\xi),X) = g(D^L(X,\xi),X) = 0$ so

$$D^L(X,\xi) = \widetilde{g}(D^L(X,\xi),Y)Y + \widetilde{g}(D^L(X,\xi),\xi)\xi.$$

Using again proposition 2.4 we get $\widetilde{g}(D^L(X,\xi),Y) = -\frac{1}{2}\mu$. On the other hand, taking into account equation (2.1) and that ξ is $\widetilde{g}$-unitary, we have

$$\widetilde{g}(D^L(X,\xi),\xi) = \widetilde{g}(\nabla^L_X\xi - \widetilde{\nabla}_X\xi,\xi) = -\tau(X).$$

In the same way we get $D^L(Y,\xi) = \frac{1}{2}\mu X - \tau(Y)\xi$. The final assertion follows from equation (2.2). □

Now we relate the Ricci curvature of ξ in the ambient Lorentzian manifold and in $(L,\widetilde{g})$. For this, recall the general formula relating the curvature tensor associated to the two connections ∇^L and $\widetilde{\nabla}$ on L.

$$R^L_{UV}W - \widetilde{R}_{UV}W = \left(\nabla_U D^L\right)(V,W) - \left(\nabla_V D^L\right)(U,W) + D^L(U,D^L(V,W)) - D^L(V,D^L(U,W)) \tag{3.2}$$

for all $U,V,W \in \mathfrak{X}(M)$.

Theorem 3.8. *Let L be an oriented null hypersurface in a four dimensional Lorentzian manifold. It holds*

$$\widetilde{Ric}(\xi,\xi) = Ric(\xi,\xi) - \tau(\xi)H + \widetilde{\operatorname{div}}\widetilde{\nabla}_\xi\xi + \frac{1}{2}\mu^2.$$

Proof. Using equation (2.4) and (3.2) we get

$$\widetilde{g}\left(R_{X\xi}\xi - \widetilde{R}_{X\xi}\xi,X\right) = \widetilde{g}\left((\widetilde{\nabla}_X D^L)(\xi,\xi),X\right) - \widetilde{g}\left((\widetilde{\nabla}_\xi D^L)(X,\xi),X\right) + \widetilde{g}\left(D^L(X,D^L(\xi,\xi)),X\right) - \widetilde{g}\left(D^L(\xi,D^L(X,\xi)),X\right). \tag{3.3}$$

We compute each term in the right hand of this equation. Using corollary 2.6, lemma 3.7 and equation (2.7), the first one is

$$\begin{aligned}\widetilde{g}\left((\widetilde{\nabla}_X D^L)(\xi,\xi),X\right) &= \widetilde{g}(\widetilde{\nabla}_X D^L(\xi,\xi),X) - 2\widetilde{g}(D^L(\widetilde{\nabla}_X\xi,\xi),X)\\ &= -\tau(\xi)\widetilde{g}(\widetilde{\nabla}_X\xi,X) - \widetilde{g}(\widetilde{\nabla}_X\widetilde{\nabla}_\xi\xi,X) - 2\sigma_{12}\widetilde{g}(D^L(Y,\xi),X)\\ &= \tau(\xi)g(A^*(X),X) - \widetilde{g}(\widetilde{\nabla}_X\widetilde{\nabla}_\xi\xi,X) - \sigma_{12}\mu.\end{aligned}$$

Since $X,\widetilde{\nabla}_\xi\xi \in \Gamma(\mathcal{S})$, from corollary 2.6 we can write the second term in (3.3) as

$$\begin{aligned}\widetilde{g}\left((\widetilde{\nabla}_\xi D^L)(X,\xi),X\right) &= -\widetilde{g}(D^L(X,\xi),\widetilde{\nabla}_\xi X) - \widetilde{g}(D^L(\widetilde{\nabla}_\xi X,\xi),X)\\ &= -\widetilde{g}(D^L(X,\xi),\widetilde{\nabla}_\xi X) - g(D^L(\widetilde{\nabla}_\xi X,\xi),X).\end{aligned}$$

But from the definition of $\widetilde{g}$ and lemma 3.7 we get

$$\begin{aligned}-\widetilde{g}(D^L(X,\xi),\widetilde{\nabla}_\xi X) &= -g(D^L(X,\xi),\widetilde{\nabla}_\xi X) - \widetilde{g}(D^L(X,\xi),\xi)\widetilde{g}(\widetilde{\nabla}_\xi X,\xi)\\ &= -g(D^L(X,\xi),\widetilde{\nabla}_\xi X) - \tau(X)\widetilde{g}(X,\widetilde{\nabla}_\xi\xi).\end{aligned}$$

Therefore the second term in (3.3) reduces to

$$\widetilde{g}\left((\widetilde{\nabla}_\xi D^L)(X,\xi),X\right)=-g(D^L(X,\xi),\widetilde{\nabla}_\xi X)-\tau(X)\widetilde{g}(X,\widetilde{\nabla}_\xi\xi)-g(D^L(\widetilde{\nabla}_\xi X,\xi),X)$$

and using proposition 2.4 the above is

$$\begin{aligned}&\frac{1}{2}\left(-\widetilde{g}(X,\widetilde{\nabla}_\xi\xi)(L_\xi g)(X,\xi)+d\omega(X,\widetilde{\nabla}_\xi X)\right)-\tau(X)\widetilde{g}(X,\widetilde{\nabla}_\xi\xi)\\&\quad+\frac{1}{2}\left(-\widetilde{g}(X,\widetilde{\nabla}_\xi\xi)d\omega(\xi,X)+d\omega(\widetilde{\nabla}_\xi X,X)\right)\\&=-\widetilde{g}(X,\widetilde{\nabla}_\xi\xi)^2-\tau(X)\widetilde{g}(X,\widetilde{\nabla}_\xi\xi).\end{aligned}$$

The third term $\widetilde{g}(X,D^L(\xi,\xi),X)$ vanishes due to corollary 2.6. For the last term in formula (3.3) we use lemma 3.7 to get

$$\begin{aligned}\widetilde{g}(D^L(\xi,D^L(X,\xi)),X)&=-\frac{1}{2}\mu\widetilde{g}(D^L(\xi,Y),X)-\tau(X)\widetilde{g}(D^L(\xi,\xi),X)\\&=-\frac{1}{4}\mu^2+\tau(X)\widetilde{g}(X,\widetilde{\nabla}_\xi\xi).\end{aligned}$$

Putting all together we arrive to

$$\widetilde{g}\left(R_{X\xi}\xi-\widetilde{R}_{X\xi}\xi,X\right)=\tau(\xi)g(A^*(X),X)-\widetilde{g}(\widetilde{\nabla}_X\widetilde{\nabla}_\xi\xi,X)+\widetilde{g}(X,\widetilde{\nabla}_\xi\xi)^2-\sigma_{12}\mu+\frac{1}{4}\mu^2.$$

In the same way, we can check that

$$\widetilde{g}\left(R_{Y\xi}\xi-\widetilde{R}_{Y\xi}\xi,Y\right)=\tau(\xi)g(A^*(Y),Y)-\widetilde{g}(\widetilde{\nabla}_Y\widetilde{\nabla}_\xi\xi,Y)+\widetilde{g}(Y,\widetilde{\nabla}_\xi\xi)^2+\sigma_{21}\mu+\frac{1}{4}\mu^2$$

and so

$$\begin{aligned}\widetilde{Ric}(\xi,\xi)&=\widetilde{g}(\widetilde{R}_{X\xi}\xi,X)+\widetilde{g}(\widetilde{R}_{Y\xi}\xi,Y)\\&=Ric(\xi,\xi)-\tau(\xi)H+\widetilde{\operatorname{div}}\widetilde{\nabla}_\xi\xi+(\sigma_{12}-\sigma_{21})\mu-\frac{1}{2}\mu^2\\&=Ric(\xi,\xi)-\tau(\xi)H+\widetilde{\operatorname{div}}\widetilde{\nabla}_\xi\xi+\frac{1}{2}\mu^2.\end{aligned}$$

□

We call $S:TL\to TL$ the endomorphism field given by $S(U)=\widetilde{\nabla}_U\xi$ and S^* its adjoint with respect to $\widetilde{g}$, i.e. $S^*:TL\to TL$ is the unique endomorphism such that $\widetilde{g}(S(U),V)=\widetilde{g}(U,S^*(V))$ for all $U,V\in\mathfrak{X}(L)$. We say that the rigged vector field ξ is orthogonally normal if

$$\widetilde{g}(S(X),S(X))=\widetilde{g}(S^{*\perp}(X),S^{*\perp}(X))$$

for all $X\in\Gamma(\mathcal{S})$, being $S^{*\perp}(X)$ the $\widetilde{g}$-orthogonal projection of $S^*(X)$ onto $\mathcal{S}$. For example, if the screen distribution $\mathcal{S}$ is integrable or if L is totally umbilical, then ξ is orthogonally normal.

If we have a null hypersurface in a Lorentzian manifold with arbitrary dimension and the rigged vector field ξ is orthogonally normal, then we can also get an explicit expression for $\widetilde{Ric}(\xi,\xi)$, which is similar (but not the same) to the one given in theorem 3.8, [11, Corollary 4.5]. This has been used to prove that, under some additional conditions, a totally umbilical null hypersurface is in fact totally geodesic, [11, Theorem 5.1].

In [2] some obstructions for the existence of a compact null hypersurface under a causality hypothesis are given. Concretely, it is proven that in a distinguishing, strongly causal, stably causal or globally hyperbolic Lorentzian manifolds there are not compact null submanifolds. However, an example of a causal Lorentzian manifold which contains a compact null hypersurface is shown. We can give an obstruction to the existence of compact null hypersurface in a causal Lorentzian manifold adding a curvature condition.

Corollary 3.9. *Let L be an oriented null hypersurface in a four-dimensional causal Lorentzian manifold. If there is a conformal rigging for L such that $\tau(X) = 0$ for all $X \in \Gamma(\mathcal{S})$ and $0 < Ric(\xi, \xi) + \frac{1}{2}\mu^2$, then L is not compact.*

Proof. Since ζ is conformal, then $\tau(\xi) = 0$ and $\tau(X) = -\frac{1}{2}\widetilde{g}(\widetilde{\nabla}_\xi \xi, X)$ for all $X \in \Gamma(\mathcal{S})$, [11]. Therefore, $\widetilde{\nabla}_\xi \xi = 0$ and theorem 3.8 gives us that

$$\widetilde{Ric}(\xi, \xi) = Ric(\xi, \xi) + \frac{1}{2}\mu^2 > 0.$$

If L is compact, then using [1, Corollary 1] or [16, Proposition 1], we have that ω is a contact form on L. Moreover, its Reeb vector field is ξ because it is $\widetilde{g}$-geodesic. From [21], ξ has a closed integral curve, which is not possible because (M, g) is causal. □

If the rigging ζ is closed, i.e. the one-form α is closed, then the endomorphism S coincides with $-A^*$ [11], but in general it has more information than A^*.

Lemma 3.10. *Let L be an orientable null hypersurface in a four-dimensional Lorentzian manifold. It holds*

$$||S||^2 = \widetilde{g}(\widetilde{\nabla}_\xi \xi, \widetilde{\nabla}_\xi \xi) + ||A^*||^2 + \frac{1}{2}\mu^2,$$
$$trace\,(S^2) = ||A^*||^2 - \frac{1}{2}\mu^2.$$

Proof. Take $\{X, Y, \xi\}$ a local $\widetilde{g}$-orthonormal positive basis. From equation (2.7) and lemma 3.7 we get

$$\widetilde{g}(S^2(X), X) + \widetilde{g}(S(X), S(X)) = 2g(A^*(X), A^*(X)) - \mu g(A^*(X), Y)$$

and analogously

$$\widetilde{g}(S^2(Y), Y) + \widetilde{g}(S(Y), S(Y)) = 2g(A^*(Y), A^*(Y)) + \mu g(A^*(X), Y).$$

Therefore

$$trace\,(S^2) + ||S||^2 = \widetilde{g}(\widetilde{\nabla}_\xi \xi, \widetilde{\nabla}_\xi \xi) + 2||A^*||^2.$$

On the other hand, using equation (3.1) it is straightforward to check that

$$||S||^2 - trace\,(S^2) = \widetilde{g}(\widetilde{\nabla}_\xi \xi, \widetilde{\nabla}_\xi \xi) + \mu^2$$

and we get the lemma. □

Now, we are ready to prove the main result of this paper.

Theorem 3.11. *Let L be an oriented compact null hypersurface in a four-dimensional Lorentzian manifold. Suppose that there is a rigging for L such that $\tau(\xi)$ is constant through the integral curves of ξ. If $\frac{1}{2}H^2 \leq Ric(\xi,\xi)$, then L is totally geodesic.*

Proof. We integrate with respect to $\widetilde{g}$ in the formula for $\widetilde{Ric}(\xi,\xi)$ in theorem 3.8. Taking into account equation (2.10) we get

$$\int_L \left(trace\,(S)^2 - trace\,(S^2)\right) d\widetilde{g} = \int_L \left(Ric(\xi,\xi) - \tau(\xi)H + \frac{1}{2}\mu^2\right) d\widetilde{g},$$

From equation (2.7) we know that $tr(S) = -H$ and using lemma 3.10 the above equation reduces to

$$\int_L \left(H^2 - ||A^*||^2\right) d\widetilde{g} = \int_L \left(Ric(\xi,\xi) - \tau(\xi)H\right) d\widetilde{g}. \tag{3.4}$$

Using equation (2.8), we have $\widetilde{\operatorname{div}}\,(\tau(\xi)\xi) = \xi(\tau(\xi)) - \tau(\xi)H$, but $\xi(\tau(\xi)) = 0$ because $\tau(\xi)$ is constant through the integral curves of ξ and so

$$\int_L \tau(\xi)H d\widetilde{g} = 0. \tag{3.5}$$

The inequality (2.3) and the hypothesis of the theorem give us

$$0 \leq \int_L \left(Ric(\xi,\xi) - \frac{1}{2}H^2\right) d\widetilde{g} \leq 0.$$

Thus from the equation (3.4) we deduce that

$$0 \leq \int_L \left(||A^*||^2 - \frac{1}{2}H^2\right) d\widetilde{g} = \int_L \left(\frac{1}{2}H^2 - Ric(\xi,\xi)\right) d\widetilde{g} \leq 0.$$

Therefore, $Ric(\xi,\xi) = \frac{1}{2}H^2$ and $||A^*||^2 = \frac{1}{2}H^2$.

Taking into account equation (2.11) we have that the null mean curvature holds the Riccati differential equation

$$\tau(\xi)H + \xi(H) - H^2 = 0.$$

This equation can be explicitly solved. In fact, if we take $\gamma : \mathbb{R} \to L$ an integral curve of ξ with $\gamma(0) = p \in L$, then

$$H_{\gamma(t)} = \begin{cases} \frac{H_p}{1-H_p t} \text{ if } \tau(\xi)_p = 0 \\ \frac{H_p\tau(\xi)}{H_p + (\tau(\xi)-H_p)e^{\tau(\xi)t}} \text{ if } \tau(\xi)_p \neq 0 \end{cases} \tag{3.6}$$

If $\tau(\xi)_p H_p < 0$, then $H_{\gamma(t)}$ is not defined for

$$t = \frac{1}{\tau(\xi)} \ln \frac{H_p}{H_p - \tau(\xi)}.$$

Therefore, we deduce that $0 \leq \tau(\xi)H$ on the whole L. From equation (3.5) we get that $\tau(\xi)H = 0$, but in view of (3.6), we have that necessarily $H = 0$. Proposition 3.1 ensures that L is totally geodesic. □

The following example holds the assumption in theorem 3.11.

Example 3.12. Take $M=\mathbb{R}\times\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1$ and

$$g=2t\cos y\left(dt^2-dx^2\right)-2dtdx+dy^2+dz^2.$$

The hypersurface $L=\{0\}\times\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1$ is null and totally geodesic. The vector field $\zeta=-\partial t$ is a rigging for it with associated rigged $\xi=\partial x$ and

$$\tau(\xi)=-g(\nabla_\xi\xi,\zeta)=g(\nabla_{\partial x}\partial x,\partial t)=\cos y,$$

which is constant through the integral curves of ξ. It is worth noting that the null geodesics of L are not complete. Indeed, the null geodesic through a point $p_0=(0,x_0,y_0,z_0)\in L$ is

$$\gamma(t)=\begin{cases}\left(0,x_0-\frac{1}{\cos y_0}\ln\left(1+t\cos y_0\right),y_0,z_0\right) \text{ for } -\frac{1}{\cos y_0}<t, \text{ if } \cos y_0>0\\ (0,x_0+t,y_0,z_0) \text{ for } t\in\mathbb{R}, \text{ if } \cos y_0=0\\ \left(0,x_0-\frac{1}{\cos y_0}\ln\left(1+t\cos y_0\right),y_0,z_0\right) \text{ for } t<-\frac{1}{\cos y_0}, \text{ if } \cos y_0<0\end{cases}$$

Observe that we can not prove theorem 3.11 as theorem 3.4. This is because given an integral curve of ξ, a reparametrization will be a future or past complete geodesic depending on the sign of $\tau(\xi)$, which can change from one integral curve of ξ to another integral curve of ξ.

On the other hand, under the conditions of the theorem 3.11, from equation (2.11) and inequality (2.3) we have that

$$0\leq\tau H_{\gamma(t)}+\frac{d}{dt}H_{\gamma(t)}-H^2_{\gamma(t)},$$

where γ is an integral curve of ξ and τ is some constant. From this differential inequality we can not deduce that $H=0$. In fact, there are complete solutions apart from the constant zero function or the constant function $H=\tau$ as for example $H(t)=\frac{\tau}{2}+\frac{\tau}{3}\cos\left(\frac{\tau}{3}t\right)$.

A null hypersurface in a three-dimensional Lorentzian manifold is always totally umbilical. In this case we can prove the following.

Theorem 3.13. *Let L be an oriented compact null hypersurface in a three-dimensional Lorentzian manifold. Suppose that there is a rigging for L such that $\tau(\xi)$ is constant through the integral curves of ξ. If $cH^2\leq Ric(\xi,\xi)$ for certain $c>0$, then L is totally geodesic.*

Proof. The existence of the rigged vector ξ implies that the Euler-Poincaré characteristic of L is zero. Therefore, the Gauss-Bonnet theorem ensures that

$$\int\widetilde{Ric}(\xi,\xi)d\widetilde{g}=0.$$

Alternatively, we can easily check that $trace\,(S^2)=trace\,(S)^2$, so equation (2.10) also gives us $\int\widetilde{Ric}(\xi,\xi)d\widetilde{g}=0$.

We can prove as in theorem 3.8 that

$$\widetilde{Ric}(\xi,\xi)=Ric(\xi,\xi)-\tau(\xi)H+\widetilde{\operatorname{div}}\widetilde{\nabla}_\xi\xi.$$

Thus

$$\int cH^2d\widetilde{g}\leq\int Ric(\xi,\xi)d\widetilde{g}=0$$

and therefore $H=0$. Now we apply proposition 3.1. □

Acknowledgements

This paper has been partially supported by the Ministry of Sciences and Innovation (MICINN) I+D+I grant number PID2020-118452GB-I00.

Funding Funding for open access publishing: Universidad Málaga/CBUA.

Data Availability Statement Not applicable.

Declarations

Conflict of Interest The author has no competing interests to declare that are relevant to the content of this article.

References

[1] Aazami, A.B.: The Newman-Penrose formalism for Riemannian 3-manifolds. J. Geom. Phys. **94**, 1–7 (2015)

[2] Atindogbé, C., Gutiérrez, M., Hounnonkpe, R.: Compact null hypersurfaces in Lorentzian manifolds. Adv. Geom. **21**, 251–263 (2021)

[3] Atindogbé, C., Olea, B.: Conformal vector fields and null hypersurfaces. Results Math. **77**, 1–22 (2022)

[4] Chen, B.Y.: On a theorem of Fenchel-Borsuk-Willmore-Chern-Lashof. Math. Ann. **194**, 19–26 (1971)

[5] Chen, B.Y.: Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions. Glasgow Math. J. **41**, 33–41 (1999)

[6] Chen, B.Y.: Pseudo-Riemannian geometry, δ-invariants and applications. Word Scientific Publishing, Singapore (2011)

[7] Duggal, K.L., Bejancu, A.: Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. Kluwer Academic Publishers (1996)

[8] Flores, J.L., Javaloyes, M.A., Piccione, P.: Periodic geodesics and geometry of compact Lorentzian manifolds with a Killing vector field. Math. Z. **267**, 221–233 (2011)

[9] Galloway, G.J.: Null geometry and the Einstein equations, pp. 379–400. Birkhäuser, Basel, The Einstein equations and the large scale behavior of gravitational fields (2004)

[10] Galloway, G.J.: Maximum principles for null hypersurfaces and null splitting theorems. Ann. Henri Poincaré **1**, 543–567 (2000)

[11] Gutiérrez, M., Olea, B.: Induced Riemannian structures on null hypersurfaces. Math. Nachr. **289**, 1219–1236 (2016)

[12] Gutiérrez, M., Olea, B.: Characterization of null cones under a Ricci curvature condition J. Math. Anal. Appl. **508** (2022), Paper No. 125906

[13] Gutiérrez, M., Olea, B.: Codimension two spacelike submanifolds through a null hypersurface in a Lorentzian manifold. Bull. Malays. Math. Sci. Soc. **44**, 2253–2270 (2021)

[14] Gutiérrez, M., Olea, B.: Conditions on a null hypersurface of a Lorentzian manifold to be a null cone. J. Geom. Phys. **145**103469, 9 pp (2019)

[15] Gutiérrez, M., Olea, B.: The rigging technique for null hypersurfaces. Axioms **10**(4), 284 (2021)

[16] Harris, A., Paternain, G.P.: Conformal great circles flows on the 3-sphere. Proc. Am. Math. Soc. **144**(4), 1725–1734 (2016)

[17] Kupeli, D.N.: On null submanifolds in spacetimes. Geom. Dedicata. **23**, 33–51 (1987)

[18] Leistner, T., Schliebner, D.: Completeness of compact Lorentzian manifolds with abelian holonomy. Math. Ann. **364**, 1469–1503 (2016)

[19] Ngakeu, F., Tetsing, H.F., Olea, B.: Rigging technique for 1-lightlike submanifolds and preferred rigged connections. Mediterr. J. Math. **16**, 139 (2019)

[20] Olea, B.: Canonical variation of a Lorentzian metric. J. Math. Anal. Appl. **419**(1), 156–171 (2014)

[21] Taubes, C.H.: The Seiberg-Witten equations and the Weinstein conjecture. Geom. Topol. **11**, 2117–2202 (2007)

[22] Romero, A., Sánchez, M.: Completeness of compact Lorentz manifolds admitting a timelike conformal Killing field. Proc. Am. Math. Soc. **123**, 2831–2833 (1995)

[23] Willmore, T.J.: Mean curvature of immersed surfaces. Ann. Sti. Univ. “Al. I. Cuza” Iasi, Sec. I **14**, 99–103 (1968)

Benjamín Olea
Dpto. Matemática Aplicada
Universidad de Málaga
Málaga
Spain
e-mail: benji@uma.es

Received: March 29, 2022.
Revised: September 7, 2022.
Accepted: January 5, 2023.