POINTWISE MULTIPLIERS BETWEEN SPACES OF ANALYTIC FUNCTIONS

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ABSTRACT. A Banach space X of analytic function in $\mathbb{D}$, the unit disc in $\mathbb{C}$, is said to be admissible if it contains the polynomials and convergence in X implies uniform convergence in compact subsets of $\mathbb{D}$.

If X and Y are two admissible Banach spaces of analytic functions in $\mathbb{D}$ and g is a holomorphic function in $\mathbb{D}$, g is said to be a multiplier from X to Y if $g \cdot f \in Y$ for every $f \in X$. The space of all multipliers from X to Y is denoted $M(X, Y)$, and $M(X)$ will stand for $M(X, X)$.

The closed graph theorem shows that if $g \in M(X, Y)$ then the multiplication operator M_g, defined by $M_g(f) = g \cdot f$, is a bounded operator from X into Y.

Is is known that $M(X) \subset H^\infty$ and that if $g \in M(X)$, then $\|g\|_{H^\infty} \leq \|M_g\|$. Clearly, this implies that $M(X, Y) \subset H^\infty$ if $Y \subset X$. If $Y \not\subset X$, the inclusion $M(X, Y) \subset H^\infty$ may not be true.

In this paper we start presenting a number of conditions on the spaces X and Y which imply that the inclusion $M(X, Y) \subset H^\infty$ holds. Next, we concentrate our attention on multipliers acting an $BMOA$ and some related spaces, namely, the Q_s-spaces ($0 < s < \infty$).

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$ and let $\mathrm{Hol}(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence in compact subsets.

Let us recall that if $f \in \mathrm{Hol}(\mathbb{D})$ and $0 \leq r < 1$, then the integral means $M_p(r, f)$ of f are defined by $M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right)^{1/p}$, if $0 < p < \infty$, and $M_\infty(r, f) = \max_{|z|=r} |f(z)|$. For $0 < p \leq \infty$ the Hardy space H^p consists of those functions f, analytic in $\mathbb{D}$, for which

$$||f||_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty.$$

We refer to [17] for the theory of Hardy spaces.

Throughout the paper we shall be dealing with linear subspaces of $\mathrm{Hol}(\mathbb{D})$ which are Banach spaces of F-spaces, contain the polynomials, and are continuously embedded in $\mathrm{Hol}(\mathbb{D})$. Such spaces X will be said to be admissible.

The Hardy spaces are admissible and so are most of the relevant classical spaces of analytic functions in $\mathbb{D}$ such as the weighted Bergman spaces A^p_α and the weighted Dirichlet spaces $\mathcal{D}^p_\alpha$ ($p > 0$, $\alpha - 1$); the space $BMOA$; the Bloch space $\mathcal{B}$; the Q^s-spaces ($0 < s < \infty$); the mean Lipschitz spaces Λ^p_α ($1 \leq p \leq \infty$,

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$0 < \alpha \le 1$). Regarding the definitions, notations, and theory of these spaces, we mention [18, 26, 46] for the Bergman spaces, [6] for the Dirichlet-type spaces, [22] for $BMOA$, [4] for the Bloch space, [39, 40] for the Q_s-spaces, and [11, 17] for the mean Lipschitz spaces.

If $g \in \text{Hol}(\mathbb{D})$, the multiplication operator M_g is defined by

$$M_g(f)(z) \stackrel{\text{def}}{=} g(z)f(z), \quad f \in \text{Hol}(\mathbb{D}),\ z \in \mathbb{D}.$$

If $g \in \text{Hol}(\mathbb{D})$ and X and Y are two admissible subspaces of $\text{Hol}(\mathbb{D})$, g is said to be a multiplier from X to Y if $M_g(X) \subset Y$. The space of all multipliers from X to Y will be denoted by $M(X, Y)$ and $M(X)$ will stand for $M(X, X)$. Using the closed graph theorem we see that if $g \in M(X, Y)$ then M_g is continuous from X into Y.

If X is an admissible subspace of $\text{Hol}(\mathbb{D})$ then it is well known that

$$M(X) \subset H^\infty, \tag{1.1}$$

(see e. g. [2] and [37]). Arévalo and Vukotić [7] have recently characterized the spaces X for which $M(X) = H^\infty$.

Using (1.1) we readily see that if X and Y are two admissible subspaces of $\text{Hol}(\mathbb{D})$ then

$$X \subset Y \implies M(X, Y) \subset H^\infty. \tag{1.2}$$

The inclusion $M(X, Y) \subset H^\infty$ is not true in general. For example, if $0 < p < \infty$ and $\alpha > -1$, we have

$$M(H^\infty, H^p) = H^p \not\subset H^\infty, \quad M(H^\infty, A^p_\alpha) = A^p_\alpha \not\subset H^\infty.$$

In Section 2 we consider the question of finding conditions on the admissible spaces X and Y which imply that $M(X, Y) \subset H^\infty$ even if $Y \not\subset X$. In Section 3 we will deal with questions relative to multipliers of the space $BMOA$ and, more generally, on the Q_s-spaces.

Let us close this section by saying that, as usual, we shall be using the convention that $C = C(p, \alpha, q, \beta, \dots)$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta \dots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Furthermore, for two real-valued functions K_1, K_2 we write $K_1 \lesssim K_2$, or $K_1 \gtrsim K_2$, if there exists a positive constant C independent of the arguments such that $K_1 \le CK_2$, respectively $K_1 \ge CK_2$. If we have $K_1 \lesssim K_2$ and $K_1 \gtrsim K_2$ simultaneously, then we say that K_1 and K_2 are equivalent and we write $K_1 \asymp K_2$.

2. Conditions which imply the inclusion of $M(X, Y)$ in H^∞

We let $\text{Aut}(\mathbb{D})$ denote the set of all disc automorphisms, that is, of all conformal mappings φ from $\mathbb{D}$ onto itself. It is well known that $\text{Aut}(\mathbb{D})$ coincides with the set of all Möbius transformations from $\mathbb{D}$ onto itself:

$$\text{Aut}(\mathbb{D}) = \{\lambda\varphi_a : |\lambda| = 1,\ a \in \mathbb{D}\},$$

where $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$ $(z \in \mathbb{D})$.

A linear subspace X of $\mathrm{Hol}(\mathbb{D})$ is said to be *conformally invariant* or *Möbius invariant* if whenever $f \in X$, then also $f \circ \varphi \in X$ for any $\varphi \in \mathrm{Aut}(\mathbb{D})$ and, moreover, X is equipped with a semi-norm ρ which is not identically zero on X for which there exists a positive constant C such that

$$\rho(f \circ \varphi) \leq C\rho(f), \quad \text{whenever } f \in X \text{ and } \varphi \in \mathrm{Aut}(\mathbb{D}).$$

Following [35], if the closed ball $X_1 = \{f \in X : \rho(f) \leq 1\}$ is a closed subset of $\mathrm{Hol}(\mathbb{D})$, we say that X is "natural". We mention [5], [33] and [46] as general references for the theory of Möbius invariant spaces which have been extensively studied in recent years (see, e.g., [3, 9, 15, 16, 24, 35, 45, 46]).

If X is an admissible subspace of $\mathrm{Hol}(\mathbb{D})$ then the point evaluation functionals are continuous in X. Then using the results [35, Proposition 1.8] and [33] we have the following result.

If X is an admissible and natural conformally invariant space of analytic functions in $\mathbb{D}$, then

$$B^1 \subset X \subset \mathcal{B}$$

and the inclusions mappings are continuous.

The following spaces are admissible and natural conformally spaces.

• The Bloch space $\mathcal{B}$ which consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\rho_{\mathcal{B}}(f) \stackrel{\mathrm{def}}{=} \sup_{z \in \mathbb{D}} (1 - |z|^2)\, |f'(z)| < \infty.$$

The Schwarz-Pick lemma easily implies that $\rho_{\mathcal{B}}$ is a conformally invariant semi-norm, thus $\mathcal{B}$ is a conformally invariant space. It is also a Banach space with the norm $\|\cdot\|_{\mathcal{B}}$ defined by $\|f\|_{\mathcal{B}} = |f(0)| + \rho_{\mathcal{B}}(f)$.

• The space $BMOA$ which consists of those $f \in H^1$ whose boundary values have bounded mean oscillation. Alternatively, a function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to $BMOA$ if and only if

$$\|f\|_{BMOA} \stackrel{\mathrm{def}}{=} |f(0)| + \|f\|_{\star} < \infty,$$

where

$$\|f\|_{\star} \stackrel{\mathrm{def}}{=} \sup_{T \in \mathrm{Aut}(\mathbb{D})} \|f \circ T - f(T(0))\|_{H^2}.$$

The seminorm $\|\cdot\|_{\star}$ is conformally invariant.

• The Besov spaces B^p, $1 \leq p < \infty$, which are defined as follows:

If $1 < p < \infty$, $B^p = \mathcal{D}^p_{p-2}$. Thus a function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to B^p if and only if

$$\|f\|_{B^p} \stackrel{\mathrm{def}}{=} |f(0)| + \rho_p(f) < \infty,$$

where

$$\rho_p(f) = \left(\int_{\mathbb{D}} (1 - |z|^2)^{p-2} |f'(z)|^p \, dA(z) \right)^{1/p}.$$

All B^p spaces $(1 < p < \infty)$ are conformally invariant with respect to the semi-norm ρ_p.

The minimal Besov space B^1 consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that $f'' \in A^2$. Another definition of B^1 is given in [5]. Using this definition, it becomes clear that B^1 is conformally invariant (see also [9]).

• The Q_s-spaces, $0 < s < \infty$. A function $f \in \mathrm{Hol}(\mathbb{D})$ is said to belong to Q_s if

$$\|f\|_{Q_s} \stackrel{\text{def}}{=} |f(0)| + \rho_{Q_s}(f) < \infty,$$

where

$$\rho_{Q_s}(f) \stackrel{\text{def}}{=} \sup_{T\in \mathrm{Aut}(\mathbb{D})} \left(\int_{\mathbb{D}} |f'(z)|^2 (1-|T(z)|^2)^s \, dA(z) \right)^{1/2} < \infty.$$

For $0 < s < \infty$, the space Q_s is conformally invariant with respect to the seminrm ρ_{Q_s}. Let us recall that $Q_s = \mathcal{B}$ for all $s > 1$ and that $Q_1 = BMOA$. The Q_s spaces, $0 < s < 1$, form a family of proper subspaces of $BMOA$, which increase with s.

The authors proved in [24] the following result.

Theorem A. *If X and Y are two admissible and natural conformally invariant spaces of analytic functions in $\mathbb{D}$ then $M(X, Y) \subset H^\infty$.*

Next we are going to obtain another result of the same kind. In order do so we need to introduce the class of normal functions. A function $f \in \mathrm{Hol}(\mathbb{D})$ is said to be a normal function in the sense of Lehto and Virtanen [29] if

$$\sup_{z\in\mathbb{D}} (1-|z|^2) \frac{|f'(z)|}{1+|f(z)|^2} < \infty.$$

The set of all normal holomorphic functions in $\mathbb{D}$ will be denoted by $\mathcal{N}$. It is clear that any Bloch function is a normal function, that is, we have $\mathcal{B} \subset \mathcal{N}$. We refer to [4], [29] and [32] for the theory of normal. In particular, we remark here that if $f \in \mathcal{N}$, $\xi \in \partial\mathbb{D}$ and f has the asymptotic value L at ξthen f has the non-tangential limit L at ξ.

Let us recall that if a sequence of points $\{a_n\}$ in the unit disc satisfies the *Blaschke condition*, $\sum_{n=1}^{\infty}(1-|a_n|) < \infty$, the corresponding Blaschke product B is defined as

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1-\overline{a}_n z}.$$

Such a product is analytic in $\mathbb{D}$. In fact, it is an inner function, that is, an H^∞-function with radial limit of absolute value 1 at almost every point of $\partial\mathbb{D}$ [17, Chapter 2].

The Blaschke sequence $\{a_n\}$ is said to be *uniformly separated* if there exists $\delta > 0$ such that

$$\prod_{m\neq n} \left| \frac{a_n - a_m}{1-\overline{a}_n a_m} \right| \geq \delta, \quad \text{for all } n.$$

In such a case the associated Blaschke product B is an *interpolating Blaschke product.* Equivalently,

$$B \text{ is an interpolating Blaschke product} \iff \inf_{n\geq 1} (1-|a_n|^2)|B'(a_n)| > 0. \tag{2.1}$$

We refer to [17, Chapter 9] and [20, Chapter VII] for the basic properties of interpolating Blaschke products. In particular, we remark that an exponential sequence is uniformly separated and that the converse holds if all the a_k's are positive. Let us recall that a sequence $\{a_n\}$ of points in $\mathbb{D}$ is said to be exponential if there exists $\lambda \in [0, 1)$ such that

$$(1 - |a_{n+1}|) \leq \lambda (1 - |a_{n+}|), \quad \text{for all } n.$$

Lappan [28, Theorem 3] proved that if B is an interpolating Blaschke product and f is a normal analytic function in $\mathbb{D}$, the product $B \cdot f$ need not be normal. Lappan used this to show that $\mathcal{N}$ is not a vector space.

Lappan's result is a consequence of the following easy fact: if B is an interpolating Blaschke product whose sequence of zeros is $\{a_n\}$ and G is an analytic function in $\mathbb{D}$ with $G(a_n) \to \infty$, then $f = B \cdot G$ is not a normal function (and hence it is not a Bloch function either). This result has been used to construct distinct classes of non-normal functions in a good number of articles such as [10, 13, 21, 42, 43] and lead us to see that if X and Y are two admissible subspaces of $\mathrm{Hol}(\mathbb{D})$ such that $Y \subset \mathcal{N}$ and X contains "enough" Blaschke products, then $M(X, Y) \subset H^\infty$. To be precise, we have the following result.

Theorem 2.1. *Let X and Y be two admissible spaces of holomorphic functions in $\mathbb{D}$ with $Y \subset \mathcal{N}$ and such that X contains all the Blaschke products whose sequence of zeros is exponential. Then $M(X, Y) \subset H^\infty$.*

Proof. Take $g \in \mathrm{Hol}(\mathbb{D}) \setminus H^\infty$ and let us see that $g \notin M(X, Y)$.

Take a sequence $\{a_n\} \subset \mathbb{D}$ so that $g(a_n) \to \infty$. Let $\{b_n\}$ be a subsequence of $\{a_n\}$ which is an exponential sequence and let B be the Blaschke product with zeros $\{b_n\}$. Then $B \in X$ and $g \cdot B$ is not normal and, hence, $g \cdot B \notin Y$. This implies that $g \notin M(X, Y)$ as desired. □

The classes of spaces considered in Theorem A and in Theorem 2.1 are somewhat 'close' to each other, but they are different. We have:

- H^∞, $BMOA$ and the Bloch space are natural and admissible conformally invariant spaces which contain all Blaschke products.
- For $0 < s < 1$, the space Q_s is also a natural and admissible conformally invariant space not containing all Blaschke products but which contains all Blaschke products B whose sequence of zeros is exponential [14].
- The Besov spaces B^p $(1 \leq p < \infty)$ are natural and admissible conformally invariant spaces but they do not contain infinite Blaschke products (see, e.g. [15]).

We can find examples of non conformally invariant spaces which contain all Blaschke products with an exponential sequence of zeros among the mean Lipschitz spaces Λ^p_α.

For $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, we let Λ^p_α be the space of those $f \in \mathrm{Hol}(\mathbb{D})$ having a non-tangential limit at almost every point of $\partial\mathbb{D}$ and so that $\omega_p(\cdot, f)$, the integral modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f, satisfies $\omega_p(\delta, f) = \mathrm{O}(\delta^\alpha)$, as $\delta \to 0$. When $p = \infty$ we write Λ_α instead of Λ^∞_α. This is the usual space of order α. More precisely, a function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to Λ_α if and only if it has a continuous extension to the closed unit disc and its boundary values satisfy a Lipschitz condition of order α.

Classical results of Hardy and Littlewood (see [11] and [17, Chapter 5]) show that whenever $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$ we have that $\Lambda^p_\alpha \subset H^p$ and that

$$\Lambda^p_\alpha = \{f \text{ analytic in } \mathbb{D}\colon \, M_p(r, f') = \mathrm{O}\left((1-r)^{\alpha-1}\right), \quad \text{as } r \to 1\right). \tag{2.2}$$

Thus, in particular, Λ^1_1 is the space of those $f \in \mathrm{Hol}(\mathbb{D})$ such that $f' \in H^1$.

Let us recall that $B^p \subset \Lambda^p 1/p$ for all $p \in [1, \infty)$.

The condition $M_p(r, f') = \mathrm{O}\left((1-r)^{-1}\right)$ does not imply that $f \in H^p$ (see [25]). In spite of this we can use (2.2) to define the spaces Λ^p_0. Thus:

For $1 \leq p \leq \infty$ and $0 \leq \alpha \leq 1$, we let Λ^p_α be the space of those $f \in \mathrm{Hol}(\mathbb{D})$ such that $M_p(r, f') = \mathrm{O}\left((1-r)^{\alpha-1}\right)$, as $r \to 1$.

If $1 < p < \infty$ and $(1/p) < \alpha \leq 1$, then $\Lambda^p_\alpha \subset \Lambda_{\alpha-(1/p)}$ and, hence, each function in Λ^p_α has a continuous extension to the closed unit disc [11, p. 88]. This is not true for the space $\Lambda^p_{1/p}$. This follows easily noticing that the function $f(z) = \log(1-z)$ is an unbounded function which belongs to $\Lambda^p_{1/p}$ for all $p \in (1, \infty)$. Bourdon, Shapiro, and Sledd [11] proved that $\Lambda^p_{1/p} \subset BMOA$ for all $p \in (1, \infty)$. Using a result of Verbitskii [36] (see also [1]), we can state the following result.

If either $1 < p < \infty$ and $0 \leq \alpha \leq \frac{1}{p}$, or if $p = 1$ and $0 \leq \alpha < 1$, then the space Λ^p_α is an admissible space which contains all Blaschke products with an exponential sequence of zeros.

3. Multipliers of $BMOA$ and related spaces

The logarithmic Bloch space $\mathcal{B}_{\log}$ is the Banach space of those functions $f \in \mathrm{Hol}(\mathbb{D})$ which satisfy

$$\|f\|_{\log} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)\left(\log\frac{2}{1-|z|^2}\right)|f'(z)| < \infty. \tag{3.1}$$

Brown and Shields [12] proved that

$$M(\mathcal{B}) \,=\, B_{\log} \cap H^\infty.$$

Now we can improve Theorem A as follows.

Theorem 3.1. *Let X and Y be two admissible and natural conformally invariant spaces of analytic functions in $\mathbb{D}$ and suppose that the function L defined by $L(z) = \log\frac{1}{1-z}$ ($z \in \mathbb{D}$) belongs to X. Then $M(X, Y) \subset \mathcal{B}_{log} \cap H^\infty$.*

Proof. Let ρ_X and ρ_Y be the conformally invariant seminorms defining X and Y, respectively. For $\theta \in \mathbb{R}$, set $L_\theta(z) = L(e^{i\theta} z)$ ($z \in \mathbb{D}$). Since (X, ρ_X) is conformally invariant, $L_\theta \in X$ for all $\theta \in \mathbb{R}$ and, furthermore, there exists $C > 0$ such that

$$\rho_X(L_\theta) \,\leq\, C, \quad \theta \in \mathbb{R}.$$

Take $g \in M(X, Y)$. Theorem A implies that $g \in H^\infty$.

The multiplication operator M_g is continuous from (X, ρ_X) into (Y, ρ_Y), Since Y is continuously embedded in the Bloch space, it follows that M_g is continuous from (X, ρ_X) into $(\mathcal{B}, \rho_{\mathcal{B}})$. Then it follows that there exists $A > 0$ such that

$$\rho_{\mathcal{B}}(gL_\theta) \leq A, \quad \theta \in \mathbb{R}. \tag{3.2}$$

Thus,

$$(1-|z|^2)\left|g'(z)\log\frac{1}{1-e^{i\theta}z} + g(z)\frac{e^{i\theta}}{1-e^{i\theta}z}\right| \leq A, \quad z \in \mathbb{D},\ \theta \in \mathbb{R}.$$

Taking $z \in \mathbb{D}$ and choosing $\theta \in \mathbb{R}$ such that $e^{i\theta}z = |z|$, we obtain

$$(1-|z|^2)\left|g'(z)\log\frac{1}{1-|z|} + g(z)\frac{e^{i\theta}}{1-|z|}\right| \leq A, \quad z \in \mathbb{D}.$$

Using this and the fact that $g \in H^\infty$, we deduce that

$$\begin{aligned}(1-|z|^2)|g'(z)|\log\frac{1}{1-|z|} &\leq A + (1-|z|^2)\left|g(z)\frac{e^{i\theta}}{1-|z|}\right| \\ &\leq A + 2\|g\|_{H^\infty}, \quad z \in \mathbb{D}.\end{aligned}$$

Thus $g \in B_{\log}$ and, hence, $g \in B_{\log} \cap H^\infty$. □

Let us remark that the space X in Theorem 3.1 can be taken to any of the Q_s spaces. However, the function L does not belong to any of the Besov spaces B^p $(1 \leq p < \infty)$.

The space $M(BMOA)$ was characterized by Stegenga [34] and by Ortega and Fàbrega [30]. Pau and Peláez [31] and Xiao [41] characterized the spaces $M(Q_s)$ $(0 < s < 1)$ solving a conjecture formulated in [38]. The authors of this article generalized these results obtaining in [24] a characterization of the spaces $M(Q_{s_1}, Q_{s_2})$ $(0 < s_1, s_2 \leq 1)$. Namely, they proved the following result.

Theorem B. *Suppose that* $0 < s_1, s_2 \leq 1$.

- (i) *If* $s_2 < s_1$ *then the only multiplier* g *from* Q_{s_1} *into* Q_{s_2} *is the trivial one,* $g \equiv 0$.
- (ii) *If* $0 < s_1 \leq s_2 \leq 1$, *then*

$$M(Q_{s_1}, Q_{s_2}) = Q_{s_2,\log} \cap H^\infty.$$

Here, for $0 < s \leq 1$, $Q_{s,\log}$ is the space of those $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\sup_{T \in \mathrm{Aut}(\mathbb{D})}\left(\log\frac{2}{1-|T(0)|}\right)^2\int_{\mathbb{D}}|f'(z)|^2(1-|T(z)|^2)^s\,dA(z) < \infty. \tag{3.3}$$

It is difficult to check whether or not a function $g \in \mathrm{Hol}(\mathbb{D})$ satisfies (3.3). Hence, it is desirable to find simple conditions on g which are sufficient to ensure that $g \in M(Q_{s_1}, Q_{s_2})$.

For $a \in \mathbb{D}$, set

$$L_a(z) = \log\frac{1}{1-\bar{a}z}, \quad z \in \mathbb{D}.$$

Using the characterization of $M(BMOA)$ obtained by Ortega and Fàbrega, that is, the result in Theorem B in the case $s_1 = s_2 = 1$, Jevtić and Karapetrović proved in [27] that a function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to $M(BMOA)$ if and only if there exists a positive constant A such that $\|gL_a\|_{BMOA} \le A$ for all $a \in \mathbb{D}$. We extend this result as follows.

Theorem 3.2. *Suppose that $0 < s_1 \le s_2 \le 1$. A function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to $M(Q_{s_1}, Q_{s_2})$ if and only if there exists a positive constant A such that*

$$\|gL_a\|_{Q_{s_2}} \le A, \quad \text{for all } a \in \mathbb{D}. \tag{3.4}$$

Another simple sufficient condition is included in the following theorem.

Theorem 3.3. *Suppose that $0 < s_1 \le s_2 \le 1$ and $\frac{1-s_2}{2} < \alpha \le 1$. Then $\Lambda_\alpha \subset M(Q_{s_1}, Q_{s_2})$.*

In particular, if $0 \le \alpha < 1$, then $\Lambda_\alpha \subset M(BMOA)$.

In order to prove these results we shall use a number of results connecting the Q_s-spaces and the $Q_{s,\log}$-spaces with measures of Carleson type.

If $I \subset \partial\mathbb{D}$ is an interval, $|I|$ will denote the length of I. The *Carleson square* $S(I)$ is defined as $S(I) = \{re^{it} : e^{it} \in I, \quad 1 - \frac{|I|}{2\pi} \le r < 1\}$. Also, for $a \in \mathbb{D}$, the Carleson box $S(a)$ is defined by

$$S(a) = \left\{ z \in \mathbb{D} : 1 - |z| \le 1 - |a|, \left| \frac{\arg(a\bar{z})}{2\pi} \right| \le \frac{1 - |a|}{2} \right\}.$$

If $s > 0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu\left(S(I)\right) \le C|I|^s, \quad \text{for any interval } I \subset \partial\mathbb{D},$$

or, equivalently, if there exists $C > 0$ such that

$$\mu\left(S(a)\right) \le C(1 - |a|)^s, \quad \text{for all } a \in \mathbb{D}.$$

These concepts were generalized in [44] as follows: If μ is a positive Borel measure in $\mathbb{D}$, $0 \le \alpha < \infty$, and $0 < s < \infty$, we say that μ is an α-logarithmic s-Carleson measure if there exists a positive constant C such that

$$\frac{\mu\left(S(I)\right)\left(\log \frac{2\pi}{|I|}\right)^\alpha}{|I|^s} \le C, \quad \text{for any interval } I \subset \partial\mathbb{D}$$

or, equivalently, if

$$\sup_{a \in \mathbb{D}} \frac{\mu\left(S(a)\right)\left(\log \frac{2}{1-|a|^2}\right)^\alpha}{(1 - |a|^2)^s} < \infty.$$

The Q_s-spaces can be characterized in terms of Carleson measures.

Theorem C. *If $0 < s \le 1$ and $f \in \mathrm{Hol}(\mathbb{D})$, then $f \in Q_s$ if and only if the Borel measure μ on D defined by*

$$d\mu(z) \,=\, (1 - |z|^2)^s |f'(z)|^2 \, dA(z)$$

is an s-Carleson measure. Furthermore

$$\|f\|^2_{Q_s} \,\asymp\, \sup_{a \in \mathbb{D}} \mu\left((S(a))(1 - |a|)^{-s}\right).$$

Let us remark for $s = 1$ this is essentially equivalent to the Fefferman's duality theorem $(H^1)^\star = BMOA$ (see [22]) and that for $0 < s < 1$ this was proved in [8].

The analogue of Theorem C for the $Q_{s,\log}$-spaces is the following (see [24]).

Theorem D. *If $0 < s \leq 1$ and $f \in \mathrm{Hol}(\mathbb{D})$, then $f \in Q_{s,\log}$ if and only if the Borel measure μ on D defined by*

$$d\mu(z) \;=\; (1-|z|^2)^s |f'(z)|^2 \, dA(z)$$

is an 2-logarithmic s-Carleson measure.

Proof of Theorem 3.2. Suppose (3.4). Since Q_{s_2} is continuously embedded into $BMOA$, there exists $B > 0$ such that

$$\|gL_a\|_{BMOA} \leq B, \quad \text{for all } a \in \mathbb{D}. \tag{3.5}$$

Now, (see e. g. [22, p. 95]) it is well known that there exists a positive constant M such that, for every $f \in BMOA$,

$$|f(z)| \leq M\|f\|_{BMOA} \log \frac{2}{1-|z|}, \quad z \in \mathbb{D}.$$

This and (3.5) gives that

$$\left| g(z) \log \frac{1}{1-\bar{a}z} \right| \;\leq\; MB \log \frac{2}{1-|z|}, \quad a, z \in \mathbb{D}.$$

Taking $z = a$ we obtain that $g \in H^\infty$.

It remains to prove that $g \in Q_{s_{s_2},\log}$. This is equivalent to saying that the measure μ on $\mathbb{D}$ defined by

$$d\mu(z) = (1-|z|^2)^s |g'(z)|^2 \, dA(z)$$

is a 2-logarithmic s_2-Carleson measure. Clearly, (3.4) implies that $g \in Q_{s_2}$ and, hence,

$$\mu \text{ is an } s_2\text{-Carleson measaure.} \tag{3.6}$$

It is also clear that

$$|L_a(z)| \;\asymp\; \log \frac{1}{1-|a|^2}, \text{ and } |1-\bar{a}z| \;\asymp\; |1-|a|^2, \quad z \in S(a).$$

Then we see that

$$\begin{aligned}
&\mu(S(a)) \left(\log \frac{2}{1-|a|^2} \right)^2 \;\asymp\; \int_{S(a)} (1-|z|)^{s_2} |g'(z) L_a(z)|^2 \, dA(z) \\
&\lesssim \int_{S(a)} (1-|z|)^{s_2} |(gL_a)'(z)|^2 \, dA(z) \;+\; \int_{S(a)} (1-|z|)^{s_2} |g(z)|^2 \frac{1}{|1-az|^2} \, dA(z) \\
&\lesssim \|gL_a\|^2_{Q_{s_2}} (1-|a|^2)^{s_2} \;+\; \|g\|^2_{H^\infty} (1-|a|^2)^{s_2} \\
&\lesssim A^2 (1-|a|^2)^{s_2} \;+\; \|g\|^2_{H^\infty} (1-|a|^2)^{s_2}.
\end{aligned}$$

□

Proof of Theorem 3.2. Suppose s_1, s_2, and α are in the conditions of the theorem and take $g \in \Lambda_\alpha$. Let us prove that $g \in M(Q_{s_1}, Q_{s_2})$.

Take $f \in Q_{s_1}$. We have to show that $g \cdot f \in Q_{s_2}$. Let μ be the measure on $\mathbb{D}$ defined by $d\,\mu(z) = (1-|z|^2)^{s_2}|(g\cdot f)'(z)|^2 dA(z)$.

Take $\varepsilon > 0$ such that $\alpha > \frac{1-s_2}{2} + \varepsilon$. Since $g \in \Lambda_\alpha$ and $f \in Q_{s_1} \subset Q_{s_2} \subset BMOA$, we have

$$g \in H^\infty, \quad |g'(z)| \lesssim \frac{1}{(1-|z|)^{1-\alpha}}, \quad |f(z)| \lesssim \log\frac{2}{1-|z|} \lesssim \frac{1}{(1-|z|)^{\varepsilon}}.$$

Then

$$\begin{aligned}(1-|z|^2)^{s_2}|(g\cdot f)'(z)|^2 &\lesssim (1-|z|)^{s_2}|g'(z)|^2|f(z)|^2 + (1-|z|)^{s_2}|g(z)|^2|f'(z)|^2\\ &\lesssim \frac{1}{(1-|z|)^{\beta}} + \|g\|^2_{H^\infty}(1-|z|)^{s_2}|^2|f'(z),\end{aligned}$$

where $\beta = 2 - 2\alpha - s_2 + 2\varepsilon$.

Let I be an interval in $\partial\mathbb{D}$, say that $I = \{e^{i\theta} : \theta_0 \le \theta \le \theta_0 + h\}$ with $0 < h \le 2\pi$. Then

$$\begin{aligned}\mu(S(I)) &= \int_{S(I))}(1-|z|^2)^{s_2}|(g\cdot f)'(z)|^2\,dA(z) \qquad (3.7)\\ &\lesssim \int_{\theta_0}^{\theta_0+h}\int_{1-\frac{h}{2\pi}}^{1}\frac{1}{(1-r)^{\beta}}\,dr\,d\theta + \int_{S(I))}(1-|z|)^{s_2}|^2|f'(z)|^2\,dA(z).\end{aligned}$$

Bearing in mind that $\beta = 2 - 2\alpha - s_2 + 2\varepsilon < 1$, we see that

$$\int_{\theta_0}^{\theta_0+h}\int_{1-\frac{h}{2\pi}}^{1}\frac{1}{(1-r)^{\beta}}\,dr\,d\theta \lesssim h^{2-\beta} = h^{s_2+2(\alpha-\varepsilon)} \lesssim h^{s_2} \qquad (3.8)$$

Also, since $f \in Q_{s_1}$ and $s_1 \le s_2$ we have that $f \in Q_{s_2}$ and, hence, the measure $(1-|z|)^{s_2}||f'(z)|^2\,dA(z)$ is an s_2-Carleson measure. Using this, (3.8), and (3.7) we see that μ is and s_2-Carleson measure and, hence $g\cdot f \in Q_{s_2}$. □

It is natural to ask whether the condition $g \in \Lambda_\alpha$ can be substituted by another weaker one:

• Theorem 1 of [19] shows the existence of a Jordan domain Ω with rectifiable boundary and such that a conformal mapping g from $\mathbb{D}$ onto Ω does not belong to $\mathcal{B}_{\log}$. For this function g we have that $g \in \Lambda^1_1$ but g is not a multiplier of $\mathcal{B}$.

• Theorem 1. 4 of [23] shows that $B^1 \not\subset M(\mathcal{B})$.

Using Theorem 3.1 we can extend this results as follows.

Theorem 3.4. *Suppose that* $0 < s_1 < s_2 < \infty$. *Then:*

(i) *There exists a Jordan domain* Ω *with rectifiable boundary such that if* g *is a conformal mapping from* $\mathbb{D}$ *onto* Ω, *then* $g \in \Lambda^1_1$ *and* $g \notin M(Q_{s_1}, Q_{s_2})$.

(ii) *There exists a function* $g \in B^1$ *such that* $g \notin M(Q_{s_1}, Q_{s_2})$.

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