A FAMILY OF DIRICHLET-MORREY SPACES

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Abstract. To each weighted Dirichlet space $\mathcal{D}_p$, $0 < p < 1$, we associate a family of Morrey-type spaces $\mathcal{D}_p^\lambda$, $0 < \lambda < 1$, constructed by imposing growth conditions on the norm of hyperbolic translates of functions. We indicate some of the properties of these spaces, mention the characterization in terms of boundary values, and study integration and multiplication operators on them.

1. Introduction

Let $\mathbb{D}$ be the unit disc in the complex plane and $\mathcal{H}ol(\mathbb{D})$ be the space of analytic functions in $\mathbb{D}$. A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the Hardy space H^2 if

$$\|f\|_{H^2}^2 = \sup_{r\in(0,1)} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty \, .$$

If $f \in H^2$ then the radial limits

$$\lim_{r\to 1} f(re^{i\theta}) = f(e^{i\theta})$$

exist almost everywhere on the unit circle $\mathbb{T}$ and

$$\|f\|_{H^2}^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta. \tag{1.1}$$

Recall that the space $BMOA$ consists of all functions $f \in H^2$ whose boundary values $f(e^{i\theta})$ have bounded mean oscillation, that is,

$$\sup_{I\subseteq\mathbb{T}} \frac{1}{|I|} \int_I \left|f(e^{i\theta}) - f_I\right|^2 \, d\theta < \infty, \tag{1.2}$$

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where $f_I = \frac{1}{|I|}\int_I f(e^{it})\,dt$ is the average of f over I, and the supremum is taken over all subarcs I of $\mathbb{T}$ with length $|I|$. There are other equivalent definitions for $BMOA$ one of which is the following. For $a \in \mathbb{D}$ let

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in \mathbb{D},$$

be the analytic automorphism of $\mathbb{D}$ which exchanges 0 with a, and for $f \in H^2$ consider the set of hyperbolic translates of f,

$$S(f) = \{f \circ \varphi_a - f(a) : a \in \mathbb{D}\}.$$

$BMOA$ then can be defined as the space of all $f \in H^2$ such that

$$\|f\|_{BMOA} = |f(0)| + \sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^2} < \infty, \tag{1.3}$$

i.e. $f \in BMOA$ if and only if the set $S(f)$ is bounded in the norm of H^2. The above quantity defines a norm making $BMOA$ a Banach space. More information on $BMOA$ can be found in [9].

Morrey spaces. Morrey spaces were introduced in the 1930's in connection to partial differential equations, and were subsequently studied as function classes in harmonic analysis on Euclidean spaces. The analytic Morrey spaces were introduced recently and studied by several authors, see for example [18], [22], [23] and [11].

We recall the definition and some of their properties. Observe that

$$\int_I \left|f(e^{i\theta}) - f_I\right|^2 d\theta \to 0 \qquad \text{as } |I| \to 0,$$

for every $f \in H^2$, and the rate of this convergence to 0 depends clearly on the degree of oscillation of f around its average f_I on I. Given a $\lambda \in (0,1)$ we can isolate functions f for which this rate of convergence is comparable to $|I|^\lambda$. Thus for $f \in H^2$ we set

$$\|f\|_{\lambda,*} = \sup_{I\subseteq\mathbb{T}} \left(\frac{1}{|I|^\lambda}\int_I \left|f(e^{i\theta}) - f_I\right|^2 d\theta\right)^{1/2}, \tag{1.4}$$

and define the space

$$H^{2,\lambda} = \{f \in H^2 : \|f\|_{\lambda,*} < \infty\}.$$

This is a linear space. The seminorm $\| \,\|_{\lambda,*}$ can be completed to a norm by adding $|f(0)|$ to it, making $H^{2,\lambda}$ into a Banach space.

It is clear that for $\lambda = 0$ or $\lambda = 1$, $H^{2,\lambda}$ reduces to H^2 and $BMOA$ respectively, and for $0 < \lambda < 1$,

$$BMOA \subset H^{2,\lambda} \subset H^2.$$

Furthermore the following Carleson measure characterization holds

$$f \in H^{2,\lambda} \Leftrightarrow \sup_{I \subset \mathbb{T}} \frac{1}{|I|^\lambda} \int_{S(I)} |f'(z)|^2 (1-|z|^2)\, dm(z) < \infty, \tag{1.5}$$

where

$$S(I) = \{re^{i\theta} : 1-|I| \leq r < 1,\, e^{i\theta} \in I\}$$

are the Carleson boxes based on arcs $I \subset \mathbb{T}$ and $dm(z)$ is the planar Lebesgue measure normalized so that the area of the unit disc is 1. This and several other properties of Morrey spaces can be found in [22], [23] or [20]. A characterization of $H^{2,\lambda}$ analogous to (1.3) says that if $f \in H^2$ then $f \in H^{2,\lambda}$ if and only if

$$\|f\|_{\lambda,**} = \sup_{a \in \mathbb{D}} (1-|a|^2)^{\frac{1}{2}(1-\lambda)} \|f \circ \varphi_a - f(a)\|_{H^2} < \infty, \tag{1.6}$$

and $\|f\|_{H^{2,\lambda}} = |f(0)| + \|f\|_{\lambda,**}$ is a norm, equivalent to $|f(0)| + \|f\|_{\lambda,*}$. Equivalently $H^{2,\lambda}$ can be constructed as the subspace of H^2 containing the functions whose conformal translates have H^2 norms of restricted growth,

$$\|f \circ \varphi_a - f(a)\|_{H^2} \leq \frac{C}{(1-|a|^2)^{\frac{1}{2}(1-\lambda)}}, \ |a| \to 1, \tag{1.7}$$

with the constant C depending only on f.

General Morrey-type spaces. We take the opportunity to notice that the above construction can be carried out in more general terms. Suppose X is a Banach space of analytic functions on $\mathbb{D}$ which contains the constant functions and such that point evaluations $f \to f(a)$, $a \in \mathbb{D}$, are continuous linear functionals on X. Suppose also that $w : \mathbb{D} \to (0,\infty)$ is an appropriate weight function. Consider the Morrey-type space generated by (X, w) which is defined to be the space $M(X, w)$ of functions $f \in X$ such that $f \circ \varphi_a \in X$ for $a \in \mathbb{D}$ and for which there is a constant $C = C(f)$ such that

$$\|f \circ \varphi_a - f(a)\|_X \leq C w(a), \quad a \in \mathbb{D}.$$

Without any restrictions on w the space $M(X, w)$ may reduce to $M(X, w) = \{0\}$ or $M(X, w) = X$ or it may consist only of constant functions. But generally there are weights, appropriate for the base space X, for which $M(X, w)$ is a nontrivial proper subspace of X. For example if $w \equiv 1$ on $\mathbb{D}$ then this construction gives the Möbius invariant spaces $M(X)$ generated by X considered in [2], a particular case of which is $M(H^2) = BMOA$. A convenient class of weights that can be considered in this construction are the radial weights, $w(a) = w(|a|)$, and it may be assumed further that $w(|a|)$ is nondecreasing in $|a|$. A particular such family is

$$w(a) = (1-|a|)^{-s}, \quad a \in \mathbb{D}, \tag{1.8}$$

with $s \geq 0$. The specific choice $w_\lambda(a) = (1-|a|)^{-\frac{1}{2}(1-\lambda)}$ gives

$$M(H^2, w_\lambda) = H^{2,\lambda}, \quad 0 \leq \lambda \leq 1.$$

In the general case, if (X, w) is a pair for which the resulting space $M(X, w)$ is nontrivial, then the quantity

$$\|f\|_{M(X,w)} = |f(0)| + \sup_{a \in \mathbb{D}} \frac{1}{w(a)} \|f \circ \varphi_a - f(a)\|_X$$

is a norm on $M(X, w)$ and makes it into a Banach space. Interesting questions arise such as characterizing $M(X, w)$ by Carleson measure type conditions or in terms of boundary values of its functions. We will not pursue this further here, but will concentrate instead on a family of spaces obtained when X is a weighted Dirichlet space and w is of the form (1.8).

Dirichlet spaces and $\mathcal{Q}_p$ spaces. Recall the following estimate for the norm of a function $f \in H^2$,

$$\|f\|^2_{H^2} \sim |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)\, dm(z),$$

where $\sim$ means that each of the two quantities is dominated by a constant multiple of the other for all $f \in H^2$. If the weight $1-|z|^2$ inside the integral is replaced by $\log(1/|z|^2)$ then the above estimate becomes an identity valid for all $f \in H^2$, known as the Littlewood-Paley identity.

The weighted Dirichlet spaces $\mathcal{D}_p$, $0 \leq p < \infty$, are defined to contain those $f \in \mathcal{H}ol(\mathbb{D})$ for which

$$\|f\|^2_{\mathcal{D}_p} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^p\, dm(z) < \infty.$$

This quantity is a norm. Clearly $\mathcal{D}_1 = H^2$ with equivalence of norms, and $\mathcal{D}_0$ is the classical Dirichlet space denoted by $\mathcal{D}$. For $p > 1$, $\mathcal{D}_p$ coincides with a weighted Bergman space with weight $(1-|z|)^{p-2}$. If $0 < p < q$ then

$$\mathcal{D} \subset \mathcal{D}_p \subset \mathcal{D}_q,$$

and there is a constant $C = C(p, q)$ such that $\|f\|_{\mathcal{D}_q} \leq C\|f\|_{\mathcal{D}_p}$ for each $f \in \mathcal{D}_p$.

As in the case of H^2 we can consider the Möbius invariant version of $\mathcal{D}_p$, that is the subspace of $\mathcal{D}_p$ which consists of all $f \in \mathcal{D}_p$ such that the set $S(f) = \{f \circ \varphi_a - f(a) : a \in \mathbb{D}\}$ is bounded in $\mathcal{D}_p$. These are the spaces $\mathcal{Q}_p$, originally defined and studied by Aulaskari, Xiao and Zhao [6]. Under the norm

$$\|f\|_{\mathcal{Q}_p} = |f(0)| + \sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{\mathcal{D}_p}, \tag{1.9}$$

they are Banach spaces and we have $\mathcal{Q}_0 = \mathcal{D}$, $\mathcal{Q}_1 = BMOA$, while for all $p > 1$, $\mathcal{Q}_p$ coincides with the Bloch space $\mathcal{B}$ of functions that satisfy

$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty$, see [19]. For the remaining values $0<p<1$ the resulting spaces satisfy

$$\mathcal{D}\subset\mathcal{Q}_p\subset BMOA$$

and they form a strictly increasing chain of Möbius invariant spaces, characterized by the Carleson measure condition,

$$f\in\mathcal{Q}_p\Leftrightarrow\sup_{I\subset\mathbb{T}}\frac{1}{|I|^p}\int_{S(I)}|f'(z)|^2(1-|z|^2)^p\,dm(z)<\infty.$$

For information on these spaces see [19] and [20] and the references therein.

Dirichlet-Morrey spaces. Let $0\le p\le 1$ and $f\in\mathcal{D}_p$. The following estimate is valid

$$\|f\circ\varphi_a-f(a)\|_{\mathcal{D}_p}\le\frac{C\|f\|_{\mathcal{D}_p}}{(1-|a|^2)^{\frac{p}{2}}},\quad a\in\mathbb{D},$$

with the constant C depending only on p. In analogy with the construction of $H^{2,\lambda}$ from the Hardy space $H^2=\mathcal{D}_1$ we define the Dirichlet-Morrey spaces as follows.

Definition 1.1. *Let* λ, $p\in[0,1]$. *We say that an* $f\in\mathcal{H}ol(\mathbb{D})$ *belongs to the Dirichlet-Morrey space* $\mathcal{D}_p^\lambda$ *if*

$$\|f\|_{\mathcal{D}_p^\lambda}=|f(0)|+\sup_{a\in\mathbb{D}}(1-|a|^2)^{\frac{p}{2}(1-\lambda)}\|f\circ\varphi_a-f(a)\|_{\mathcal{D}_p}<\infty.\tag{1.10}$$

It is clear $\mathcal{D}_p^\lambda$ is a linear space and the above quantity is a norm, under which $\mathcal{D}_p^\lambda$ is a Banach space. We see that $\mathcal{D}_1^\lambda=H^{2,\lambda}$ and that for each p, $\mathcal{D}_p^1=\mathcal{Q}_p$ and $\mathcal{D}_p^0=\mathcal{D}_p$ and we have

$$\mathcal{Q}_p\subseteq\mathcal{D}_p^\lambda\subseteq\mathcal{D}_p,\qquad 0<\lambda<1.$$

In the rest of the article we will state some basic properties of Dirichlet-Morrey spaces, discuss briefly their characterization in terms of boundary values and concentrate in Section 3 on the boundedness of integration operators and pointwise multipliers.

We will write $A\lesssim B$ between two quantities if there is a constant C such that $A\le CB$ for all values of the parameter involved in the quantities A,B. If both $A\lesssim B$ and $B\lesssim A$ are valid we write $A\sim B$. When a constant C appears, its value may be different from one step to the next.

2. Some basic properties

The following proposition gives a Carleson measure characterization of $\mathcal{D}_p^\lambda$, which is analogous to (1.5) for Hardy-Morrey spaces

Proposition 1. *Let* $0 < p, \lambda < 1$ *and* $f \in \mathcal{H}ol(\mathbb{D})$. *Then the following are equivalent,*

(1) $f \in \mathcal{D}_p^\lambda$.

(2) $\|f\|_{p,\lambda,*} = \sup\limits_{I \subset \mathbb{T}} \left(\frac{1}{|I|^{p\lambda}} \int_{S(I)} |f'(z)|^2 (1-|z|^2)^p \, dm(z) \right) < \infty$,

and the norm $\|f\|_{\mathcal{D}_p^\lambda}$ *is comparable to* $|f(0)| + \|f\|_{p,\lambda,*}$.

Proof. Assume $f \in \mathcal{D}_p^\lambda$. For an interval $I \subset \mathbb{T}$ let ζ be the midpoint of I and let $a = a_I = (1 - |I|)\zeta$. Note that

$$|1 - \overline{a}z| \sim |I| = 1 - |a| \sim 1 - |a|^2, \qquad z \in S(I),$$

thus

$$\begin{aligned}
&\frac{1}{|I|^{p\lambda}} \int_{S(I)} |f'(z)|^2 (1-|z|^2)^p \, dm(z) \\
&\sim \frac{1}{|I|^{p\lambda}} \int_{S(I)} |f'(z)|^2 (1-|z|^2)^p \frac{(1-|a|^2)^{2p}}{|1-\overline{a}z|^{2p}} \, dm(z) \\
&\sim |I|^{p(1-\lambda)} \int_{S(I)} |f'(z)|^2 (1-|\varphi_a(z)|^2)^p \, dm(z) \\
&\lesssim (1-|a|^2)^{p(1-\lambda)} \int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_a(z)|^2)^p \, dm(z) \\
&= (1-|a|^2)^{p(1-\lambda)} \int_{\mathbb{D}} |(f \circ \varphi_a)'(z)|^2 (1-|z|^2)^p \, dm(z) \\
&= (1-|a|^2)^{p(1-\lambda)} \|f \circ \varphi_a - f(a)\|_{\mathcal{D}_p}^2 \\
&\leq \|f\|_{\mathcal{D}_p^\lambda}^2.
\end{aligned}$$

This is valid for each interval $I \subset \mathbb{T}$ and taking supremum shows that (1) implies (2).

Conversely suppose (2) holds. That is, for the nonnegative measure $d\mu(z) = |f'(z)|^2(1-|z|^2)^p \, dm(z)$ there is a constant C such that

$$\mu(S(I)) = \int_{S(I)} d\mu(z) \leq C|I|^{p\lambda}$$

for all $I \subset \mathbb{T}$, i.e. $d\mu$ is a $p\lambda$-Carleson measure. Then for $a \in \mathbb{D}$,

$$\begin{aligned}
\|f \circ \varphi_a - f(a)\|_{\mathcal{D}_p}^2 &= \int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_a(z)|^2)^p \, dm(z) \\
&= \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|z|^2)^p (1-|a|^2)^p}{|1-\overline{a}z|^{2p}} \, dm(z) \\
&= \int_{\mathbb{D}} \frac{(1-|a|^2)^p}{|1-\overline{a}z|^{2p}} \, d\mu(z).
\end{aligned}$$

Thus we have

$$\begin{aligned}\sup_{a\in\mathbb{D}}(1-|a|^2)^{p(1-\lambda)}\|f\circ\varphi_a-f(a)\|^2_{\mathcal{D}_p} &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^{2p-p\lambda}}{|1-\overline{a}z|^{2p}}\,d\mu(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^{q}}{|1-\overline{a}z|^{q+p\lambda}}\,d\mu(z)\\ &< \infty,\end{aligned}$$

by using the characterization of Carleson measures in [20, Lemma 3.1.1] with $q=2p-p\lambda>0$, completing the proof. □

Proposition 2. *Let $0<p,\lambda<1$ then,*

1. *There is a constant $C=C(p,\lambda)$ such that any $f\in\mathcal{D}_p^\lambda$ satisfies*

$$|f(z)|\le\frac{C\|f\|_{\mathcal{D}_p^\lambda}}{(1-|z|)^{\frac{p}{2}(1-\lambda)}},\quad z\in\mathbb{D} \tag{2.1}$$

2. *The function $f_{p,\lambda}(z)=(1-z)^{-\frac{p}{2}(1-\lambda)}$ belongs to $\mathcal{D}_p^\lambda$.*

Proof. (i) Suppose $f\in\mathcal{D}_p^\lambda$. We apply the inequality

$$|g(0)|^2\le(p+1)\int_{\mathbb{D}}|g(z)|^2(1-|z|^2)^p\,dm(z),$$

see [25, Lemma 4.12], valid for all analytic g on $\mathbb{D}$, to the function $g=(f\circ\varphi_w-f(w))'$ to obtain

$$\begin{aligned}|f'(w)|^2(1-|w|^2)^2 &\le (p+1)\int_{\mathbb{D}}|(f\circ\varphi_w)'(z)|^2(1-|z|^2)^p\,dm(z)\\ &= \frac{p+1}{(1-|w|^2)^{p(1-\lambda)}}\left((1-|w|^2)^{p(1-\lambda)}\|f\circ\varphi_w-f(w)\|^2_{\mathcal{D}_p}\right)\\ &\le \frac{p+1}{(1-|w|^2)^{p(1-\lambda)}}\|f\|^2_{\mathcal{D}_p^\lambda},\end{aligned}$$

for each $w\in\mathbb{D}$. Thus

$$|f'(w)|\le\frac{(p+1)^{1/2}}{(1-|w|^2)^{1+\frac{p}{2}(1-\lambda)}}\|f\|_{\mathcal{D}_p^\lambda},\quad w\in\mathbb{D}.$$

Using this and the integration $f(z)-f(0)=\int_0^z f'(\zeta)\,d\zeta$ we obtain the desired growth inequality.

(ii) We will verify that $|f'_{p,\lambda}(z)|^2(1-|z|^2)^p\,dm(z)$ is a $p\lambda$-Carleson measure and then Proposition 1 gives the conclusion. In doing so, it is more convenient to work with the equivalent family of Carleson lune-shaped sets

$S(b,h) = \{z \in \mathbb{D} : |b - z| < h\}$, where $b \in \mathbb{T}$ and $0 < h < 1$, than with the Carleson boxes $S(I), I \subset \mathbb{T}$. Thus it suffices to show that

$$\sup_{b\in\mathbb{T},0<h<1} \frac{1}{h^{p\lambda}} \int_{S(b,h)} |f'_{p,\lambda}(z)|^2(1-|z|^2)^p \, dm(z) < \infty. \tag{2.2}$$

We have

$$\begin{aligned}
\int_{S(b,h)} |f'_{p,\lambda}(z)|^2(1-|z|^2)^p \, dm(z) &= C_1 \int_{S(b,h)} \frac{(1-|z|^2)^p}{|1-z|^{2+p(1-\lambda)}} \, dm(z) \\
&\lesssim \int_{S(b,h)} \frac{1}{|1-z|^{2-p\lambda}} \, dm(z) \\
&\lesssim \int_{S(1,h)} \frac{1}{|1-z|^{2-p\lambda}} \, dm(z) \\
&\lesssim \int_{|w|<h} \frac{1}{|w|^{2-p\lambda}} \, dm(w) \\
&= \int_0^h \frac{1}{r^{1-p\lambda}} dr \\
&= h^{p\lambda}.
\end{aligned}$$

Thus (2.2) holds and the proof is finished. $\square$

Observe that both parts of the above Proposition are also valid when $p = 1, 0 < \lambda < 1$.

Proposition 3. *Let $\lambda_1, p_1, \lambda_2, p_2 \in (0,1)$. Then*

$$\mathcal{D}^{\lambda_1}_{p_1} \subseteq \mathcal{D}^{\lambda_2}_{p_2} \iff p_1 \le p_2 \text{ and } p_1(1-\lambda_1) \le p_2(1-\lambda_2).$$

Proof. Assume $p_1 \le p_2$ and $p_1(1-\lambda_1) \le p_2(1-\lambda_2)$ and let $f \in \mathcal{D}^{\lambda_1}_{p_1}$ and $I \subset \mathbb{T}$. Then

$$\begin{aligned}
&\frac{1}{|I|^{p_2\lambda_2}} \int_{S(I)} |f'(z)|^2(1-|z|^2)^{p_2} \, dm(z) \\
&\quad = \frac{1}{|I|^{p_2\lambda_2}} \int_{S(I)} |f'(z)|^2(1-|z|^2)^{p_1}(1-|z|^2)^{p_2-p_1} \, dm(z) \\
&\quad \le \frac{|I|^{p_2-p_1}}{|I|^{p_2\lambda_2}} \int_{S(I)} |f'(z)|^2(1-|z|^2)^{p_1} \, dm(z) \\
&\quad = |I|^{p_2(1-\lambda_2)-p_1(1-\lambda_1)} \left(\frac{1}{|I|^{p_1\lambda_1}} \int_{S(I)} |f'(z)|^2(1-|z|^2)^{p_1} \, dm(z) \right)
\end{aligned}$$

and by Proposition 1, it follows $\mathcal{D}^{\lambda_1}_{p_1} \subseteq \mathcal{D}^{\lambda_2}_{p_2}$.

Assume now that $\mathcal{D}^{\lambda_1}_{p_1} \subseteq \mathcal{D}^{\lambda_2}_{p_2}$. Then it is necessary that $p_1 \le p_2$. The easiest way to see this is to use the class HG of functions in $\mathcal{H}ol(\mathbb{D})$

whose Taylor series with center at 0 has Hadamard gaps. According to [19, Theorem 1.2.1] for $0 < p < 1$ we have $HG \cap \mathcal{Q}_p = HG \cap \mathcal{D}_p$, and for $0 < p < q < 1$ we have $HG \cap \mathcal{D}_p \subsetneq HG \cap \mathcal{D}_q$. If we assume that $p_2 < p_1$ then $\mathcal{D}_{p_2} \subseteq \mathcal{D}_{p_1}$ and using the assumption $\mathcal{D}_{p_1}^{\lambda_1} \subseteq \mathcal{D}_{p_2}^{\lambda_2}$ we will have further $\mathcal{Q}_{p_1} \subseteq \mathcal{D}_{p_1}^{\lambda_1} \subseteq \mathcal{D}_{p_2}^{\lambda_2} \subseteq \mathcal{D}_{p_2} \subseteq \mathcal{D}_{p_1}$. This would imply that $HG \cap \mathcal{D}_{p_1} = HG \cap \mathcal{D}_{p_2}$ which contradicts part of the above mentioned theorem. In addition from (2.1) it follows easily that $p_1(1-\lambda_1) \leq p_2(1-\lambda_2)$. $\square$

We next discuss the boundary values characterization of Dirichlet-Morrey spaces. Recall the corresponding result for Dirichlet spaces and $\mathcal{Q}_p$ spaces from [19, Lemma 6.1.1.]. If $f \in H^2$ and $0 < p < 1$, then $f \in \mathcal{D}_p$ if and only if

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f(u) - f(v)|^2}{|u - v|^{2-p}} \, |du||dv| < \infty,$$

where the simplified notation is $u = e^{i\theta} \in \mathbb{T}$ and $|du| = d\theta$. This result together with the fact that $\mathcal{Q}_p$ is the Möbius invariant version of $\mathcal{D}_p$, is used to prove Theorem 6.1.1. in [19] which says that for $0 < p < 1$, $f \in \mathcal{Q}_p$ if and only if

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^p} \int_I \int_I \frac{|f(u) - f(v)|^2}{|u - v|^{2-p}} \, |du||dv| < \infty,$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$. The proof of this result is rather long and technical. But it is easily adapted without any new conceptual or technical requirements to obtain the following characterization of Dirichlet-Morrey spaces.

Theorem 1. *Suppose* $f \in H^2$ *and let* $0 < p, \lambda < 1$. *Then* $f \in \mathcal{D}_p^\lambda$ *if and only if*

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^{p\lambda}} \int_I \int_I \frac{|f(u) - f(v)|^2}{|u - v|^{2-p}} \, |du| \, |dv| < \infty.$$

3. Pointwise multipliers

Let X be a Banach space of analytic functions on $\mathbb{D}$. A function $g \in \mathcal{H}ol(\mathbb{D})$ is said to be a multiplier of X if the multiplication operator

$$M_g(f)(z) = g(z)f(z), \quad f \in X$$

is a bounded operator on X. For this it is usually enough to check that $M_g(X) \subset X$ and apply the closed graph theorem. The space of all multipliers of X is denoted by $M(X)$. Multiplication operators are closely related to integration operators J_g and I_g. These are induced by symbols $g \in \mathcal{H}ol(\mathbb{D})$ as follows

$$J_g(f)(z) = \int_0^z f(w)g'(w) \, dw, \quad z \in \mathbb{D},$$

and

$$I_g(f)(z) = \int_0^z f'(w)g(w)\,dw, \quad z \in \mathbb{D},$$

and act on functions $f \in \mathcal{H}ol(\mathbb{D})$. The operators I_g, J_g have been studied in a number of papers, see for example [1], [3], [7], [9] and [10]. Their relation to M_g comes from the integration by parts formula

$$J_g(f)(z) = M_g(f)(z) - f(0)g(0) - I_g(f)(z). \tag{3.1}$$

This essentially says that if g is a symbol for which two of the operators I_g, J_g, M_g are bounded on a space X so is the third. It also says that it is possible for two of the operators to be unbounded but the third is bounded due to cancellation.

The space of multipliers is known for several of the classical spaces such as Hardy and Bergman spaces. In particular for $H^2 = \mathcal{D}_1$ the space of multipliers is $M(H^2) = H^\infty$, the algebra of bounded analytic functions. For other Dirichlet spaces $\mathcal{D}_p$, $p \in (0,1)$, the situation is more complicated. The description of $M(\mathcal{D}_p)$ is in terms of $\mathcal{D}_p$-Carleson measures. Recall that a positive Borel measure μ on the disc is a $\mathcal{D}_p$-Carleson measure if there is a constant $C = C(\mu)$ such that

$$\int_{\mathbb{D}} |f(z)|^2\,d\mu(z) \le C\|f\|^2_{\mathcal{D}_p}, \qquad f \in \mathcal{D}_p.$$

These measures were described initially by Stegena [16] with the help of Bessel capacities, and similar characterizations were given by other authors.

In another approach, Arcozzi, Rochberg and Sawyer [4] described these measures by a different condition, a simplified form of which is given in [8]. Accordingly a finite μ is $\mathcal{D}_p$-Carleson if and only if

$$\sup_{w\in\mathbb{D}} \frac{1}{\mu(S(w))} \int_{S(w)} \frac{(\mu(S(z)\cap S(w)))^2}{(1-|z|^2)^{2+p}}\,dm(z) < \infty, \tag{3.2}$$

where for $w \in \mathbb{D}$ the set $S(w)$ on which integration takes place is the Carleson box $S(w) = \{z \in \mathbb{D} : 1-|z| \le 1-|w|,\ |arg(\bar{z}w)| \le \pi(1-|w|)\}$.

It is convenient at this point to use the space $\mathcal{W}_p$ of functions $g \in \mathcal{H}ol(\mathbb{D})$ such that the measure

$$d\mu_g(z) = |g'(z)|^2(1-|z|^2)^p\,dm(z)$$

is a $\mathcal{D}_p$-Carleson measure. This space has been studied [15] and [17]. The multipliers of $\mathcal{D}_p$ were described in [16] as follows.

Theorem A. *Suppose* $0 < p < 1$ *and* $g \in \mathcal{H}ol(\mathbb{D})$. *Then* $g \in M(\mathcal{D}_p)$ *if and only if* $g \in H^\infty$ *and* $d\mu_g(z) = |g'(z)|^2(1-|z|^2)^p\,dm(z)$ *is a* $\mathcal{D}_p$*-Carleson* *measure. In other words,*

$$M(\mathcal{D}_p) = H^\infty \cap \mathcal{W}_p.$$

On the other hand the multipliers of $\mathcal{Q}_p$ are completely described in [13], [21] as follows.

Theorem B. *Suppose* $0 < p < 1$ *and* $g \in \mathcal{H}ol(\mathbb{D})$. *Then* $g \in M(\mathcal{Q}_p)$ *if and only if* $g \in H^\infty$ *and*

$$\sup_{I \subseteq \mathbb{T}} \frac{\left(\log \frac{1}{|I|}\right)^2}{|I|^p} \int_{S(I)} |g'(z)|^2 (1-|z|^2)^p \, dm(z) < \infty. \tag{3.3}$$

Thus if we denote by $\mathcal{Q}_{p,\log}$ the space of functions that satisfy (3.3) then the above theorem says

$$M(\mathcal{Q}_p) = H^\infty \cap \mathcal{Q}_{p,\log}.$$

It is not difficult to check that $\mathcal{Q}_{p,\log} \subset \mathcal{W}_p$. On the other hand it was shown in [4] that $\mathcal{W}_p \subset \mathcal{Q}_p$ and there is a simplified proof of this in [12, Lemma 4]. Thus we have

$$\mathcal{Q}_{p,\log} \subset \mathcal{W}_p \subset \mathcal{Q}_p, \quad 0 < p < 1.$$

In what follows we study the action of the operators I_g, J_g on the spaces $\mathcal{D}_p^\lambda$, and obtain information on pointwise multipliers. We will need the following technical lemma from [14] (p. 488). We state only the part of it that we need.

Lemma A. *Let* $u \in \mathbb{D}$, $|v| \le 1$ *and* $s > -1$, r, $t > 0$. *Then*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\bar{u}z|^r |1-\bar{v}z|^t} \, dm(z) \le \frac{C}{(1-|u|^2)^{r+t-s-2}}, \quad 0 < r+t-s-2 < r$$

where C *is an absolute, positive constant.*

Using this estimate we obtain a family of test functions in $\mathcal{D}_p^\lambda$.

Lemma 1. *Let* $0 < p, \lambda < 1$ *and* $c \in \mathbb{D}$. *Then the functions*

$$f_c(z) = \frac{1}{(1-\bar{c}z)^{p(1-\lambda)/2}} \tag{3.4}$$

belong to $\mathcal{D}_p^\lambda$ *and* $K = \sup_{c \in \mathbb{D}} \|f_c\|_{\mathcal{D}_p^\lambda} < \infty$.

Proof. Fix $c \in \mathbb{D}$. Then for $a \in \mathbb{D}$,

$$\begin{aligned}(1-|a|^2)^{p(1-\lambda)} &\int_{\mathbb{D}} |f_c'(z)|^2 (1-|\varphi_a(z)|^2)^p \, dm(z) \\ &= (1-|a|^2)^{p(2-\lambda)} \int_{\mathbb{D}} \frac{(1-|z|^2)^p}{|1-\bar{c}z|^{2+p(1-\lambda)} |1-\bar{a}z|^{2p}} \, dm(z)\,.\end{aligned}$$

Now for $r = 2p$, $t = 2 + p(1-\lambda)$, $s = p$, Lemma A gives the desired result. $\square$

Theorem 2. *Let* $0 < p, \lambda < 1$ *and* $g \in \mathcal{H}ol(\mathbb{D})$. *Then* $I_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ *is bounded if and only if* $g \in H^\infty$.

Proof. Let $g \in H^\infty$ then

$$\begin{aligned}\|I_g(f)\|^2_{\mathcal{D}_p^\lambda} &\sim \sup_{I \subset \mathbb{T}} \frac{1}{|I|^{p\lambda}} \int_{S(I)} |I_g(f)'(z)|^2 (1-|z|^2)^p \, dm(z) \\ &= \sup_{I \subset \mathbb{T}} \frac{1}{|I|^{p\lambda}} \int_{S(I)} |f'(z)|^2 |g(z)|^2 (1-|z|^2)^p \, dm(z) \\ &\lesssim \|g\|^2_\infty \|f\|^2_{\mathcal{D}_p^\lambda}\end{aligned}$$

for every $f \in \mathcal{D}_p^\lambda$. So $\|I_g\| \leq C\|g\|_\infty$ where C is a constant.

On the other hand, assume that I_g is bounded on $\mathcal{D}_p^\lambda$. We will use the test functions f_c of Lemma 1 for $\{|c| > \frac{1}{2}\}$. Then from the Lemma there is a constant C such that $1 \leq \|f_c\|_{\mathcal{D}_p^\lambda} \leq C$ for all c, so that $\|I_g\|^2 \geq \frac{1}{C^2}\|I_g(f_c)\|^2_{\mathcal{D}_p^\lambda}$ and,

$$\begin{aligned}\|I_g(f_c)\|^2_{\mathcal{D}_p^\lambda} &= \sup_{a \in \mathbb{D}} (1-|a|^2)^{p(1-\lambda)} \int_{\mathbb{D}} |I_g(f_c)'(z)|^2 (1-|\varphi_a(z)|^2)^p \, dm(z) \\ &\gtrsim (1-|c|^2)^{p(1-\lambda)} \int_{\mathbb{D}} |I_g(f_c)'(z)|^2 (1-|\varphi_c(z)|^2)^p \, dm(z) \\ &= (1-|c|^2)^{p(1-\lambda)} \int_{\mathbb{D}} |f_c'(z)|^2 |g(z)|^2 (1-|\varphi_c(z)|^2)^p \, dm(z) \\ &\sim |c|(1-|c|^2)^{p(1-\lambda)} \int_{\mathbb{D}} \frac{|g(z)|^2(1-|\varphi_c(z)|^2)^p}{|1-\bar{c}z|^{2+p(1-\lambda)}} \, dm(z)\,,\end{aligned}$$

now by restricting the above integral on a disc with center the point c and radius $\frac{1-|c|}{2}$ and by applying the mean value property of subharmonic functions we get that

$$\|I_g\|^2 \gtrsim |g(c)|^2$$

for any $\{|c| > \frac{1}{2}\}$. It follows that g is a bounded analytic function on $\mathbb{D}$. $\square$

Concerning the action of J_g on $\mathcal{D}_p^\lambda$ we have the following necessary condition.

Theorem 3. *Let* $0 < p, \lambda < 1$ *and* $g \in \mathcal{H}ol(\mathbb{D})$. *If* $J_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ *is bounded then* $g \in \mathcal{Q}_p$.

Proof. We use the test functions $f_c(z) = (1-\bar{c}z)^{-p(1-\lambda)/2}$ of Lemma 1. From the hypothesis there is a constant C such that

$$\|J_g(f_c)\|_{\mathcal{D}_p^\lambda} \le C\|f_c\|_{\mathcal{D}_p^\lambda} \le C \sup_{c\in\mathbb{D}} \|f_c\|_{\mathcal{D}_p^\lambda} = CK < \infty,$$

for all $c \in \mathbb{D}$. This means that

$$\sup_{I\subset\mathbb{T}} \frac{1}{|I|^{p\lambda}} \int_{S(I)} |f_c(z)|^2|g'(z)|^2(1-|z|^2)^p \, dm(z) \le K' < \infty$$

for all $c \in \mathbb{D}$. For each interval I choose $c = c_I = (1-|I|)e^{i\theta}$ where $e^{i\theta}$ is the center of I, then $|1-\bar{c}z| \sim |I|$ for $z \in S(I)$ and we have

$$\begin{aligned} K' \ge& \frac{1}{|I|^{p\lambda}} \int_{S(I)} \frac{1}{|1-\bar{c}z|^{p(1-\lambda)}} |g'(z)|^2(1-|z|^2)^p \, dm(z) \\ \sim& \frac{1}{|I|^{p}} \int_{S(I)} |g'(z)|^2(1-|z|^2)^p \, dm(z) \end{aligned}$$

with K' independent of I. Taking the supremum of the last integral over all $I \subset \mathbb{T}$ we see that $g \in \mathcal{Q}_p$. □

We now find sufficient conditions on g for J_g to be bounded on $\mathcal{D}_p^\lambda$.

Theorem 4. *Suppose* $0 < p < 1$.

1. *If* $0 < q < p$ *and* $g \in \mathcal{Q}_q$ *then* $J_g : \mathcal{D}_p^{q/p} \to \mathcal{D}_p^{q/p}$ *is bounded.*
2. *If* $0 < \lambda < 1$ *and* $g \in \mathcal{W}_p$ *then* $J_g : \mathcal{D}_p^{\lambda} \to \mathcal{D}_p^{\lambda}$ *is bounded.*

Proof. (1) Set $\lambda = q/p < 1$ and suppose $I \subset \mathbb{T}$ is an interval. Using the growth condition (2.1) for $f \in \mathcal{D}_p^\lambda$ we have

$$\begin{aligned} &\frac{1}{|I|^{p\lambda}} \int_{S(I)} |J_g(f)'(z)|^2(1-|z|^2)^p \, dm(z) \\ =& \frac{1}{|I|^{q}} \int_{S(I)} |f(z)|^2|g'(z)|^2(1-|z|^2)^p \, dm(z) \\ \lesssim& \frac{1}{|I|^{q}} \int_{S(I)} \frac{1}{(1-|z|^2)^{p(1-\lambda)}} |g'(z)|^2(1-|z|^2)^p \, dm(z) \, \|f\|_{\mathcal{D}_p^\lambda}^2 \\ =& \frac{1}{|I|^{q}} \int_{S(I)} |g'(z)|^2(1-|z|^2)^q \, dm(z) \, \|f\|_{\mathcal{D}_p^\lambda}^2 \\ \lesssim& \|g\|_{\mathcal{Q}_q}^2 \|f\|_{\mathcal{D}_p^\lambda}^2, \end{aligned}$$

and the assertion follows by taking supremum on the left.

(2) Let $f \in \mathcal{D}_p^\lambda$. For an interval $I \subset \mathbb{T}$ let $w = w_I = (1-|I|)e^{i\theta}$ where $e^{i\theta}$ is the center of I. Then

$$\begin{aligned}
&\frac{1}{|I|^{p\lambda}} \int_{S(I)} |J_g(f)'(z)|^2 (1-|z|^2)^p \, dm(z) \\
=&\frac{1}{|I|^{p\lambda}} \int_{S(I)} |f(z)|^2 |g'(z)|^2 (1-|z|^2)^p \, dm(z) \\
\leq&\frac{2}{|I|^{p\lambda}} \int_{S(I)} |f(w)|^2 |g'(z)|^2 (1-|z|^2)^p \, dm(z) \\
&+ \frac{2}{|I|^{p\lambda}} \int_{S(I)} |f(z)-f(w)|^2 |g'(z)|^2 (1-|z|^2)^p \, dm(z) \\
=&A_I + B_I.
\end{aligned}$$

For the first integral, using (2.1) and recalling that $\mathcal{W}_p \subset \mathcal{Q}_p$ we have

$$A_I \lesssim \|f\|^2_{\mathcal{D}_p^\lambda} \frac{1}{|I|^p} \int_{S(I)} |g'(z)|^2 (1-|z|^2)^p \, dm(z) \lesssim \|f\|^2_{\mathcal{D}_p^\lambda} \|g\|^2_{\mathcal{Q}_p}.$$

For the second integral we write

$$\begin{aligned}
B_I &= \frac{2}{|I|^{p\lambda}} \int_{S(I)} \left| \frac{f(z)-f(w)}{(1-\overline{w}z)^p} \right|^2 |1-\overline{w}z|^{2p} |g'(z)|^2 (1-|z|^2)^p \, dm(z) \\
&\lesssim |I|^{p(2-\lambda)} \int_{S(I)} \left| \frac{f(z)-f(w)}{(1-\overline{w}z)^p} \right|^2 |g'(z)|^2 (1-|z|^2)^p \, dm(z) \\
&= (1-|w|)^{p(2-\lambda)} \int_{S(I)} \left| \frac{f(z)-f(w)}{(1-\overline{w}z)^p} \right|^2 |g'(z)|^2 (1-|z|^2)^p \, dm(z). \\
&\lesssim (1-|w|)^{p(2-\lambda)} |f(0)-f(w)|^2 \\
&\quad + (1-|w|)^{p(2-\lambda)} \int_{\mathbb{D}} \left| \frac{d}{dz} \left(\frac{f(z)-f(w)}{(1-\overline{w}z)^p} \right) \right|^2 (1-|z|^2)^p \, dm(z) \\
&= (1-|w|)^{p(2-\lambda)} |f(0)-f(w)|^2 + C_w,
\end{aligned}$$

where we have used the hypothesis that $d\mu_g(z) = |g'(z)|^2 (1-|z|^2)^p \, dm(z)$ is a $\mathcal{D}_p$-Carleson measure. The first term in the last sum is

$$(1-|w|)^{p(2-\lambda)} |f(0)-f(w)|^2 < (1-|w|)^{p(1-\lambda)} |f(0)-f(w)|^2 \lesssim \|f\|^2_{\mathcal{D}_p^\lambda}$$

by using (2.1) once more. For the second term we have

$$\begin{aligned}
C_w =&(1-|w|)^{p(2-\lambda)}\int_{\mathbb{D}}\left|\frac{d}{dz}\left(\frac{f(z)-f(w)}{(1-\overline{w}z)^p}\right)\right|^2(1-|z|^2)^p\,dm(z)\\
&= (1-|w|)^{p(2-\lambda)}\int_{\mathbb{D}}|f'(z)|^2\frac{(1-|z|^2)^p}{|1-\overline{w}z|^{2p}}\,dm(z)\\
&\quad + |w|^2(1-|w|)^{p(2-\lambda)}\int_{\mathbb{D}}\left|\frac{f(z)-f(w)}{(1-\overline{w}z)^{1+p}}\right|^2(1-|z|^2)^p\,dm(z)\\
&\sim (1-|w|)^{p(1-\lambda)}\int_{\mathbb{D}}|f'(z)|^2(1-|\varphi_w(z)|^2)^p\,dm(z)\\
&\quad + |w|^2(1-|w|)^{p(1-\lambda)}\int_{\mathbb{D}}\left|\frac{f(z)-f(w)}{1-\overline{w}z}\right|^2(1-|\varphi_w(z)|^2)^p\,dm(z)\\
&\lesssim \|f\|^2_{\mathcal{D}^\lambda_p} + (1-|w|)^{p(1-\lambda)}\int_{\mathbb{D}}\left|\frac{f(z)-f(w)}{1-\overline{w}z}\right|^2(1-|\varphi_w(z)|^2)^p\,dm(z)\\
&= \|f\|^2_{\mathcal{D}^\lambda_p} + D_w.
\end{aligned}$$

Observe that

$$\begin{aligned}
D_w &= (1-|w|)^{p(1-\lambda)}\int_{\mathbb{D}}\left|\frac{f(z)-f(w)}{1-\overline{w}z}\right|^2(1-|\varphi_w(z)|^2)^p\,dm(z)\\
&= (1-|w|)^{p(1-\lambda)}\int_{\mathbb{D}}\left|\frac{f\circ\varphi_w(z)-f\circ\varphi_w(0)}{1-\overline{w}\varphi_w(z)}\right|^2|\varphi_w'(z)|^2(1-|z|^2)^p\,dm(z)\\
&= (1-|w|)^{p(1-\lambda)}\int_{\mathbb{D}}\left|\frac{f\circ\varphi_w(z)-f\circ\varphi_w(0)}{1-\overline{w}z}\right|^2(1-|z|^2)^p\,dm(z)
\end{aligned}$$

To find an upper estimate for D_w, we follow the argument of [13], pages 551-552. See also [21, page 2080]. The argument consists of applying a reproducing formula from [15], the Cauchy-Schwarz inequality, Fubini's theorem and the estimate [25, Lemma 3.10(b)]. We refrain from writing all the details since the argument applies *mutatis mutandis.* The final steps of the calculation are as follows

$$\begin{aligned}
D_w &\lesssim (1-|w|)^{p(1-\lambda)}\int_{\mathbb{D}}|(f\circ\varphi_w)'(z)|^2\,\frac{(1-|z|^2)^{2+p}}{|1-\overline{w}z|^2}\,dm(z)\\
&\lesssim (1-|w|)^{p(1-\lambda)}\int_{\mathbb{D}}|(f\circ\varphi_w)'(z)|^2\,(1-|z|^2)^p\,dm(z)\\
&\lesssim (1-|w|)^{p(1-\lambda)}\int_{\mathbb{D}}|f'(z)|^2\,(1-|\varphi_w(z)|^2)^p\,dm(z)\\
&\lesssim \|f\|^2_{\mathcal{D}^\lambda_p}.
\end{aligned}$$

Collecting all the above estimates gives $\|J_g(f)\|_{\mathcal{D}_p^\lambda} \leq C\|f\|_{\mathcal{D}_p^\lambda}$ which is the desired conclusion. $\square$

The above theorems in combination with (3.1) give the following corollary for multipliers of $\mathcal{D}_p^\lambda$.

Corollary 1. *Suppose* $0 < p, \lambda < 1$ *and* $g \in \mathcal{H}ol(\mathbb{D})$. *Then*

1. *If* $g \in \mathcal{W}_p \cap H^\infty$ *then* $M_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ *is bounded.*
2. *If* $g \in \mathcal{Q}_{p\lambda} \cap H^\infty$ *then* $M_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ *is bounded*
3. *If* $M_g : \mathcal{D}_p^\lambda \to \mathcal{D}_p^\lambda$ *is bounded then* $g \in \mathcal{Q}_p \cap H^\infty$.

Remark. Let $0 < p < 1$. We know that $\mathcal{W}_p \subset \mathcal{Q}_p$, and this inclusion is strict [20, Theorem 6.3.4]. At the same time for $0 < q < p$ we have $\mathcal{Q}_q \subset \mathcal{Q}_p$ with strict inclusion. For each $q < p$ we give an example of a function f such that $f \in \mathcal{W}_p$ but f does not belong to $\mathcal{Q}_q$. Thus $\mathcal{W}_p \nsubseteq \mathcal{Q}_q$ for any $q < p$.

Indeed with q, p as above consider the function

$$f(z) = \sum_{k=1}^{\infty} a_k z^{2^k} \tag{3.5}$$

where $a_k = 1/2^{k(1-q)/2}$. By a theorem of Yamashita [24, Theorem 1(i)] for such Hadamard gap series, and since

$$\limsup |a_k| 2^{k\left(1-\frac{1+q}{2}\right)} = 1 < \infty,$$

it follows that f satisfies the growth condition

$$\sup_{z\in\mathbb{D}} |f'(z)|(1-|z|)^{\frac{1+q}{2}} < \infty.$$

Applying Proposition 4.2 of [5] (after adjusting the parameters involved to our notation) we find that this function is a multiplier of $\mathcal{D}_p$ because $q < p$. Thus the bounded function f belongs to $\mathcal{W}_p$.

On the other hand

$$\sum_{k=0}^{\infty} 2^{k(1-q)} \left(\sum_{2^k \leq n_j < 2^{k+1}} |a_j|^2 \right) = \sum_{k=0}^{\infty} 1 = \infty,$$

and therefore by [19, Theorem 1.2.1] for such Hadamard gap series, $f \notin \mathcal{Q}_q$.

The complete description of the multiplier space $M(\mathcal{D}_p^\lambda)$ and of the symbols g for which J_g is bounded on $\mathcal{D}_p^\lambda$, seems to be a hard problem.

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