THE CYCLINE SUBALGEBRA OF A KUMJIAN-PASK ALGEBRA

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Abstract. Let Λ be a row-finite higher-rank graph with no sources. We identify a maximal commutative subalgebra $\mathcal{M}$ inside the Kumjian-Pask algebra $\mathrm{KP}_R(\Lambda)$. We also prove a generalized Cuntz-Krieger uniqueness theorem for Kumjian-Pask algebras which says that a representation of $\mathrm{KP}_R(\Lambda)$ is injective if and only if it is injective on $\mathcal{M}$.

1. Introduction

The Leavitt path algebra of a directed graph over a field is a specific type of path algebra associated to a graph modulo some relations. Leavitt path algebras were introduced in [1] and [3], and they are the purely algebraic version of Cuntz-Krieger graph C^*-algebras; on the other hand, they generalize the algebras without invariant basis number studied by Leavitt in [11]. The relationship between the algebraic and analytic theories has been mutually beneficial. Both families of algebras have proved to be rich sources of interesting examples and have attracted interest from a broad range of researchers. In this paper we study analogues of Leavitt path algebras associated to higher-rank graphs; these algebras are called Kumjian-Pask algebras. Concretely we extend the results given in [9] to Kumjian-Pask algebras.

Kumjian and Pask first introduced the notion of a higher-rank graph or k-graph Λ (in which paths have a k-dimensional degree and a 1-graph reduces to a directed graph) and the associated C^*-algebras $C^*(\Lambda)$ in [10]. These C^*-algebras provide a visualisable model for higher-rank versions of the Cuntz-Krieger algebras studied by Robertson and Steger in [12]. The *Kumjian-Pask algebra* $\mathrm{KP}_R(\Lambda)$, defined and studied in [2], is an algebraic version of $C^*(\Lambda)$. Kumjian-Pask algebras have a universal property based on a family of generators satisfying suitable relations. The study of basic ideals and simplicity of $\mathrm{KP}_R(\Lambda)$ is done in [2], the socle and semisimplicity are considered in [4] and the center is analysed in [5]. Kumjian-Pask algebras for more general graphs are considered in [7] and [8].

A central topic in k-graphs algebras is to determine when a given homomorphism from $\mathrm{KP}_R(\Lambda)$ (or $C^*(\Lambda)$ in the analytic case) is injective; this is the content of the so-called uniqueness theorems. In [2] a graded-uniqueness theorem and a Cuntz-Krieger uniqueness theorem are proved for $\mathrm{KP}_R(\Lambda)$. Both require some conditions: the first one considers only $\mathbb{Z}^k$-graded homomorphisms, while the second requires the extra hypothesis on the graph, that is, Λ is 'aperiodic'.

In [6], a more general version of the Cuntz-Krieger uniqueness theorem is proved in the C^*-algebraic setting that has no additional hypotheses on the homomorphism or

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the graph. Here we translate to Kumjian-Pask algebras the analytic result given in [6]: a representation of $\mathrm{KP}_R(\Lambda)$ is injective if and only if it is injective on a distinguished subalgebra, called the *cycline subalgebra*.

At the same time we prove a more general version of the main results of [9] in the context of 1-graphs. In [9] the second and third named authors prove a uniqueness theorem for Leavitt path algebras which establishes that the injectivity of a representation depends only on its injectivity on a certain commutative subalgebra [9, Theorem 5.2]. Note that these results are not corollaries of the ones we obtain in this paper, since in [9] arbitrary graphs are considered and here we suppose that k-graphs are row-finite with no sources.

The paper is organized as follows. We begin with a section where we give the background material, including the definition of $\mathrm{KP}_R(\Lambda)$ and some basic properties. In Section 3 we establish some properties of the diagonal. In Section 4 we study the cycline subalgebra. Analogous to the definition given in [6], the cycline subalgebra $\mathcal{M}$ is generated by elements of the form $s_\alpha s_{\beta^*}$ where $(\alpha, \beta) \in \Lambda \times \Lambda$ is a *cycline pair* (Proposition 4.1). In Theorem 4.6, we prove that $\mathcal{M}$ is a maximal commutative subalgebra inside $\mathrm{KP}_R(\Lambda)$.

Finally in Section 5 we give our main result Theorem 5.4: for Λ a row-finite k-graph without sources, $\Phi : \mathrm{KP}_R(\Lambda) \to A$ is an injective ring homomorphism if and only if $\Phi|_{\mathcal{M}}$ is injective.

2. Preliminaries

First we give some necessary background which will be used later and we fix our notation. Let k be a positive integer. We consider the additive semigroup $\mathbb{N}^k$ as a category with one object. We say a countable category $\Lambda = (\Lambda^0, \Lambda, r, s)$ with objects Λ^0, morphisms Λ, range map r and source map s, is a *k-graph* if there exists a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, satisfying the *unique factorization property*: if $d(\lambda) = m + n$ for some $m, n \in \mathbb{N}^k$, then there exist unique $\mu, \nu \in \mathbb{N}^k$ such that $r(\nu) = s(\mu)$ and $d(\mu) = m$, $d(\nu) = n$ with $\lambda = \mu\nu$. Since we think of Λ as a generalized graph, we call $\lambda \in \Lambda$ a *path* in Λ and $v \in \Lambda^0$ a *vertex*.

For $n \in \mathbb{N}^k$ define $\Lambda^n = d^{-1}(\{n\})$ and call the elements λ of Λ^n *paths of degree n*; by the factorization property we identify Λ^0 as the paths of degree 0 (or the set of vertices). For any $v \in \Lambda^0$ and $X \subseteq \Lambda$ we denote $vX = \{\lambda \in X \mid r(\lambda) = v\}$ and $Xv = \{\lambda \in X \mid s(\lambda) = v\}$. A k-graph Λ has *no sources* if for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ the set $v\Lambda^n$ is nonempty; Λ is *row-finite* if for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ the set $v\Lambda^n$ is finite. In this paper we are concerned only with row-finite k-graphs without sources.

Example 2.1. Let $E = (E^0, E^1, r, s)$ be a directed graph. Then the *path category* Λ_E has object set E^0, and the morphisms in Λ_E from $v \in E^0$ to $w \in E^0$ are finite paths μ with $s(\mu) = v$ and $r(\mu) = w$; composition is defined by concatenation, and the identity morphisms obtained by viewing the vertices as paths of length 0. With the degree functor $d : \Lambda_E \to \mathbb{N}$ as the length function, the path category (Λ_E, d) is a 1-graph. Note that this requires us to use the convention where a path is a sequence of edges $e_1 \dots e_n$ such that $s(e_i) = r(e_{i+1})$.

For $m, n \in \mathbb{N}^k$, $m \leq n$ means $m_i \leq n_i$ for all $1 \leq i \leq k$ and $m \vee n$ denotes the pointwise maximum.

Example 2.2. Let $\Omega_k^0 = \mathbb{N}^k$, $\Omega_k = \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q\}$, define $r, s : \Omega_k \to \Omega_k^0$ by $r(p,q) = p$ and $s(p,q) = q$, define composition by $(p,q)(q,r) = (p,r)$, and define $d : \Omega_k \to \mathbb{N}^k$ by $d(p,q) = q - p$. Then $\Omega_k = (\Omega_k, r, s, d)$ is a k-graph.

An *infinite path* in Λ is a degree-preserving functor $x : \Omega_k \to \Lambda$. We denote the set of all infinite paths by Λ^∞. We denote $x(0,0)$ by $r(x)$ and refer to this vertex as the *range* of x.

For $\mu \in \Lambda$ define the *cylinder set*

$$Z(\mu) := \{x \in \Lambda^\infty \mid x(0, d(\mu)) = \mu\}.$$

For $\alpha \in \Lambda$ and $x \in Z(s(\alpha))$, define αx to be the unique infinite path such that for any $n \geq d(\alpha)$, we have that $(\alpha x)(0,n) = \alpha(x(0, n - d(\alpha)))$. We denote $Z(\alpha)$ by $\alpha\Lambda^\infty$ when α is a vertex. The collection of cylinder sets is a basis of compact sets for a Hausdorff topology on Λ^∞ (see [10, Proposition 2.8]).

For $p \in \mathbb{N}^k$ we define a map $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by $\sigma^p(x)(m,n) = x(m+p, n+p)$ for every $x \in \Lambda^\infty$, $(m,n) \in \Omega_k$. Note that by unique factorization $x = x(0,p)\sigma^p(x)$.

We say that a path $x \in \Lambda^\infty$ is *periodic* if there exists $p \neq q \in \mathbb{N}^k$ such that $\sigma^p(x) = \sigma^q(x)$. That is, for all $m, n \in \mathbb{N}^k$ $x(m+p, n+p) = x(m+q, n+q)$. If x is not periodic, we say x is *aperiodic.* A row-finite k-graph with no sources is called *aperiodic* if for every vertex $v \in \Lambda^0$, there exists an aperiodic path in $v\Lambda^\infty$.

If Λ is a k-graph, we let $\Lambda^{\neq 0} := \{\lambda \in \Lambda : d(\lambda) \neq 0\}$. For each $\lambda \in \Lambda^{\neq 0}$ we introduce a *ghost path* λ^* and for $v \in \Lambda^0$ define $v^* := v$. We write $G(\Lambda)$ for the set of ghost paths, or $G(\Lambda^{\neq 0})$ if we wish to exclude vertices. We define d, r and s on $G(\Lambda)$ by $d(\lambda^*) = -d(\lambda)$, $r(\lambda^*) = s(\lambda)$, $s(\lambda^*) = r(\lambda)$; we then define composition on $G(\Lambda)$ by setting $\lambda^*\mu^* = (\mu\lambda)^*$ for $\lambda, \mu \in \Lambda^{\neq 0}$ with $r(\mu^*) = s(\lambda^*)$. The factorization property of Λ induces a similar factorization property on $G(\Lambda)$.

Definition 2.3. Let Λ be a row-finite k-graph without sources and let R be a commutative ring with 1. A *Kumjian-Pask Λ-family* (P, S) in an R-algebra A consists of two functions $P : \Lambda^0 \to A$ and $S : \Lambda^{\neq 0} \cup G(\Lambda^{\neq 0}) \to A$ such that:

(KP1) $\{P_v : v \in \Lambda^0\}$ is a family of mutually orthogonal idempotents;

(KP2) for all $\lambda, \mu \in \Lambda^{\neq 0}$ with $r(\mu) = s(\lambda)$, we have

$$S_\lambda S_\mu = S_{\lambda\mu},\ S_{\mu^*}S_{\lambda^*} = S_{(\lambda\mu)^*},\ P_{r(\lambda)}S_\lambda = S_\lambda = S_\lambda P_{s(\lambda)},\ P_{s(\lambda)}S_{\lambda^*} = S_{\lambda^*} = S_{\lambda^*}P_{r(\lambda)};$$

(KP3) for all $\lambda, \mu \in \Lambda^{\neq 0}$ with $d(\lambda) = d(\mu)$, we have $S_{\lambda^*}S_\mu = \delta_{\lambda,\mu}P_{s(\lambda)}$; and

(KP4) for all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k \setminus \{0\}$, we have $P_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_{\lambda^*}$.

In [2, Theorem 3.4] it is proved that there is an R-algebra $\mathrm{KP}_R(\Lambda)$ generated by a Kumjian-Pask Λ-family (p, s) with the following universal property: whenever (Q, T) is a Kumjian-Pask Λ-family in an R-algebra A, there is a unique R-algebra homomorphism $\pi_{Q,T} : \mathrm{KP}_R(\Lambda) \to A$ such that $\pi_{Q,T}(p_v) = Q_v$, $\pi_{Q,T}(s_\lambda) = T_\lambda$, $\pi_{Q,T}(s_{\mu^*}) = T_{\mu^*}$ for $v \in \Lambda^0$ and $\lambda, \mu \in \Lambda^{\neq 0}$. We call $\mathrm{KP}_R(\Lambda)$ the *Kumjian-Pask algebra* of Λ and the generating family (p, s) the *universal Kumjian-Pask family.*

With the convention that $s_v = p_v$ and $s_{v^*} = p_v$ it follows from the *Kumjian-Pask relations* (KP1)-(KP4) that

$$\mathrm{KP}_R(\Lambda) = \mathrm{span}_R\{s_\mu s_{\nu^*} : \mu, \nu \in \Lambda \text{ with } s(\mu) = s(\nu)\},$$

and $s_\mu s_{\nu^*} \neq 0$ in $\mathrm{KP}_R(\Lambda)$ if $s(\mu) = s(\nu)$. For every nonzero $a \in \mathrm{KP}_R(\Lambda)$ and $n \in \mathbb{N}^k$ there exist $m \geq n$ and a finite subset $F \subseteq \Lambda \times \Lambda^m$ such that $s(\alpha) = s(\beta)$ for all $(\alpha, \beta) \in F$ and

$$a = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*}$$

with $r_{\alpha,\beta} \in R \setminus \{0\}$. In this situation, we say that a is written in *normal form* [2, Lemma 4.2]. We can define an R-linear involution $a \mapsto a^*$ on $\mathrm{KP}_R(\Lambda)$ as follows: if $a = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*}$ then $a^* = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\beta s_{\alpha^*}$.

It is proved in [2, Theorem 3.4] that $\mathrm{KP}_R(\Lambda)$ is graded by $\mathbb{Z}^k$ such that for each $n \in \mathbb{Z}^k$,

$$\mathrm{KP}_R(\Lambda)_n = \mathrm{span}_R\{s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda \text{ and } d(\lambda) - d(\mu) = n\}.$$

[2, Theorem 3.4] also says that $rp_v \neq 0$ for $v \in \Lambda^0$ and $r \in R \setminus \{0\}$.

3. THE DIAGONAL SUBALGEBRA

In this section we establish some properties of the *diagonal subalgebra* $\mathcal{D}$ of the Kumjian-Pask algebra $\mathrm{KP}_R(\Lambda)$. Define

$$\mathcal{D} := \langle s_\mu s_{\mu^*} : \mu \in \Lambda \rangle;$$

that is, $\mathcal{D}$ is the R-subalgebra of $\mathrm{KP}_R(\Lambda)$ generated by the set $\{s_\mu s_{\mu^*} : \mu \in \Lambda\}$. Observe that for $v \in \Lambda^0$, $s_v s_{v^*} = p_v$.

Lemma 3.1. *The diagonal subalgebra $\mathcal{D}$ is commutative.*

Proof. Using the Kumjian-Pask relations it follows that for each n, $\{s_\mu s_{\mu^*} : \mu \in \Lambda^n\}$ is a set of mutually orthogonal idempotents (see [2, Remarks 3.2]). Now if we consider $\lambda, \mu \in \Lambda$ then by [2, Lemma 3.3], for each $q \geq d(\lambda) \vee d(\mu)$ we have that

$$s_{\lambda^*} s_\mu = \sum_{d(\lambda\alpha)=q,\ \lambda\alpha=\mu\beta} s_\alpha s_{\beta^*}.$$

Then $(s_\lambda s_{\lambda^*})(s_\mu s_{\mu^*}) = (s_\mu s_{\mu^*})(s_\lambda s_{\lambda^*})$ because:

$$\begin{aligned}
(s_\lambda s_{\lambda^*})(s_\mu s_{\mu^*}) &= s_\lambda (s_{\lambda^*} s_\mu) s_{\mu^*} \\
&= s_\lambda \Big(\sum_{d(\lambda\alpha)=q,\ \lambda\alpha=\mu\beta} s_\alpha s_{\beta^*} \Big) s_{\mu^*} \\
&= \sum_{d(\lambda\alpha)=q,\ \lambda\alpha=\mu\beta} s_\lambda s_\alpha s_{\beta^*} s_{\mu^*} \\
&= \sum_{d(\lambda\alpha)=q,\ \lambda\alpha=\mu\beta} s_{\lambda\alpha} s_{(\mu\beta)^*} \\
&= \sum_{d(\mu\beta)=q,\ \mu\beta=\lambda\alpha} s_{\mu\beta} s_{(\lambda\alpha)^*} \\
&= \sum_{d(\mu\beta)=q,\ \mu\beta=\lambda\alpha} s_\mu s_\beta s_{\alpha^*} s_{\lambda^*}
\end{aligned}$$

$$\begin{aligned}
&= s_\mu\Big(\sum_{d(\mu\beta)=q,\ \mu\beta=\lambda\alpha} s_\beta s_{\alpha^*}\Big)s_{\lambda^*}\\
&= s_\mu(s_{\mu^*}s_\lambda)s_{\lambda^*}\\
&= (s_\mu s_{\mu^*})(s_\lambda s_{\lambda^*}).
\end{aligned}$$

So $\mathcal{D}$ is commutative. □

Remark 3.2. Observe that

$$s_{\alpha\gamma}s_{(\alpha\gamma)^*}s_\alpha s_{\beta^*}s_{\beta\eta}s_{(\beta\eta)^*} = s_{\alpha\gamma}s_{\gamma^*}s_\eta s_{(\beta\eta)^*}.$$

In particular if $d(\gamma) = d(\eta)$, then

$$s_{\alpha\gamma}s_{(\alpha\gamma)^*}s_\alpha s_{\beta^*}s_{\beta\eta}s_{(\beta\eta)^*} = \begin{cases} s_{\alpha\gamma}s_{(\beta\gamma)^*} & \text{if } \ \gamma=\eta,\\ 0 & \text{otherwise.}\end{cases}$$

We may view elements of the diagonal $\mathcal{D}$ as functions from Λ^∞ to R in the following way. For $\mu \in \Lambda$ let $1_{Z(\mu)}$ denote the characteristic function association to $Z(\mu)$. That is, $1_{Z(\mu)} : \Lambda^\infty \to R$ such that

$$1_{Z(\mu)}(x) = \begin{cases} 1 & \text{if } \ x \in Z(\mu),\\ 0 & \text{otherwise.}\end{cases}$$

Let

$$A_{\mathcal{D}} := \operatorname{span}_R\{1_{Z(\mu)} : \ \mu \in \Lambda\}.$$

Then $A_{\mathcal{D}}$ as an R-algebra with addition and scalar multiplication defined pointwise and multiplication of the generators is given by

$$1_{Z(\mu)} \cdot 1_{Z(\nu)} = 1_{Z(\mu)\cap Z(\nu)}.$$

Remark 3.3. In fact $A_{\mathcal{D}}$ is the diagonal of the *Steinberg algebra* associated to Λ (see Section 5 of [8] for more details).

The following lemma follows from [8, Proposition 5.4]:

Lemma 3.4. *The map* $\pi : \mathcal{D} \to A_{\mathcal{D}}$ *such that* $\pi(s_\mu s_{\mu^*}) = 1_{Z(\mu)}$ *is an isomorphism.*

Lemma 3.5. *Let* $\mu, \nu \in \Lambda$ *with* $s(\mu) = s(\nu)$ *and* $r \in R$ *such that* $rs_\mu s_{\mu^*} = rs_\nu s_{\nu^*}$. *Then* $s_\mu s_{\mu^*} = s_\nu s_{\nu^*}$.

Proof. Let $\pi : \mathcal{D} \to A_{\mathcal{D}}$. Then $\pi(rs_\mu s_{\mu^*}) = \pi(rs_\nu s_{\nu^*})$. Therefore we have $r1_{Z(\mu)} = r1_{Z(\nu)}$, which implies $Z(\mu) = Z(\nu)$. This means $s_\mu s_{\mu^*} = s_\nu s_{\nu^*}$ because the map π of Lemma 3.4 is injective. □

4. The cycline subalgebra

We study a special class of generators for $\mathrm{KP}_R(\Lambda)$ which will be used in the construction of a distinguished subalgebra, called *the cycline subalgebra* of $\mathrm{KP}_R(\Lambda)$. First an element $a \in \mathrm{KP}_R(\Lambda)$ is said to be *normal* if $aa^* = a^*a$. The proof of the following proposition follows exactly as the one given in [6, Proposition 4.1]. Note that in [6, Proposition 4.1], they write P_α for the projection $s_\alpha s_{\alpha^*}$.

Proposition 4.1. *Let Λ be a row-finite k-graph with no sources and R be a commutative ring with 1. Then for $(\alpha,\beta)\in\Lambda\times\Lambda$ with $s(\alpha)=s(\beta)$, the following conditions are equivalent:*

1. $s_{\alpha\gamma}s_{(\alpha\gamma)^*}=s_{\beta\gamma}s_{(\beta\gamma)^*}$ *for all* $\gamma\in s(\alpha)\Lambda$*;*
2. $s_\alpha s_{\beta^*}$ *is normal and commutes with* $\mathcal{D}$*; and*
3. $\alpha\gamma=\beta\gamma$ *for all* $\gamma\in s(\alpha)\Lambda^\infty$*.*

Definition 4.2. A pair $(\alpha,\beta)\in\Lambda\times\Lambda$ with $s(\alpha)=s(\beta)$ satisfying the equivalent conditions of Proposition 4.1 is called a *cycline pair*. Define

$$\mathcal{M}:=\ \langle s_\alpha s_{\beta^*}:(\alpha,\beta)\text{ cycline}\rangle;$$

that is, the R-subalgebra of $\mathrm{KP}_R(\Lambda)$ generated by the cycline pairs. We call $\mathcal{M}$ the *cycline subalgebra* of the Kumjian-Pask algebra $\mathrm{KP}_R(\Lambda)$.

Remark 4.3. Observe that (α,α) is cycline and hence $\mathcal{D}\subseteq\mathcal{M}$. Also (α,β) is cycline if and only if (β,α) is. For $(\alpha,\beta),(\mu,\nu)\in\Lambda\times\Lambda$ and $n\in\mathbb{N}^k$, $n\geq d(\alpha)\vee d(\beta)$ we have

$$s_\alpha s_{\beta^*}s_\mu s_{\nu^*}=\sum_{\gamma,\eta\in\Lambda,\ \beta\gamma=\mu\eta,\ d(\beta\gamma)=n}s_{\alpha\gamma}s_{(\nu\eta)^*}. \tag{1}$$

If $(\alpha,\beta),(\mu,\nu)$ are cycline and $\lambda\in s(\gamma)\Lambda$, then

$$s_{\nu\eta\lambda}s_{(\nu\eta\lambda)^*}=s_{\mu\eta\lambda}s_{(\mu\eta\lambda)^*}=s_{\beta\gamma\lambda}s_{(\beta\gamma\lambda)^*}=s_{\alpha\gamma\lambda}s_{(\alpha\gamma\lambda)^*}.$$

This means that the pairs $(\alpha\gamma,\nu\eta)$ appearing in the right hand side of (1) are cycline too by Proposition 4.1 (1).

Lemma 4.4. *The cycline subalgebra $\mathcal{M}$ is commutative.*

Proof. Let (α,β) and (μ,ν) be two cycline pairs. It suffices to show that

$$(s_\alpha s_{\beta^*})(s_\mu s_{\nu^*})=(s_\mu s_{\nu^*})(s_\alpha s_{\beta^*}).$$

Now

$$\begin{aligned}
s_\alpha s_{\beta^*}\ s_\mu s_{\nu^*}&=\sum_{\beta\gamma=\mu\eta}s_{\alpha\gamma}s_{(\nu\eta)^*}\\
&=\sum_{\beta\gamma=\mu\eta,\ \nu\eta=\alpha\gamma}s_{\alpha\gamma}\ \left(s_{(\alpha\gamma)^*}s_{\nu\eta}\right)\ s_{(\nu\eta)^*}\ \text{by (KP3)}\\
&=\sum_{\beta\gamma=\mu\eta,\ \nu\eta=\alpha\gamma}\left(s_{\alpha\gamma}s_{(\alpha\gamma)^*}\right)\ \left(s_{\nu\eta}s_{(\nu\eta)^*}\right)\\
&=\sum_{\beta\gamma=\mu\eta,\ \nu\eta=\alpha\gamma}\left(s_{\beta\gamma}s_{(\beta\gamma)^*}\right)\ \left(s_{\mu\eta}s_{(\mu\eta)^*}\right)\ \text{ since }(\alpha,\beta)\text{ and }(\mu,\nu)\text{ are cycline}\\
&=\sum_{\beta\gamma=\mu\eta,\ \nu\eta=\alpha\gamma}\left(s_{\mu\eta}s_{(\mu\eta)^*}\right)\ \left(s_{\beta\gamma}s_{(\beta\gamma)^*}\right)\ \text{since }\mathcal{D}\text{ is commutative}\\
&=\sum_{\beta\gamma=\mu\eta,\ \nu\eta=\alpha\gamma}s_{\mu\eta}\ \left(s_{(\mu\eta)^*}s_{\beta\gamma}\right)\ s_{(\beta\gamma)^*}
\end{aligned}$$

$$= \sum_{\nu\eta=\alpha\gamma} s_{\mu\eta} s_{(\beta\gamma)^*}$$
$$= s_\mu s_{\nu^*}\ s_\alpha s_{\beta^*} \text{ as desired.} \qquad \square$$

Remark 4.5. Let E be a 1-graph. In [6, Example 4.9], it is seen that a standard generator $s_\alpha s_{\beta^*}$ is cycline if and only if $\alpha = \beta$, $\alpha = \beta c$ or $\beta = \alpha c$ for some cycle $c \in s(\alpha)Es(\alpha)$ without entry. Thus the cycline subalgebra $\mathcal{M}$ coincides with *the commutative core* $M_R(E)$ defined in [9, Definition 3.15].

Define $\mathcal{D}'$ such that

$$\mathcal{D}' := \{a \in \mathrm{KP}_R(\Lambda) : ad = da \text{ for every } d \in \mathcal{D}\}. \tag{2}$$

Notice that we have the inclusions $\mathcal{D} \subseteq \mathcal{M} \subseteq \mathcal{D}'$, where the second inclusion comes from Proposition 4.1 (2). In fact, $\mathcal{M} = \mathcal{D}$ if Λ is aperiodic (see Lemma 5.5). Our main goal in this section is to prove the following.

Theorem 4.6. *Let Λ be a row-finite k-graph with no sources, R be a commutative ring with* 1*, $\mathcal{M}$ be the cycline subalgebra and $\mathcal{D}'$ be defined as in* (2)*. Then $\mathcal{M} = \mathcal{D}'$. In particular, $\mathcal{M}$ is a maximal commutative subalgebra of* $\mathrm{KP}_R(\Lambda)$.

To prove Theorem 4.6, we establish three lemmas. Recall that $\mathrm{KP}_R(\Lambda)$ is a $\mathbb{Z}^k$-graded algebra such that for $n \in \mathbb{Z}^k$,

$$\mathrm{KP}_R(\Lambda)_n = \mathrm{span}_R\{s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda \text{ and } d(\lambda) - d(\mu) = n\}.$$

Lemma 4.7. *Let $a \in \mathcal{D}'$ and suppose $a = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*}$ is in normal form. Then for any $(\alpha, \beta) \in F$, $r_{\alpha,\beta} s_\alpha s_{\beta^*} \in \mathcal{D}'$.*

Proof. First we claim that $a \in \mathcal{D}'$ if and only if $a_n \in \mathcal{D}'$ for all $n \in \mathbb{Z}^k$, where a_n is the homogeneous part of a of degree n. To prove the claim note if for all $n \in \mathbb{Z}^k$, $a_n \in \mathcal{D}'$ then $a \in \mathcal{D}'$. Assume that $a \in \mathcal{D}'$. Then for any $d \in \mathcal{D}$, $ad = da$. Since elements of $\mathcal{D}$ are homogeneous of degree zero then $a_n d = (ad)_n$. Then

$$a_n d = (ad)_n = (da)_n = da_n,$$

and so $a_n \in \mathcal{D}'$ for every $n \in \mathbb{Z}^k$.

Thus to prove the lemma, we can assume that a is homogeneous of degree n. Since all β have the same degree, then all α have the same degree. For every $(\gamma, \delta) \in F$, $s_\gamma s_{\gamma^*} \in \mathcal{D}$ and $s_\delta s_{\delta^*} \in \mathcal{D}$ so then $s_\gamma s_{\gamma^*} a s_\delta s_{\delta^*} \in \mathcal{D}'$. By (KP3) we have, $s_\gamma s_{\gamma^*} a s_\delta s_{\delta^*} = r_{\gamma,\delta} s_\gamma s_{\delta^*} \in \mathcal{D}'$. $\square$

In the following lemma we adopt some ideas from [13, Lemma 3.2].

Lemma 4.8. *Let $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$ and $r \in R \setminus \{0\}$ such that $rs_\mu s_{\nu^*} \in \mathcal{D}'$. Then $s_\mu s_{\mu^*} = s_\nu s_{\nu^*}$.*

Proof. First since $rs_\mu s_{\nu^*} \in \mathcal{D}'$ we have $rs_\mu s_{\mu^*} s_\mu s_{\nu^*} = rs_\mu s_{\nu^*} s_\mu s_{\mu^*}$, that is, $rs_\mu s_{\nu^*} = rs_\mu s_{\nu^*} s_\mu s_{\mu^*}$. Then multiplying both sides of this equation by s_{μ^*} on the left, we get $rs_{\mu^*} s_\mu s_{\nu^*} = rs_{\mu^*} s_\mu s_{\nu^*} s_\mu s_{\mu^*}$, which means by (KP3) that $rs_{\nu^*} = rs_{\nu^*} s_\mu s_{\mu^*}$. Finally multiplying by s_ν again on the left gives,

$$rs_\nu s_{\nu^*} = rs_\nu s_{\nu^*} s_\mu s_{\mu^*}.$$

Since $rs_\mu s_{\nu^*} \in \mathcal{D}'$, we have $rs_\nu s_{\mu^*} \in \mathcal{D}'$. Using the same argument we get

$$rs_\mu s_{\mu^*} = rs_\mu s_{\mu^*} s_\nu s_{\nu^*}.$$

Now $s_\nu s_{\nu^*}, s_\mu s_{\mu^*} \in \mathcal{D}$ and $\mathcal{D}$ is commutative so

$$rs_\mu s_{\mu^*} = rs_\mu s_{\mu^*} s_\nu s_{\nu^*} = rs_\nu s_{\nu^*} s_\mu s_{\mu^*} = rs_\nu s_{\nu^*}.$$

That is, $rs_\mu s_{\mu^*} = rs_\nu s_{\nu^*}$. Finally by Lemma 3.5 we obtain $s_\mu s_{\mu^*} = s_\nu s_{\nu^*}$. ☐

Lemma 4.9. *Let $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$ and $r \in R \setminus \{0\}$ such that $rs_\mu s_{\nu^*} \in \mathcal{D}'$. Then (μ, ν) is a cycline pair.*

Proof. Let $\gamma \in s(\mu)\Lambda$. We show $s_{\mu\gamma} s_{(\mu\gamma)^*} = s_{\nu\gamma} s_{(\nu\gamma)^*}$. Because $rs_\mu s_{\nu^*} \in \mathcal{D}'$, $rs_\nu s_{\mu^*} \in \mathcal{D}'$. So, on the one hand, we have:

$$\begin{aligned}
(rs_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*})(s_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*})^* &= (s_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*})(rs_\nu s_{\mu^*}) s_{\mu\gamma} s_{(\mu\gamma)^*} \\
&= (s_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*}) s_{\mu\gamma} s_{(\mu\gamma)^*} (rs_\nu s_{\mu^*}) \\
&= s_{\mu\gamma} s_{(\mu\gamma)^*} (rs_\mu s_{\nu^*}) s_{\mu\gamma} s_{(\mu\gamma)^*} s_\nu s_{\mu^*} \\
&= s_{\mu\gamma} s_{(\mu\gamma)^*} s_{\mu\gamma} s_{(\mu\gamma)^*} (rs_\mu s_{\nu^*}) s_\nu s_{\mu^*} \\
&= rs_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\mu^*}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(s_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*})^* (rs_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*}) &= (rs_\nu s_{\mu^*}) s_{\mu\gamma} s_{(\mu\gamma)^*} s_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*} \\
&= s_{\mu\gamma} s_{(\mu\gamma)^*} (rs_\nu s_{\mu^*}) s_\mu s_{\nu^*} \\
&= rs_{\mu\gamma} s_{(\mu\gamma)^*} s_\nu s_{\nu^*}.
\end{aligned}$$

But by Lemma 4.8, $rs_\mu s_{\nu^*} \in \mathcal{D}'$ implies $s_\mu s_{\mu^*} = s_\nu s_{\nu^*}$. Therefore

$$(rs_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*})(s_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*})^* = (s_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*})^* (rs_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*}). \tag{3}$$

Now:

$$\begin{aligned}
rs_{\mu\gamma} s_{(\mu\gamma)^*} &= (rs_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*})(s_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*})^* \\
&= (s_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*})^* (rs_{\mu\gamma} s_{(\mu\gamma)^*} s_\mu s_{\nu^*}) \text{ by (3)} \\
&= rs_{\nu\gamma} s_{(\nu\gamma)^*}.
\end{aligned}$$

Hence $rs_{\mu\gamma} s_{(\mu\gamma)^*} = rs_{\nu\gamma} s_{(\nu\gamma)^*}$ and by Lemma 3.5, we obtain

$$s_{\mu\gamma} s_{(\mu\gamma)^*} = s_{\nu\gamma} s_{(\nu\gamma)^*}.$$

Thus by Proposition 4.1 (1), (μ, ν) is a cycline pair. ☐

We are now ready to prove the main result of this section.

Proof of Theorem 4.6. To see that $\mathcal{M} = \mathcal{D}'$, we show $\mathcal{D}' \subseteq \mathcal{M}$. Let $a \in \mathcal{D}'$. Suppose a is written in normal form

$$a = \sum_{(\alpha,\beta) \in F} r_{\alpha,\beta} s_\alpha s_{\beta^*}$$

By Lemma 4.7, for each $(\alpha, \beta) \in F$ we have $r_{\alpha,\beta} s_\alpha s_{\beta^*} \in \mathcal{D}'$. Now we apply Lemma 4.9 to see that each (α, β) in F is a cycline pair. Thus $a \in \mathcal{M}$.

In order to prove that $\mathcal{M}$ is a maximal commutative subalgebra inside $\mathrm{KP}_R(\Lambda)$, first recall that $\mathcal{M}$ is commutative by Lemma 4.4. Now consider $\mathcal{C}$ a commutative subalgebra of $\mathrm{KP}_R(\Lambda)$ such that $\mathcal{M} \subseteq \mathcal{C}$. Since we have $\mathcal{D} \subseteq \mathcal{M} \subseteq \mathcal{C}$ then in particular

$$\mathcal{C} \subseteq \{x \in \mathrm{KP}_R(\Lambda) : xd = dx \text{ for every } d \in \mathcal{D}\} = \mathcal{D}'.$$

Therefore $\mathcal{C} = \mathcal{M}$. □

The following corollary involving the center of a Kumjian-Pask algebra, (studied in [5]) is immediate.

Corollary 4.10. *Let Λ be a row-finite k-graph with no sources and R be a commutative ring with* 1. *Then the center of the Kumjian-Pask algebra*

$$\mathcal{Z}(\mathrm{KP}_R(\Lambda)) := \{a \in \mathrm{KP}_R(\Lambda) : ax = xa \textit{ for every } x \in \mathrm{KP}_R(\Lambda)\} \subseteq \mathcal{M}.$$

5. General uniqueness theorem for Kumjian-Pask algebras

In this section we give a new uniqueness theorem for Kumjian-Pask algebras that says a homomorphism on $\mathrm{KP}_R(\Lambda)$ is injective if and only if it is injective on the cycline subalgebra $\mathcal{M}$. First we adapt some of the technical innovations from [6] to our setting.

For any subset $U \subseteq X$ of a topological space X we denote its interior by $\mathrm{Int}(U)$ and its boundary by ∂U. The following definition appears in [6, Definition 5.2]. For a k-graph Λ, define

$$\Sigma := \{(\alpha, \beta) \in \Lambda \times \Lambda \mid s(\alpha) = s(\beta), \alpha \neq \beta\}.$$

Then for any pair $(\alpha, \beta) \in \Sigma$ let

$$F_{\alpha,\beta} := \{x \in \Lambda^\infty \mid x \in Z(\alpha) \cap Z(\beta) \text{ and } \sigma^{d(\alpha)}(x) = \sigma^{d(\beta)}(x)\}.$$

We define the set of *regular paths* in Λ^∞ to be

$$\mathfrak{T}_\Lambda := \Lambda^\infty - \bigcup_{(\alpha,\beta)\in\Sigma} \partial F_{\alpha,\beta}.$$

Remark 5.1. In this remark we recall some properties of $F_{\alpha,\beta}$ and $\mathfrak{T}_\Lambda$, which are given in [6, Section 5].

(a) We have $F_{\alpha,\beta} = F_{\beta,\alpha}$.
(b) We have that $F_{\alpha,\beta}$ is closed for all $(\alpha, \beta) \in \Sigma$. Indeed if $x_i \to x$ in Λ^∞ and $n \in \mathbb{N}^k$ then $\sigma^n(x_i) \to \sigma^n(x)$; in particular if $x_i \in F_{\alpha,\beta}$ for all i then $x(0, d(\alpha)) = \alpha$, $x(0, d(\beta)) = \beta$ and $\sigma^{d(\alpha)}(x) = \sigma^{d(\beta)}(x)$, so $x \in F_{\alpha,\beta}$. Therefore, $\partial F_{\alpha,\beta} \subseteq F_{\alpha,\beta}$. Note that $\partial F_{\alpha,\beta}$ is also closed.
(c) The set $\mathfrak{T}_\Lambda$ is dense in Λ^∞ (by the Baire Category Theorem). So for every $v \in \Lambda^0$ there exists an $x \in Z(v) \cap \mathfrak{T}_\Lambda$.
(d) We have $\mathfrak{T}_\Lambda \cap F_{\alpha,\beta} \subseteq \mathrm{Int}(F_{\alpha,\beta})$.
(e) The cylinder set $Z(\mu) \subseteq F_{\alpha,\beta}$ if and only if $Z(\nu\mu) \subseteq F_{\nu\alpha,\nu\beta}$.
(f) If $x \in \mathfrak{T}_\Lambda$ and $\nu \in \Lambda r(x)$, then $\nu x \in \mathfrak{T}_\Lambda$.
(g) If $x \in \mathfrak{T}_\Lambda$ and $n \in \mathbb{N}^k$, then $\sigma^n(x) \in \mathfrak{T}_\Lambda$.

Remark 5.2. Let E be a 1-graph. An infinite path in E is called *essentially aperiodic* if it is either aperiodic or of the form $\mu ccc\cdots$, where c is a cycle without entry (see [9, Definition 3.5 (B)] for more details). In [6, Example 5.4], it is seen that $\mathfrak{T}_E$ is the set of essentially aperiodic paths of E.

The next lemma corresponds to [6, Lemma 5.8] and the proof translates exactly (so we omit it).

Lemma 5.3. *For $(\alpha,\beta)\in\Sigma$ and $x\in\mathfrak{T}_\Lambda$ we have the following:*

(a) *If $x\notin F_{\alpha,\beta}$, then there exists $\mu,\nu\in\Lambda$ such that $x\in Z(\mu)\cap Z(\nu)$ and*

$$s_\mu s_{\mu^*} s_\alpha s_{\beta^*} s_\nu s_{\nu^*} = 0.$$

(b) *If $x\in F_{\alpha,\beta}$, then there exists $\gamma\in\Lambda$ with $x\in Z(\alpha\gamma)\cap Z(\beta\gamma)$,*

$$s_{\alpha\gamma}s_{(\alpha\gamma)^*}s_\alpha s_{\beta^*}s_{\beta\gamma}s_{(\beta\gamma)^*} = s_{\alpha\gamma}s_{(\beta\gamma)^*},\ \textit{and } (\alpha\gamma,\beta\gamma) \textit{ cycline.}$$

Theorem 5.4. *Let Λ be a row-finite k-graph with no sources, R be a commutative ring with 1 and $\mathcal{M}$ be the cycline subalgebra of $\mathrm{KP}_R(\Lambda)$. If $\Phi:\mathrm{KP}_R(\Lambda)\to A$ is a ring homomorphism, then Φ is injective if and only if $\Phi|_{\mathcal{M}}$ is injective.*

Proof. We show that $\Phi|_{\mathcal{M}}$ injective implies Φ injective. By way of contradiction suppose we have $0\neq a\in\mathrm{KP}_R(\Lambda)$ such that $a\in\mathrm{Ker}\ \Phi$. Applying [4, Lemma 2.3 (i)] (and writing the sum involved in normal form) we can find $(\delta,\epsilon)\in\Lambda\times\Lambda$ such that

$$0\neq s_{\delta^*}as_\epsilon = r_{\delta,\epsilon}p_{s(\delta)} + \sum_{(\alpha,\beta)\in F,\ d(\alpha)\neq d(\beta)} r_{\alpha,\beta}s_\alpha s_{\beta^*},$$

where in particular $r_{\delta,\epsilon}\neq 0$. Let $b:=s_{\delta^*}as_\epsilon\neq 0$. Notice that $b\in\mathrm{Ker}\ \Phi$ because $\mathrm{Ker}\ \Phi$ is an ideal of $\mathrm{KP}_R(\Lambda)$. Now by Remark 5.1 (c), fix $x\in Z(s(\delta))\cap\mathfrak{T}_\Lambda$. For each $(\alpha,\beta)\in F$ we have two possibilities:

- If $x\notin F_{\alpha,\beta}$, then there exists $\mu=\mu_{\alpha,\beta},\nu=\nu_{\alpha,\beta}\in\Lambda$ as in Lemma 5.3(a) such that $x\in Z(\mu)\cap Z(\nu)$ and
$$s_\mu s_{\mu^*} s_\alpha s_{\beta^*} s_\nu s_{\nu^*} = 0.$$
- Otherwise, if $x\in F_{\alpha,\beta}$ then there exists $\gamma=\gamma_{\alpha,\beta}\in\Lambda$ as in Lemma 5.3(b) such that $x\in Z(\alpha\gamma)\cap Z(\beta\gamma)$ and
$$s_{\alpha\gamma}s_{(\alpha\gamma)^*}s_\alpha s_{\beta^*}s_{\beta\gamma}s_{(\beta\gamma)^*} = s_{\alpha\gamma}s_{(\beta\gamma)^*} \text{ with } (\alpha\gamma,\beta\gamma) \text{ cycline.}$$

Now let m be the following product:

$$\Big(\prod_{(\alpha,\beta)\in F,\ x\in F_{\alpha,\beta}} s_{\alpha\gamma}s_{(\alpha\gamma)^*}\Big)\Big(\prod_{(\alpha,\beta)\in F,\ x\notin F_{\alpha,\beta}} s_\mu s_{\mu^*}\Big)\, b\, \Big(\prod_{(\alpha,\beta)\in F,\ x\notin F_{\alpha,\beta}} s_\nu s_{\nu^*}\Big)\Big(\prod_{(\alpha,\beta)\in F,\ x\in F_{\alpha,\beta}} s_{\beta\gamma}s_{(\beta\gamma)^*}\Big)$$

Observe that every $s_{\alpha\gamma}s_{(\alpha\gamma)^*}, s_{\beta\gamma}s_{(\beta\gamma)^*}, s_\mu s_{\mu^*}, s_\nu s_{\nu^*}$ is inside $\mathcal{D}$. We have that $m\in\mathcal{M}$ by construction. Since $b\in\mathrm{Ker}\ \Phi$, we have $m\in\mathrm{Ker}\ \Phi$.

Let us see that $m\neq 0$: we show that the zero-graded component of m is non-zero, that is, $m_0\neq 0$. We have

$$m = \Big(\prod s_{\alpha\gamma}s_{(\alpha\gamma)^*}\Big)\Big(\prod s_\mu s_{\mu^*}\Big)\, r_{\delta,\epsilon}p_{s(\delta)}\, \Big(\prod s_\nu s_{\nu^*}\Big)\Big(\prod s_{\beta\gamma}s_{(\beta\gamma)^*}\Big) \ +\ m',$$

where

$$m' = (\prod s_{\alpha\gamma}s_{(\alpha\gamma)^*})(\prod s_\mu s_{\mu^*})\ (\sum_{d(\alpha)\neq d(\beta)} r_{\alpha,\beta}s_\alpha s_{\beta^*})\ (\prod s_\nu s_{\nu^*})(\prod s_{\beta\gamma}s_{(\beta\gamma)^*}).$$

We claim that

$$m_0 = (\prod s_{\alpha\gamma}s_{(\alpha\gamma)^*})(\prod s_\mu s_{\mu^*})\ r_{\delta,\epsilon}p_{s(\delta)}\ (\prod s_\nu s_{\nu^*})(\prod s_{\beta\gamma}s_{(\beta\gamma)^*}) \in \mathcal{D}.$$

To see this, consider $(\prod s_{\alpha\gamma}s_{(\alpha\gamma)^*})(\prod s_\mu s_{\mu^*})\ r_{\alpha,\beta}s_\alpha s_{\beta^*}\ (\prod s_\nu s_{\nu^*})(\prod s_{\beta\gamma}s_{(\beta\gamma)^*})$ a term in m'. Since $\prod s_{\alpha\gamma}s_{(\alpha\gamma)^*}, \prod s_\mu s_{\mu^*}, \prod s_\nu s_{\nu^*}, \prod s_{\beta\gamma}s_{(\beta\gamma)^*}$ are each 0-graded,

$$(\prod s_{\alpha\gamma}s_{(\alpha\gamma)^*})(\prod s_\mu s_{\mu^*})\ r_{\alpha,\beta}s_\alpha s_{\beta^*}\ (\prod s_\nu s_{\nu^*})(\prod s_{\beta\gamma}s_{(\beta\gamma)^*})$$

is $d(\alpha) - d(\beta) \neq 0$ graded. Thus m_0 is as claimed.

Recall from Lemma 3.4, we have an isomorphism $\pi : \mathcal{D} \to A_{\mathcal{D}}$. To see that $m_0 \neq 0$, it suffices to show $\pi(m_0) \neq 0$. Now

$$\begin{aligned}\pi(m_0) &= r_{\delta,\epsilon}(\prod 1_{Z(\alpha\gamma)})(\prod 1_{Z(\mu)}\ 1_{Z(s(\delta))})\ (\prod 1_{Z(\nu)})(\prod 1_{Z(\beta\gamma)})\\ &= r_{\delta,\epsilon}1_U\end{aligned}$$

where

$$U = \left(\bigcap Z(\alpha\gamma)\right) \cap \left(\bigcap Z(\mu)\right) \cap Z(s(\delta)) \cap \left(\bigcap Z(\nu)\right) \cap \left(\bigcap Z(\beta\gamma)\right).$$

By construction $x \in U$ and hence $1_U \neq 0$. So $\pi(m_0) \neq 0$ and finally $m \neq 0$ as desired.

This contradicts the assumption that $\Phi|_{\mathcal{M}}$ is injective because we found $0 \neq m \in \mathcal{M}$ and $m \in \text{Ker } \Phi$. Therefore Φ is injective. □

From Theorem 5.4 we can recover the usual Cuntz-Krieger uniqueness theorem given in [2, Theorem 4.7]. First we need to consider the following lemma which is proved similarly to [6, Proposition 4.8].

Lemma 5.5. *If Λ is aperiodic, then (α, β) is a cycline pair if and only if $\alpha = \beta$. In particular, $\mathcal{M} = \mathcal{D}$.*

Corollary 5.6. *Let Λ be an aperiodic row-finite k-graph without sources and R be a commutative ring with 1. If $\Phi : \mathrm{KP}_R(\Lambda) \to A$ is a ring homomorphism, then Φ is injective if and only if $\Phi(rp_v) \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$.*

Proof. We verify that $\Phi(rp_v) \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$ implies Φ is injective. We show $\Phi|_{\mathcal{M}}$ is injective, which suffices by Theorem 5.4. First, by Lemma 5.5, $\mathcal{M} = \mathcal{D}$. By way of contradiction suppose there exist $r \in R \setminus \{0\}$ and $\lambda \in \Lambda$ such that $\Phi(rs_\lambda s_{\lambda^*}) = 0$. But then $\Phi(s_{\lambda^*}(rs_\lambda s_{\lambda^*})s_\lambda) = 0$. Thus $\Phi(rp_{s(\lambda)}) = 0$, which is a contradiction. □

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