On optimization over a polyhedral set and Augmented Lagrangians*

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Abstract Linear constraints have a long tradition in optimization problems. It means that the feasible set is a polyhedral set or can be described as a polytope. In Global Optimization, we use the characteristics to derive specific algorithms. In our contribution, we will focus on the so-called Augmented Lagrangian approach where constraints are captured in Lagrangian terms and penalties. We show some first numerical analysis with the easiest case of having a linear objective.

Keywords: Polytope, Polyhedral set, Augmented Lagrangian, Linear Programming

1. Introduction

Using linear constraints is a powerful modeling tool to formulate practical optimization problems. One can think of simple constraints, such as a symmetry-breaking constraint defining $x_2 \geq x_1$ or blending constraint $\sum_i x_i = 1$. Equality constraints such as the latter also add the challenge to work in a space where the dimension is less than that of the continuous objective function $f : \mathbb{R}^n \to \mathbb{R}$.

Exact methods in Global Optimization focus usually on low dimensional problems that are box constrained. Adding linear constraints will then lead to a polytope feasible set $\mathcal{P}$. There are two ways to represent $\mathcal{P}$.

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The first one is as the convex hull $\mathrm{conv}(\mathcal{V})$ of its vertex set $\mathcal{V} := \{v_1, \ldots, v_q\}$;

$$\mathcal{P} := \{x \in \mathbb{R}^n | \sum_{j=1}^{q} \lambda_j v_j, \sum_{j=1}^{q} \lambda_j = 1, \lambda_j \geq 0, j = 1, \ldots, q\}. \quad (1)$$

This so-called barycenter notation allows us to introduce the relative interior, where in (1) we have $\lambda_j > 0$. Basically a global optimum can be found in the relative interior of $\mathcal{P}$ or on one of its faces, where each face is defined by a subset of $\mathcal{I} \subset \{1, \ldots, q\}$ for which $\lambda_j > 0$. For some GO characteristics, it is easy to typify the minimum. The founding handbook [5] starts by focusing on concave optimization. This implies that minimum points are attained at the vertices. Algorithmically, this means we can go for a so-called vertex enumeration.

It is important to see this in the context where $\mathcal{P}$ is represented by a set of inequalities, i.e. the intersection of half spaces.

$$\mathcal{P} := \{x \in \mathbb{R}^n | a_j x \leq b_j, j = 1, \ldots, m\}. \quad (2)$$

In this context, the term polyhedral set is more appropriate. $\mathcal{P}$ can also have directions d_k for which it is unbounded; the term $\sum_k \mu_k d_L, \mu_k \geq 0$ is added to (1). Now for box constrained GO, $\mathcal{P}$ is bounded and with that a polytope. The dimension of $\mathcal{P}$ is defined by the dimension of its affine hull which can basically also be represented by (1), leaving out the nonnegativity constraints on λ. It can be determined by the rank of translated vertex set $\mathcal{W} := \{v_1 - v_q, \ldots, v_{q-1} - v_q\}$. For instance, the mixture design problem [4] has a reduced dimension feasible set.

In our recent investigation [1], we compared traditional Interval Arithmetic B&B (iBB, [3]) with simplicial ones (sBB). The latter ones are more effective for low-dimensional polytope feasible set problems, where the minimum is not interior. However, they suffer from slower convergence due to volume reduction. In our finding, this bad behavior of iBB has to do with working with decimal accuracies. Complexity theory focuses on the integer representation of the data; in our case in matrix A, vector b or the vertices in V. Even having exact representations of vertices or constraints, e.g. $x_1 - x_2 \geq 0$, standard routines like scipy.spatial to transform (1) into (2) and vice versa, will normalize representation introducing a non-exact representation including $\sqrt{2}$. We found that this makes it harder for iBB methods to verify feasibility of simple linear inequalities.

2. Augmented Lagrangians

In our investigation, we pose the question whether it is possible to obtain exact algorithms using Lagrangian theory. In [8] the use of Augmented Lagrangians (AL) to solve generic constrained GO was investigated. The handbook [6] provides an introduction to the concepts of nonlinear optimization including AL. Consider a NLP description with objective function $g(y)$ and set of equalities $s(y) = 0$ which we will call the dual (D): $\min_{y\geq 0} g(y), s_i(y) = 0, i = 1, \dots, n$. Let x be an estimate of the dual solution (so a solution of the primal) and ρ a penalty parameter. A quadratic penalty based AL objective with respect to the equality constraints is

$$\mathcal{L}_D(y; x, \rho) := g(y) - x^T s(y) + \frac{\rho}{2} s(y)^T s(y). \tag{3}$$

Using nonnegativity restrictions, a basic algorithm can be based on minimizing $\min_{y\geq 0} \mathcal{L}_D(y; x_k, \rho_k)$ slowly increasing the penalty parameter ρ_k and updating the dual estimate according to $x_{k+1} = x_k - \rho_k s(y_k)$.

We investigated this approach in terms of the easiest polyhedral optimization problem, which is LP in [2] to solve sparse problems. Consider the dual $\min_{y\geq 0} b^T y, Ay = c$ in [2]. To find a basic feasible dual solution y, one of the used algorithms in the software `Clp` in the open source COIN-OR software is the so-called "idiot crash" algorithm (IC). The basic iterate direction d_k in the primal space relates to the AL (3) of the dual:

$$y_{k+1} = \text{argmin}_{y\geq 0} \left(-b^T y - d_k^T (c - Ay) + \frac{\rho_k}{2} (c - Ay)^T (c - Ay) \right). \tag{4}$$

We found that surprisingly IC decreases the penalty parameter. A much older AL algorithm (in Russian) is due to [7] (PT), which increases the penalty. We investigated what happens if we add to PT a first stage solving (3) with $b = 0$. This usually does not provide a basic solution, but generates a certificate of infeasibility of the dual (unboundedness of the primal) in case no feasible solution exists. A first sketch of numerical results based on 28 netlib problems is given in Figures 1 and 2.

3. Summary

The target of the workshop presentation is to sketch findings with respect to optimization over polyhedral sets. On one hand sketching challenges for B&B based methods and on the other hand questions on applying AL.

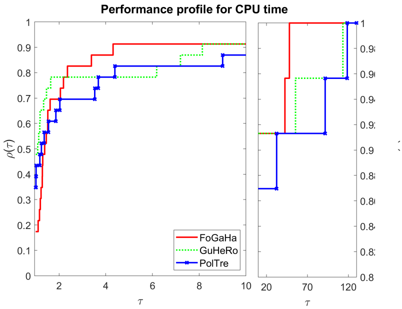

Figure 1: Comparing IC (red), PT (blue) and added first stage (green) in CPU.

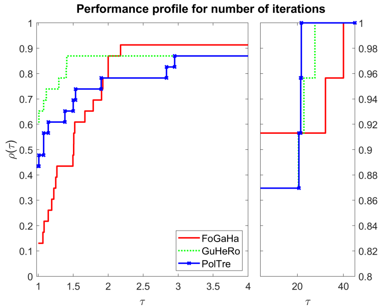

Figure 2: Comparing IC (red), PT (blue) and added first stage (green) in iterations.

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