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ORIGINAL PAPER

Linear models of the exceptional Lie algebra $\mathfrak{e}_8$

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Abstract
This work presents five explicit constructions of the exceptional Lie algebra $\mathfrak{e}_8$, each associated with a semisimple subalgebra of maximal rank. The provided models are based on gradings by finite abelian groups, namely, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, $\mathbb{Z}_3^2$, and $\mathbb{Z}_2 \times \mathbb{Z}_4$. In all cases, the neutral component is a direct sum of special linear algebras, while the remaining homogeneous components are irreducible modules over it.

Keywords Exceptional Lie algebra · Gradings · Constructions · Multilinear algebra · Irreducible modules

Mathematics Subject Classification Primary 17B25 · Secondary 17B70

1 Introduction

Exceptional Lie algebras, and in particular the largest among them, $\mathfrak{e}_8$, are extensively studied and highly sought-after objects, yet they are often difficult to work with. Recall that the lowest-dimensional representation of $\mathfrak{e}_8$ is the adjoint one, so if we wish to realize $\mathfrak{e}_8$ inside an algebra of $n \times n$ matrices, the minimal possible value of n is 248. Alternatively, one can study the split form of $\mathfrak{e}_8$ via its root space decomposition. While this decomposition plays a central role in Lie theory, it may not be practical for certain purposes. For example, the bracket of two non-opposite root spaces, $[\mathcal{L}_\alpha, \mathcal{L}_\beta]$, lies in the one-dimensional space $\mathcal{L}_{\alpha+\beta}$; however, to compute the bracket explicitly, one must determine the precise scalar involved—a process that can be lengthy and technically involved. A third approach to all the exceptional Lie

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algebras (except for G_2) is the unified construction introduced by Tits in the 1960s [33]. This elegant and surprising construction involves two main ingredients: the Albert algebra, an exceptional simple Jordan algebra, and a composition algebra. However, Jordan algebras tend to be technically demanding and less intuitive to work with.

Moreover, in many situations–especially in physics–one is interested in specific symmetries of the object under study. In our context, a symmetry refers to an automorphism of the Lie algebra of finite order m, which is equivalent to a concrete grading of the algebra (namely, the $\mathbb{Z}_m$-grading defined by the eigenspaces of the automorphism). This is one of the reasons why models of exceptional Lie algebras based on multilinear algebra have a long-standing tradition: they are naturally related to gradings, and therefore exhibit a high degree of symmetry, while also relying—at least in principle—on multilinear algebra objects, which tend to be more manageable in practice.

To be more precise regarding the literature on explicit constructions of exceptional Lie algebras, a classical reference is the encyclopedia [32]: [32, Chapter 5, §1] is devoted to models arising from the octonion algebra, while [32, Chapter 5, §2] develops models associated with gradings over cyclic groups (not necessarily finite). Another influential source throughout this work has been [1], a posthumous book based on lecture notes from Adams' talks on exceptional Lie groups delivered at the University of Cambridge. This book provides constructions of all complex exceptional Lie algebras, as well as some of their real forms. A third reference, arguably the most accessible, is the section on algebraic constructions of the exceptional Lie algebras in [23, §22.4]. The *linear models* of $\mathfrak{e}_8$ developed in the above references are those related to gradings with very few components:

- The $\mathbb{Z}_2$-grading on $\mathfrak{e}_8$ whose neutral homogeneous component is isomorphic to the orthogonal algebra $\mathfrak{so}(16)$, as in [1, Theorem 6.3]. Here, the other homogeneous component corresponds to the half-spin representation of $\mathcal{L}_{\bar{0}}$.
- The $\mathbb{Z}_3$-grading on $\mathfrak{e}_8$ with neutral component isomorphic to the special linear algebra $\mathfrak{sl}(9)$. This construction dates back to Freudenthal [22] in the 1950s, but due to its significance it also appears in [32, p. 181] and [23, pp. 360–361]. Starting with a 9-dimensional vector space V, the Lie algebra $\mathfrak{e}_8$ is thus realized as the direct sum of the algebra of traceless endomorphisms of V, the space of trivectors on V, and the space of cotrivectors. In Chevalley's words, "the bracket operation is given in a very condensed form which exhibits a large degree of symmetry."

In fact, [32, pp. 180–181] also presents: a $\mathbb{Z}_2$-grading on E_7 whose neutral homogeneous component is a subalgebra isomorphic to $\mathfrak{sl}(8)$; a $\mathbb{Z}_2$-grading on E_6 with neutral homogeneous component isomorphic to $\mathfrak{sp}(8)$; and several $\mathbb{Z}$-gradings on E_6, E_7, and E_8 whose neutral homogeneous components are isomorphic to the general linear algebras $\mathfrak{gl}(6)$, $\mathfrak{gl}(7)$, and $\mathfrak{gl}(8)$, respectively. The structure constants needed to fully describe the Lie brackets in these models are listed in [32] without proof. They were computed by Katanova and include large values such as 40320 and 1080. (See Remark 2.4 for a comparison, showing how a suitable description of the invariant products can reduce such scalars to ± 1.)

A nice survey of both classical and recent results on exceptional structures of type E_8—including the Lie algebra itself—is [24]. Section 4 of that work recalls several constructions of the Lie algebra $\mathfrak{e}_8$ via gradings, which Garibaldi refers to as *an* $\mathcal{L}_{\bar{0}}$*-construction of* $\mathcal{L}$. He also discusses the most notable examples (the $\mathbb{Z}_2$- and $\mathbb{Z}_3$-gradings mentioned above), and briefly mentions the $\mathbb{Z}_5$- and $\mathbb{Z}_2^2$-gradings, although without providing complete constructions with explicit scalars. What he does include is a wealth of references (170!) as well as many interesting connections and applications.

The procedure of constructing simple Lie algebras based on $\mathbb{Z}_m$-gradings is, in some sense, part of the folklore: as mentioned, such gradings arise from automorphisms of order m. These automorphisms are classified in [28, Chapter 8], which also describes key properties of the homogeneous components of the grading as modules over the neutral component $\mathcal{L}_{\bar{0}}$ (see Sect. 2.1). Once the components are identified, the Lie brackets between them are determined up to scalar multiples (see Theorem 1.1 (d) below). These scalars can then be fixed by requiring that the resulting $\mathbb{Z}_m$-graded algebra satisfies the Jacobi identity. However, the outcome of this computation depends on the initial choice of invariant bilinear products between homogeneous components.

A more general situation is considered by the second author in [14], which deals with gradings over not necessarily cyclic groups. That work provides a description of the exceptional Lie algebra $\mathfrak{f}_4$ for each of its semisimple subalgebras of maximal rank, along with a constructive procedure that can be applied to any simple Lie algebra and any reductive subalgebra of maximal rank, using its irreducible modules. The key idea behind this procedure is that there always exists a grading of the Lie algebra by a (finitely generated) abelian group such that the neutral homogeneous component is the given reductive subalgebra, while the remaining components are irreducible modules for it. Once again, the products between homogeneous components are determined up to scalar multiples. The main result in that paper, inspired by [7], is the following:

Theorem 1.1 [14] *Let $\mathcal{L}$ be a (finite-dimensional) simple Lie algebra over an algebraically closed field $\mathbb{F}$ of zero characteristic, Φ a root system of $\mathcal{L}$ relative to a Cartan subalgebra H of $\mathcal{L}$, and Φ' a subset of Φ verifying $-\Phi' = \Phi'$ (*symmetric*) and $(\Phi' + \Phi') \cap \Phi \subset \Phi'$ (*closed*). Let G be the abelian group $\mathbb{Z}\Phi/\mathbb{Z}\Phi'$. Then:*

(a) $\Phi \cap \mathbb{Z}\Phi' = \Phi'$.
(b) $\mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ *is G-graded, being* $\mathcal{L}_e = \mathfrak{h} := H \oplus \sum_{\alpha\in\Phi'} L_\alpha$ *a reductive subalgebra and, for any $e \neq g \in G$, either $\mathcal{L}_g = 0$ or $\mathcal{L}_g$ is an $\mathfrak{h}$-irreducible module.*
(c) If $\mathcal{L} = \oplus_{h\in\widetilde{G}}M_h$ is another grading by an abelian group $\widetilde{G}$, where $\mathfrak{h} \subset M_e$ and $\widetilde{G}$ is generated by $\{h \in \widetilde{G} : M_h \neq 0\}$, then there is a group epimorphism $\pi : G \to \widetilde{G}$ such that $\mathcal{L}_g \subset M_{\pi(g)}$ for any $g \in G$.
(d) If $g_1, g_2, g_1 + g_2 \in G \setminus \{e\}$, then $\mathrm{Hom}_{\mathfrak{h}}(\mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2}, \mathcal{L}_{g_1+g_2}) = \mathbb{F}[\ ,\ ]|_{\mathcal{L}_{g_1}\otimes\mathcal{L}_{g_2}}$.
(e) If $g_1, g_2 \in G$, $g_1 + g_2 \neq e$, $[\mathcal{L}_{g_1}, \mathcal{L}_{g_2}] = \mathcal{L}_{g_1+g_2}$.
(f) $\mathfrak{h}$ is semisimple if and only if G is a finite group.
(g) $\mathcal{L}_g \neq 0$ for all $g \in G$ if and only if the bracket of any two irreducible $\mathfrak{h}$-submodules of $\mathcal{L}$ which are not contained in $\mathfrak{h}$ is not zero. In this case, $\mathfrak{h}$ is semisimple.

We are interested in obtaining new constructions of the largest exceptional Lie algebra, $\mathfrak{e}_8$, through the application of this result. In [17, Chapter II, 5, Table 10], one finds the list of maximal rank semisimple subalgebras of $\mathcal{L} = \mathfrak{e}_8$, that is, those to which Theorem 1.1 can be applied. For each such semisimple subalgebra $\mathfrak{h}$ of $\mathcal{L}$, we indicate the associated grading group $G = \mathbb{Z}\Phi/\mathbb{Z}\Phi'$, and present the resulting pairs $(G, \mathfrak{h})$ below:

$$\begin{gathered}(\mathbb{Z}_3, A_8),\quad (\mathbb{Z}_4, A_7 \oplus A_1),\quad (\mathbb{Z}_6, A_5 \oplus A_2 \oplus A_1),\quad (\mathbb{Z}_5, 2A_4),\\ (\mathbb{Z}_2, D_8),\quad (\mathbb{Z}_4, D_5 \oplus A_3),\quad (\mathbb{Z}_3, E_6 \oplus A_2),\quad (\mathbb{Z}_2, E_7 \oplus A_1),\end{gathered} \tag{1}$$

when the group is cyclic, and

$$\begin{gathered}(\mathbb{Z}_2^2, D_6 \oplus 2A_1),\quad (\mathbb{Z}_2^2, 2D_4),\quad (\mathbb{Z}_4 \times \mathbb{Z}_2, 2A_3 \oplus 2A_1),\\ (\mathbb{Z}_3^2, 4A_2),\quad (\mathbb{Z}_2^3, D_4 \oplus 4A_1),\quad (\mathbb{Z}_2^4, 8A_1),\end{gathered}$$

otherwise. For any of these pairs, $\mathfrak{e}_8$ can be constructed as a G-graded algebra with $\mathcal{L}_e = \mathfrak{h}$. Surprisingly, the large number of homogeneous components does not compromise the symmetry or simplicity of the resulting constructions. All fourteen models were previously presented in concise form at a conference in Spain in 2005 [13]. However, we now believe it is worthwhile to present, expand, and complete these results, given their potential usefulness to the mathematical community. The fact that this work takes the aforementioned theorem as a starting point does not imply that applying it is straightforward; what the theorem guarantees is merely the existence of such constructions. We will focus on the cases in which all the simple ideals of the neutral component are of type A, as the actions on their irreducible modules are considerably easier to describe than those involving simple ideals of type D, which require the use of spin modules. This situation occurs in 7 out of the 14 semisimple subalgebras of $\mathfrak{e}_8$ of rank 8, as listed above. More precisely, we will concentrate on providing a complete description of the models based on five of these subalgebras, since the other two have already been developed in the literature. Namely, in addition to the $\mathbb{Z}_3$-grading on $\mathfrak{e}_8$ based on A_8, a model of $\mathfrak{e}_8$ associated with the pair $(\mathbb{Z}_2^4, 8A_1)$ was proposed by Alberto Elduque in his paper [20], which deals with magic squares and symmetric composition algebras. In that work, he applied $\mathbb{Z}_2^2$- and $\mathbb{Z}_2^3$-gradings on the para-Hurwitz algebras to construct all the exceptional Lie algebras, using only copies of the symplectic Lie algebra $\mathfrak{sp}(V)$ and tensor products of copies of a two-dimensional vector space V endowed with an alternating bilinear form. In the specific case of $\mathfrak{e}_8$, this construction yields a neutral homogeneous component $\mathcal{L}_e \cong 8\,\mathfrak{sp}(V)$, and all 14 remaining nonzero homogeneous components are $\mathcal{L}_e$-modules isomorphic to $V^{\otimes 4}$. Moreover, the scalars involved in the multiplication between any pair of homogeneous components were explicitly computed in [20, Table 5]. Thus, the present manuscript is devoted to developing the models based on the remaining five subalgebras: those of types $A_7 \oplus A_1$, $A_5 \oplus A_2 \oplus A_1$, $2A_4$, $4A_2$, and $2A_3 \oplus 2A_1$, each of them with its own beauty. While the procedure may not be new, the resulting constructions are. In our view, the main difficulty in producing fully explicit models has been to establish a suitable notation, as introduced in Sect. 2.2. This notation has allowed us to compute all the relevant structure constants in a fully explicit and uniform way. It is worth noting that the scalars involved in our models are significantly simpler than those previously computed in the literature, despite the fact that the grading groups we consider are, in general, more intricate.

For completeness, let us briefly recall some of the appearances in the literature of the remaining cases in the list (1), that is, those with $\mathfrak{h}$ not of type A. Along the way, we would like to emphasize the importance of having explicit models and of maintaining a panoramic view of the most well-known constructions. The (complex) model associated with the pair $(\mathbb{Z}_2, E_7 \oplus A_1)$ has been used in [15] to describe the real symplectic triple systems whose standard enveloping algebras are of type $\mathfrak{e}_{8,8}$ or $\mathfrak{e}_{8,-24}$. Examples of irreducible Lie-Yamaguti algebras related to this model are also given in [5]. A beautiful construction of $\mathfrak{e}_8$ based on the pair $(\mathbb{Z}_2^2, 2D_4)$ appears in [3, Eq. (25)]. The structure constants are not computed there, as [3] is a broad narrative that connects ideas and areas, providing at most hints rather than complete proofs. A full construction, including the Lie brackets, is given in [19, §3], where $\mathfrak{e}_8$ is built from two copies of the para-octonion algebra; note that the triality Lie algebra of the para-octonion algebra is precisely of type D_4. In fact, the constructions in [19] also yield several other realizations from our list. For instance, the model of $\mathfrak{e}_8$ associated with the pair $(\mathbb{Z}_3, E_6 \oplus A_2)$: Elduque constructs $\mathfrak{e}_8$ using a para-octonion algebra and an Okubo algebra (in fact, any two symmetric composition algebras of dimension 8 suffice), and the natural $\mathbb{Z}_3$-grading on the Okubo algebra extends trivially to the whole $\mathfrak{e}_8$. Similarly, when two Okubo algebras are used in the construction, the extended $\mathbb{Z}_3$-grading on $\mathfrak{e}_8$ corresponds to the already mentioned pair $(\mathbb{Z}_3, A_8)$. The entire work [19] can be seen as a reinterpretation of the Freudenthal magic square, in which the classical composition (Hurwitz) algebras are replaced by two symmetric composition algebras, yielding a unified construction of all exceptional Lie algebras. An approach based on Hurwitz algebras is adopted in [4], where Barton and Sudbery describe the exceptional Lie superalgebras using the triality principle. A construction in the same spirit is presented by Landsberg and Manivel in [29].

Finally, let us mention one of the applications of the approach based on constructing a Lie algebra via its gradings. The graded contractions are introduced in [9] as an alternative to Inönü-Wigner contractions in Physics. Instead of a limit of sequences of mutually isomorphic Lie algebras (*continuos contractions*), the commutation relations among graded subspaces are multiplied by contraction parameters. This provides a system of quadratic equations resulting from the Jacobi identities. Of course, it is not difficult to read our results from this viewpoint. Equations as (16), (31), (42), (61) and (71) are, evidently, contraction equations. Throughout this work, we have restricted ourselves to using only nonzero scalars in order to simplify the proofs of the main propositions (our goal has always been to provide explicit constructions of $\mathfrak{e}_8$). This restriction implies that any two sets of scalars satisfying one of these systems are related via a symmetric 2-cocycle (see Remark 3.4 for more details). In contrast, a *graded contraction* of a G-graded algebra $\mathcal{L}$ is a map $\sigma: G \times G \to \mathbb{F}$ (note that it may take zero values) such that the algebra $\mathcal{L}^\sigma$, with multiplication defined as in (28), is a Lie algebra. The different isomorphism classes that arise in this setting depend heavily on the support of σ, that is, on the subset of $G \times G$ where σ takes nonzero values. An example of work on this topic is [26], which studies the algebras not isomorphic to $\mathfrak{sl}(3, \mathbb{C})$ which result from applying graded contractions to its fine $\mathbb{Z}_3^2$-grading given by the Gell-Mann matrices. Some examples of graded contractions motivated by Physics appear in [10]. Although we have not pursued that line of work here, it is closely related to ours. If one knows the graded contractions of a G-graded algebra $\mathcal{L}$, together with an explicit model of $\mathcal{L}$ adapted to the G-grading, then explicit models of the (possibly nilpotent or solvable) Lie algebras obtained after graded contraction can be immediately derived. For instance, after dealing in [12] with graded contractions of the $\mathbb{Z}_2^3$-graded exceptional Lie algebra $\mathfrak{g}_2$, the explicit construction of $\mathfrak{g}_2$ adapted to its $\mathbb{Z}_2^3$-grading given in [8, 11] has allowed us to provide concrete expressions for the resulting new Lie algebras in [8], as well as to study their properties in a rather unified way. This kind of approach could lead to a fairly straightforward application of our $\mathfrak{e}_8$-constructions as well.

The structure of the paper is as follows. We begin by presenting the general tools required to construct our linear models of $\mathfrak{e}_8$. In Sect. 2.1, we recall the relation between automorphisms of order m and $\mathbb{Z}_m$-gradings, since this is the mechanism that determines what can actually be constructed and leads us to the list (1). To apply Theorem 1.1, we need a precise description of the invariant products between pairs of irreducible $\mathfrak{sl}(V)$-modules, which is developed throughout Sect. 2.2. For convenience, we include Lemma 2.3, where these products are described in terms of basic elements, facilitating their use in the subsequent sections. We then turn to the explicit constructions of $\mathcal{L} = \mathfrak{e}_8$ based on the following pairs $(G, \mathfrak{h})$: the case $(\mathbb{Z}_5, 2A_4)$ in Sect. 3.1; the case $(\mathbb{Z}_4, A_7 \oplus A_1)$ in Sect. 3.2; and a third model with six homogeneous components in Sect. 3.3, corresponding to the pair $(\mathbb{Z}_6, A_5 \oplus A_2 \oplus A_1)$. The remaining two models are associated with non-cyclic grading groups: Sect. 3.4 is devoted to the particularly elegant case $(\mathbb{Z}_3^2, 4A_2)$, while Sect. 3.5 addresses the more intricate case $(\mathbb{Z}_2 \times \mathbb{Z}_4, 2A_3 \oplus 2A_1)$. The proofs are presented in a constructive spirit, with the aim of making the models not only verifiable but also usable in practice. In particular, the first proof—Proposition 3.1—is given in full detail, so that the reader can follow and reproduce each step, including the explicit computations of the products. As the underlying philosophy becomes clearer, the subsequent proofs—Propositions 3.6, 3.8—gradually adopt a more abstract style, with fewer intermediate computations, but still grounded in the same guiding principles. In particular, the last two models are especially demanding from a technical point of view. The main goal of Propositions 3.9 and 3.10 is to provide all the structure constants explicitly, a task that becomes substantially more involved in these cases.

Throughout the paper, the ground field $\mathbb{F}$ is assumed to be algebraically closed and of characteristic zero, although the models can be extended to Lie algebras over more general fields.

2 Tools for dealing with subalgebras of type *A*

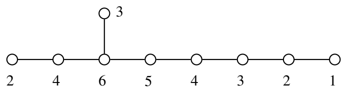
We begin by recalling how to determine the neutral homogeneous component of a grading over a finite cyclic group on the simple Lie algebra of type E_8. Applying the same procedure a second time leads to the cases associated with non-cyclic grading groups. Subsequently, in Sect. 2.2, we will give explicit descriptions of the irreducible $\mathfrak{sl}(V)$-modules and the invariant products between them, which serve as the fundamental tools in all our constructions.

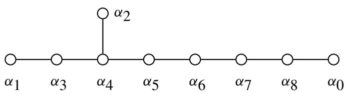
To simplify the notation, we will denote by $[p]_n$ the unique element in $\{0, 1, \dots, n-1\}$ congruent to $p \in \mathbb{N}$ modulo n, and we will write $\equiv_n$ for this congruence relation. This notation will allow us to describe the homogeneous components in a uniform way; see, for instance, Eq. (59).

2.1 $\mathbb{Z}_m$-gradings on simple Lie algebras

Any $\mathbb{Z}_m$-grading on a simple Lie algebra $\mathcal{L} = \bigoplus_{\bar{j}\in\mathbb{Z}_m} \mathcal{L}_{\bar{j}}$ is the eigenspace decomposition of an automorphism θ of $\mathcal{L}$ of order m. (Here we assume that the support $\{\bar{j} \in \mathbb{Z}_m : \mathcal{L}_{\bar{j}} \neq 0\}$ generates the grading group $\mathbb{Z}_m$.) This automorphism is given by $\theta|_{\mathcal{L}_{\bar{j}}} = \xi^j\,\mathrm{id}$, where $\xi \in \mathbb{F}$ is a fixed primitive mth root of unity. The converse also holds: any automorphism of $\mathcal{L}$ of order m determines a $\mathbb{Z}_m$-grading on $\mathcal{L}$ via its eigenspace decomposition. In particular, the neutral component $\mathcal{L}_{\bar{0}}$ is the fixed-point subalgebra of θ. Moreover, each $\mathcal{L}_{\bar{j}}$ is an irreducible $\mathcal{L}_{\bar{0}}$-module.

The finite order automorphisms of $\mathfrak{e}_8$ are classified, up to conjugation, in [28, §8.6], in terms of affine Dynkin diagrams and sequences of pairwise relatively prime nonnegative integers $(s_0, \dots, s_8)$. If the fixed-point subalgebra of a nontrivial finite order automorphism θ is semisimple, then [28, Proposition 8.6] shows that the associated sequence is of the form $(0, \dots, 1, \dots, 0)$, with a single entry $s_i = 1$ for some $i > 0$. In this case, the order of θ is m_i, where the coefficients $m_1, \dots, m_8$ are defined by the expression of the highest root $-\alpha_0 := 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 = \sum_{j=1}^{8} m_j\alpha_j$ in a root system Φ of $\mathfrak{e}_8$, relative to a Cartan subalgebra H and a choice of simple roots $\{\alpha_1, \dots, \alpha_8\}$. Moreover, the Dynkin diagram of the fixed-point subalgebra of θ is obtained by removing the ith node from the extended (affine) Dynkin diagram $E_8^{(1)}$,

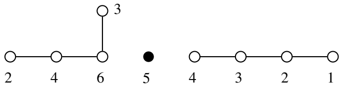




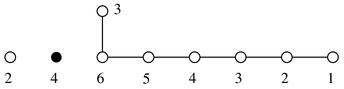
since $\{\alpha_0, \alpha_j : j \neq i\}$ is a set of simple roots of $\mathcal{L}_{\bar{0}}$. This entails that, up to conjugation, there are eight finite order automorphisms of $\mathfrak{e}_8$ whose fixed-point subalgebra is semisimple: two of order 2, two of order 3, two of order 4, one of order 5, and one of order 6. The choice $i = 2$ yields the automorphism of $\mathcal{L} = \mathfrak{e}_8$ of order 3 whose fixed-point subalgebra is isomorphic to

$\mathfrak{sl}(9)$, corresponding to the model described in Remark 2.4. The choices $i = 1, 6, 7, 8$ lead to gradings whose neutral components have direct summands of type D and E. There remain only three other cases, all of which produce gradings on $\mathfrak{e}_8$ over cyclic groups:

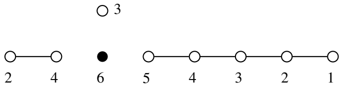
- If $i = 5$, we have a $\mathbb{Z}_5$-grading on $\mathfrak{e}_8$ with $\mathcal{L}_{\bar{0}} \cong \mathfrak{sl}(5) \oplus \mathfrak{sl}(5)$, since the removed diagram is



- If $i = 3$, we have a $\mathbb{Z}_4$-grading on $\mathcal{L} = \mathfrak{e}_8$ with $\mathcal{L}_{\bar{0}} \cong \mathfrak{sl}(8) \oplus \mathfrak{sl}(2)$, since the removed diagram is



- If $i = 4$, we have a $\mathbb{Z}_6$-grading on $\mathfrak{e}_8$ with $\mathcal{L}_{\bar{0}} \cong \mathfrak{sl}(6) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(3)$, since the removed diagram is



In order to be even more precise, the homogeneous components of the $\mathbb{Z}_{m_i}$-grading related to an index i can be written as a sum of root spaces:

$$\begin{aligned} \mathcal{L}_{\bar{0}} &= H \oplus \big(\oplus \{\mathcal{L}_\alpha : \alpha = \textstyle\sum_{k=0}^{8} t_k \alpha_k \in \Phi,\ t_i \equiv_{m_i} 0\}\big), \\ \mathcal{L}_{\bar{j}} &= \oplus \{\mathcal{L}_\alpha : \alpha = \textstyle\sum_{k=0}^{8} t_k \alpha_k \in \Phi,\ t_i \equiv_{m_i} j\}. \end{aligned} \tag{2}$$

Here we have used the fact that every root $\alpha \in \Phi$ can be uniquely written as $\alpha = \sum_{k=0}^{8} t_k \alpha_k$, with $0 \le t_k \le m_k$, where $t_0 = 0$ if $\alpha \in \Phi^+$ and $t_0 = 1$ if $\alpha \in \Phi^-$. The closed symmetric subset to which we apply Theorem 1.1 in this context is $\Phi' = \left\{ \alpha = \sum_{k=0}^{8} t_k \alpha_k \in \Phi \;\middle|\; t_i \equiv_{m_i} 0 \right\}$, which is the root system of $\mathcal{L}_{\bar{0}}$. Although the description of the grading in (2) is very concrete, working with root spaces is not always convenient, as it is often difficult to determine the precise scalars involved when multiplying two root vectors. In Sects. 3.1, 3.2, and 3.3, we address the $\mathbb{Z}_5$-, $\mathbb{Z}_4$-, and $\mathbb{Z}_6$-gradings introduced above, respectively, by providing models that avoid the use of root spaces altogether.

According to the list (1) extracted from the classical Dynkin's paper [17, Table 10], we can apply Theorem 1.1 to root subsystems of types $4A_2$ and $2A_1 \oplus 2A_3$ too. The first case is tackled in Sect. 3.4: there we consider two order 3 commuting automorphisms, both related to the 7th node. The subalgebra fixed by one of these automorphisms is of type $E_6 \oplus A_2$; restricting the other automorphism to the E_6-factor yields a fixed subalgebra of type $3A_2$. However, following this combinatorial process to construct a suitable model is not particularly useful. It is more effective to consider directly the possible irreducible modules, based on dimension constraints. Finally, Sect. 3.5 deals with a grading over $\mathbb{Z}_2 \times \mathbb{Z}_4$, obtained by considering two commuting automorphisms of finite order, associated with the 8th and 6th nodes of the extended Dynkin diagram, respectively.

2.2 Invariant products

The main reference used here for results in multilinear algebra is [23, Appendix B] (see also [6, Chapter 3]). Although this material is well known, fixing the notation is crucial for our goal of presenting unified models of $\mathfrak{e}_8$.

If V is a vector space over $\mathbb{F}$ of dimension n, and $\mathcal{L} = \mathfrak{sl}(V)$ denotes the simple Lie algebra of traceless matrices, of rank $n-1$, recall that $\bigwedge^r V$ is an $\mathcal{L}$-irreducible module for any $r = 0, \dots, n$, where the action of $f \in \mathfrak{sl}(V)$ is given by $f \cdot v_1 \wedge \cdots \wedge v_r = \sum_{i=1}^{r} v_1 \wedge \cdots \wedge f(v_i) \wedge \cdots \wedge v_r$. More precisely, $\bigwedge^r V$ is the $\mathcal{L}$-module of fundamental weight ϖ_r if $1 \le r < n$ [27]. (If $r > n$, what happens is $\bigwedge^r V = 0$. And, if either $r = 0$ or $r = n$, then $\bigwedge^0 V \cong \mathbb{F}$ and $\bigwedge^r V \cong \mathbb{F}$ are one-dimensional trivial modules.) Also, the dual module V^* has fundamental weight ϖ_{n-1} and, more generally, $\bigwedge^r V^*$ has fundamental weight ϖ_{n-r} if $1 \le r < n$. The adjoint module has as highest weight $\varpi_1 + \varpi_{n-1}$. In fact, $V(\varpi_1) \otimes V(\varpi_{n-1}) \cong V(\varpi_1 + \varpi_{n-1}) \oplus V(0)$ because $\Psi\colon V^* \otimes V \to \mathfrak{gl}(V)$ given as $\Psi(\alpha \otimes v)(w) = \alpha(w)v$ for any $v, w \in V$ and any $\alpha \in V^*$, is an $\mathcal{L}$-module isomorphism.

Take some concrete $\mathcal{L}$-invariant maps among these modules. First, the wedge product

$$\bigwedge^r V \times \bigwedge^s V \to \bigwedge^{r+s} V, \qquad (v_1 \wedge \cdots \wedge v_r, v_{r+1} \wedge \cdots \wedge v_{r+s}) \mapsto v_1 \wedge \cdots \wedge v_{r+s} \tag{3}$$

is an $\mathcal{L}$-invariant map (equal to zero if $r + s > n$). Second, for any $r = 0, \dots, n$, we have the non-degenerate bilinear form

$$\langle\ \rangle_r\colon \bigwedge^r V \times \bigwedge^r V^* \to \mathbb{F},$$

given by

$$\langle v_1 \wedge \cdots \wedge v_r, \varphi_1 \wedge \cdots \wedge \varphi_r \rangle_r := \det(\varphi_i(v_j)) = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \varphi_{\sigma(1)}(v_1) \dots \varphi_{\sigma(r)}(v_r),$$

again an $\mathcal{L}$-invariant map. Here, S_r denotes the usual symmetric group or permutation group. Also, if $s, t \in \mathbb{F}$, we are taking $\langle s, t \rangle_0 = st$. We simply use the notation $\langle\ \rangle$ in case there is no ambiguity. This pairing provides the identification

$$\bigwedge^r V^* \to (\bigwedge^r V)^*, \qquad \alpha \in \bigwedge^r V^* \mapsto \langle -, \alpha \rangle\colon \bigwedge^r V \to \mathbb{F}. \tag{4}$$

Now fix any non-zero linear map

$$\phi\colon \bigwedge^n V \xrightarrow{\cong} \mathbb{F}, \tag{5}$$

which clearly is an isomorphism between trivial $\mathcal{L}$-modules. It provides another $\mathcal{L}$-module isomorphism,

$$\begin{aligned} \bigwedge^{n-r} V &\to (\bigwedge^r V)^* \\ x \quad &\mapsto [y \in \bigwedge^r V \mapsto \phi(x \wedge y) \in \mathbb{F}]. \end{aligned} \tag{6}$$

Here the operation $\wedge$ refers to the map in Eq. (3). Be careful with this symbol, which can lead to confusion, since $\wedge$ is not necessarily skew-symmetric, but $x \wedge y = (-1)^{r(n-r)} y \wedge x$. Now compose the map in Eq. (6) with the inverse of the map in Eq. (4) to get the $\mathcal{L}$-module isomorphism

$$\begin{aligned} \bigwedge^{n-r} V &\equiv \bigwedge^r V^* \\ x \quad &\mapsto \tilde{x} \end{aligned}$$

given by $\langle y, \tilde{x} \rangle = \phi(x \wedge y)$ for any $x \in \bigwedge^{n-r} V$ and $y \in \bigwedge^r V$.

There are related contraction maps, also called *internal products*, denoted by $\lrcorner$ and $\llcorner$. For p and q positive integers with $p+q \leq n$, the contraction

$$\begin{array}{ccc} \bigwedge^p V \otimes \bigwedge^{p+q} V^* & \to & \bigwedge^q V^* \\ x \otimes \alpha & \mapsto & x \lrcorner \alpha \end{array}$$

is determined by $\langle z, x \lrcorner \alpha\rangle = \langle z \wedge x, \alpha\rangle$, for any $z \in \bigwedge^q V$. Similarly, the contraction

$$\begin{array}{ccc} \bigwedge^{p+q} V \otimes \bigwedge^p V^* & \to & \bigwedge^q V \\ x \otimes \alpha & \mapsto & x \llcorner \alpha \end{array}$$

is determined by

$$\langle x \llcorner \alpha, \beta\rangle = \langle x, \alpha \wedge \beta\rangle,$$

for any $\beta \in \bigwedge^q V^*$. We will make an extensive use of $\llcorner$, so that we should have some way in mind to handle it. For $v_i \in V$, $\varphi_i \in V^*$, $i \leq p+q$, we have

$$(v_1 \wedge \cdots \wedge v_{p+q}) \llcorner (\varphi_1 \wedge \cdots \wedge \varphi_p) = \sum_{\sigma \in \hat{S}_{p+q}} \operatorname{sgn}(\sigma)\varphi_1(v_{\sigma(1)}) \ldots \varphi_p(v_{\sigma(p)})\, v_{\sigma(p+1)} \wedge \cdots \wedge v_{\sigma(q+p)}$$

where $\hat{S}_{p+q}$ denotes the set of permutations of $\{1, \ldots, p+q\}$ that preserve the order of $\{p+1, \ldots, p+q\}$. Observe that if $q = 0$, then $x \llcorner \alpha = x \lrcorner \alpha = \langle x, \alpha\rangle$, for any $x \in \bigwedge^p V$ and $\alpha \in \bigwedge^p V^*$.

With these tools, we introduce a product on the exterior algebra $\bigwedge V = \bigoplus_{i=0}^n \bigwedge^i V$, which does not always coincide with the usual exterior (wedge) product. We will use this operation throughout the manuscript to give a unified description of various models of $\mathfrak{e}_8$. Thus, we define an $\mathcal{L}$-invariant bilinear map

$$\begin{array}{rcl} *\colon \bigwedge V \times \bigwedge V & \longrightarrow & \bigwedge V \\ (x, y) & \longmapsto & x * y, \end{array}$$

where, for $x \in \bigwedge^i V$ and $y \in \bigwedge^j V$,

$$x * y := \begin{cases} x \wedge y, & \text{if } i + j < n, \\ x \llcorner \tilde{y}, & \text{if } i + j \geq n. \end{cases} \tag{7}$$

Note that $x * y \in \bigwedge^{[i+j]_n} V$, where, as previously mentioned, $[p]_n$ denotes the unique element in $\{0, 1, \ldots, n-1\}$ congruent to $p \in \mathbb{N}$ modulo n.

This operation $*$ gives a remarkable $\mathcal{L}$-invariant map in case $i+j = n$, since for $x \in \bigwedge^i V$, $y \in \bigwedge^{n-i} V$, we have $x * y = x \llcorner \tilde{y} = \langle x, \tilde{y}\rangle = \phi(y \wedge x) \in \mathbb{F}$. We denote the related bilinear map given by this restriction of $*$ by

$$(\ ,\ )\colon \bigwedge^i V \times \bigwedge^{n-i} V \to \mathbb{F}, \qquad (x, y) := x * y = \phi(y \wedge x). \tag{8}$$

Note that $(x, y) = -(y, x)$ if both i and $n-i$ are odd, while $(x, y) = (y, x)$ otherwise. Next, in order to consider a second $\mathcal{L}$-invariant map $[\ ,\ ]\colon \bigwedge^i V \times \bigwedge^{n-i} V \to \mathfrak{sl}(V)$, we need to recall some facts about the dualization of an action, essentially extracted from [1].

Remark 2.1 If $\mathcal{L}$ is a Lie algebra and U and W are $\mathcal{L}$-modules, denote by $U^{\mathcal{L}} := \{u \in U : x \cdot u = 0\ \forall x \in \mathcal{L}\}$, the trivial submodule of U. It is easy to check that $\operatorname{Hom}_{\mathcal{L}}(U, W) = \operatorname{Hom}(U, W)^{\mathcal{L}}$. Now consider the usual $\mathcal{L}$-module isomorphism $U^* \otimes W \to \operatorname{Hom}(U, W)$ given by $\alpha \otimes w \mapsto (u \mapsto \alpha(u)w)$ for any $u \in U$, $w \in W$ and $\alpha \in U^*$, whose restriction to the trivial submodule allows to identify $(U^* \otimes W)^{\mathcal{L}}$ with $\mathrm{Hom}_{\mathcal{L}}(U, W)$. Take also in mind the identifications

$$\mathrm{Hom}(\mathcal{L} \otimes W, W) \cong (\mathcal{L} \otimes W)^* \otimes W \cong (W \otimes W^*)^* \otimes \mathcal{L}^* \cong \mathrm{Hom}(W \otimes W^*, \mathcal{L}^*). \quad (9)$$

Now the action of $\mathcal{L}$ on W provides a map in $\mathrm{Hom}_{\mathcal{L}}(\mathcal{L} \otimes W, W)$, which in turn gives a map in $\mathrm{Hom}_{\mathcal{L}}(W \otimes W^*, \mathcal{L}^*)$ by Eq. (9) and the above considerations. The $\mathcal{L}$-invariant map $W \times W^* \to \mathcal{L}^*$ obtained in this way is usually referred to as the map *dualizing the action* of $\mathcal{L}$ on W. If besides $\mathcal{L}$ is a simple Lie algebra, its Killing form provides a convenient isomorphism between the adjoint module $\mathcal{L}$ and its dual, and then we also get an $\mathcal{L}$-invariant map $W \times W^* \to \mathcal{L}$. In the case $\mathcal{L} = \mathfrak{sl}(V)$, we can replace the Killing form with the bilinear form on $\mathfrak{sl}(V)$ given by the trace, which also provides a way of identifying $\mathcal{L}$ with $\mathcal{L}^*$ by means of $f \mapsto \mathrm{tr}(f \circ -)$.

Particularizing Remark 2.1 for $\mathcal{L} = \mathfrak{sl}(V)$ and $W = \bigwedge^i V$, we get the mentioned $\mathcal{L}$-invariant map

$$[\ ,\ ] \colon \bigwedge^i V \times \bigwedge^{n-i} V \to \mathfrak{sl}(V)$$

determined by the condition

$$\mathrm{tr}(f \circ [x, y]) = (f \cdot x, y) \quad (10)$$

for all $f \in \mathfrak{sl}(V), x \in \bigwedge^i V, y \in \bigwedge^{n-i} V$. Again, note that $[x, y] = [y, x]$ if both i and $n - i$ are odd, while $[x, y] = -[y, x]$ otherwise. (Although the bracket notation $[\ ,\ ]$ is commonly associated with Lie algebras, this should not lead to confusion: in this context, $[\ ,\ ]$ is *skew-symmetric* if and only if the bilinear form $(\ ,\ )$ is *symmetric*.) For later use, observe that if $i = 0$, then $[x, y] = 0$ for any $x \in \bigwedge^0 V = \mathbb{F}$ and any $y \in \bigwedge^n V \cong \mathbb{F}$, since $f \cdot x = 0$ for all $f \in \mathfrak{sl}(V)$.

Recall from representation theory (see, for instance, [27]) that $\dim\mathrm{Hom}_{\mathfrak{sl}(V)}(\bigwedge^r V \otimes \bigwedge^{n-r} V, \mathfrak{sl}(V)) = 1$ for $0 < r < n$. It follows that any other $\mathcal{L}$-invariant bilinear map $\bigwedge^r V \times \bigwedge^{n-r} V \to \mathfrak{sl}(V)$ must be a scalar multiple of $[\ ,\ ]$. Similarly, $\dim\mathrm{Hom}_{\mathfrak{sl}(V)}(\bigwedge^r V \otimes \bigwedge^s V, \bigwedge^{r+s} V) = 1$ whenever $0 \le r + s \le n$, so that any $\mathcal{L}$-invariant map $\bigwedge^r V \otimes \bigwedge^s V \to \bigwedge^{r+s} V$ must be a scalar multiple of the map defined in Eq. (3). Both results will be crucial in constructing $\mathfrak{e}_8$ from its gradings, as outlined in Theorem 1.1.

A remark concerning this type of construction is worth noting.

Remark 2.2 In order to construct a Lie algebra from a "simpler"Lie algebra $\mathfrak{h}$ and an $\mathfrak{h}$-module $\mathfrak{m}$, define in $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the bracket $[h + x, h' + x'] := [h, h'] + h \cdot x' - h' \cdot x + \mu(x, x')$ for some fixed $\mathfrak{h}$-invariant skew-symmetric map $\mu \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{h}$, and any elements $h, h' \in \mathfrak{h}$, $x, x' \in \mathfrak{m}$, where the action of $\mathfrak{h}$ on $\mathfrak{m}$ is denoted with $\cdot$. We wonder when $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra, that is, when $J(\mathfrak{g}, \mathfrak{g}, \mathfrak{g}) = 0$, for J the Jacobian operator defined by $J(x, y, z) := [[x, y], z] + [[y, z], x] + [[z, x], y]$. Recall that $J(\mathfrak{h}, \mathfrak{h}, \mathfrak{h}) = 0$ is a consequence of the fact that $\mathfrak{h}$ is a Lie subalgebra; and $J(\mathfrak{h}, \mathfrak{h}, \mathfrak{m}) = 0$ is a consequence of $\mathfrak{m}$ being an $\mathfrak{h}$-module. The condition $J(\mathfrak{h}, \mathfrak{m}, \mathfrak{m}) = 0$ is equivalent to the $\mathfrak{h}$-invariance of μ. All this means that $\mathfrak{g}$ is a Lie algebra if and only if $J(\mathfrak{m}, \mathfrak{m}, \mathfrak{m}) = 0$.

At some point, concrete computations will be necessary to verify whether $J(\mathfrak{m}, \mathfrak{m}, \mathfrak{m}) = 0$, in order to apply the above remark. To that end, it is convenient to work with the operations $*$, $\llcorner$, and $[\ ,\ ]$ on elements of a chosen basis.

Take $(e_1, \dots, e_n)$ a basis of V and $(e^1, \dots, e^n)$ its dual basis of V^*. That is, $e^i(e_j) = \delta_{ij}$ for δ_{ij} the Kronecker delta (1 if the variables are equal, and 0 otherwise). Denote by

$$e_{i_1 \dots i_k} := e_{i_1} \wedge \cdots \wedge e_{i_k} \in \bigwedge^k V, \qquad e^{i_1 \dots i_k} := e^{i_1} \wedge \cdots \wedge e^{i_k} \in \bigwedge^k V^*,$$

for any $i_1, \dots, i_k \in \{1, \dots, n\}$. Thus $\mathcal{B}_k := \{e_{i_1 \dots i_k} : 1 \le i_1 < \cdots < i_k \le n\}$ is a basis of $\bigwedge^k V$. Denote by $e_i^j : V \to V$ the linear map such that $e_i^j(e_k) := \delta_{jk} e_i$. (With the notation at the beginning of the section, $e_i^j = \Psi(e^j \otimes e_i)$.) The set $\{e_i^j : 1 \le i, j \le n\}$ provides a basis of $\mathfrak{gl}(V)$ such that $\operatorname{tr}(e_i^j) = \delta_{ij}$.

Take as the map ϕ in (5) the one such that $\phi(e_{1\dots n}) = 1$. If $\sigma = (\sigma(1), \dots, \sigma(n))$ denotes a permutation of $\{1, \dots, n\}$, we have $e_{\sigma(1)\dots\sigma(n)} = \operatorname{sgn}(\sigma)\, e_{1\dots n}$, so that $\phi(e_{\sigma(1)\dots\sigma(n)}) = \operatorname{sgn}(\sigma)$.

Lemma 2.3 *Fix $(e_1, \dots, e_n)$ a basis of V and let $\phi\colon \bigwedge^n V \to \mathbb{F}$ be the linear map given by $\phi(e_{1\dots n}) = 1$. Let $\sigma = (i_1, \dots, i_n)$ and $(j_1, \dots, j_n)$ be permutations of $\{1, \dots, n\}$. Then we have*

(a) $\widetilde{e_{i_1 \dots i_k}} = \operatorname{sgn}(\sigma)\, e^{i_{k+1} \dots i_n}$;

(b) *The map $[\ ,\ ]$ defined in Eq. (10) is given, for elements in $\mathcal{B}_k$ and $\mathcal{B}_{n-k}$, by:*

  (1) $[e_{i_1 \dots i_k}, e_{i_{k+1} \dots i_n}] = (-1)^{k(n-k)} \operatorname{sgn}(\sigma) \left(\sum_{j=1}^k e_{i_j}^{i_j} - \frac{k}{n} \mathrm{id}_V \right)$*;*

  (2) $[e_{i_1 \dots i_k}, e_{i_1 i_{k+1} \dots i_{n-1}}] = (-1)^{k(n-k)} (-1)^{n+k} \operatorname{sgn}(\sigma)\, e_{i_1}^{i_n}$*;*

  (3) $[e_{i_1 \dots i_k}, e_{j_1 \dots j_{n-k}}] = 0$ *if* $\{i_1, \dots, i_k\} \cap \{j_1, \dots, j_{n-k}\}$ *has at least cardinal 2.*

(c) $e_{i_1 \dots i_{p+q}} \llcorner e^{j_1 \dots j_p} = 0$ *if* $J = \{j_1, \dots, j_p\} \not\subset I = \{i_1, \dots, i_{p+q}\}$. *Otherwise, if* $k_1 < \cdots < k_p \in \{1, \dots, p+q\}$, *then,*

$$e_{i_1 \dots i_{p+q}} \llcorner e^{i_{k_1} \dots i_{k_p}} = \operatorname{sgn} \begin{pmatrix} i_1 & \cdots & i_p & i_{p+1} & & & \cdots & & & i_{p+q} \\ i_{k_1} & \cdots & i_{k_p} & i_1 & \cdots & \widehat{i_{k_1}} & \cdots & \widehat{i_{k_p}} & \cdots & i_{p+q} \end{pmatrix} e_{i_1 \dots \widehat{i_{k_1}} \dots \widehat{i_{k_p}} \dots i_{p+q}}.$$

(d) *If* $x \in \bigwedge^i V$, $y \in \bigwedge^j V$, *then* $x * y = \begin{cases} (-1)^{ij}\, y * x & \text{if } i + j \le n, \\ (-1)^{ij + (i+j-n)}\, y * x & \text{if } i + j > n. \end{cases}$

As usual, the hat over an index in item (c) indicates that the corresponding index is omitted.

Proof For (a) we have to check that $\langle y, \operatorname{sgn}(\sigma)\, e^{i_{k+1} \dots i_n} \rangle = \phi(e_{i_1 \dots i_k} \wedge y)$ for any $y \in \bigwedge^{n-k} V$. If $y = e_{i_{k+1} \dots i_n}$, clearly $\langle y, e^{i_{k+1} \dots i_n} \rangle = \det I_{n-k} = 1$, so that both sides of the equation equal $\operatorname{sgn}(\sigma)$. If y is any other basic vector in $\mathcal{B}_k$, both sides of the equation annihilate.

(b1) Note that, for any $f \in \mathfrak{sl}(V)$,

$$e_{i_1} \wedge \cdots \wedge f(e_{i_k}) \wedge \cdots \wedge e_{i_n} = \operatorname{tr}(f \circ e_{i_k}^{i_k}) e_{i_1 \dots i_n}, \tag{11}$$

since, if we write $f(e_j) = \sum_{i=1}^n a_{ij} e_i$, then $\operatorname{tr}(f \circ e_i^i) = a_{ii}$ and $e_{i_1} \wedge \cdots \wedge f(e_{i_k}) \wedge \cdots \wedge e_{i_n} = \sum_{i=1}^n a_{i i_k} e_{i_1 \dots i_{k-1} i i_{k+1} \dots i_n} = a_{i_k i_k} e_{i_1 \dots i_n}$. Now,

$$\begin{aligned} (-1)^{k(n-k)} \operatorname{tr}(f \circ [e_{i_1 \dots i_k}, e_{i_{k+1} \dots i_n}]) &= \phi(f \cdot e_{i_1 \dots i_k} \wedge e_{i_{k+1} \dots i_n}) \\ &= \phi(f(e_{i_1}) \wedge e_{i_2 \dots i_n} + \cdots + e_{i_1 \dots i_{k-1}} \wedge f(e_{i_k}) \wedge e_{i_{k+1} \dots i_n}) \end{aligned}$$

coincides, by (11), with

$$\phi\left(\sum_{j=1}^k \operatorname{tr}(f \circ e_{i_j}^{i_j}) e_{i_1 \dots i_n} \right) = \operatorname{sgn}(\sigma) \operatorname{tr}\left(f \circ \sum_{j=1}^k e_{i_j}^{i_j} \right) = \operatorname{sgn}(\sigma) \operatorname{tr}\left(f \circ \Big(\sum_{j=1}^k e_{i_j}^{i_j} - \frac{k}{n} \mathrm{id}_V \Big) \right),$$

since f has zero trace. As $\left(\sum_{j=1}^k e_{i_j}^{i_j} - \frac{k}{n} \mathrm{id}_V \right) \in \mathfrak{sl}(V)$ and Eq. (10) determines a unique element in $\mathfrak{sl}(V)$, so this traceless map coincides with $(-1)^{k(n-k)} \operatorname{sgn}(\sigma)\, [e_{i_1 \dots i_k}, e_{i_{k+1} \dots i_n}]$.

(b2) Observe first that

$$\begin{aligned}(-1)^{k(n-k)}\mathrm{tr}(f\circ[e_{i_1\dots i_k}, e_{i_1 i_{k+1}\dots i_{n-1}}]) &= \phi(f\cdot e_{i_1\dots i_k}\wedge e_{i_1 i_{k+1}\dots i_{n-1}})\\ &= \phi(f(e_{i_1})\wedge e_{i_2\dots i_k i_1 i_{k+1}\dots i_{n-1}}),\end{aligned} \tag{12}$$

since the index i_1 appears twice in each term $e_{i_1\dots i_{j-1}}\wedge f(e_{i_j})\wedge e_{i_{j+1}\dots i_k i_1 i_{k+1}\dots i_{n-1}}$, $j\le k$. Since $\mathrm{tr}(f\circ e_i^j) = a_{ji}$, the expression in (12) coincides with

$$\phi\left(\sum_{i=1}^{n} a_{ii_1} e_{ii_2\dots i_k i_1 i_{k+1}\dots i_{n-1}}\right) = \phi(e_{i_n i_2\dots i_k i_1 i_{k+1}\dots i_{n-1}})a_{i_n i_1} = (-1)^{n-1+k-1}\mathrm{sgn}(\sigma)\mathrm{tr}(f\circ e_{i_1}^{i_n}),$$

and hence Eq. (10) gives the required expression for $[e_{i_1\dots i_k}, e_{i_1 i_{k+1}\dots i_{n-1}}]$.

(b3) For any $f\in\mathfrak{sl}(V)$, we have

$$(f\cdot e_{i_1\dots i_k})\wedge e_{j_1\dots j_{n-k}} = \sum_{j=1}^{k} e_{i_1\dots i_{j-1}}\wedge f(e_{i_j})\wedge e_{i_{j+1}\dots i_k j_1\dots j_{n-k}} = 0.$$

In fact, each term in the sum vanishes, as it is the wedge product of basis vectors with at least two of them repeated.

Item (c) is clear. Moreover, $e_I \llcorner e^J = \mathrm{sgn}\left(\begin{smallmatrix} & I \\ J & I\setminus J\end{smallmatrix}\right) e_{I\setminus J}$ if $J\subset I$, even though J does not appear preserving the order in I. Here I and J are ordered subsets of different indices and we have used the notation e_I for $e_{i_1\dots i_k}$ if $I=(i_1,\dots,i_k)$.

(d) If $i+j<n$, we have already mentioned that $x\wedge y = (-1)^{ij}y\wedge x$. Also, if $i+j=n$, then $x*y = \phi(y\wedge x) = (-1)^{ij}\phi(x\wedge y) = (-1)^{ij}y*x$. Finally consider the case $i+j>n$. There is no loss of generality in assuming that both x and y are wedge products of basic vectors in $\{e_1,\dots,e_n\}$. If the union of the indices involved in x and y is not the whole set $\{1,\dots,n\}$, then $x*y = x\llcorner\tilde{y} = 0$ by (a) and (c), so there is nothing to prove, since similarly $y*x=0$. Otherwise, we can write $x = e_{i_1\dots i_s j_1\dots j_p}$ and $y = e_{i_1\dots i_s k_1\dots k_q}$ with $\{1,\dots,n\} = \{i_1,\dots,i_s,j_1,\dots,j_p,k_1,\dots,k_q\}$, $n=s+p+q$, with the indices possibly not ordered. We compute

$$\begin{aligned} x*y &= x\llcorner\tilde{y} \overset{(a)}{=} x\llcorner\mathrm{sgn}(\sigma_1)e^{j_1\dots j_p} \overset{(c)}{=} \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)e_{i_1\dots i_s},\\ y*x &= y\llcorner\tilde{x} = y\llcorner\mathrm{sgn}(\sigma_3)e^{k_1\dots k_q} = \mathrm{sgn}(\sigma_3)\mathrm{sgn}(\sigma_4)e_{i_1\dots i_s},\end{aligned}$$

for

$$\begin{aligned} &\sigma_1 = (i_1,\dots,i_s,k_1,\dots,k_q,j_1,\dots,j_p), &&\sigma_2 = (j_1,\dots,j_p,i_1,\dots,i_s),\\ &\sigma_3 = (i_1,\dots,i_s,j_1,\dots,j_p,k_1,\dots,k_q), &&\sigma_4 = (k_1,\dots,k_q,i_1,\dots,i_s).\end{aligned}$$

Now $\mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_3) = (-1)^{pq}$ and $\mathrm{sgn}(\sigma_2)\mathrm{sgn}(\sigma_4) = (-1)^{ps+qs}$. Taking in mind that $i=p+s$, $j=q+s$ and $i+j=n+s$, then $pq+(p+q)s = ij-(i+j-n)^2 \equiv_2 ij+i+j-n$. □

Remark 2.4 With the above introduced notations, there is a concise but complete description of the model of $\mathfrak{e}_8$ based on the $\mathbb{Z}_3$-grading with neutral homogeneous component isomorphic to $\mathfrak{sl}(9)$. Take V a vector space over $\mathbb{F}$ of dimension 9, and take $\mathcal{L} = \mathcal{L}_{\bar{0}}\oplus\mathcal{L}_{\bar{1}}\oplus\mathcal{L}_{\bar{2}}$ as

$$\begin{aligned}\mathcal{L}_{\bar{0}} &= \mathfrak{sl}(V),\\ \mathcal{L}_{\bar{i}} &= \textstyle\bigwedge^{[3i]_9} V,\end{aligned}$$

for $i=1,2$. Fix $a_{11}, a_{22}, b_1\in\mathbb{F}$ and define on $\mathcal{L}$ the product such that $\mathcal{L}_{\bar{0}}$ is subalgebra, the action of $\mathcal{L}_{\bar{0}}$ on each $\mathcal{L}_{\bar{i}}$ is the usual one, and, if $i,j\in\{1,2\}$, $i\le j$ $x\in\mathcal{L}_{\bar{i}}$, $y\in\mathcal{L}_{\bar{j}}$,

$$[x,y]_{\mathcal{L}} = \begin{cases} a_{ij}\,x*y & i+j\neq 3,\\ b_i[x,y] & i+j=3.\end{cases} \tag{13}$$

Then $\mathcal{L}$ is a Lie algebra if and only if $a_{11}a_{22}+b_1=0$, following Remark 2.2 for $\mathfrak{h}=\mathcal{L}_{\bar{0}}$ and $\mathfrak{m}=\mathcal{L}_{\bar{1}}\oplus\mathcal{L}_{\bar{2}}$. If besides the three scalars are nonzero, then the obtained $\mathbb{Z}_3$-graded Lie algebra is isomorphic to $\mathfrak{e}_8$, because there is only one simple Lie algebra of dimension 248 over $\mathbb{F}$, up to isomorphism. For instance, this is the case for the choice of scalars $a_{11}=a_{22}=-b_1=1$.

This description of $\mathfrak{e}_8$ for $(G,\mathfrak{h})=(\mathbb{Z}_3,A_8)$ is essentially the same as the constructions based on A_8 mentioned in the Introduction, but the comparison remains worthwhile. First, the model in [32] and [34] gives, for $S, S_1, S_2 \in \bigwedge^3 V$ and $T, T^1, T^2 \in \bigwedge^3 V^*$, the products

$$[S,T]^i_j=\tfrac{1}{2}S^{ijk}T_{jkl}-\tfrac{1}{18}S^{klm}T_{klm}\delta^i_j,\qquad [S_1,S_2]_{ijk}=\tfrac{1}{36}\varepsilon_{i_1j_1k_1i_2j_2k_2ijk}S_1^{i_1j_1k_1}S_2^{i_2j_2k_2},$$
$$[T^1,T^2]^{ijk}=-\tfrac{1}{36}\varepsilon^{i_1j_1k_1i_2j_2k_2ijk}T^1_{i_1j_1k_1}T^2_{i_2j_2k_2}.$$

Here the products are expressed in coordinates, which would be more difficult to write when increasing the number of pieces of the considered grading. Second, the description in [23, §22.4] uses two fixed trilinear maps $T\colon\ \bigwedge^3(\bigwedge^3 V)\to\mathbb{C}$ and $T\colon\ \bigwedge^3(\bigwedge^3 V)^*\to\mathbb{C}$ to give a pair of identifications $\wedge\colon\ \bigwedge^2(\bigwedge^3 V)\to(\bigwedge^3 V)^*$ and $\wedge\colon\ \bigwedge^2(\bigwedge^3 V)^*\to\bigwedge^3 V$ to define thereafter (for a, b and c fixed scalars)

$$[v,w]=a\,v\wedge w,\quad [\varphi,\psi]=b\,\varphi\wedge\psi,\quad [v,\varphi]=c\,v\star\varphi,$$

where $v\star\varphi$ is determined by $B(v\star\varphi,Z)=\varphi(Z\cdot v)$, for B the Killing form, $v\in\bigwedge^3 V$, $\varphi\in\bigwedge^3 V^*$ and $Z\in\mathfrak{sl}(V)$. With this notation, the Jacobi identity is equivalent to $18ab+c=0$ ([23, Exercise 22.21]). Our version is quite inspired in this last one, but we have needed to establish a notation that would allow the procedure to be easily generalised (observe the similarity between Eq. (13) and Proposition 3.1).

Remark 2.5 In the specific case of U a vector space of dimension 2, we can give more concrete descriptions of the invariant maps $(\ ,\ )\colon U\times U\to\mathbb{F}$ and $[\ ,\ ]\colon U\times U\to\mathfrak{sl}(U)$ considered in Eqs. (8) and (10). If $\varphi\colon U\times U\to\mathbb{F}$ is a nonzero alternating map, and $\mathfrak{sp}(U,\varphi)=\{f\in\mathfrak{gl}(U):\varphi(f(u),v)+\varphi(u,f(v))=0\}$ is the corresponding symplectic Lie algebra, then the map $\varphi_{u,v}:=\varphi(u,-)v+\varphi(v,-)u$ belongs to $\mathfrak{sp}(U,\varphi)$. Moreover $\mathfrak{sp}(U,\varphi)=\{\sum_i\varphi_{u_i,v_i}:u_i,v_i\in U\}$ coincides with the algebra $\mathfrak{sl}(U)$. If we choose (e_1,e_2) a basis of U with $\varphi(e_1,e_2)=1$, and the isomorphism ϕ in (5) is taken as $\phi(e_1\wedge e_2)=1$, then straightforward computations give:

$$(u,v)=u*v=u\llcorner\tilde{v}=-\varphi(u,v),\qquad [u,v]=\frac{1}{2}\varphi_{u,v}.$$

3 Models based on subalgebras of type A

3.1 Model based on the subalgebra of type $2A_4$

As mentioned, a $\mathbb{Z}_5$-grading on $\mathcal{L}=\mathfrak{e}_8$ arises by removing the node labeled 5 from the extended Dynkin diagram. In this case, the neutral component $\mathcal{L}_{\bar{0}}$ is of type $2A_4$, and the homogeneous components are $\mathcal{L}_{\bar{0}}$-irreducible modules, each necessarily a tensor product of two irreducible $\mathfrak{sl}_5(\mathbb{F})$-modules. The precise description of the irreducible modules involved can be deduced from their dimensions and from the dimensions of their tensor products. Alternatively, this decomposition appears explicitly in [31, Table 5], a work devoted to the classification of semisimple subalgebras, as well as in [21], where an interesting refinement of this $\mathbb{Z}_5$-grading is studied. According to these references, there is a vector space V of

$\mathbb{F}$-dimension 5, such that, up to isomorphism, the $\mathcal{L}_{\bar{0}}$-modules $\mathcal{L}_{\bar{i}}$ can be identified to:

$$\begin{aligned} \mathcal{L}_{\bar{0}} &= \mathfrak{sl}(V) \oplus \mathfrak{sl}(V) \\ \mathcal{L}_{\bar{i}} &= \textstyle\bigwedge^{[i]_5} V \otimes \bigwedge^{[2i]_5} V \qquad i = 1, 2, 3, 4. \end{aligned} \tag{14}$$

Thus $\dim \mathcal{L}_{\bar{0}} = 48$ and $\dim \mathcal{L}_{\bar{i}} = 50$ for any $i = 1, 2, 3, 4$. Keep in mind that the action of $\mathcal{L}_{\bar{0}}$ on $\mathcal{L}_{\bar{i}}$ is given by $((f, g), x \otimes y) \mapsto f \cdot x \otimes y + x \otimes g \cdot y$ for any $x \in \bigwedge^i V$ and $y \in \bigwedge^{[2i]_5} V$, that is, each simple component of $\mathcal{L}_{\bar{0}}$ acts on the corresponding module. We slightly abuse notation by writing $f + g$ for $(f, g) \in \mathcal{L}_{\bar{0}}$ when there is no risk of confusion regarding the copy of $\mathfrak{sl}(V)$ to which f and g belong. When this is not clear, we will write $f_1 + g_2$ to indicate the respective components. Next, let us change the viewpoint and begin with a graded vector space $\mathcal{L}$ with homogeneous components as in Eq. (14). We will endow this graded vector space $\mathcal{L}$ with a Lie algebra structure, to recover our exceptional Lie algebra.

Proposition 3.1 *Let $\mathcal{L}$ be the $\mathbb{Z}_5$-graded vector space given in Eq.* (14) *for V a vector space of $\mathbb{F}$-dimension 5. Consider the product of $\mathcal{L}_{\bar{0}}$ with $\mathcal{L}_{\bar{i}}$ given by the action of the jth copy of $\mathfrak{sl}(V)$ on the jth slot ($j = 1, 2$). Fix some nonzero scalars $a_{ij}, b_k^{(1)}, b_k^{(2)} \in \mathbb{F}^\times$ for any $1 \le i \le j \le 4$, $i + j \neq 5$, $k = 1, 2$. Define the bracket of $\mathcal{L}_{\bar{i}}$ with $\mathcal{L}_{\bar{j}}$, for any $1 \le i \le j \le 4$, by*

$$[x_1 \otimes x_2, y_1 \otimes y_2] := \begin{cases} a_{ij}\, x_1 * y_1 \otimes x_2 * y_2 & i + j \neq 5, \\ b_i^{(1)}\, (x_2, y_2)[x_1, y_1]_1 \,+\, b_i^{(2)}\, (x_1, y_1)[x_2, y_2]_2 & i + j = 5, \end{cases} \tag{15}$$

for any $x_1 \in \bigwedge^i V$, $x_2 \in \bigwedge^{[2i]_5} V$, $y_1 \in \bigwedge^j V$ and $y_2 \in \bigwedge^{[2j]_5} V$. Extend the bracket to the whole $\mathcal{L}$ such that it is skew-symmetric and the restriction to $\mathcal{L}_{\bar{0}}$ is the usual bracket in $2\mathfrak{sl}(V)$. Then, $\mathcal{L}$ endowed with the product $[\ ,\ ]$ is a ($\mathbb{Z}_5$-graded) Lie algebra if and only if

$$\begin{aligned} b_1^{(1)} &= b_1^{(2)} = -a_{11}a_{24} = -a_{12}a_{34} = -a_{13}a_{44}, \\ b_2^{(1)} &= b_2^{(2)} = -a_{12}a_{33} = -a_{22}a_{34} = -a_{13}a_{24}. \end{aligned} \tag{16}$$

Moreover, such Lie algebra $(\mathcal{L}, [\ ,\ ])$ is simple and isomorphic to $\mathfrak{e}_8$. A solution to this system of equations is, for instance,

$$\begin{aligned} 1 &= a_{11} = a_{22} = a_{33} = a_{34} = a_{44} = a_{12} = a_{13} = a_{24}, \\ -1 &= b_1^{(1)} = b_1^{(2)} = b_2^{(1)} = b_2^{(2)}. \end{aligned} \tag{17}$$

In particular, $(\mathcal{L}, [\ ,\ ])$ with the bracket extending the one in (15)*, for the scalars in* (17)*, gives a model of the split Lie algebra of type E_8.*

Remark 3.2 As in the paragraph above the theorem, the split Lie algebra of type E_8 has a decomposition like (14) given by its $\mathbb{Z}_5$-grading. Now, taking into account item d) in Theorem 1.1, the product in such Lie algebra has to be the one considered in (15) for some choice of scalars. This implies that there exist scalars $a_{ij}, b_i^{(1)}, b_i^{(2)} \in \mathbb{F}^\times$ such that the graded vector space $\mathcal{L}$, endowed with the bracket defined in (15), becomes a Lie algebra. In particular, the system of equations with variables $a_{ij}, b_i^{(1)}, b_i^{(2)} \in \mathbb{F}^\times$ that ensures $\mathcal{L}$ satisfies the Lie bracket identities always has a solution. This a priori knowledge of the existence of solutions will allow us to avoid many computations in the proof of Proposition 3.1, as well as elsewhere in the manuscript.

Proof First, note that it makes sense to deal with the skew-symmetric extension of the bracket because, for $i = j$, it is true that $[x_1 \otimes x_2, y_1 \otimes y_2] = -[y_1 \otimes y_2, x_1 \otimes x_2]$. In fact, for

$x, y \in \bigwedge^i V$, Lemma 2.3 says that $x * y = (-1)^{i^2} y * x$ if $2i \leq 5$ and $x * y = (-1)^{i^2-5} y * x$ if $2i > 5$. So $x * y = -y * x$ if $i = 1, 4$ and $x * y = y * x$ if $i = 2, 3$, but if $i \in \{1, 4\}$ then $2i \in \{2, 3\}$ and if $i \in \{2, 3\}$ then $[2i]_5 \in \{1, 4\}$.

The constructed skew-symmetric algebra $(\mathcal{L}, [\ ,\ ])$ is a Lie algebra if and only if $J(\mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{j}}, \mathcal{L}_{\bar{k}}) = 0$ for $0 \leq i, j, k \leq 4$, because the Jacobian operator is trilinear. Taking into consideration Remark 2.2, that condition reduces to $J(\mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{j}}, \mathcal{L}_{\bar{k}}) = 0$ for $1 \leq i \leq j \leq k \leq 4$.

A relevant observation is that $J(\mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{i}}) = 0$ always holds for all $0 \leq i \leq 4$, independently of the choice of scalars. Although that fact could be checked directly for each i, an indirect argument immediately gives the result: by Remark 3.2, the system of equations with variables $a_{ij}, b_i^{(1)}, b_i^{(2)} \in \mathbb{F}^\times$ always has at least one solution. Now note that, for any $x, y, z \in \mathcal{L}_{\bar{i}}$, $1 \leq i \leq 4$, the expression $\frac{J(x,y,z)}{a_{ii}a_{[2i]_5,i}}$ does not involve any of such variables, but it is only written in terms of $x, y, z, *, \sim$ and $\llcorner$. (Here we use the notation $a_{ij} = a_{ji}$ if $j > i$; for avoiding to distinguish the value of i.) This means that either it vanishes for any $x, y, z \in \mathcal{L}_{\bar{i}}$, or it does not vanish for some choice of homogeneous elements x, y, z. In the latter case, $J(x, y, z)$ would not vanish for any values of the variables a_{ij}, which is impossible due to the mentioned existence of solutions (Remark 3.2).

Denote by (J_{ijk}) the identity $J(\mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{j}}, \mathcal{L}_{\bar{k}}) = 0$. We proceed by checking that the identity (J_{ijk}) holds case by case. It is enough to check it by taking elements in $\mathcal{L}_{\bar{l}} = \bigwedge^l V \otimes \bigwedge^{[2l]_5} V$ which are tensor product of elements in $\bigwedge^l V$ and $\bigwedge^{[2l]_5} V$, since the Jacobian operator J is trilinear. The same multilinearity says that we can assume that these elements are wedge products of elements in V. So, take $\{u_i, v_i : i \in \mathbb{N}\}$ arbitrary elements in V, and use the notation $u_{i_1 \ldots i_k}$ for $u_{i_1} \wedge \cdots \wedge u_{i_k}$ as in Lemma 2.3 (and similarly for v_i's). Sometimes we will restrict to basic elements of $\bigwedge^l V$, obtained from a fixed basis $\{e_i : i = 1, \ldots, 5\}$ of V such that $\phi(e_{12345}) = 1$.

$\boxed{(J_{1,1,2})}$ Take $x = u_1 \otimes v_{12} \in \mathcal{L}_{\bar{1}}$, $y = u_2 \otimes v_{34} \in \mathcal{L}_{\bar{1}}$, $z = u_{34} \otimes v_{5678} \in \mathcal{L}_{\bar{2}}$. Then

$$\begin{aligned} J(x, y, z) = a_{11}a_{22}\, & u_{1234} \otimes (v_{1234} \llcorner \alpha) \\ & -a_{12}a_{13}\, (u_{1234} \otimes (v_{12} \wedge (\alpha(v_3)v_4 - \alpha(v_4)v_3)) \\ & -a_{12}a_{13}\, (u_{1234} \otimes (v_{34} \wedge (\alpha(v_1)v_2 - \alpha(v_2)v_1)), \end{aligned}$$

for $\alpha = \widetilde{v_{5678}}$. But $(v_{1234}) \llcorner \alpha = \alpha(v_1)v_2 \wedge v_3 \wedge v_4 - \alpha(v_2)v_1 \wedge v_3 \wedge v_4 + \alpha(v_3)v_1 \wedge v_2 \wedge v_4 - \alpha(v_4)v_1 \wedge v_2 \wedge v_3$, so that

$$J(x, y, z) = (a_{11}a_{22} - a_{12}a_{13})(u_{1234} \otimes (v_{1234} \llcorner \alpha)).$$

This vanishes (for any x, y and z) if and only if $a_{11}a_{22} = a_{12}a_{13}$.

$\boxed{(J_{1,1,3})}$

Take $x = u_1 \otimes v_{12} \in \mathcal{L}_{\bar{1}}$, $y = u_2 \otimes v_{34} \in \mathcal{L}_{\bar{1}}$, $z = u_{345} \otimes v_5 \in \mathcal{L}_{\bar{3}}$. We compute

$$\begin{aligned} J(x, y, z) = {} & \phi(v_{12345})\big(a_{11}b_2^{(1)}[u_{12}, u_{345}] - a_{13}b_1^{(1)}([u_1, u_{2345}] - [u_2, u_{1345}])\big)_1 \\ & + \phi(u_{12345})\big(a_{11}b_2^{(2)}[v_{1234}, v_5] - a_{13}b_1^{(2)}([v_{34}, v_{125}] + [v_{12}, v_{345}])\big)_2 \in \mathcal{L}_{\bar{0}}. \end{aligned}$$

This expression is zero if and only its projections on each copy of $\mathfrak{sl}(V)$ are zero. But, these projections are zero if and only if

$$a_{11}b_2^{(1)}\mathrm{tr}(f \circ [u_{12}, u_{345}]) - a_{13}b_1^{(1)}\mathrm{tr}(f \circ ([u_1, u_{2345}] - [u_2, u_{1345}])) = 0, \tag{18}$$

and

$$a_{11}b_2^{(2)}\mathrm{tr}(f \circ [v_{1234}, v_5]) - a_{13}b_1^{(2)}\mathrm{tr}(f \circ ([v_{12}, v_{345}] + [v_{34}, v_{125}])) = 0, \tag{19}$$

for every $f \in \mathfrak{sl}(V)$, since it is always possible to choose u's such that $\phi(u_{12345}) \neq 0$ (similarly for v's). According to the definition of [ ] in Eq. (10), we can rewrite (18) as

$$a_{11}b_2^{(1)}(f \cdot u_{12}, u_{345}) - a_{13}b_1^{(1)}\big((f \cdot u_1, u_{2345}) - (f \cdot u_2, u_{1345})\big) = 0.$$

Taking in mind that $f \cdot u_{12} \wedge u_{345} = (f \cdot u_1) \wedge u_{2345} - (f \cdot u_2) \wedge u_{1345}$, the above expression can be written as

$$(a_{11}b_2^{(1)} - a_{13}b_1^{(1)})(f \cdot u_{12}, u_{345}) = 0,$$

which holds for any choice of u's if and only if $a_{11}b_2^{(1)} = a_{13}b_1^{(1)}$. Also $f \cdot v_{1234} \wedge v_5 = (f \cdot v_{12}) \wedge v_{345} + (f \cdot v_{34}) \wedge v_{125}$, so that (19) is equivalent to

$$(a_{11}b_2^{(2)} - a_{13}b_1^{(2)})(f \cdot v_{1234}, v_5) = 0,$$

which is always true when $a_{11}b_2^{(2)} = a_{13}b_1^{(2)}$.

$\boxed{(J_{1,1,4})}$ If $x = u_1 \otimes v_{12},\ y = u_2 \otimes v_{34} \in \mathcal{L}_{\bar{1}}$ and $z = u_{3456} \otimes v_{567} \in \mathcal{L}_{\bar{4}}$, we have that

$$\begin{aligned} J(x, y, z) = \quad & a_{11}a_{24}(u_{12} \llcorner \widetilde{u_{3456}} \otimes v_{1234} \llcorner \widetilde{v_{567}}) \\ & + b_1^{(1)}\phi(v_{56734})[u_2, u_{3456}](u_1) \otimes v_{12} + b_1^{(2)}\phi(u_{34562})u_1 \otimes [v_{34}, v_{567}](v_{12}) \\ & - b_1^{(1)}\phi(v_{56712})[u_1, u_{3456}](u_2) \otimes v_{34} - b_1^{(2)}\phi(u_{34561})u_2 \otimes [v_{12}, v_{567}](v_{34}). \end{aligned} \tag{20}$$

We want necessary and sufficient conditions on the scalars a_{11}, a_{24}, $b_1^{(1)}$ and $b_1^{(2)}$ which guarantee that $J(\mathcal{L}_{\bar{1}}, \mathcal{L}_{\bar{1}}, \mathcal{L}_{\bar{4}}) = 0$. Necessary conditions are easy to obtain by using concrete elements. In fact,

$$0 = J(e_1 \otimes e_{12}, e_2 \otimes e_{34}, e_{1345} \otimes e_{125}) = \left(a_{11}a_{24} + \frac{1}{5}b_1^{(1)} + \frac{4}{5}b_1^{(2)}\right) e_1 \otimes e_{12},$$

and

$$0 = J(e_1 \otimes e_{12}, e_2 \otimes e_{12}, e_{1345} \otimes e_{345}) = \frac{6}{5}\left(b_1^{(1)} - b_1^{(2)}\right) e_1 \otimes e_{12},$$

which imply that some necessary conditions are

$$b_1^{(1)} = b_1^{(2)} = -a_{11}a_{24}. \tag{21}$$

Now let us check that these conditions are enough too. Assume we have taken the scalars satisfying Eq. (21). Then Eq. (20) becomes,

$$\begin{aligned} J(x, y, z) = b_1^{(1)}\big(& - u_{12} \llcorner \widetilde{u_{3456}} \otimes v_{1234} \llcorner \widetilde{v_{567}} \\ & + \phi(v_{56734})[u_2, u_{3456}](u_1) \otimes v_{12} + \phi(u_{34562})u_1 \otimes [v_{34}, v_{567}](v_{12}) \\ & - \phi(v_{56712})[u_1, u_{3456}](u_2) \otimes v_{34} - \phi(u_{34561})u_2 \otimes [v_{12}, v_{567}](v_{34})\big). \end{aligned}$$

Now we are sure that this expression is zero without the need to compute the value of the right side in full detail: simply note that the vanishing of the above expression does not depend on the choice of the nonzero scalar $b_1^{(1)}$, and recall that the system of equations with variables $a_{ij}, b_i^{(1)}, b_i^{(2)} \in \mathbb{F}^\times$ always has at least one solution as in Remark 3.2. So we can conclude that $J(\mathcal{L}_{\bar{1}}, \mathcal{L}_{\bar{1}}, \mathcal{L}_{\bar{4}})$ is identically zero.

From now on, we can assume that the scalars satisfy

$$a_{11}a_{22} = a_{12}a_{13}, \quad a_{11}b_2^{(1)} = a_{13}b_1^{(1)}, \quad a_{11}b_2^{(2)} = a_{13}b_1^{(2)}, \quad b_1^{(1)} = b_1^{(2)} = -a_{11}a_{24}, \tag{22}$$

all of them necessary conditions for $\mathcal{L}$ to be a Lie algebra. Since $a_{11}b_2^{(1)} = a_{13}b_1^{(1)} = a_{13}b_1^{(2)} = a_{11}b_2^{(2)}$ and a_{11} is non zero, so $b_2^{(1)} = b_2^{(2)}$. Then, to simplify notation, we denote by

$$b_1 := b_1^{(1)} = b_1^{(2)}, \qquad b_2 := b_2^{(1)} = b_2^{(2)}.$$

$\boxed{(J_{1,2,2})}$ Now take $x = u_1 \otimes v_{12} \in \mathcal{L}_{\bar{1}}$ and $y = u_{23} \otimes v_{3456}$, $z = u_{45} \otimes v_{789\,10} \in \mathcal{L}_{\bar{2}}$. Note that Eq. (22) tells $(a_{12}a_{13})(b_2a_{11}) = (a_{11}a_{22})(b_1a_{13})$, so that

$$a_{12}b_2 = a_{22}b_1.$$

This allows to write

$$\begin{aligned} J(x, y, z) = a_{12}b_2\big(&-[u_{45}, u_{123}]_1(v_{789\,10}, v_{12} \llcorner \widetilde{v_{3456}}) - (u_{45}, u_{123})[v_{789\,10}, v_{12} \llcorner \widetilde{v_{3456}}]_2 \\ &+[u_{23}, u_{145}]_1(v_{3456}, v_{12} \llcorner \widetilde{v_{789\,10}}) + (u_{23}, u_{145})[v_{3456}, v_{12} \llcorner \widetilde{v_{789\,10}}]_2 \\ &-[u_1, u_{2345}]_1(v_{12}, v_{3456} \llcorner \widetilde{v_{789\,10}}) - (u_1, u_{2345})[v_{12}, v_{3456} \llcorner \widetilde{v_{789\,10}}]_2\big). \end{aligned}$$

We conclude that this expression is always zero due to the fact that being zero or not does not depend on the choice of scalars. That is, we use the same arguments as those ones when checking the identity $(J_{1,1,4})$.

$\boxed{(J_{1,3,3})}$ Let $x = u_1 \otimes v_{12} \in \mathcal{L}_{\bar{1}}$ and $y = u_{234} \otimes v_3$, $z = u_{567} \otimes v_4 \in \mathcal{L}_{\bar{3}}$. We get that

$$J(x, y, z) = \Big(a_{13}a_{34}(u_{567} \llcorner \widetilde{u_{1234}} + u_{234} \llcorner \widetilde{u_{1567}}) + a_{33}a_{11}(u_{234} \llcorner \widetilde{u_{567}}) \wedge u_1\Big) \otimes v_{1234}.$$

For instance, for $x = e_1 \otimes e_{12} \in \mathcal{L}_{\bar{1}}$, $y = e_{234} \otimes e_3$ and $z = e_{125} \otimes e_4 \in \mathcal{L}_{\bar{3}}$, it is easy to check that

$$J(x, y, z) = (a_{13}a_{34} - a_{33}a_{11})e_{12} \otimes e_{1234},$$

so the fact $J(x, y, z) = 0$ gives

$$a_{13}a_{34} = a_{33}a_{11}. \tag{23}$$

This necessary condition on the scalars is of course sufficient for getting $(J_{1,3,3})$, since the above general expression becomes

$$J(x, y, z) = a_{33}a_{11}\big(u_{567} \llcorner \widetilde{u_{1234}} + u_{234} \llcorner \widetilde{u_{1567}} + (u_{234} \llcorner \widetilde{u_{567}}) \wedge u_1\big) \otimes v_{1234}.$$

As it has to vanish for some choice of scalars, and $a_{33}a_{11} \neq 0$, the only way is that it vanishes for any choice of the scalars.

$\boxed{(J_{2,2,4})}$ A first computation,

$$J(e_{12} \otimes e_{1234}, e_{34} \otimes e_{1235}, e_{1235} \otimes e_{345}) = (a_{22}a_{44} - a_{24}a_{12})e_{123} \otimes e_3,$$

gives, as a necessary condition,

$$a_{22}a_{44} = a_{24}a_{12}. \tag{24}$$

With this requisite on the scalars, $(J_{2,2,4})$ holds, since for $x = u_{12} \otimes v_{1234}$, $y = u_{34} \otimes v_{5678} \in \mathcal{L}_{\bar{2}}$ and $z = u_{5678} \otimes v_{9\,10\,11} \in \mathcal{L}_{\bar{4}}$,

$$\begin{aligned} J(x, y, z) = a_{22}a_{44}\Big(&u_{1234} \llcorner \widetilde{u_{5678}} \otimes (v_{1234} \llcorner \widetilde{v_{5678}}) \llcorner \widetilde{v_{9\,10\,11}} \\ &+(u_{34} \llcorner \widetilde{u_{5678}}) \wedge u_{12} \otimes (v_{5678} \llcorner \widetilde{v_{9\,10\,11}}) \llcorner \widetilde{v_{1234}} \\ &-(u_{12} \llcorner \widetilde{u_{5678}}) \wedge u_{34} \otimes (v_{1234} \llcorner \widetilde{v_{9\,10\,11}}) \llcorner \widetilde{v_{5678}}\Big), \end{aligned}$$

which is zero independently of the values of a_{22} and a_{44}.

$\boxed{(J_{2,2,3})}$ Let $x = e_{12} \otimes e_{1234} \in \mathcal{L}_{\bar{2}}$, $y = e_{34} \otimes e_{1235} \in \mathcal{L}_{\bar{2}}$ and $z = e_{125} \otimes e_4 \in \mathcal{L}_{\bar{3}}$. From

$$J(x, y, z) = (a_{22}a_{34} + b_2)e_{12} \otimes e_{1234},$$

we get

$$a_{22}a_{34} = -b_2. \tag{25}$$

As in the above cases, this condition is sufficient to guarantee $(J_{2,2,3})$.

Compiling the information of (22), (23), (24) and (25), we have that all the restrictions on the scalars in (16) are necessary for $\mathcal{L}$ to be a Lie algebra, since

$$\begin{array}{ll}
a_{13}a_{24}a_{12} = a_{24}a_{11}a_{22} = -b_1a_{22} = -a_{12}b_2 & \Rightarrow a_{13}a_{24} = -b_2, \\
-b_1a_{13} = a_{11}a_{24}a_{13} = -b_2a_{11} = a_{22}a_{34}a_{11} = a_{13}a_{34}a_{12} & \Rightarrow a_{34}a_{12} = -b_1, \\
-b_2a_{11} = a_{22}a_{34}a_{11} = a_{13}a_{34}a_{12} = a_{33}a_{11}a_{12} & \Rightarrow a_{33}a_{12} = -b_2, \\
a_{22}a_{44}a_{13} = a_{24}a_{12}a_{13} = a_{22}a_{11}a_{24} = -a_{22}b_1 & \Rightarrow a_{44}a_{13} = -b_1.
\end{array}$$

From now on, we asume that our scalars satisfy all the conditions in (16). In particular they satisfy all the relations in (22), (23), (24) and (25). Then we only have to test that the remaining identities $(J_{i,j,k})$ hold too.

$\boxed{(J_{1,2,3})}$ For elements $x = u_1 \otimes v_{12} \in \mathcal{L}_{\bar{1}}$, $y = u_{23} \otimes v_{3456} \in \mathcal{L}_{\bar{2}}$ and $z = u_{456} \otimes v_7 \in \mathcal{L}_{\bar{3}}$, and taking into account that $a_{12}a_{33} = -b_2 = a_{13}a_{24}$, we obtain that

$$\begin{aligned} J(x, y, z) = b_2\Big(&[u_{23}, u_{456}](u_1) \otimes (v_{3456}, v_7)v_{12} + (u_{23}, u_{456})u_1 \otimes [v_{3456}, v_7](v_{12}) \\ &-u_{123} \llcorner \widetilde{u_{456}} \otimes (v_{12} \llcorner \widetilde{v_{3456}}) \wedge v_7 - u_{23} \llcorner \widetilde{u_{1456}} \otimes v_{3456} \llcorner \widetilde{v_{127}}\Big); \end{aligned}$$

which is necessarily zero as in the above cases.

$\boxed{(J_{1,2,4})}$ Consider now $x = u_1 \otimes v_{12} \in \mathcal{L}_{\bar{1}}$, $y = u_{23} \otimes v_{3456} \in \mathcal{L}_{\bar{2}}$ and $z = u_{4567} \otimes v_{789} \in \mathcal{L}_{\bar{4}}$ and take into account $a_{12}a_{34} = a_{24}a_{11} = -b_1$ to get

$$\begin{aligned} J(x, y, z) = -b_1\Big(&u_{123} \llcorner \widetilde{u_{4567}} \otimes (v_{12} \llcorner \widetilde{v_{3456}}) \wedge v_{789} + (u_{23} \llcorner \widetilde{u_{4567}}) \wedge u_1 \otimes (v_{3456} \llcorner \widetilde{v_{789}}) \wedge v_{12} \\ &+[u_1, u_{4567}](u_{23}) \otimes (v_{12}, v_{789})v_{3456} + (u_1, u_{4567})u_{23} \otimes [v_{12}, v_{789}](v_{3456})\Big). \end{aligned}$$

Arguments as in the previous cases permit us to conclude that it vanishes.

$\boxed{(J_{1,3,4})}$ Recall that $a_{13}a_{44} = a_{34}a_{12} = -b_1$, so that, for $x = u_1 \otimes v_{12} \in \mathcal{L}_{\bar{1}}$, $y = u_{234} \otimes v_3 \in \mathcal{L}_{\bar{3}}$ and $z = u_{5678} \otimes v_{456} \in \mathcal{L}_{\bar{4}}$, we have

$$\begin{aligned} J(x, y, z) = b_1\Big(&u_1 \wedge (u_{234} \llcorner \widetilde{u_{5678}}) \otimes v_{12} \llcorner \widetilde{v_{3456}} - u_{1234} \llcorner \widetilde{u_{5678}} \otimes v_{123} \llcorner \widetilde{v_{456}} \\ &-[u_1, u_{5678}](u_{234}) \otimes \phi(v_{12456})v_3 - \phi(u_{15678})u_{234} \otimes [v_{12}, v_{456}](v_3)\Big), \end{aligned}$$

which is identically zero.

Clearly, we do not have to compute all the expressions to be completely sure that the rest of the identities hold too:

- $(J_{1,4,4})$ holds since, for any $x \in \mathcal{L}_{\bar{1}}$, $y, z \in \mathcal{L}_{\bar{4}}$, we can write $\frac{J(x,y,z)}{b_1}$ without using any scalar, taking into account that $b_1 = -a_{44}a_{13}$.
- Similarly $\frac{J(x,y,z)}{b_2}$ is independent of the choice of scalars for any $x \in \mathcal{L}_{\bar{2}}$, $y, z \in \mathcal{L}_{\bar{3}}$, since $b_2 = -a_{33}a_{12}$. That is, we get that $(J_{2,3,3})$ is true.
- $(J_{2,3,4})$ holds, since $b_2 = -a_{24}a_{13} = -a_{22}a_{34}$.
- $(J_{2,4,4})$ follows from $a_{24}b_1 = a_{44}b_2$.
- $(J_{3,3,4})$ follows from $a_{33}b_1 = a_{34}b_2$.

- The identity $(J_{3,4,4})$ is a consequence of $a_{34}a_{24} = a_{44}a_{33}$.

All this implies that $\mathcal{L}$ is a Lie algebra under our assumptions in (16). In fact, it is very easy to find concrete solutions of the system (16), for instance, the one provided in (17).

The following step is to check that the obtained Lie algebras are mutually isomorphic. To do that, take $\mathcal{L}$ and $\mathcal{L}'$ the $\mathbb{Z}_5$-graded vector spaces as in (14) endowed with the Lie bracket as in (15) with the coefficients given by (16) and (17) respectively. Let $f\colon \mathcal{L} \to \mathcal{L}'$ be the map given by $f(x) = \alpha_i x$, for any $x \in \mathcal{L}_{\bar{i}}$, for any choice of scalars $\{\alpha_i : i = 0, \dots, 4\} \subset \mathbb{F}^\times$ such that

$$\alpha_0 = 1, \quad (\alpha_1)^5 = -b_1 a_{11} a_{12} a_{13}, \quad \alpha_2 = -\frac{b_1 a_{12} a_{13}}{\alpha_1^3}, \quad \alpha_3 = \frac{-b_1 a_{13}}{\alpha_1^2}, \quad \alpha_4 = -\frac{b_1}{\alpha_1}. \tag{26}$$

An straightforward computation shows that

$$\frac{\alpha_i \alpha_j}{\alpha_{[i+j]_5}} = a_{ij}, \quad \alpha_1\alpha_4 = -b_1, \quad \alpha_2\alpha_3 = -b_2, \tag{27}$$

for any $0 \leq i \leq j \leq 4$ such that $(i, j) \in \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (2, 4), (3, 4)\}$, where b_i is used for both $b_i^{(1)}$ and $b_i^{(2)}$. Thus f is an isomorphism between $\mathcal{L}$ and $\mathcal{L}'$. Recall from Remark 3.2 that there is some solution of the system of quadratic equations (16) such that the corresponding Lie algebra $\mathcal{L}$ is the split algebra $\mathfrak{e}_8$. Hence all the obtained Lie algebras are isomorphic to $\mathfrak{e}_8$ and so, in particular, they are simple. □

Remark 3.3 The construction in Proposition 3.1 yields simple Lie algebras of type E_8, even over fields that are not algebraically closed. To prove that all Lie algebras obtained in this way are simple, it suffices to verify this property after extending the base field. If the field is algebraically closed, then all choices of parameters give rise to isomorphic Lie algebras (as shown in the proof above), and one of them is simple of type E_8 by Theorem 1.1. It follows that all such extensions must also be simple.

Alternatively, over an arbitrary field, simplicity can be verified by a direct argument that relies only on the grading and the irreducibility of the homogeneous components. If an ideal I of $\mathcal{L}$ contains a nonzero homogeneous element in $\mathcal{L}_{\bar{i}}$ for some $i \neq 0$, then, since $\mathcal{L}_{\bar{i}}$ is an irreducible $\mathcal{L}_{\bar{0}}$-module, it follows that $\mathcal{L}_{\bar{i}} \subset I$, and hence $I = \mathcal{L}$. If instead I contains a nonzero element in $\mathcal{L}_{\bar{0}}$, then at least one of the copies of $\mathfrak{sl}(V)$ lies in I. Multiplying, for instance, with elements of $\mathcal{L}_{\bar{1}}$, we obtain $\mathcal{L}_{\bar{1}} \subset I$, and as before, this implies $I = \mathcal{L}$. Finally, aiming for a contradiction, suppose that a certain nonzero ideal $I \subset \mathcal{L}$ contains no nonzero homogeneous elements. Let us first show that any nonzero element $x = \sum x_i \in I$, with $x_i \in \mathcal{L}_{\bar{i}}$, gives rise to another nonzero element $x' = \sum_{i \neq 0} x_i' \in I$, such that $x_i' = 0$ for all indices i with $x_i = 0$. Indeed, since each copy of $\mathfrak{sl}(V)$ acts faithfully on $\mathcal{L}_{\bar{i}}$ for $i \neq 0$, but centralizes the other copy of $\mathfrak{sl}(V)$, we can first multiply x by elements from one copy of $\mathfrak{sl}(V)$, and then by elements from the other, to obtain a nonzero element $x' \in I$ with no component in $\mathcal{L}_{\bar{0}}$. Next, we multiply x' with elements in $\mathcal{L}_{\bar{1}}$, obtaining a nonzero element in $[\sum_{i \neq \bar{0}} \mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{1}}] \subset \mathcal{L}_{\bar{2}} + \mathcal{L}_{\bar{3}} + \mathcal{L}_{\bar{4}} + \mathcal{L}_{\bar{0}}$, which still lies in I. Applying the same reduction process to eliminate the $\bar{0}$-component again, we obtain a nonzero element in $(\mathcal{L}_{\bar{2}} + \mathcal{L}_{\bar{3}} + \mathcal{L}_{\bar{4}}) \cap I$. Repeating this procedure and multiplying with $\mathcal{L}_{\bar{1}}$ as before, we eventually obtain a nonzero homogeneous element in I, contradicting our assumption. This contradiction shows that every nonzero ideal of $\mathcal{L}$ must contain a nonzero homogeneous element, and hence $\mathcal{L}$ is simple.

Remark 3.4 A second interesting question is whether all the simple Lie algebras obtained via Proposition 3.1 are isomorphic over arbitrary fields, and, if not, how many isomorphism classes may arise. At first glance, this does not seem to be the case, since the scalars used in (26) cannot be chosen in the same way when the base field does not contain fifth roots of all its elements.

A remarkable example where fifth roots can be taken is the field of real numbers. In this case, only one of the three real forms of the complex Lie algebra of type E_8 can be obtained via Proposition 3.1—namely, the split real form of type E_8, that is, the one whose Killing form has signature 8, taking into account Sect. 2.1).

To clarify the picture, it is helpful to interpret the above in the language of group cohomology. Consider G a group and the abelian group $M = \mathbb{F}^\times$ as a G-module (with the trivial group action $G \times \mathbb{F}^\times \to \mathbb{F}^\times$, $g \cdot m = m$ for all $m \in \mathbb{F}^\times$). Recall that the n-cochains form an abelian group $C^n(G, M) = \{\varphi\colon G^n \to M : \text{function}\}$ and the coboundary homomorphisms are defined by $\delta \equiv \delta^{n+1}\colon C^n(G, M) \to C^{n+1}(G, M)$,

$$\begin{aligned}&\delta^{n+1}\varphi(g_1, \dots, g_{n+1}) =\\ &\quad = \varphi(g_2, \dots, g_{n+1})\big(\textstyle\prod_{i=1}^{n} \varphi(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1})^{(-1)^i}\big)\varphi(g_1, \dots, g_n)^{(-1)^{n+1}}.\end{aligned}$$

Since $\delta^{n+1}\delta^n = 0$, this defines a cochain complex whose cohomology is given by $H^n(G, M) = \frac{Z^n(G,M)}{B^n(G,M)}$, where the group of n-cocycles is $Z^n(G, M) = \ker(\delta^{n+1})$, and the group of n-coboundaries is $B^n(G, M) = \operatorname{im}(\delta^n)$.

For each choice of nonzero scalars satisfying (16), we can define a symmetric 2-cochain $\varphi\colon G \times G \to \mathbb{F}^\times$, where $G = \mathbb{Z}_5$, by setting $\varphi(\bar{i}, \bar{0}) = \varphi(\bar{0}, \bar{i}) = 1$ for all $\bar{i} \in G$; and $\varphi(\bar{i}, \bar{j}) = \varphi(\bar{j}, \bar{i}) = a_{ij}$ for all $1 \leq i \leq j \leq 4$. Note that φ can be directly verified to be a 2-cocycle; that is, it satisfies $\varphi(\bar{i}, \bar{j})\, \varphi(\bar{i}+\bar{j}, \bar{k}) = \varphi(\bar{j}, \bar{k})\, \varphi(\bar{j}+\bar{k}, \bar{i})$ for all $\bar{i}, \bar{j}, \bar{k} \in G = \mathbb{Z}_5$. Remarkably, the proof of Proposition 3.1 establishes more than what is explicitly stated. The 2-cocycle corresponding to the solution (17) is simply $\varphi_0 \equiv 1$. Moreover, if $\mathbb{F}$ is algebraically closed, Equation (27) shows that any other 2-cochain φ associated with a second solution is of the form $\delta^2\alpha$, where α is the 1-cochain defined in (26); in other words, φ is a 2-coboundary. In particular, this implies that φ is automatically a symmetric 2-cocycle, and no further verification is required.

All this can be viewed from a wider perspective. If G is any abelian group and $\Gamma : \mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ is a graded algebra, then for any 2-cochain $\sigma\colon G \times G \to \mathbb{F}^\times$ we can define the twisted algebra $\mathcal{L}^\sigma = (\mathcal{L}; [\ ,\ ]^\sigma)$ by means of

$$[x, y]^\sigma = \sigma(g, h)[x, y] \tag{28}$$

for any $g, h \in G$; $x \in \mathcal{L}_g$, $y \in \mathcal{L}_h$. The algebra $\mathcal{L}^\sigma$ is trivially G-graded, with $(\mathcal{L}^\sigma)_g := \mathcal{L}_g$. If σ is a symmetric 2-cocycle, then $\mathcal{L}^\sigma$ is a Lie algebra: the symmetry of σ ensures that $[x, y]^\sigma = -[y, x]^\sigma$, and the 2-cocycle condition guarantees that the Jacobi identity holds. Under mild assumptions, the converse is also true. Indeed, suppose that the homogeneous components of the grading Γ satisfy the following:

i) $[\mathcal{L}_g, \mathcal{L}_h] \neq 0$ for all $g, h \in G$;
ii) for all $g, h, k \in G$, there exist elements $x \in \mathcal{L}_g$, $y \in \mathcal{L}_h$, and $z \in \mathcal{L}_k$ such that the set $\{\operatorname{ad}_x \operatorname{ad}_y(z),\ \operatorname{ad}_y \operatorname{ad}_x(z)\}$ is linearly independent.

Then, if $\mathcal{L}^\sigma$ is a Lie algebra, it is not difficult to verify that σ is a symmetric 2-cocycle. (This confirms that the situation described for our $\mathbb{Z}_5$-graded algebra in Proposition 3.1 is exactly as expected, since it is clear that conditions *i)* and *ii)* are satisfied in that construction.) Along the same lines, we may note another general fact: if σ is a 2-coboundary, then the Lie algebra $\mathcal{L}^\sigma$ it defines is isomorphic to the original one $\mathcal{L}$. The argument is straightforward:

if $\sigma = \delta^2\alpha$ for some 1-cochain $\alpha\colon G \to \mathbb{F}^\times$, then the map $\mathcal{L} \to \mathcal{L}^\sigma$, given by $x \mapsto \alpha(g)x$ for all $x \in \mathcal{L}_g$, is an isomorphism of Lie algebras.

Furthermore, for any finite abelian group G and any algebraically closed field $\mathbb{F}$, the subgroup of $H^2(G, \mathbb{F}^\times)$ represented by symmetric 2-cocycles is trivial; see, for instance, [30, Chapter XIX, 6]. It follows that twisting the multiplication of our G-graded Lie algebra by a symmetric 2-cocycle—necessarily a 2-coboundary under our assumptions on the base field—does not change its isomorphism class. This provides an alternative, more conceptual argument for proving the uniqueness of the Lie algebra constructed in Proposition 3.1—an argument that can be applied in the same way to the other constructions in this article. (Another subtle point we are relying on is that the solutions of (16) satisfy $b_i^{(1)} = b_i^{(2)}$; this property likewise holds in the remaining models.)

Remark 3.5 It is well-known the existence of a $\mathbb{Z}_p^3$-graded Lie algebra with zero neutral homogeneous component and all the remaining $p^3 - 1$ homogeneous components of dimension 2 (pieces of Cartan subalgebras), for $p = 2, 3, 5$ and the obtained Lie algebra $\mathfrak{g}_2$, $\mathfrak{f}_4$ and $\mathfrak{e}_8$ respectively (dimensions $2(p^3 - 1)$). These are examples of the Jordan gradings described in [21] produced by the Jordan subgroups in [2] (see also [32, 3.8]). The related highly symmetric model of $\mathfrak{g}_2$ as a twisted ring group based on this $\mathbb{Z}_2^3$-grading has been very recently obtained in [11]. The challenge of getting a similar construction of $\mathfrak{e}_8$ by doubling and twisting the group $\mathbb{Z}_5^3$ could have as a convenient starting point the concrete products in Proposition 3.1, as explained in [21, Theorem 2.1].

3.2 Model based on the subalgebra of type $A_1 \oplus A_7$

Recall from Sect. 2.1 the existence of vector spaces U and W over $\mathbb{F}$, with $\dim U = 2$ and $\dim W = 8$, such that the split Lie algebra $\mathcal{L} = \mathfrak{e}_8 = \sum_{\bar{i}\in\mathbb{Z}_4} \mathcal{L}_{\bar{i}}$ is $\mathbb{Z}_4$-graded, with $\mathcal{L}_{\bar{0}} \cong \mathfrak{sl}(U) \oplus \mathfrak{sl}(W)$, and

$$\mathcal{L}_{\bar{1}} \cong U \otimes \textstyle\bigwedge^2 W, \quad \mathcal{L}_{\bar{2}} \cong \mathbb{F} \otimes \bigwedge^4 W, \quad \mathcal{L}_{\bar{3}} \cong U \otimes \bigwedge^6 W,$$

as $\mathcal{L}_{\bar{0}}$-modules.

According to Theorem 1.1, we can reconstruct the Lie algebra $\mathfrak{e}_8$ from these components, since the brackets $[\mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{j}}] \subset \mathcal{L}_{\bar{i}+\bar{j}}$ are determined up to scalars. We again make use of the invariant products described in Sect. 2.2—namely, $*$ in Eq. (7), $(\ ,\ )$ in Eq. (8), and $[\ ,\ ]$ in Eq. (10). In Remark 2.5, these three products are expressed for the vector space U in terms of an alternating form $\varphi\colon U \times U \to \mathbb{F}$.

Proposition 3.6 *Let $\mathcal{L}$ be the $\mathbb{Z}_4$-graded vector space given, for any $i = 1, 2, 3$, by*

$$\begin{aligned} \mathcal{L}_{\bar{0}} &= \mathfrak{sl}(U) \oplus \mathfrak{sl}(W) \\ \mathcal{L}_{\bar{i}} &= \textstyle\bigwedge^{[i]_2} U \otimes \bigwedge^{[2i]_8} W \end{aligned} \tag{29}$$

where U and W denote vector spaces over $\mathbb{F}$ of dimensions 2 and 8, respectively. Consider the product of $\mathcal{L}_{\bar{0}}$ with $\mathcal{L}_{\bar{i}}$ given by the action of the jth ideal of $\mathcal{L}_{\bar{0}}$ on the jth slot ($j = 1, 2$). Consider the product of $\mathcal{L}_{\bar{i}}$ with $\mathcal{L}_{\bar{j}}$, for $1 \le i \le j \le 3$, given by

$$[x_1 \otimes x_2, y_1 \otimes y_2] = \begin{cases} a_{ij}\, x_1 * y_1 \otimes x_2 * y_2 & i + j \neq 4, \\ b_1^{(1)}[x_1, y_1](x_2, y_2) + b_1^{(2)}(x_1, y_1)[x_2, y_2] & (i, j) = (1, 3), \\ b_2^{(2)}(x_1, y_1)[x_2, y_2] & (i, j) = (2, 2), \end{cases} \tag{30}$$

for any $x_1 \in \bigwedge^{[i]_2} U$, $x_2 \in \bigwedge^{[2i]_8} W$, $y_1 \in \bigwedge^{[j]_2} U$ and $y_2 \in \bigwedge^{[2j]_8} W$.

Extend the bracket to $\mathcal{L}$ making that the product of $\mathcal{L}_{\bar{i}}$ with $\mathcal{L}_{\bar{j}}$ if $i > j$ is skew-symmetric, and making $(\mathcal{L}_{\bar{0}}, [\ ,\ ])$ a subalgebra. Assume that all the scalars are nonzero. Then, $\mathcal{L}$ endowed with the product $[\ ,\ ]$ is a Lie algebra if and only if

$$\begin{aligned} b_1^{(1)} &= b_1^{(2)} = -a_{11}a_{23} = -a_{12}a_{33}, \\ b_2^{(2)} &= -a_{12}a_{23}. \end{aligned} \tag{31}$$

Moreover, $(\mathcal{L}, [\ ,\ ])$ is a simple exceptional Lie algebra isomorphic to $\mathfrak{e}_8$. A solution to this system of quadratic equations that, in particular, gives a model of $\mathfrak{e}_8$, is

$$\begin{aligned} 1 &= a_{11} = a_{12} = a_{23} = a_{33}, \\ -1 &= b_2^{(2)} = b_1^{(1)} = b_1^{(2)}. \end{aligned} \tag{32}$$

Proof Again it makes sense to deal with the skew-symmetric extension of the bracket because, for $i = j$ it is true that $[x_1 \otimes x_2, y_1 \otimes y_2] = -[y_1 \otimes y_2, x_1 \otimes x_2]$. In fact, if $i = 1, 3$, $x_1, y_1 \in U$, $x_2, y_2 \in \bigwedge^{[2i]_8} W$, and Lemma 2.3 says that $x_1 * y_1 = -y_1 * x_1$ while $x_2 * y_2 = y_2 * x_2$. For $i = 2$, $x_1, y_1 \in \mathbb{F}$ so that $(x_1, y_1) = x_1 y_1 = (y_1, x_1)$. In this case $x_2, y_2 \in \bigwedge^4 W$, and $[x_2, y_2] = -[y_2, x_2]$ since $(\ ,\ ) \colon \bigwedge^4 W \times \bigwedge^4 W \to \mathbb{F}$ is symmetric.

Similarly to the proof of Proposition 3.1, we have to prove the identity $(J_{i,j,k})$:

$$J(\mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{j}}, \mathcal{L}_{\bar{k}}) = 0 \text{ for } 0 \le i, j, k \le 3.$$

Again we can assume that $1 \le i, j, k \le 3$ by Remark 2.2, we can also assume $i \le j \le k$ by trilinearity, and the cases $i = j = k$ are consequence of Remark 3.2 adapted to our setting: the veracity of $(J_{i,i,i})$ does not depend on the scalars and the system of equations with variables $a_{ij}, b_i^{(1)}, b_i^{(2)} \in \mathbb{F}^\times$ always has at least one solution.

For checking the identities, we can restrict the considered elements in $\bigwedge^{[2l]_8} W$ to those which are wedge products of elements in W. Take $\{v_i : i \in \mathbb{N}\}$ arbitrary elements in W, and use the notation $v_{i_1 \dots i_k}$ for $v_{i_1} \wedge \cdots \wedge v_{i_k}$ as in Lemma 2.3 (and similarly for u_i's, elements in U). Also fix $\{e_i : i = 1, \dots, 8\}$ a basis of W and $\{f_1, f_2\}$ a basis of U. There is no ambiguity with the notation $f + g$ in $\mathcal{L}_{\bar{0}}$, for $f \in \mathfrak{sl}(U)$ and in $g \in \mathfrak{sl}(W)$. (We will not use subindices $f_1 + g_2$ in this case.)

$\boxed{(J_{1,1,2})}$ Take $x = u_1 \otimes v_{12}$, $y = u_2 \otimes v_{34} \in \mathcal{L}_{\bar{1}}$ and $z = 1 \otimes v_{5678} \in \mathcal{L}_{\bar{2}}$. Compute

$$\begin{aligned} J(x, y, z) = {} & -a_{11} b_2^{(2)} \phi(u_{12})[v_{1234}, v_{5678}] \\ & -a_{12} b_1^{(1)} \phi(v_{1\dots8})[u_1, u_2] + a_{12} b_1^{(2)} \phi(u_{12})[v_{12}, v_{345678}] \\ & +a_{12} b_1^{(1)} \phi(v_{1\dots8})[u_2, u_1] + a_{12} b_1^{(2)} \phi(u_{12})[v_{34}, v_{125678}]. \end{aligned}$$

As $[u_1, u_2] = [u_2, u_1]$; then

$$J(x, y, z) = \phi(u_{12})\big(-a_{11} b_2^{(2)} [v_{1234}, v_{5678}] + a_{12} b_1^{(2)} ([v_{12}, v_{345678}] + [v_{34}, v_{125678}])\big). \tag{33}$$

If we choose concrete elements, for instance, $x = f_1 \otimes e_{12}$, $y = f_2 \otimes e_{34}$ and $z = 1 \otimes e_{5678}$, we use Lemma 2.3 (b) to get

$$[e_{12}, e_{345678}] + [e_{34}, e_{125678}] = [e_{1234}, e_{5678}],$$

so that Eq. (33) becomes

$$J(x, y, z) = \big(-a_{11} b_2^{(2)} + a_{12} b_1^{(2)}\big)[e_{1234}, e_{5678}],$$

which gives a necessary condition to have the identity $(J_{1,1,2})$:

$$a_{11}b_2^{(2)} = a_{12}b_1^{(2)}. \tag{34}$$

Note that Eq. (34) is also sufficient to get $(J_{1,1,2})$, since this condition placed on Eq. (33) gives

$$J(x, y, z) = a_{11}b_2^{(2)}\phi(u_{12})\big([v_{12}, v_{345678}] + [v_{34}, v_{125678}] - [v_{1234}, v_{5678}]\big),$$

so that to be zero (for all $x, y \in \mathcal{L}_{\bar{1}}$ and all $z \in \mathcal{L}_{\bar{2}}$) or not does not depend on the choice of the scalars. That is, as so many times in the proof of Proposition 3.1, knowing the existence of solutions has made it possible for us to avoid the direct verification of

$$[v_{12}, v_{345678}] + [v_{34}, v_{125678}] = [v_{1234}, v_{5678}],$$

having had to check it only for one concrete basic element. (The direct verification is not difficult at all, it is enough to use (10) and the concrete action of $\mathfrak{sl}(W)$ on $\bigwedge^r W$, but it is very convenient for us to mechanize and shorten the arguments, realizing that we can often omit the computation with arbitrary elements.)

$\boxed{(J_{1,2,2})}$ For $x = u \otimes x_2 \in \mathcal{L}_{\bar{1}}$ and $y = 1 \otimes y_2$, $z = 1 \otimes z_2 \in \mathcal{L}_{\bar{2}}$,

$$J(x, y, z) = u \otimes \big(b_2^{(2)}[y_2, z_2](x_2) + a_{12}a_{23}\big(y_2 \llcorner \widetilde{x_2 \wedge z_2} - z_2 \llcorner \widetilde{x_2 \wedge y_2}\big)\big). \tag{35}$$

This is easy to compute using concrete elements, taking into account Lemma 2.3. For instance,

$$J(u \otimes e_{12}, 1 \otimes e_{3456}, 1 \otimes e_{1278}) = (-a_{12}a_{23} - b_2^{(2)})u \otimes e_{12}.$$

Hence, we get another necessary condition

$$a_{12}a_{23} = -b_2^{(2)}. \tag{36}$$

This is sufficient too, since under this restriction on the scalars, Eq. (35) becomes

$$J(x, y, z) = b_2^{(2)}u \otimes \big([y_2, z_2](x_2) + y_2 \llcorner \widetilde{x_2 \wedge z_2} - z_2 \llcorner \widetilde{x_2 \wedge y_2}\big),$$

and now the identity $(J_{1,2,2})$ is independent of the choice of the scalars, and, as above, it has to be always true.

$\boxed{(J_{1,2,3}), (J_{1,1,3})}$ As the identity $(J_{1,2,3})$ a priori involves $a_{12}a_{33}$, $a_{23}a_{11}$, $b_1^{(1)}$ and $b_1^{(2)}$, we focus first on concrete elements. For instance,

$$J(f_1 \otimes e_{12}, 1 \otimes e_{3456}, f_2 \otimes e_{345678}) = (-a_{12}a_{33} - b_1^{(2)})1 \otimes e_{3456}.$$

This implies that a necessary condition to get $(J_{1,2,3})$ is

$$a_{12}a_{33} = -b_1^{(2)}. \tag{37}$$

Also, $J(f_1 \otimes e_{12}, f_2 \otimes e_{34}, f_1 \otimes e_{125678}) = \big(-a_{23}a_{11} - \frac{1}{2}(b_1^{(1)} + b_1^{(2)})\big)f_1 \otimes e_{12}$; so that

$$a_{11}a_{23} = -\frac{1}{2}(b_1^{(1)} + b_1^{(2)}) \tag{38}$$

is necessary to get $(J_{1,1,3})$. In particular, Eqs. (34), (36), (37) and (38) are all necessary for $\mathcal{L}$ to be a Lie algebra. Let us check that these four equations imply Eq. (31). First, from Eq. (38) we have that $a_{11}a_{23}a_{12} = -\frac{1}{2}(b_1^{(1)} + b_1^{(2)})a_{12}$ and from Eq. (36) we get that

$-a_{11}b_2^{(2)} = -\frac{1}{2}(b_1^{(1)} + b_1^{(2)})a_{12}$. Now, applying Eq. (34) we obtain $b_1^{(1)} = b_1^{(2)}$. We rewrite (38) as $a_{11}a_{23} = -b_1^{(1)}$ and then we have all the conditions required in Eq. (31).

For the converse, assume that the scalars satisfy Eq. (31) and let us check that $\mathcal{L}$ satisfies the Jacobi identity. This gives immediately (36), (37) and (38), and, as $a_{11}a_{23}b_2^{(2)} = -b_1^{(2)}b_2^{(2)} = b_1^{(2)}a_{12}a_{23}$, we have (34) too. This guarantees $(J_{1,1,2})$ and $(J_{1,2,2})$, at the moment. But also $(J_{1,2,3})$ holds, since $\frac{J(x,y,z)}{b_1^{(2)}}$ can be written without involving any of the variables, for any $x \in \mathcal{L}_{\bar{1}}$, $y \in \mathcal{L}_{\bar{2}}$ and $z \in \mathcal{L}_{\bar{3}}$. In a similar way, $(J_{1,1,3})$ is true, since it only involves $a_{11}a_{23}$, $b_1^{(1)}$ and $b_1^{(2)}$, which are all related. The rest of the identities hold:

- $(J_{3,3,3})$ follows from $a_{33}a_{12} = -b_1^{(1)} = -b_1^{(2)}$;
- $(J_{2,2,3})$ follows from $a_{23}a_{12} = -b_2^{(2)}$;
- $(J_{2,3,3})$ holds, since $a_{23}b_1^{(2)} = a_{23}b_1^{(1)} = b_2^{(2)}a_{33}$;

which finish the proof that $\mathcal{L}$ is a Lie algebra. Moreover, it is evident that the choice of scalars in (32),

$$1 = a_{11} = a_{12} = a_{23} = a_{33}, \qquad -1 = b_2^{(2)} = b_1^{(1)} = b_1^{(2)},$$

provides a concrete solution of the system of equations (31).

Finally we prove that all the Lie algebras obtained for different solutions of (31) are isomorphic. To do that, take $\mathcal{L}$ and $\mathcal{L}'$ the $\mathbb{Z}_4$-graded vector spaces as in (29) endowed with the Lie bracket as in (30) with the coefficients given by (31) and (32) respectively. Let $f : \mathcal{L} \to \mathcal{L}'$ be the map given by $f(x) = \alpha_i x$, for any $x \in \mathcal{L}_{\bar{i}}$, where $\{\alpha_i : i = 0, \dots, 3\} \subset \mathbb{F}^\times$ are chosen such that

$$\alpha_0 = 1, \quad \alpha_1^4 = -b_2 a_{11}^2, \quad \alpha_2 = \frac{-b_1 a_{12}}{\alpha_1^2}, \quad \alpha_3 = \frac{-b_1}{\alpha_1}. \tag{39}$$

An straightforward computation shows that

$$\frac{\alpha_i \alpha_j}{\alpha_{[i+j]_4}} = a_{ij}, \quad \alpha_1\alpha_3 = -b_1, \quad \alpha_2\alpha_2 = -b_2,$$

for any $0 \le i \le j \le 3$ such that $(i, j) \in \{(1, 1), (1, 2), (2, 3), (3, 3)\}$, where b_i is used for both $b_i^{(1)}$ and $b_i^{(2)}$. Thus f is an isomorphism between $\mathcal{L}$ and $\mathcal{L}'$. Since one of the solutions of the system (31) makes the corresponding algebra Lie algebra $\mathcal{L}$ isomorphic to $\mathfrak{e}_8$, then all the obtained Lie algebras are simple and all of them are isomorphic to $\mathfrak{e}_8$. □

Remark 3.7 Over arbitrary fields, simplicity can be argued as in Remark 3.3, but, once again, uniqueness no longer holds. This time, not even over the real numbers: the arguments in (39) fail due to the nonexistence of fourth roots in $\mathbb{R}$, which suggests that the Lie algebras obtained for different values in (31) may not be isomorphic, and that we might obtain both real forms corresponding to signatures 8 and −24 from different solutions of (31). Further calculations would be needed to make this more precise.

3.3 Model based on the subalgebra of type $A_1 \oplus A_2 \oplus A_5$

In this case, Sect. 2.1 implies the existence of U, V and W vector spaces over $\mathbb{F}$ with $\dim U = 2$, $\dim V = 3$ and $\dim W = 6$ such that the Lie algebra $\mathcal{L} = \mathfrak{e}_8 = \sum_{\bar{i}\in\mathbb{Z}_6} \mathcal{L}_{\bar{i}}$ is $\mathbb{Z}_6$-graded with $\mathcal{L}_{\bar{0}} \cong \mathfrak{sl}(U) \oplus \mathfrak{sl}(V) \oplus \mathfrak{sl}(W)$ and

$$\mathcal{L}_{\bar{1}} \cong U \otimes V \otimes W, \quad \mathcal{L}_{\bar{2}} \cong \mathbb{F} \otimes \bigwedge^2 V \otimes \bigwedge^2 W, \quad \mathcal{L}_{\bar{3}} \cong U \otimes \mathbb{F} \otimes \bigwedge^3 W,$$
$$\mathcal{L}_{\bar{4}} \cong \mathbb{F} \otimes V \otimes \bigwedge^4 W, \quad \mathcal{L}_{\bar{5}} \cong U \otimes \bigwedge^2 V \otimes \bigwedge^5 W,$$

where $\cong$ means isomorphisms as $\mathcal{L}_{\bar{0}}$-modules. According to Theorem 1.1, the Lie algebra $\mathfrak{e}_8$ can be recovered from these pieces since the brackets $[\mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{j}}] \subset \mathcal{L}_{\bar{i}+\bar{j}}$ are determined up to scalars. We again make use of $*$ in Eq. (7), $(\ ,\ )$ in Eq. (8) and $[\ ,\ ]$ in Eq. (10).

Proposition 3.8 *Let U, V and W be vector spaces over $\mathbb{F}$ of dimension 2, 3 and 6 respectively. Let $\mathcal{L}$ be the $\mathbb{Z}_6$-graded vector space given by*

$$\begin{aligned} \mathcal{L}_{\bar{0}} &= \mathfrak{sl}(U) \oplus \mathfrak{sl}(V) \oplus \mathfrak{sl}(W), \\ \mathcal{L}_{\bar{i}} &= \textstyle\bigwedge^{[i]_2} U \otimes \bigwedge^{[i]_3} V \otimes \bigwedge^{[i]_6} W, \end{aligned} \tag{40}$$

for any $i = 1, \dots, 5$. Consider the product of $\mathcal{L}_{\bar{0}}$ with $\mathcal{L}_{\bar{i}}$ given by the action of the jth ideal of $\mathcal{L}_{\bar{0}}$ on the jth slot ($j = 1, 2, 3$). Fix some nonzero scalars $a_{ij}, b_k^{(l)} \in \mathbb{F}^\times$ for any $1 \le i \le j \le 5$, $i + j \neq 6$, $k, l = 1, 2, 3$.[1] *Consider the product of $x = x_1 \otimes x_2 \otimes x_3 \in \mathcal{L}_{\bar{i}}$ with $y = y_1 \otimes y_2 \otimes y_3 \in \mathcal{L}_{\bar{j}}$, for $1 \le i \le j \le 5$, given by*

$$[x, y] = \begin{cases} a_{ij}\, x_1 * y_1 \otimes x_2 * y_2 \otimes x_3 * y_3 & i + j \neq 6, \\ \sum_{k=1}^3 b_i^{(k)} [x_k, y_k] \Pi_{r=1, r\neq k}^3 (x_r, y_r) & i + j = 6. \end{cases} \tag{41}$$

Extend the bracket to $\mathcal{L}$ making that the product of $\mathcal{L}_{\bar{i}}$ with $\mathcal{L}_{\bar{j}}$ if $i > j$ is skew-symmetric, and making $(\mathcal{L}_{\bar{0}}, [\ ,\ ])$ a subalgebra. Then, $\mathcal{L}$ endowed with the product $[\ ,\ ]$ is a Lie algebra if and only if

$$\begin{aligned} &b_1^{(1)} = b_1^{(2)} = b_1^{(3)} = -a_{11}a_{25} = -a_{12}a_{35} = -a_{13}a_{45} = -a_{14}a_{55}, \\ &b_2^{(2)} = b_2^{(3)} = a_{12}a_{34} = a_{14}a_{25} = -a_{22}a_{44} = -a_{23}a_{45}, \\ &b_3^{(1)} = b_3^{(3)} = a_{13}a_{34}, \\ &a_{11}a_{22} = -a_{12}a_{13}. \end{aligned} \tag{42}$$

Moreover, $(\mathcal{L}, [\ ,\ ])$ is a simple exceptional Lie algebra isomorphic to $\mathfrak{e}_8$. A solution of this system of equations that, in particular, gives a model of $\mathfrak{e}_8$, is

$$\begin{aligned} 1 &= a_{11} = a_{13} = a_{22} = a_{23} = a_{25} = a_{34} = a_{44} = a_{45} = b_3^{(1)} = b_3^{(3)}, \\ -1 &= a_{12} = a_{14} = a_{35} = a_{55} = b_1^{(1)} = b_1^{(2)} = b_1^{(3)} = b_2^{(2)} = b_2^{(3)}. \end{aligned} \tag{43}$$

Proof First of all, we will check that all the conditions in (42) are necessary for $\mathcal{L}$ to be a Lie algebra, simply by applying the Jacobi identity to several triples of elements to achieve the required conditions. We abuse a little bit of the notations, by using $\{e_1, e_2\}$, $\{e_1, e_2, e_3\}$ and $\{e_1, \dots, e_6\}$ bases of U, V and W, respectively. When we do a tensor product, there is no confusion in the position of the elements. Also, in case there is some ambiguity, we denote a linear map $F \colon U \to U$ with a subindex, F_U, and similarly for endomorphisms of V and W. Hence assume now that $\mathcal{L}$ is a Lie algebra.

$\star$ We compute, for $x = e_1 \otimes e_1 \otimes e_1$, $y = e_2 \otimes e_2 \otimes e_2$ and

$$\begin{aligned} z = e_2 \otimes e_{13} \otimes e_{13456} &\Rightarrow J(x, y, z) = (a_{11}a_{25} + b_1^{(1)}) e_2 \otimes e_1 \otimes e_1, \\ z = e_1 \otimes e_{23} \otimes e_{13456} &\Rightarrow J(x, y, z) = (a_{11}a_{25} + b_1^{(2)}) e_1 \otimes e_2 \otimes e_1, \\ z = e_2 \otimes e_{23} \otimes e_{13456} &\Rightarrow J(x, y, z) = (a_{11}a_{25} + b_1^{(3)}) e_2 \otimes e_2 \otimes e_1, \\ z = 1 \otimes e_{13} \otimes e_{34} &\Rightarrow J(x, y, z) = -(a_{11}a_{22} + a_{12}a_{13}) 1 \otimes e_1 \otimes e_{1234}, \\ z = e_1 \otimes 1 \otimes e_{345} &\Rightarrow J(x, y, z) = -(a_{11}a_{23} + a_{13}a_{14}) e_1 \otimes e_{12} \otimes e_{12345}. \end{aligned}$$

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[1] Actually, there is no need to consider $b_2^{(1)}$ and $b_3^{(2)}$, since the following brackets vanish: $[\ ,\ ] \colon \bigwedge^i U \times \bigwedge^{n-i} U \to \mathfrak{sl}(U)$ if $i = 2$ and $[\ ,\ ] \colon \bigwedge^i V \times \bigwedge^{n-i} V \to \mathfrak{sl}(V)$ if $i = 3$. But this makes easier to write (41).

The Jacobi identity so implies that $b_1^{(1)}=b_1^{(2)}=b_1^{(3)}$, which we will denote as b_1, as well as

$$a_{11}a_{25}=-b_1, \tag{44}$$

$$a_{11}a_{22}=-a_{12}a_{13}, \tag{45}$$

$$a_{11}a_{23}=-a_{13}a_{14}. \tag{46}$$

Use Lemma 2.3 (b) to check that $[e_1,e_{23}]_V-[e_2,e_{13}]_V=[e_{12},e_3]_V$ and $[e_1,e_{23456}]_W-[e_2,e_{13456}]_W=-[e_{12},e_{3456}]_W$, and recall that $[\ ,\ ]_U$ is symmetric, to get

$$J(x,y,1\otimes e_3\otimes e_{3456})=-(a_{11}b_2^{(2)}+a_{14}b_1)[e_{12},e_3]_V-(a_{11}b_2^{(3)}+a_{14}b_1)[e_{12},e_{3456}]_W.$$

Again its projections on $\mathfrak{sl}(V)$ and $\mathfrak{sl}(W)$ are both zero, so that $b_2^{(2)}=b_2^{(3)}=:b_2$ and

$$a_{11}b_2=-a_{14}b_1. \tag{47}$$

⋆ Now take $x=e_1\otimes e_1\otimes e_1$, $y=1\otimes e_{23}\otimes e_{23}$, and compute

$$\begin{aligned}J(x,y,e_2\otimes 1\otimes e_{456})=&\,(a_{23}b_1-a_{12}b_3^{(1)})[e_1,e_2]_U+(a_{13}b_2-a_{23}b_1)[e_1,e_{23}]_V\\&+(a_{23}b_1-a_{12}b_3^{(3)})[e_1,e_{23456}]_W-(a_{13}b_2-a_{12}b_3^{(3)})[e_{23},e_{1456}]_W;\end{aligned}$$

taking into account that $[e_1,e_{23}]_V=-[e_{23},e_1]_V$, and $[e_1,e_{23456}]_W-[e_{23},e_{1456}]_W=[e_{123},e_{456}]_W$. The three projections are zero, so that $b_3^{(1)}=b_3^{(3)}=:b_3$ and

$$a_{23}b_1=a_{12}b_3=a_{13}b_2. \tag{48}$$

Choosing a different third element,

$$\begin{aligned}z&=1\otimes e_1\otimes e_{1456} &&\Rightarrow J(x,y,z)=(a_{12}a_{34}-b_2)e_1\otimes e_1\otimes e_1,\\ z&=e_2\otimes e_{12}\otimes e_{12456} &&\Rightarrow J(x,y,z)=(-a_{12}a_{35}+a_{11}a_{25})1\otimes e_{12}\otimes e_{12},\end{aligned}$$

we obtain

$$a_{12}a_{34}=b_2,\qquad a_{12}a_{35}=a_{11}a_{25}\overset{(44)}{=}-b_1. \tag{49}$$

⋆ More necessary conditions appear when considering $x=e_1\otimes e_1\otimes e_1$, $y=e_2\otimes 1\otimes e_{234}$. We compute

$$\begin{aligned}z&=1\otimes e_2\otimes e_{2356} &&\Rightarrow J(x,y,z)=-(a_{13}a_{44}-a_{14}a_{35})1\otimes e_{12}\otimes e_{23},\\ z&=e_1\otimes e_{23}\otimes e_{12356} &&\Rightarrow J(x,y,z)=(a_{13}a_{45}-a_{35}a_{12})e_1\otimes 1\otimes e_{123},\end{aligned}$$

which yields

$$a_{13}a_{44}=a_{14}a_{35},\qquad a_{13}a_{45}=a_{35}a_{12}. \tag{50}$$

⋆ Finally $J(e_1\otimes e_1\otimes e_1,1\otimes e_2\otimes e_{2345},e_2\otimes e_{13}\otimes e_{12346})=(a_{14}a_{55}-a_{45}a_{13})1\otimes e_1\otimes e_{1234}$, so that

$$a_{14}a_{55}=a_{45}a_{13}. \tag{51}$$

Keeping in mind this information, next we prove that all the equations in (42) are necessary. From $a_{12}b_3\overset{(48)}{=}a_{13}b_2\overset{(49)}{=}a_{13}a_{12}a_{34}$, we get

$$b_3=a_{13}a_{34}. \tag{52}$$

Also, $a_{11}b_2\overset{(47)}{=}-a_{14}b_1\overset{(44)}{=}a_{14}a_{11}a_{25}$, so that

$$b_2=a_{14}a_{25}. \tag{53}$$

Note $a_{11}a_{23}a_{45} \overset{(46)}{=} -a_{14}a_{13}a_{45} \overset{(50)}{=} -a_{14}a_{35}a_{12} \overset{(49)}{=} -a_{14}a_{11}a_{25}$, getting

$$a_{23}a_{45} = -a_{14}a_{25} \overset{(53)}{=} -b_2. \tag{54}$$

Take also into account $a_{35}a_{12}a_{23} \overset{(49)}{=} -b_1a_{23} \overset{(48)}{=} -a_{12}b_3$, so

$$b_3 = -a_{23}a_{35}; \tag{55}$$

and $a_{22}a_{13}a_{14} \overset{(46)}{=} -a_{11}a_{22}a_{23} \overset{(45)}{=} a_{12}a_{13}a_{23}$, which gives

$$a_{22}a_{14} = a_{12}a_{23}. \tag{56}$$

And we also need $a_{13}b_2 \overset{(48)}{=} a_{12}b_3 \overset{(55)}{=} -a_{35}a_{12}a_{23} \overset{(56)}{=} -a_{22}a_{14}a_{35} \overset{(50)}{=} -a_{22}a_{13}a_{44}$, which yields

$$b_2 = -a_{22}a_{44}. \tag{57}$$

Finally observe that Eqs. (44), (45), (49), (50),(51), (52), (53), (54), (57) give all the conditions in (42). That is, the conditions in (42) are necessary for $\mathcal{L}$ to be a Lie algebra.

Conversely, assume that we have nonzero scalars satisfying (42) and let us prove that $\mathcal{L}$ is a Lie algebra. We can talk about the skew-symmetric extension because, for $i = j$, the defined bracket already satisfies $[x, y] = -[y, x]$. In fact,

- If $i = j \in \{1, 5\}$, then $x_1 * y_1 = -y_1 * x_1$, $x_2 * y_2 = -y_2 * x_2$ and $x_3 * y_3 = -y_3 * x_3$;
- If $i = j \in \{2, 4\}$, then $x_1 * y_1 = y_1 * x_1$, $x_2 * y_2 = -y_2 * x_2$ and $x_3 * y_3 = y_3 * x_3$;
- If $i = j = 3$, then $[x_1, y_1] = [y_1, x_1]$, $(x_1, y_1) = -(y_1, x_1)$, $(x_2, y_2) = (y_2, x_2)$, and $[x_3, y_3] = [y_3, x_3]$, $(x_3, y_3) = -(y_3, x_3)$.

The constructed skew-symmetric algebra $(\mathcal{L}, [\ ,\ ])$ is a Lie algebra if and only if $J(\mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{j}}, \mathcal{L}_{\bar{k}}) = 0$ for $0 \le i, j, k \le 5$. We will again denote this identity by $(J_{i,j,k})$. By Remark 2.2 applied to $\mathfrak{m} = \oplus_{i=1}^{5}\mathcal{L}_{\bar{i}}$, we have only to check $(J_{i,j,k})$ for $1 \le i \le j \le k \le 5$. Furthermore, if one establishes some relations between $a_{ij}a_{[i+j]_6,k}$, $a_{jk}a_{[j+k]_6,i}$, and $a_{ki}a_{[k+i]_6,j}$, (where we abuse of the notation and understand $a_{i,6-i} = b_i$ if $i = 1, 2, 3$, and $a_{ij} = a_{ji}$ if $i > j$), then the identity $(J_{i,j,k})$ holds. In fact, $\frac{J(x,y,z)}{a_{ij}a_{[i+j]_6,k}}$ would be written without any appearance of the scalars, only in terms of x, y, z, $*$, $\sim$ and $\llcorner$; so that it would be always zero due to the existence of scalars making $\mathcal{L}$ a Lie algebra, as in Remark 3.2. For instance, $(J_{i,i,i})$ is always true since the three quantities are obviously equal, $a_{ii}a_{[2i]_6,i}$. In case two of the indices i, j, k are repeated, we will only have one equation relating the scalars, instead of two. All this means that, when we check that (42) implies all the identities on the left of Table 1, this will guarantee the corresponding identities of type $(J_{i,j,k})$ and then the Jacobi identity.

Proving all the identities in Table 1 is a tedious but fairly straightforward computation, starting from (42). First we obtain immediately the following useful equations:

$$\frac{a_{44}}{a_{34}} = \frac{-a_{12}}{a_{22}} = \frac{a_{11}}{a_{13}} = \frac{a_{45}}{a_{25}} = \frac{-a_{14}}{a_{23}}. \tag{58}$$

Next proceed case by case. The identity related to $(J_{1,1,2})$, (45), is one of the identities in the list (42). The one related to $(J_{1,1,3})$ follows from (58). Third, $a_{11}a_{25}b_2 = -b_1b_2 = -a_{14}a_{25}b_1$, so that we get $a_{11}b_2 = -a_{14}b_1$, the identity necessary for $(J_{1,1,4})$. The restriction for $(J_{1,1,5})$ appears in (42). For $(J_{1,2,2})$, we need $a_{12}a_{23} = a_{14}a_{22}$, which follows from (58). The two identities necessary for $(J_{1,2,3})$ are easy to recover: $a_{23}b_1 = -a_{23}a_{13}a_{45} = a_{13}b_2$ and $a_{13}b_2 = a_{13}a_{12}a_{34} = a_{12}b_3$. The identities for $(J_{1,2,4})$, $(J_{1,2,5})$, $(J_{1,3,3})$, $(J_{1,3,5})$,

Table 1 Equivalent conditions to $J(\mathcal{L}_{\bar{i}}, \mathcal{L}_{\bar{j}}, \mathcal{L}_{\bar{k}}) = 0$

Identity	Gives
$a_{11}a_{22} = -a_{12}a_{13}$	$(J_{1,1,2})$
$a_{11}a_{23} = -a_{13}a_{14}$	$(J_{1,1,3})$
$a_{11}b_2 = -a_{14}b_1$	$(J_{1,1,4})$
$a_{11}a_{25} = -b_1$	$(J_{1,1,5})$
$a_{12}a_{23} = a_{14}a_{22}$	$(J_{1,2,2})$
$a_{13}a_{34} = b_3$	$(J_{1,3,3})$
$a_{14}a_{45} = a_{44}a_{12}$	$(J_{1,4,4})$
$a_{14}a_{55} = -b_1$	$(J_{1,5,5})$
$a_{22}a_{34} = a_{23}a_{25}$	$(J_{2,2,3})$
$a_{22}a_{44} = -b_2$	$(J_{2,2,4})$
$a_{22}a_{45} = -a_{25}a_{12}$	$(J_{2,2,5})$
$a_{23}a_{35} = -b_3$	$(J_{2,3,3})$
$a_{22}a_{44} = -b_2$	$(J_{2,4,4})$
$a_{55}b_2 = -a_{25}b_1$	$(J_{2,5,5})$
$a_{13}a_{34} = b_3$	$(J_{3,3,4})$
$a_{35}a_{23} = -b_3$	$(J_{3,3,5})$
$a_{23}a_{44} = -a_{34}a_{14}$	$(J_{3,4,4})$
$a_{35}a_{25} = a_{55}a_{34}$	$(J_{3,5,5})$
$a_{44}a_{25} = a_{45}a_{34}$	$(J_{4,4,5})$
$a_{45}a_{35} = a_{55}a_{44}$	$(J_{4,5,5})$
$a_{23}b_1 = a_{12}b_3 = a_{13}b_2$	$(J_{1,2,3})$
$b_2 = a_{12}a_{34} = a_{14}a_{25}$	$(J_{1,2,4})$
$a_{12}a_{35} = a_{11}a_{25} = -b_1$	$(J_{1,2,5})$
$a_{13}a_{44} = a_{14}a_{35} = a_{11}a_{34}$	$(J_{1,3,4})$
$a_{13}a_{45} = a_{35}a_{12} = -b_1$	$(J_{1,3,5})$
$a_{14}a_{55} = a_{45}a_{13} = -b_1$	$(J_{1,4,5})$
$a_{23}a_{45} = -a_{34}a_{12} = -b_2$	$(J_{2,3,4})$
$a_{23}a_{55} = a_{35}a_{22} = -a_{25}a_{13}$	$(J_{2,3,5})$
$a_{25}a_{14} = -a_{23}a_{45} = b_2$	$(J_{2,4,5})$
$-a_{34}b_1 = a_{35}b_2 = a_{45}b_3$	$(J_{3,4,5})$

$(J_{1,4,5})$ and $(J_{1,5,5})$ just appear in the list. Now, $a_{13}a_{44} = a_{11}a_{34}$ comes from (58), but also $a_{11}a_{34}a_{12} = a_{11}a_{14}a_{25} = a_{14}a_{35}a_{12}$ gives $a_{11}a_{34} = a_{14}a_{35}$; so that $(J_{1,3,4})$ holds. Also $(a_{14}a_{25})(a_{13}a_{45}) = (-a_{44}a_{22})(a_{11}a_{25}) = a_{44}a_{12}a_{13}a_{25}$ gives $(J_{1,4,4})$. The identity for $(J_{2,2,3})$, $a_{34}a_{22} = a_{25}a_{23}$, follows from $a_{34}a_{22}a_{11} = -a_{34}a_{12}a_{13} = -a_{14}a_{25}a_{13} \overset{(58)}{=} a_{25}a_{11}a_{23}$. Next, $a_{22}a_{45}a_{14} \overset{J_{144}}{=} a_{22}a_{44}a_{12} = -a_{12}a_{25}a_{14}$, and, removing a_{14}, we get the identity in $(J_{2,2,5})$. Note $a_{12}b_3 = b_1a_{23} = -a_{12}a_{35}a_{23}$, which gives $b_3 = -a_{23}a_{35}$ and $(J_{2,3,3})$. The identities $(J_{2,2,4})$ and $(J_{2,3,4})$ belong to our original list. As regards the two equations in $(J_{2,3,5})$, simply remove a_{12} from $a_{12}a_{23}a_{55} \overset{(58)}{=} a_{22}a_{14}a_{55} = a_{22}a_{12}a_{35}$, as well as remove a_{14} from $a_{14}a_{55}a_{23} = a_{25}a_{11}a_{23} \overset{(58)}{=} -a_{25}a_{13}a_{14}$. Again $(J_{2,4,4})$, $(J_{2,4,5})$ and $(J_{3,3,4})$ are direct from (42). We easily get $(J_{2,5,5})$ as $a_{55}b_2 = a_{55}a_{25}a_{14} = -a_{25}b_1$. Note $(J_{3,3,5}) = (J_{2,3,3})$. Both $(J_{3,4,4})$ and $(J_{4,4,5})$ are direct from (58). From $a_{55}a_{12}a_{34} =$

$a_{55}a_{25}a_{14} = a_{25}a_{12}a_{35}$, we derive $(J_{3,5,5})$. Multiply the two identities above to have $(a_{35}a_{25})(a_{45}a_{34}) = (a_{55}a_{34})(a_{44}a_{25})$ and simplify $a_{25}a_{34}$ to get $(J_{4,5,5})$. Only $(J_{3,4,5})$ is left, which is achieved from $a_{45}b_3 = a_{45}a_{13}a_{34} = -b_1a_{34}$ and $a_{35}b_2 = -a_{35}a_{45}a_{23} \overset{J_{335}}{=} a_{45}b_3$. This finishes the proof that $\mathcal{L}$ is a Lie algebra. The fact that (43) is a solution is plain.

Finally, we show that all the Lie algebras obtained through this procedure are isomorphic. Let us give an isomorphism between $\mathcal{L}$ and $\mathcal{L}'$ the $\mathbb{Z}_6$-graded vector spaces as in (40) endowed with the Lie bracket as in (41) with the coefficients given by (42) and (43) respectively. An isomorphism can be provided by $f\colon \mathcal{L} \to \mathcal{L}'$, $f|_{\mathcal{L}_{\bar{i}}} = \alpha_i \mathrm{id}_{\mathcal{L}_{\bar{i}}}$, by choosing some scalars $\{\alpha_i : i = 0, \dots, 5\} \subset \mathbb{F}^\times$ such that $\alpha_1^6 = -a_{11}a_{12}a_{13}a_{14}b_1$ and

$$\alpha_0 = 1, \quad \alpha_2 = \frac{\alpha_1^2}{a_{11}}, \quad \alpha_3 = \frac{a_{13}a_{14}b_1}{\alpha_1^3}, \quad \alpha_4 = \frac{a_{14}b_1}{\alpha_1^2}, \quad \alpha_5 = \frac{-b_1}{\alpha_1}.$$

If we denote by $a_{i,6-i} = b_i$, the condition for f to be isomorphism follows from

$$\frac{\alpha_i\alpha_j}{\alpha_{[i+j]_6}} = \begin{cases} a_{ij}, & \text{if } (i,j) = (1,1), (1,3), (2,2), (2,3), (2,5), (3,3), (3,4), (4,4), (4,5), \\ -a_{ij}, & \text{if } (i,j) = (1,2), (1,4), (3,5), (5,5), (1,5), (2,4); \end{cases}$$

which is a routine calculation. Since one of the solutions of (42) makes the corresponding Lie algebra $\mathcal{L}$ isomorphic to $\mathfrak{e}_8$, then all the obtained Lie algebras are simple and all of them are isomorphic to $\mathfrak{e}_8$. □

3.4 Model based on the subalgebra of type $4A_2$

The maximal elementary abelian p-subgroups of E_8 (from the viewpoint of algebraic groups) have been obtained in [25], where a $\mathbb{Z}_3^5$ subgroup of the group of type E_8, automorphisms of $\mathfrak{e}_8$, is described. This 3-group coincides with its centralizer, so producing a fine $\mathbb{Z}_3^5$-grading on the Lie algebra $\mathfrak{e}_8$. If we consider the coarsening produced by any two of the order 3 automorphisms involved, the $\mathbb{Z}_3^2$-grading so obtained is toral, and hence it satisfies the hypothesis in Theorem 1.1.

A concrete description of this $\mathbb{Z}_3^5$-grading on $\mathfrak{e}_8$ can be found on [18, §6.3], which starts with two $\mathbb{Z}_3^2$-graded Okubo algebras and constructs $\mathfrak{e}_8$ from these algebras. Thus $\mathfrak{e}_8$ is naturally endowed with the grading produced when combining the two pairs of related order three automorphisms with the triality automorphism. This nice construction, due to A. Elduque, makes use of symmetric composition algebras. It is clear that we can describe our $\mathbb{Z}_3^2$-grading as a coarsening of the above $\mathbb{Z}_3^5$-grading, simply by taking the $\mathbb{Z}_3^2$-grading induced on $\mathfrak{e}_8$ when taking only the $\mathbb{Z}_3^2$-grading on one of the Okubo algebras involved. But we would like to describe a model similar to all the others in this work. The first step is to describe the homogeneous components of the $\mathbb{Z}_3^2$-grading. Recall from Sect. 2.1 that there are two types of order three automorphisms, which fix an algebra of type $\mathfrak{a}_8$ and an algebra of type $\mathfrak{e}_6 \oplus \mathfrak{a}_2$, respectively. So the dimensions of the neutral homogeneous component are respectively, 80 and 86. Both the order three automorphisms involved now are of the second type. The $\mathbb{Z}_3$-grading is given by $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}} \oplus \mathcal{L}_{\bar{2}}$, for

$$\mathcal{L}_{\bar{0}} = \mathfrak{e}_6 \oplus \mathfrak{sl}(V), \qquad \mathcal{L}_{\bar{1}} = V(\varpi_1) \otimes V, \qquad \mathcal{L}_{\bar{2}} = V(\varpi_1)^* \otimes V^*,$$

if V is a 3-dimensional vector space and $V(\varpi_1)$ denotes the $\mathfrak{e}_6$-irreducible module of highest weight ϖ_1, the first fundamental weight. Now, recall the nice $\mathbb{Z}_3$-grading on $\mathcal{M} = \mathfrak{e}_6$, described in a very symmetric way as

$$\mathfrak{e}_6 = \mathfrak{sl}(V) \oplus \mathfrak{sl}(V) \oplus \mathfrak{sl}(V) \,\oplus\, (V \otimes V \otimes V) \,\oplus\, (V^* \otimes V^* \otimes V^*).$$

According to [16], the module $\mathcal{V} = V(\varpi_1)$ can be $\mathbb{Z}_3$-graded in a way compatible with the $\mathbb{Z}_3$-grading on $\mathfrak{e}_6$, that is, $\mathcal{M}_{\bar{i}} \cdot \mathcal{V}_{\bar{j}} \subset \mathcal{V}_{\bar{i}+\bar{j}}$, for

$$\mathcal{V}_{\bar{0}} = \mathbb{F} \otimes V \otimes V^*, \qquad \mathcal{V}_{\bar{1}} = V \otimes V^* \otimes \mathbb{F}, \qquad \mathcal{V}_{\bar{2}} = V^* \otimes \mathbb{F} \otimes V.$$

All this together gives the homogeneous components of the $\mathbb{Z}_3^2$-grading on $\mathfrak{e}_8$ as modules for the neutral component $\mathcal{L}_{(\bar{0},\bar{0})} = \mathfrak{sl}(V) \oplus \mathfrak{sl}(V) \oplus \mathfrak{sl}(V) \oplus \mathfrak{sl}(V)$:

$$\begin{array}{ll}
\mathcal{L}_{(\bar{0},\bar{1})} = \mathbb{F} \otimes V \otimes V^* \otimes V\,, & \mathcal{L}_{(\bar{0},\bar{2})} = \mathbb{F} \otimes V^* \otimes V \otimes V^*\,, \\
\mathcal{L}_{(\bar{1},\bar{0})} = V \otimes V \otimes V \otimes \mathbb{F}\,, & \mathcal{L}_{(\bar{2},\bar{0})} = V^* \otimes V^* \otimes V^* \otimes \mathbb{F}\,, \\
\mathcal{L}_{(\bar{1},\bar{1})} = V \otimes V^* \otimes \mathbb{F} \otimes V\,, & \mathcal{L}_{(\bar{2},\bar{1})} = V^* \otimes \mathbb{F} \otimes V \otimes V\,, \\
\mathcal{L}_{(\bar{1},\bar{2})} = V \otimes \mathbb{F} \otimes V^* \otimes V^*\,, & \mathcal{L}_{(\bar{2},\bar{2})} = V^* \otimes V \otimes \mathbb{F} \otimes V^*\,.
\end{array}$$

Thus we have the $\mathbb{Z}_3^2$-graded vector space to be endowed with a Lie algebra structure.

Proposition 3.9 *Let $\mathcal{L}$ be the graded vector space given, for any $i, j = 0, 1, 2$, by*

$$\begin{aligned}
\mathcal{L}_{(\bar{0},\bar{0})} &= \mathfrak{sl}(V) \oplus \mathfrak{sl}(V) \oplus \mathfrak{sl}(V) \oplus \mathfrak{sl}(V), \\
\mathcal{L}_{(\bar{i},\bar{j})} &= \bigwedge^{[i]_3} V \otimes \bigwedge^{[i+j]_3} V \otimes \bigwedge^{[i+2j]_3} V \otimes \bigwedge^{[j]_3} V,
\end{aligned} \tag{59}$$

where V denotes a vector space over $\mathbb{F}$ of dimension 3.

Note that, for any $(\bar{0}, \bar{0}) \neq \alpha = (\bar{i}, \bar{j}) \in \mathbb{Z}_3^2$, there is just one index $k_\alpha \in \{1, 2, 3, 4\}$ such that $\alpha_{k_\alpha} = 0$, where $\alpha_1 = [i]_3$, $\alpha_2 = [i + j]_3$, $\alpha_3 = [i + 2j]_3$, $\alpha_4 = [j]_3$, that is, $\mathcal{L}_\alpha = \bigwedge^{\alpha_1} V \otimes \bigwedge^{\alpha_2} V \otimes \bigwedge^{\alpha_3} V \otimes \bigwedge^{\alpha_4} V$.

Consider the product of $\mathcal{L}_{(\bar{0},\bar{0})}$ with $\mathcal{L}_{(\bar{i},\bar{j})}$ given by the action of the kth ideal of $\mathcal{L}_{\bar{0}}$ on the kth slot ($k = 1, 2, 3, 4$). Consider the product of $x = x_1 \otimes x_2 \otimes x_3 \otimes x_4 \in \mathcal{L}_\alpha$ and $y = y_1 \otimes y_2 \otimes y_3 \otimes y_4 \in \mathcal{L}_\beta$, for $(\bar{0}, \bar{0}) \neq \alpha, \beta \in \mathbb{Z}_3^2$, given by

$$[x, y] = \begin{cases} a_{\alpha,\beta}\, x_1 * y_1 \otimes x_2 * y_2 \otimes x_3 * y_3 \otimes x_4 * y_4 & \text{if } \beta \neq 2\alpha, \\ \sum_{i=1, i\neq k_\alpha}^{4} b_\alpha^{(i)} [x_i, y_i]_i \Pi_{j=1, j\neq i}^{4} (x_j, y_j) & \text{if } \beta = 2\alpha, \end{cases} \tag{60}$$

for some nonzero scalars $a_{\alpha,\beta}$ and $b_\alpha^{(i)}$ (for any $i \neq k_\alpha$).
Then, $\mathcal{L}$ endowed with the product $[\ ,\ ]$ is a Lie algebra if and only if

$$\begin{aligned}
a_{\alpha,\beta} &= a_{\beta,\alpha}, \\
-b_\alpha^{(i)} &= a_{\alpha,\alpha} a_{2\alpha,2\alpha}, \\
-a_{\alpha,\alpha} a_{2\alpha,\beta} &= a_{\alpha,\beta} a_{\alpha,\alpha+\beta}, \\
-a_{\alpha,\beta} a_{\alpha+\beta,\alpha+\beta} &= a_{\beta,\alpha+\beta} a_{\alpha+2\beta,\alpha}, \\
-b_\alpha^{(i)} &= a_{\alpha,\beta} a_{\alpha+\beta,2\alpha},
\end{aligned} \tag{61}$$

for all $\beta \neq \alpha, 2\alpha$ and for all $i \neq k_\alpha$.

Moreover, $(\mathcal{L}, [\ ,\ ])$ is a simple exceptional Lie algebra isomorphic to $\mathfrak{e}_8$. A solution of this system of equations that, in particular, gives a model of $\mathfrak{e}_8$, is

$$\begin{aligned}
1 &= a_{\alpha,\beta}, & \forall \alpha \neq \beta, \\
-1 &= b_\alpha^{(i)} = a_{\alpha,\alpha}, & \forall i \neq k_\alpha.
\end{aligned} \tag{62}$$

Here we consider more scalars and more equations than necessary, unlike the above models based on cyclic groups. This permits us to take advantage of a greater symmetry, and so to handle a big number of equations simultaneously.

Proof Recall that, for all $u, v \in V$ and all $\mu, \eta \in \bigwedge^2 V$,

$$\begin{array}{rl} u * v = -v * u, & \mu * \eta = -\eta * \mu, \\ (u, \eta) = u * \eta = \eta * u = (\eta, u), & [u, \eta] = -[\eta, u]. \end{array} \tag{63}$$

Take also into account that $1 * u = u = u * 1$ and that $1 * \eta = \eta = \eta * 1$. Then, for any $x = \otimes_{i=1}^4 x_i \in \mathcal{L}_\alpha$ and $y = \otimes_{i=1}^4 y_i \in \mathcal{L}_\beta$, we have:

- if $\beta = \alpha$, then $x_i * y_i = -y_i * x_i$ for just three indices $i \in \{1, 2, 3, 4\}$ (precisely, if and only if $i \neq k_\alpha$), so that $[x, y] = -[y, x]$;
- if $\beta \neq \alpha, 2\alpha$, then $x_i * y_i = -y_i * x_i$ for just one index $i \in \{1, 2, 3, 4\}$, so that $\otimes_{i=1}^4 x_i * y_i = -\otimes_{i=1}^4 y_i * x_i$. This implies that $[x, y] = -[y, x]$ if and only if $a_{\alpha,\beta} = a_{\beta,\alpha}$;
- if $\beta = 2\alpha$, then $\alpha = 2\beta$. For any $i \neq k_\alpha$ $(= k_\beta)$, one of the elements x_i, y_i belongs to V and the other one to $\bigwedge^2 V$, so that $[x_i, y_i]\Pi_{j=1,j\neq i}^4 (x_j, y_j) = -[y_i, x_i]\Pi_{j=1,j\neq i}^4 (y_j, x_j)$. This gives that $[x, y] = -[y, x]$ if and only if $b_\alpha^{(i)} = b_{2\alpha}^{(i)}$ for all $i \neq k_\alpha$.

Consequently, the bracket defined in $\mathcal{L}$ is skew-symmetric if and only if $a_{\alpha,\beta} = a_{\beta,\alpha}$ (for any $\beta \neq 2\alpha$) and $b_\alpha^{(i)} = b_{2\alpha}^{(i)}$ for all $i \neq k_\alpha$. Assume these conditions for the rest of the proof. Thus we have to determine 32 scalars type $a_{\alpha,\beta}$ (8 with $\alpha = \beta$ and 24 with $\beta \notin \langle\alpha\rangle$) and 12 type $b_\alpha^{(i)}$.

Now, the condition for $\mathcal{L}$ to be a Lie algebra reduces to check that the Jacobi identities $J(\mathcal{L}_\alpha, \mathcal{L}_\beta, \mathcal{L}_\gamma) = 0$ hold for any choice of $\alpha, \beta, \gamma \in \mathbb{Z}_3^2$. We denote such identities as $(J_{\alpha,\beta,\gamma})$. Again they are always satisfied whenever $(\bar{0}, \bar{0}) \in \{\alpha, \beta, \gamma\}$, and also when $\alpha = \beta = \gamma$. Of course, the order is not important in the choice of the triple $\{\alpha, \beta, \gamma\}$. Note that we have just five different kind of identities to study:

(i) $(J_{\alpha,\alpha,2\alpha})$;
(ii) $(J_{\alpha,\alpha,\beta})$;
(iii) $(J_{\alpha,\beta,\alpha+\beta})$;
(iv) $(J_{\alpha,2\alpha,\beta})$;
(v) $(J_{\alpha,\beta,2(\alpha+\beta)})$;

for any $(\bar{0}, \bar{0}) \neq \alpha, \beta \in \mathbb{Z}_3^2$ with $\beta \neq \alpha, 2\alpha$. Indeed, if there are two repeated indices in $\{\alpha, \beta, \gamma\}$, of course the situations (i) and (ii) appear, otherwise we can assume the three α, β and γ different. If the sum of two of them is $(\bar{0}, \bar{0})$; then we have (iv) with $\beta \neq \alpha, 2\alpha$. If this is not the case but one element is the sum of the other two, then we have (iii), again with $\beta \neq \alpha, 2\alpha$. If the situation is not any of these ones, $\gamma \in \{2\alpha + 2\beta, 2\alpha + \beta, \alpha + 2\beta\}$, but in the two last cases the elements could also be labelled to be in the situation (iii); so that $\gamma = 2\alpha + 2\beta$. To summarize, we have 8 identities of type (i), 48 of type (ii), 24 of type (iii) (α and β interchangeable), 24 of type (iv) and 8 of type (v) (each of the three elements is twice the sum of the other ones), in total, 112 identities. Some of the resultant equations are redundant, but this does not make it more difficult to find a solution. First we focus on writing the equations related to the identities $J(\mathcal{L}_\alpha, \mathcal{L}_\beta, \mathcal{L}_\gamma) = 0$ according to this distribution of cases (i) $- \cdots -$(v).

$\boxed{(J_{\alpha,\alpha,2\alpha})}$ Consider first the case $\alpha = (\bar{1}, \bar{0})$, so that $\mathcal{L}_\alpha = V \otimes V \otimes V \otimes \mathbb{F}$. Let $x = x_1 \otimes x_2 \otimes x_3 \otimes 1$, $z = z_1 \otimes z_2 \otimes z_3 \otimes 1 \in \mathcal{L}_\alpha$ and take any element $y = \otimes_{i=1}^3 y_i \otimes 1 \in \mathcal{L}_{2\alpha}$ such that $(x_1, y_1) = 0$. Then

$$[[x, y], z] = b_\alpha^{(1)}[x_1, y_1]_1(z_1) \otimes (x_2, y_2)z_2 \otimes (x_3, y_3)z_3 \otimes 1.$$

(Recall that, for $f \in \mathfrak{sl}(V)$, f_1 denotes $(f, 0, 0, 0) \in 4\mathfrak{sl}(V)$.) If besides the elements have been chosen such that $(y_2, z_2) = 0 = (y_3, z_3)$, then $[[y, z], x] = 0$. As usual, let us fix a basis $\{e_1, e_2, e_3\}$ such that $\phi(e_{123}) = 1$ in order to make concrete computations with the help of Lemma 2.3. For instance, if we take

$$x_1 = z_2 = z_3 = e_1, \quad x_2 = x_3 = z_1 = e_2, \quad y_1 = y_2 = y_3 = e_{31},$$

thus $[[x, y], z] = b^{(1)}_\alpha [e_1, e_{31}]_1(e_2) \otimes (e_2, e_{31})e_1 \otimes (e_2, e_{31})e_1 \otimes 1 = b^{(1)}_\alpha e_1 \otimes e_1 \otimes e_1 \otimes 1$ and

$$\begin{aligned}[[z, x], y] &= a_{\alpha,\alpha} a_{2\alpha,2\alpha} (z_1 * x_1) * y_1 \otimes (z_2 * x_2) * y_2 \otimes (z_3 * x_3) * y_3 \otimes 1 \\ &= a_{\alpha,\alpha} a_{2\alpha,2\alpha} e_1 \otimes e_1 \otimes e_1 \otimes 1.\end{aligned}$$

That is, for our choice $x = e_1 \otimes e_2 \otimes e_2 \otimes 1$, $z = e_2 \otimes e_1 \otimes e_1 \otimes 1$, $y = e_{31} \otimes e_{31} \otimes e_{31} \otimes 1$, we have

$$J(x, y, z) = \big(b^{(1)}_\alpha + a_{\alpha,\alpha} a_{2\alpha,2\alpha}\big) e_1 \otimes e_1 \otimes e_1 \otimes 1.$$

If $\mathcal{L}$ is a Lie algebra, this implies that $b^{(1)}_\alpha = -a_{\alpha,\alpha} a_{2\alpha,2\alpha}$. The same computation proves $b^{(i)}_\alpha = -a_{\alpha,\alpha} a_{2\alpha,2\alpha}$ for all $i = 1, 2, 3$, when we *permute the indices*, i.e., for $x_i = z_j = e_1$ and $x_j = z_i = e_2$ for any $j \neq i$, $j \in \{1, 2, 3\}$ and the same y; in particular $b^{(i)}_\alpha$ does not depend on the superindex i.

Furthermore, these conditions $b^{(i)}_\alpha = -a_{\alpha,\alpha} a_{2\alpha,2\alpha}$ for any $i \neq 4$ are not only necessary but sufficient to guarantee the identity $(J_{\alpha,\alpha,2\alpha})$, due to the preliminary knowledge of the $\mathbb{Z}_3^2$-grading on $\mathfrak{e}_8$, with similar arguments to those ones used in the above models (see Remark 3.2).

We have checked the identity only for $\alpha = (\bar{1}, \bar{0})$. For the remaining values of $\alpha \in \mathbb{Z}_3^2$ ($\alpha \neq (\bar{0}, \bar{0})$), some concrete choices of elements will be particularly useful.

(a) There are $x, z \in V$, $y \in \bigwedge^2 V$ such that

$$[x, y](z) = (z * x) * y = x, \qquad (x, y) = 0.$$

In fact, $x = e_1$, $z = e_2$, $y = e_{31}$ provides a solution. (Always in mind Lemma 2.3.)

(b) There are $x, z \in V$, $y \in \bigwedge^2 V$ such that

$$(x, y)z = -(z * x) * y = z, \qquad (z, y) = 0.$$

In fact, $x = e_2$, $z = e_1$, $y = e_{31}$ provides a solution.

(c) There are $x, z \in \bigwedge^2 V$, $y \in V$ such that

$$[x, y](z) = (z * x) * y = x, \qquad (x, y) = 0.$$

In fact, $x = e_{12}$, $z = e_{23}$, $y = e_1$ gives a solution.

(d) There are $x, z \in \bigwedge^2 V$, $y \in V$ such that

$$(x, y)z = -(z * x) * y = z, \qquad (z, y) = 0.$$

In fact, $x = e_{23}$, $z = e_{12}$, $y = e_1$ is a solution.

With these elements, it is not difficult to prove that necessarily $b^{(i)}_\alpha + a_{\alpha,\alpha} a_{2\alpha,2\alpha} = 0$ for any $i \neq k_\alpha$. In fact, let j and k be the two indices in $\{1, 2, 3, 4\}$ different from i and k_α. Take $x_{k_\alpha} = y_{k_\alpha} = z_{k_\alpha} = 1$. Take $\{x_i, y_i, z_i\}$ as in item (a) if $\alpha_i = 1$ and as in item (c) if $\alpha_i = 2$. Take $\{x_j, y_j, z_j\}$ as in item (b) if $\alpha_j = 1$ and as in item (d) if $\alpha_j = 2$. Similarly, take $\{x_k, y_k, z_k\}$ as in item (b) if $\alpha_k = 1$ and as in item (d) if $\alpha_k = 2$. Now $[[y, z], x] = 0$, and $\otimes_{t=1}^4 (z_t * x_t) * y_t$ coincides with $1 \otimes x_i \otimes (-z_j) \otimes (-z_k)$ but suitably ordered, which in particular is nonzero and coincides with $1\otimes[x_i,y_i]_i(z_i)\otimes(x_j,y_j)z_j\otimes(x_k,y_k)z_k$, also suitably ordered. Hence

$$J(x,y,z)=\big(b^{(i)}_\alpha+a_{\alpha,\alpha}a_{2\alpha,2\alpha}\big)\big(\otimes_{t=1}^{4}(z_t*x_t)*y_t\big),$$

getting the required equations. Of course, the equations $-a_{\alpha,\alpha}a_{2\alpha,2\alpha}=b^{(i)}_\alpha=b^{(j)}_\alpha=b^{(k)}_\alpha$ are sufficient to get that $J(x,y,z)=0$ for all $x,z\in\mathcal{L}_\alpha$, $y\in\mathcal{L}_{2\alpha}$. Moreover, these equations evidently imply $b^{(i)}_\alpha=b^{(i)}_{2\alpha}$.

$\boxed{(J_{\alpha,\alpha,\beta})}$ with $\beta\neq\alpha,2\alpha$. Take for instance $\alpha=(\bar{1},\bar{0})$ and $\beta=(\bar{0},\bar{1})$. So we can assume that the elements $x,y\in\mathcal{L}_\alpha$ and $z\in\mathcal{L}_\beta$ are $x=u_1\otimes u_2\otimes u_3\otimes 1$, $y=v_1\otimes v_2\otimes v_3\otimes 1$ and $z=1\otimes w_2\otimes\eta_3\otimes w_4$, for some $u_i,v_i,w_i\in V$ and $\eta_i\in\bigwedge^2 V$. Trivially we have

$$\begin{aligned}
[[x,y],z]&=a_{\alpha,\alpha}a_{2\alpha,\beta}\,u_1*v_1\otimes(u_2*v_2)*w_2\otimes(u_3*v_3)*\eta_3\otimes w_4,\\
[[y,z],x]&=a_{\alpha,\beta}a_{\alpha,\alpha+\beta}\,v_1*u_1\otimes(v_2*w_2)*u_2\otimes(v_3*\eta_3)*u_3\otimes w_4,\\
[[z,x],y]&=a_{\alpha,\beta}a_{\alpha,\alpha+\beta}\,u_1*v_1\otimes(w_2*u_2)*v_2\otimes(\eta_3*u_3)*v_3\otimes w_4.
\end{aligned}$$

First take into account that $(u*v)*w=\phi(u\wedge v\wedge w)$, so that

$$(u*v)*w=(v*w)*u=(w*u)*v,$$

and

$$\begin{aligned}
J(x,y,z)=u_1*v_1\otimes(u_2*v_2)*w_2\otimes\big(&a_{\alpha,\alpha}a_{2\alpha,\beta}(u_3*v_3)*\eta_3+\\
&+a_{\alpha,\beta}a_{\alpha,\alpha+\beta}(-(v_3*\eta_3)*u_3+(\eta_3*u_3)*v_3)\big)\otimes w_4.
\end{aligned}$$

This is identically zero if and only if

$$a_{\alpha,\alpha}a_{2\alpha,\beta}(u_3*v_3)*\eta_3+a_{\alpha,\beta}a_{\alpha,\alpha+\beta}\big(-(v_3*\eta_3)*u_3+(\eta_3*u_3)*v_3\big)=0. \tag{64}$$

Second, observe $(u\wedge v)\llcorner\tilde{\eta}=\tilde{\eta}(u)v-\tilde{\eta}(v)u$ for any $u,v\in V$ and $\eta\in\bigwedge^2 V$, according to Lemma 2.3 or to [23, Exercise B.15]. So,

$$(u*v)*\eta=-(v*\eta)*u+(\eta*u)*v, \tag{65}$$

and Eq. (64) becomes $\big(a_{\alpha,\alpha}a_{2\alpha,\beta}+a_{\alpha,\beta}a_{\alpha,\alpha+\beta}\big)(u_3*v_3)*\eta_3=0$. This always holds if and only if

$$a_{\alpha,\alpha}a_{2\alpha,\beta}+a_{\alpha,\beta}a_{\alpha,\alpha+\beta}=0.$$

In general, for any choice of $\alpha,\beta\in\mathbb{Z}_3^2$, $\beta\neq\pm\alpha$, there is just one index $i\in\{1,2,3,4\}$ such that $\alpha_i=\beta_i\in\{1,2\}$, and there is just one index $j\neq k_\alpha,k_\beta$ such that $\alpha_j\neq\beta_j$ (both different from 0, so $\{\alpha_j,\beta_j\}=\{1,2\}$). If $\alpha_i=1$, the situation is completely analogous to the above, replacing j with the index 3 and i with the index 2. In order to deal with the case $\alpha_i=2$, we have to take into consideration

$$(\mu*\nu)*\eta=(\nu*\eta)*\mu=(\eta*\mu)*\nu, \tag{66}$$

$$(\mu*\eta)*w=-(\eta*w)*\mu+(w*\mu)*\eta, \tag{67}$$

for any $w\in V$ and $\mu,\nu,\eta\in\bigwedge^2 V$. Equation (66) follows from $(e_{ij}*e_{jk})*e_{ki}=1$, and from $(e_{ji}*e_{jk})*e_{jl}=0$, for any $\{i,j,k\}$ a permutation of $\{1,2,3\}$ and $l\neq j$, again by Lemma 2.3. Equation (67) is trivial if $\mu=\eta$, since both sides vanish. Otherwise we can assume $\mu=e_{ij}$ and $\eta=e_{ik}$ for $\sigma=\{i,j,k\}$ a permutation of $\{1,2,3\}$. Then $\mu*\eta=\operatorname{sgn}(\sigma)e_i$. If $w=e_i$, both sides vanish; if $w=e_j$, both sides equal $\operatorname{sgn}(\sigma)e_{ij}$; and if $w=e_k$, both sides are equal to $\operatorname{sgn}(\sigma)e_{ik}$.

$\boxed{(J_{\alpha,\beta,\alpha+\beta})}$ Take for instance $\alpha = (\bar{1}, \bar{0})$ and $\beta = (\bar{0}, \bar{1})$. So we can assume that the elements $x \in \mathcal{L}_\alpha$, $y \in \mathcal{L}_\beta$, $z \in \mathcal{L}_{(\bar{1},\bar{1})}$ are $x = u_1 \otimes u_2 \otimes u_3 \otimes 1$, $y = 1 \otimes v_2 \otimes \eta_3 \otimes v_4$, and $z = w_1 \otimes \eta_2 \otimes 1 \otimes w_4$, for some $u_i, v_i, w_i \in V$ and $\eta_i \in \bigwedge^2 V$. Trivially we have

$$\begin{aligned}
[[x, y], z] &= a_{\alpha,\beta} a_{\alpha+\beta,\alpha+\beta}\, u_1 * w_1 \otimes (u_2 * v_2) * \eta_2 \otimes u_3 * \eta_3 \otimes v_4 * w_4, \\
[[y, z], x] &= a_{\beta,\alpha+\beta} a_{\alpha+2\beta,\alpha}\, w_1 * u_1 \otimes (v_2 * \eta_2) * u_2 \otimes \eta_3 * u_3 \otimes v_4 * w_4, \\
[[z, x], y] &= a_{\alpha,\alpha+\beta} a_{\beta+2\alpha,\beta}\, w_1 * u_1 \otimes (\eta_2 * u_2) * v_2 \otimes u_3 * \eta_3 \otimes w_4 * v_4,
\end{aligned}$$

which gives, by (63),

$$\begin{aligned}
J(x, y, z) = u_1 * w_1 \otimes \Big(& a_{\alpha,\beta} a_{\alpha+\beta,\alpha+\beta} (u_2 * v_2) * \eta_2 - a_{\beta,\alpha+\beta} a_{\alpha+2\beta,\alpha} (v_2 * \eta_2) * u_2 \\
& + a_{\alpha,\alpha+\beta} a_{\beta+2\alpha,\beta} (\eta_2 * u_2) * v_2 \Big) \otimes u_3 * \eta_3 \otimes v_4 * w_4.
\end{aligned}$$

This is identically zero if and only if,

$$\begin{aligned}
&-\big(a_{\alpha,\beta} a_{\alpha+\beta,\alpha+\beta} + a_{\beta,\alpha+\beta} a_{\alpha+2\beta,\alpha}\big)(v_2 * \eta_2) * u_2 \\
&+\big(a_{\alpha,\beta} a_{\alpha+\beta,\alpha+\beta} + a_{\alpha,\alpha+\beta} a_{\beta+2\alpha,\beta}\big)(\eta_2 * u_2) * v_2 = 0,
\end{aligned}$$

by (65). And of course this holds just when

$$-a_{\alpha,\beta} a_{\alpha+\beta,\alpha+\beta} = a_{\beta,\alpha+\beta} a_{\alpha+2\beta,\alpha} = a_{\alpha,\alpha+\beta} a_{\beta+2\alpha,\beta}.$$

The second identity does not give any extra information, if we swap the roles of α and β.

What happens for another choice of α and β? As mentioned above, there is just one index $i \in \{1, 2, 3, 4\}$ such that $\alpha_i = \beta_i \in \{1, 2\}$, and there is just one index $j \neq k_\alpha, k_\beta$ such that $\alpha_j \neq \beta_j$ (both different from 0). Now k_α and k_β play the role of the indices 1 and 4 (it does not mind if its value is either 1 or 2, since in both cases $*$ is skew-symmetric (see Eq. (63)). The index j plays the role of the index 3 in the above example, so that we use again $u * \eta = \eta * u$. And the index i plays the role of the index 2 in the example. If $\alpha_i = 1$, the above computations work without any change; while if $\alpha_i = 2$, the identity we need is not more (65) but (67).

$\boxed{(J_{\alpha,2\alpha,\beta})}$ Again we begin with $\alpha = (\bar{1}, \bar{0})$ and $\beta = (\bar{0}, \bar{1})$, to have an example to figure out how to deal with the remaining 23 cases. Take $x = u_1 \otimes u_2 \otimes u_3 \otimes 1 \in \mathcal{L}_\alpha$, $y = 1 \otimes v_2 \otimes \eta_3 \otimes v_4 \in \mathcal{L}_\beta$, and $z = \mu_1 \otimes \mu_2 \otimes \mu_3 \otimes 1 \in \mathcal{L}_{2\alpha}$, for some $u_i, v_i \in V$ and $\mu_i, \eta_i \in \bigwedge^2 V$. Recalling, from the above cases, that the scalars $b_\alpha^{(i)}$ do not depend on i, we write

$$\begin{aligned}
[[x, y], z] &= a_{\alpha,\beta} a_{\alpha+\beta,2\alpha}\, u_1 * \mu_1 \otimes (u_2 * v_2) * \mu_2 \otimes (u_3 * \eta_3) * \mu_3 \otimes v_4, \\
[[y, z], x] &= a_{\beta,2\alpha} a_{2\alpha+\beta,\alpha}\, \mu_1 * u_1 \otimes (v_2 * \mu_2) * u_2 \otimes (\eta_3 * \mu_3) * u_3 \otimes v_4, \\
[[z, x], y] &= b_\alpha^{(1)} \big((\mu_1, u_1) 1 \otimes [\mu_2, u_2]_2(v_2) \otimes (\mu_3, u_3)\eta_3 \otimes v_4 + \\
&\qquad + (\mu_1, u_1) 1 \otimes (\mu_2, u_2) v_2 \otimes [\mu_3, u_3]_3 \cdot \eta_3 \otimes v_4\big).
\end{aligned}$$

Since $u_1 * \mu_1 = (u_1, \mu_1) = \mu_1 * u_1$, we focus on the elements in positions 2 and 3 of the tensor product. If we choose (as in item (b)),

$$(u_2, v_2, \mu_2) = (e_2, e_1, e_{31}) \Rightarrow (u_2 * v_2) * \mu_2 = e_1,\ (v_2 * \mu_2) * u_2 = 0,\ (\mu_2, u_2) v_2 = e_1,$$

and (as in item (d)),

$$(\eta_3, \mu_3, u_3) = (e_{23}, e_{12}, e_1) \Rightarrow (u_3 * \eta_3) * \mu_3 = e_{12},\ (\mu_3, u_3) = 0,\ [\mu_3, u_3]_3 \cdot \eta_3 = e_{12},$$

then

$$J(x, y, z) = \big(a_{\alpha,\beta} a_{\alpha+\beta,2\alpha} + b_\alpha^{(1)}\big)\, u_1 * \mu_1 \otimes e_1 \otimes e_{12} \otimes v_4,$$

and the Jacobi identity implies

$$a_{\alpha,\beta}a_{\alpha+\beta,2\alpha}+b_{\alpha}^{(1)}=0. \tag{68}$$

The identity $a_{\beta,2\alpha}a_{2\alpha+\beta,\alpha}+b_{\alpha}^{(1)}=0$ is achieved as a direct consequence of (68) by replacing α with 2α (recall that $a_{\alpha,\beta}=a_{\beta,\alpha}$ and $b_{\alpha}^{(1)}=b_{2\alpha}^{(1)}$). Of course, the two equations are sufficient to get $(J_{\alpha,2\alpha,\beta})$. For other values of α and β, we proceed as in the above cases, exploiting the symmetry.

$\boxed{(J_{\alpha,\beta,2(\alpha+\beta)})}$ Until now we have proved that all the conditions in Eq. (61) are necessary. Let us check that they are sufficient since this new identity holds too. For any $x\in\mathcal{L}_{\alpha}$, $y\in\mathcal{L}_{\beta}$, and $z\in\mathcal{L}_{2(\alpha+\beta)}$, the three expressions

$$\frac{[[x,y],z]}{a_{\alpha,\beta}b_{\alpha+\beta}^{(1)}},\quad \frac{[[y,z],x]}{a_{\beta,2\alpha+2\beta}b_{2\alpha}^{(1)}},\quad \frac{[[z,x],y]}{a_{\alpha,2\alpha+2\beta}b_{2\beta}^{(1)}},$$

do not depend on the scalars. Observe that

$$a_{\alpha,2(\alpha+\beta)}b_{\beta}^{(i)}\overset{(68)}{=}-a_{\alpha,2(\alpha+\beta)}a_{\beta,\alpha}a_{\beta+\alpha,2\beta}=-a_{\alpha,\beta}a_{\beta+\alpha,2\beta}a_{\alpha,2(\alpha+\beta)}\overset{(68)}{=}a_{\alpha,\beta}b_{\alpha+\beta}^{(i)}.$$

Changing the role of α and β, and recalling $b_{\beta}^{(i)}=b_{2\beta}^{(i)}$, we also have

$$a_{\alpha,\beta}b_{\alpha+\beta}^{(1)}=a_{\beta,2\alpha+2\beta}b_{2\alpha}^{(1)}=a_{\alpha,2\alpha+2\beta}b_{2\beta}^{(1)},$$

and so $\frac{J(x,y,z)}{a_{\alpha,\beta}b_{\alpha+\beta}^{(1)}}$ has to annihilate and hence $\mathcal{L}$ is a Lie algebra.

It is besides evident that the set of scalars in Eq. (62) provides a solution.

The fact that the obtained Lie algebra is isomorphic to $\mathfrak{e}_8$ is a direct consequence of its simplicity. And this simplicity comes from the fact that all the chosen scalars are nonzero. We can provide a proof by adapting the arguments in Remark 3.3. Suppose there were an ideal I of $\mathcal{L}$ without any homogeneous element. Denote by $\pi_{\alpha}\colon\mathcal{L}\to\mathcal{L}_{\alpha}$ the projection given by the decomposition (59). First, for any $x\in I$, we find $x'\in I$ with $\pi_{(\bar{0},\bar{0})}(x')=0$ and such that $\pi_{\alpha}(x')\neq 0$ if and only if $\pi_{\alpha}(x)\neq 0$ for any α different from the neutral element. And second, for any $0\neq x\in I$ with $\pi_{(\bar{0},\bar{0})}(x)=0$, we find $x'\in I$ with $\pi_{(\bar{0},\bar{0})}(x')\neq 0$ such that the number of α's (including $(\bar{0},\bar{0})$) with $\pi_{\alpha}(x')\neq 0$ is at most the number of α's with $\pi_{\alpha}(x)\neq 0$. The first step is achieved by multiplying alternatively with two convenient copies of $\mathfrak{sl}(V)$, while for the second step we only have to multiply with one homogeneous component. This leads to contradiction. Next, assume we have an ideal I with $0\neq\mathcal{L}_{\alpha}\cap I$ for some $\alpha\neq(\bar{0},\bar{0})$. As $\mathcal{L}_{\alpha}$ is an irreducible $\mathcal{L}_{(\bar{0},\bar{0})}$-module, then $\mathcal{L}_{\alpha}\subset I$ and we deduce $I=\mathcal{L}$. It is easy to get the same conclusion if $0\neq\mathcal{L}_{(\bar{0},\bar{0})}\cap I$. □

3.5 Model based on the subalgebra of type $2A_1\oplus 2A_3$

Our last model will be based on the simultaneous diagonalization relative to two commuting automorphisms F and G, with F of order two (relative to the 8th node) and G of order four (relative to the 6th node), as in Sect. 2.1. The $\mathbb{Z}_2$-grading induced by F is (up to isomorphism)

$$\mathcal{L}_{\bar{0}}=\mathfrak{sl}(U)\oplus\mathfrak{e}_7,\qquad \mathcal{L}_{\bar{1}}=U\otimes V(\varpi_7);$$

where U is a 2-dimensional vector space and $V(\varpi_7)$ is the $\mathfrak{e}_7$-irreducible module of dimension 56. Now apply Sect. 2.1 to $\mathcal{M} = \mathfrak{e}_7$ to describe its $\mathbb{Z}_4$-grading as

$$\begin{aligned} \mathcal{M}_{\bar 0} &= \mathfrak{sl}(U) \oplus \mathfrak{sl}(W) \oplus \mathfrak{sl}(W), & \mathcal{M}_{\bar 2} &= \mathbb{F} \otimes \textstyle\bigwedge^2 W \otimes \bigwedge^2 W,\\ \mathcal{M}_{\bar 1} &= U \otimes W \otimes W, & \mathcal{M}_{\bar 3} &= U \otimes \textstyle\bigwedge^3 W \otimes \bigwedge^3 W, \end{aligned}$$

for W a 4-dimensional vector space. (Here only the dimensions are not enough to distinguish at a first glance whether the module $\mathcal{M}_{\bar 1}$ -which determines the others- is either the above or $U \otimes W \otimes \bigwedge^3 W$. Note that this is not a problem: W and its dual module $\bigwedge^3 W$ are not isomorphic but there is an -outer- automorphism of $\mathfrak{sl}(W)$ interchanging them. This means that we can recover the Lie algebra $\mathcal{M}$ in both ways.) According to [16], the module $\mathcal{V} = V(\varpi_7)$ is compatible with the $\mathbb{Z}_4$-grading on $\mathfrak{e}_7$, that is, $\mathcal{M}_{\bar i} \cdot \mathcal{V}_{\bar j} \subset \mathcal{V}_{\bar i + \bar j}$, for

$$\begin{aligned} \mathcal{V}_{\bar 0} &= U \otimes \textstyle\bigwedge^2 W \otimes \mathbb{F}, & \mathcal{V}_{\bar 1} &= \mathbb{F} \otimes \textstyle\bigwedge^3 W \otimes W,\\ \mathcal{V}_{\bar 2} &= U \otimes \mathbb{F} \otimes \textstyle\bigwedge^2 W, & \mathcal{V}_{\bar 3} &= \mathbb{F} \otimes W \otimes \textstyle\bigwedge^3 W. \end{aligned}$$

Again, shifts are possible (but, once $\mathcal{V}_{\bar 0}$ is fixed, all is determined). Gathering the information, we have the descriptions (descriptions as modules for the neutral component) of all the homogeneous components of a $\mathbb{Z}_2 \times \mathbb{Z}_4$-grading on the split algebra $\mathfrak{e}_8$ with neutral homogeneous component isomorphic to $2\mathfrak{sl}(U) \oplus 2\mathfrak{sl}(W)$. This is the starting point for the model given in the next proposition.

Proposition 3.10 *Take U and W vector spaces over $\mathbb{F}$ of dimensions 2 and 4 respectively, and let $\mathcal{L}$ be the $\mathbb{Z}_2 \times \mathbb{Z}_4$-graded vector space given, for any $i = 0, 1$ and $j = 0, 1, 2, 3$, by*

$$\begin{aligned} \mathcal{L}_{(\bar 0,\bar 0)} &= \mathfrak{sl}(U) \oplus \mathfrak{sl}(U) \oplus \mathfrak{sl}(W) \oplus \mathfrak{sl}(W),\\ \mathcal{L}_{(\bar i,\bar j)} &= \textstyle\bigwedge^{[i]_2} U \otimes \bigwedge^{[i+j]_2} U \otimes \bigwedge^{[2i+j]_4} W \otimes \bigwedge^{[j]_4} W. \end{aligned} \tag{69}$$

That is, for $(\bar 0, \bar 0) \neq \alpha = (\bar i, \bar j)$, we write $\mathcal{L}_\alpha = \bigwedge^{\alpha_1} U \otimes \bigwedge^{\alpha_2} U \otimes \bigwedge^{\alpha_3} W \otimes \bigwedge^{\alpha_4} W$ for $\alpha_1 = [i]_2$, $\alpha_2 = [i + j]_2$, $\alpha_3 = [2i + j]_4$, $\alpha_4 = [j]_4$. Denote by $I_\alpha = \{k \in \{1, 2, 3, 4\} : \alpha_k = 0\}$.

Take the product of $\mathcal{L}_{(\bar 0,\bar 0)}$ with $\mathcal{L}_{(\bar i,\bar j)}$ given by the action of the kth ideal of $\mathcal{L}_{\bar 0}$ on the kth slot ($k = 1, 2, 3, 4$). Order $(\mathbb{Z}_2 \times \mathbb{Z}_4, \prec)$ lexicographically, and consider, for $(\bar 0, \bar 0) \neq \alpha, \beta \in \mathbb{Z}_2 \times \mathbb{Z}_4$, $\alpha \prec \beta$, the product of $x = x_1 \otimes x_2 \otimes x_3 \otimes x_4 \in \mathcal{L}_\alpha$ and $y = y_1 \otimes y_2 \otimes y_3 \otimes y_4 \in \mathcal{L}_\beta$ given by

$$[x, y] = \begin{cases} a_{\alpha,\beta}\, x_1 * y_1 \otimes x_2 * y_2 \otimes x_3 * y_3 \otimes x_4 * y_4 & \text{if } \beta + \alpha \neq (\bar 0, \bar 0),\\ \sum_{i=1, i \notin I_\alpha}^{4} b_\alpha^{(i)} [x_i, y_i]_i \Pi_{j=1, j\neq i}^{4} (x_j, y_j) & \text{if } \beta + \alpha = (\bar 0, \bar 0), \end{cases} \tag{70}$$

for some nonzero scalars $a_{\alpha,\beta}$ and $b_\alpha^{(i)}$. Extend the bracket to $\mathcal{L}$ making that the product of $\mathcal{L}_\alpha$ with $\mathcal{L}_\beta$ is skew-symmetric if $\beta \prec \alpha$, and making $(\mathcal{L}_{(\bar 0,\bar 0)}, [\ ,\ ])$ a subalgebra. Then, $\mathcal{L}$

endowed with the product [,] *is a Lie algebra if and only if*

$$\begin{array}{ll}
b_1^{(2)} = b_1^{(3)} = b_1^{(4)} = -a_{11}a_{23} = a'_{10}a'_{31}, & a_{12}a'_{32} = a'_{10}a'_{22}, \\
b_2^{(3)} = b_2^{(4)} = -a_{12}a_{23}, & a_{12}a'_{33} = -a'_{11}a'_{23}, \\
a_{11}a'_{20} = a'_{10}a'_{11}, & a'_{10}a''_{11} = a'_{11}a''_{02} = -a_{11}a''_{01}, \\
a_{11}a'_{21} = -a'_{11}a'_{12}, & a'_{10}a''_{12} = -a_{12}a''_{02} = -a'_{12}a''_{03}, \\
a_{11}a'_{22} = a'_{12}a'_{13}, & b_0^{'(1)} = b_0^{'(2)} = b_0^{'(3)} = a'_{10}a''_{01}, \\
a_{11}a'_{23} = -a'_{10}a'_{13}, & b_1^{'(1)} = b_1^{'(3)} = b_1^{'(4)} = a'_{31}a''_{03} = a'_{11}a''_{23}, \\
a_{12}a_{33} = a_{11}a_{23}, & b_2^{'(1)} = b_2^{'(2)} = b_2^{'(4)} = -a''_{12}a'_{32}, \\
a_{12}a'_{30} = a'_{12}a'_{20}, & a'_{12}a''_{33} = -a''_{23}a_{11}. \\
a_{12}a'_{31} = -a'_{13}a'_{21}, &
\end{array} \tag{71}$$

To abbreviate the notation a bit, we have used

$$a_{ij} := a_{(\bar{0},\bar{i}),(\bar{0}.\bar{j})}, \quad a'_{ij} := a_{(\bar{0},\bar{i}),(\bar{1}.\bar{j})}, \quad a''_{ij} := a_{(\bar{1},\bar{i}),(\bar{1}.\bar{j})}, \quad b_i^{(s)} := b^{(s)}_{(\bar{0},\bar{i})}, \quad b_i^{'(s)} := b^{(s)}_{(\bar{1},\bar{i})}.$$

Moreover, $(\mathcal{L}, [\ ,\ ])$ *is a simple exceptional Lie algebra isomorphic to* $\mathfrak{e}_8$. *A solution of this system of equations that, in particular, gives a model of* $\mathfrak{e}_8$, *is, for instance,*

$$\begin{array}{c}
1 = a_{11} = a_{12} = a'_{10} = a'_{11} = a'_{12} = a'_{13} = a'_{20} = a'_{22} = a'_{30}, \\
1 = a'_{31} = a'_{32} = a'_{33} = a''_{01} = a''_{12} = a''_{33} = b_1^{(i)} = b_2^{(i)} = b_0^{'(i)}, \\
-1 = a_{23} = a'_{21} = a'_{23} = a_{33} = a''_{02} = a''_{03} = a''_{11} = a''_{23} = b_1^{'(i)} = b_2^{'(i)}.
\end{array} \tag{72}$$

Proof Choose $\{e_1, e_2\}$ a basis of U such that $\phi(e_{12}) = 1$ and $\{e_1, \dots, e_4\}$ a basis of W such that $\phi(e_{1234}) = 1$. (The slight abuse of notation does not produce confusion.) Denote by $\mathrm{pr}_i : \mathcal{L}_{(\bar{0},\bar{0})} \to \mathfrak{sl}(U)$, $i = 1, 2$, and by $\mathrm{pr}_i : \mathcal{L}_{(\bar{0},\bar{0})} \to \mathfrak{sl}(W)$, $i = 3, 4$, the projections onto the four simple ideals of the neutral component, and use $\mathrm{pr}_i(f) \equiv f_i$ if convenient.

First we will check that all the conditions in (71) are necessary by imposing the Jacobi identity to concrete elements.

⋆ Take $x = 1 \otimes e_1 \otimes e_1 \otimes e_1$, $\tilde{x} = 1 \otimes e_1 \otimes e_1 \otimes e_2$, $y = 1 \otimes e_2 \otimes e_2 \otimes e_2$. We compute,

$$\begin{aligned}
z = 1 \otimes e_1 \otimes e_{134} \otimes e_{134} \Rightarrow\ & J(x, y, z) = -\big(a_{11}a_{23} + (\tfrac{1}{2}b_1^{(2)} + \tfrac{1}{4}b_1^{(3)} + \tfrac{1}{4}b_1^{(4)})\big)\, x, \\
& J(\tilde{x}, y, z) = \big(-(\tfrac{1}{2}b_1^{(2)} + \tfrac{1}{4}b_1^{(3)} + \tfrac{1}{4}b_1^{(4)}) + b_1^{(4)}\big)\, \tilde{x}, \\
z = 1 \otimes e_2 \otimes e_{134} \otimes e_{134} \Rightarrow\ & J(x, y, z) = (-a_{11}a_{23} - b_1^{(2)}) 1 \otimes e_2 \otimes e_1 \otimes e_1;
\end{aligned}$$

so we have

$$-a_{11}a_{23} = b_1^{(2)} = b_1^{(3)} = b_1^{(4)} =: b_1. \tag{73}$$

Moreover,

$$J(x, 1 \otimes e_2 \otimes e_{123} \otimes e_{234}, e_1 \otimes e_2 \otimes e_{34} \otimes 1) = (-b_1 + a'_{10}a'_{31}) e_1 \otimes e_2 \otimes e_{13} \otimes 1,$$

which gives

$$b_1 = a'_{10}a'_{31}. \tag{74}$$

We continue with a few more calculations,

$$\begin{aligned}
z = 1 \otimes 1 \otimes e_{13} \otimes e_{34} &\Rightarrow \mathrm{pr}_3(J(x, y, z)) = (-a_{11}b_2^{(3)} + a_{12}b_1)e_1^4, \\
z = 1 \otimes 1 \otimes e_{34} \otimes e_{13} &\Rightarrow \mathrm{pr}_4(J(x, y, z)) = (-a_{11}b_2^{(4)} + a_{12}b_1)e_1^4.
\end{aligned}$$

This implies $a_{11}b_2^{(3)} = a_{12}b_1 = a_{11}b_2^{(4)}$, so that we get

$$b_2^{(3)} = b_2^{(4)} := b_2, \qquad b_2 = -a_{12}a_{23}, \tag{75}$$

from $a_{11}b_2 = a_{12}b_1 = -a_{12}a_{11}a_{23}$. Carrying on with the same choice of x and y,

$$\begin{aligned}
z = e_1 \otimes e_1 \otimes e_{34} \otimes 1 &\Rightarrow J(x,y,z) = (-a_{11}a'_{20} + a'_{10}a'_{11})\, e_1 \otimes e_1 \otimes 1 \otimes e_{12},\\
z = e_1 \otimes 1 \otimes e_{134} \otimes e_4 &\Rightarrow J(x,y,z) = (a_{11}a'_{21} + a'_{11}a'_{12})\, e_1 \otimes 1 \otimes e_1 \otimes e_{124},\\
z = e_1 \otimes e_1 \otimes 1 \otimes e_{34} &\Rightarrow J(x,y,z) = (-a_{11}a'_{22} + a'_{12}a'_{13})\, e_1 \otimes e_1 \otimes e_{12} \otimes 1,\\
z = e_1 \otimes 1 \otimes e_4 \otimes e_{134} &\Rightarrow J(x,y,z) = (a_{11}a'_{23} + a'_{13}a'_{10})\, e_1 \otimes 1 \otimes e_{124} \otimes e_1;
\end{aligned}$$

so that we achieve

$$a_{11}a'_{20} = a'_{10}a'_{11}, \quad a_{11}a'_{21} = -a'_{11}a'_{12}, \quad a_{11}a'_{22} = a'_{12}a'_{13}, \quad a_{11}a'_{23} = -a'_{13}a'_{10}. \tag{76}$$

⋆ Next, take $x = 1 \otimes e_1 \otimes e_1 \otimes e_1$ and $y = 1 \otimes 1 \otimes e_{23} \otimes e_{23}$. From

$$\begin{aligned}
z = 1 \otimes e_2 \otimes e_{134} \otimes e_{134} &\Rightarrow J(x,y,z) = (-a_{12}a_{33} + a_{23}a_{11})1 \otimes 1 \otimes e_{13} \otimes e_{13},\\
z = e_2 \otimes e_2 \otimes e_{14} \otimes 1 &\Rightarrow J(x,y,z) = (-a_{12}a'_{30} + a'_{20}a'_{12})e_2 \otimes 1 \otimes e_1 \otimes e_{123},\\
z = e_2 \otimes 1 \otimes e_{134} \otimes e_4 &\Rightarrow J(x,y,z) = (a_{12}a'_{31} + a'_{21}a'_{13})e_2 \otimes e_1 \otimes e_{13} \otimes 1,\\
z = e_2 \otimes e_2 \otimes 1 \otimes e_{14} &\Rightarrow J(x,y,z) = (-a_{12}a'_{32} + a'_{22}a'_{10})e_2 \otimes 1 \otimes e_{123} \otimes e_1,\\
z = e_2 \otimes 1 \otimes e_4 \otimes e_{134} &\Rightarrow J(x,y,z) = (a_{12}a'_{33} + a'_{23}a'_{11})e_2 \otimes e_1 \otimes 1 \otimes e_{13};
\end{aligned}$$

we follow

$$\begin{aligned}
a_{12}a_{33} = a_{23}a_{11}, \quad a_{12}a'_{30} &= a'_{20}a'_{12}, \quad a_{12}a'_{31} = -a'_{21}a'_{13},\\
a_{12}a'_{32} &= a'_{22}a'_{10}, \quad a_{12}a'_{33} = -a'_{23}a'_{11}.
\end{aligned} \tag{77}$$

In particular,

$$\frac{a'_{32}}{a'_{31}} = \frac{a_{11}a_{12}a'_{32}}{a_{11}a_{12}a'_{31}} = \frac{a_{11}a'_{22}a'_{10}}{-a_{11}a'_{21}a'_{13}} = \frac{a'_{12}a'_{13}a'_{10}}{a'_{11}a'_{12}a'_{13}} = \frac{a'_{10}}{a'_{11}} \Rightarrow b_1 = a'_{10}a'_{31} = a'_{11}a'_{32}. \tag{78}$$

⋆ Take $x = 1 \otimes e_1 \otimes e_1 \otimes e_1$, $y = e_2 \otimes e_2 \otimes e_{23} \otimes 1$, and $\tilde{y} = e_2 \otimes e_2 \otimes e_{12} \otimes 1$. Then

$$\begin{aligned}
z = e_1 \otimes 1 \otimes e_{124} \otimes e_2 &\Rightarrow J(x,y,z) = (a'_{10}a''_{11} + a_{11}a''_{01})1 \otimes 1 \otimes e_{12} \otimes e_{12},\\
z = e_1 \otimes 1 \otimes e_{234} \otimes e_2 &\Rightarrow J(x,\tilde{y},z) = -(a'_{11}a''_{02} + a_{11}a''_{01})1 \otimes 1 \otimes e_{12} \otimes e_{12},\\
z = e_1 \otimes e_1 \otimes 1 \otimes e_{23} &\Rightarrow J(x,y,z) = -(a'_{10}a''_{12} + a_{12}a''_{02})1 \otimes e_1 \otimes e_{123} \otimes e_{123},\\
z = e_1 \otimes e_2 \otimes 1 \otimes e_{23} &\Rightarrow J(x,y,z) = -(a'_{10}a''_{12} + a'_{12}a''_{03})1 \otimes e_2 \otimes e_{123} \otimes e_{123}.
\end{aligned}$$

So we have

$$a'_{10}a''_{11} = a'_{11}a''_{02} = -a_{11}a''_{01}, \qquad a'_{10}a''_{12} = -a_{12}a''_{02} = -a'_{12}a''_{03}. \tag{79}$$

Moreover, evaluating the Jacobian operator for more choices of z, we get

$$\begin{aligned}
z = e_1 \otimes e_1 \otimes e_{34} \otimes 1 &\Rightarrow J(x,y,z) = (-b_0^{\prime(3)} + a'_{10}a''_{01})1 \otimes e_1 \otimes e_3 \otimes e_1,\\
z = e_1 \otimes e_2 \otimes e_{14} \otimes 1 &\Rightarrow J(x,y,z) = (-b_0^{\prime(2)} + a'_{10}a''_{01})1 \otimes e_2 \otimes e_1 \otimes e_1;
\end{aligned}$$

thus the identities $b_0^{\prime(3)} = b_0^{\prime(2)} = a'_{10}a''_{01}$ follow. Also,

$$J(e_1 \otimes e_1 \otimes e_{12} \otimes 1, e_1 \otimes e_2 \otimes e_{34} \otimes 1, e_2 \otimes 1 \otimes e_{123} \otimes e_4) = (-b_0^{\prime(1)} + a'_{10}a''_{01})e_1 \otimes 1 \otimes e_{123} \otimes e_4,$$

and gathering the information,

$$a'_{10}a''_{01} = b_0^{\prime(3)} = b_0^{\prime(2)} = b_0^{\prime(1)} =: b'_0. \tag{80}$$

⋆ Now, take arbitrary elements $x = \otimes x_i \in \mathcal{L}_{(0,1)}$, $y = \otimes y_i \in \mathcal{L}_{(1,0)}$, and $z = \otimes z_i \in \mathcal{L}_{(1,3)}$, (with $x_1 = y_4 = z_2 = 1$) and compute the four projections of $J(x, y, z) \in \mathcal{L}_{(\bar{0},\bar{0})}$:

$$\begin{aligned}
\mathrm{pr}_1(J(x,y,z)) &= -\big(b_1^{'(1)} a'_{10} + a'_{13} b'_0\big)\phi(x_2 \wedge y_2)\phi(x_3 \wedge y_3 \wedge z_3)\phi(x_4 \wedge z_4)[y_1, z_1]_1,\\
\mathrm{pr}_2(J(x,y,z)) &= \big(a'_{13} b'_0 + a''_{03} b_1\big)\phi(y_1 \wedge z_1)\phi(x_3 \wedge y_3 \wedge z_3)\phi(x_4 \wedge z_4)[x_2, y_2]_2,\\
\mathrm{pr}_3(J(x,y,z)) &= \phi(y_1 \wedge z_1)\phi(x_2 \wedge y_2)\phi(x_4 \wedge z_4)\Big((a''_{03} b_1 - a'_{10} b_1^{'(3)})[x_3, y_3 \wedge z_3]_3\\
&\qquad\qquad + (a'_{10} b_1^{'(3)} + a'_{13} b'_0)[y_3, x_3 \wedge z_3]_3\Big),\\
\mathrm{pr}_4(J(x,y,z)) &= \big(a''_{03} b_1 - a'_{10} b_1^{'(4)}\big)\phi(y_1 \wedge z_1)\phi(x_2 \wedge y_2)\phi(x_3 \wedge y_3 \wedge z_3)[x_4, z_4]_4.
\end{aligned}$$

(Here we have used $[x_3 \wedge y_3, z_3] = [x_3, y_3 \wedge z_3] - [y_3, x_3 \wedge z_3]$ for any $x_3, z_3 \in W$, $y_3 \in \bigwedge^2 W$.) This gives $a'_{10} b_1^{'(1)} = -a'_{13} b'_0 = a''_{03} b_1 = a'_{10} b_1^{'(3)} = a'_{10} b_1^{'(4)}$, and hence

$$b'_1 := b_1^{'(1)} = b_1^{'(3)} = b_1^{'(4)} = a''_{03} a'_{31}, \tag{81}$$

since $a'_{10} b'_1 = a''_{03} b_1 \overset{(74)}{=} a''_{03} a'_{10} a'_{31}$. Now we calculate

$$J(1 \otimes e_1 \otimes e_1 \otimes e_1, e_2 \otimes 1 \otimes e_{234} \otimes e_2, e_1 \otimes 1 \otimes e_1 \otimes e_{234}) = (a'_{11} a''_{23} - b'_1) 1 \otimes e_1 \otimes e_1 \otimes e_2,$$

which immediately gives

$$a'_{11} a''_{23} = b'_1. \tag{82}$$

⋆ Similarly, for any $x = \otimes x_i \in \mathcal{L}_{(0,1)}$, $y = \otimes y_i \in \mathcal{L}_{(1,1)}$, and $z = \otimes z_i \in \mathcal{L}_{(1,2)}$ with $x_1 = y_2 = z_3 = 1$, we find the four projections of $J(x, y, z) \in \mathcal{L}_{(\bar{0},\bar{0})}$:

$$\begin{aligned}
\mathrm{pr}_1(J(x,y,z)) &= \big(a'_{11} b_2^{'(1)} - a'_{12} b'_1\big)\phi(x_2 \wedge z_2)\phi(x_3 \wedge y_3)\phi(x_4 \wedge y_4 \wedge z_4)[y_1, z_1]_1,\\
\mathrm{pr}_2(J(x,y,z)) &= \big(a'_{11} b_2^{'(2)} + a''_{12} b_1\big)\phi(y_1 \wedge z_1)\phi(x_3 \wedge y_3)\phi(x_4 \wedge y_4 \wedge z_4)[x_2, z_2]_2,\\
\mathrm{pr}_3(J(x,y,z)) &= \big(a''_{12} b_1 + a'_{12} b'_1\big)\phi(y_1 \wedge z_1)\phi(x_2 \wedge z_2)\phi(x_4 \wedge y_4 \wedge z_4)[x_3, y_3]_3,\\
\mathrm{pr}_4(J(x,y,z)) &= \phi(y_1 \wedge z_1)\phi(x_2 \wedge z_2)\phi(x_3 \wedge y_3)\Big(\big(a'_{12} b'_1 - a'_{11} b_2^{'(4)}\big)[x_4 \wedge y_4, z_4]_4\\
&\qquad\qquad + \big(a'_{12} b'_1 + a''_{12} b_1\big)[x_4, y_4 \wedge z_4]_4\Big).
\end{aligned}$$

This yields

$$b'_2 := b_2^{'(1)} = b_2^{'(2)} = b_2^{'(4)} = -a''_{12} a'_{32}, \tag{83}$$

taking into account $a'_{11} b'_2 = -a''_{12} b_1 \overset{(78)}{=} -a''_{12} a'_{11} a'_{32}$.

⋆ Finally, we need one more equation: for $x = 1 \otimes e_1 \otimes e_1 \otimes e_1$, $y = e_2 \otimes e_2 \otimes 1 \otimes e_{23}$ and $z = e_1 \otimes 1 \otimes e_2 \otimes e_{124}$,

$$J(x, y, z) = (a'_{12} a''_{33} + a_{11} a''_{23}) 1 \otimes 1 \otimes e_{12} \otimes e_{12};$$

giving

$$a'_{12} a''_{33} = -a_{11} a''_{23}. \tag{84}$$

At last we have achieved all the necessary conditions in (71), which follow immediately from (73), (74), (75), (76), (77), (79), (80), (81), (82), (83), and (84). That is, if $\mathcal{L}$ is a Lie algebra, the scalars have to satisfy (71).

Table 2 Equivalent conditions in the $\mathbb{Z}_2 \times \mathbb{Z}_4$-model

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{1}),(\bar{0},\bar{2})})$: $a_{11}b_2 = a_{12}b_1$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{1}),(\bar{1},\bar{0})})$: $a_{11}a'_{20} = a'_{10}a'_{11}$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{1}),(\bar{1},\bar{2})})$: $a_{11}a'_{22} = a'_{12}a'_{13}$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{2}),(\bar{0},\bar{2})})$: $a_{12}a_{23} = -b_2$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{2}),(\bar{1},\bar{0})})$: $a_{12}a'_{30} = a'_{12}a'_{20} = -a'_{10}a'_{21}$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{2}),(\bar{1},\bar{2})})$: $a_{12}a'_{32} = a'_{10}a'_{22} = -a'_{12}a'_{23}$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{3}),(\bar{0},\bar{3})})$: $b_1 = -a_{12}a_{33}$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{3}),(\bar{1},\bar{1})})$: $b_1 = a'_{10}a'_{31} = a'_{11}a'_{32}$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{3}),(\bar{1},\bar{3})})$: $b_1 = a'_{33}a'_{12} = a'_{13}a'_{30}$

$(J_{(\bar{0},\bar{1}),(\bar{1},\bar{0}),(\bar{1},\bar{1})})$: $a'_{10}a''_{11} = -a_{11}a''_{01} = a'_{11}a''_{02}$

$(J_{(\bar{0},\bar{1}),(\bar{1},\bar{0}),(\bar{1},\bar{3})})$: $a'_{10}b'_1 = -a'_{13}b'_0 = a''_{03}b_1$

$(J_{(\bar{0},\bar{1}),(\bar{1},\bar{1}),(\bar{1},\bar{2})})$: $a'_{11}b'_2 = a'_{12}b'_1 = -a''_{12}b_1$

$(J_{(\bar{0},\bar{1}),(\bar{1},\bar{2}),(\bar{1},\bar{2})})$: $b'_2 = a'_{12}a''_{23}$

$(J_{(\bar{0},\bar{1}),(\bar{1},\bar{3}),(\bar{1},\bar{3})})$: $a'_{13}a''_{03} = -a''_{33}a_{12}$

$(J_{(\bar{0},\bar{2}),(\bar{0},\bar{2}),(\bar{1},\bar{0})})$: $a'_{22}a'_{20} = b_2$

$(J_{(\bar{0},\bar{2}),(\bar{0},\bar{2}),(\bar{1},\bar{2})})$: $a'_{20}a'_{22} = b_2$

$(J_{(\bar{0},\bar{2}),(\bar{0},\bar{3}),(\bar{0},\bar{3})})$: $b_1a_{23} = b_2a_{33}$

$(J_{(\bar{0},\bar{2}),(\bar{0},\bar{3}),(\bar{1},\bar{1})})$: $a_{23}a'_{11} = -a'_{31}a'_{20} = a'_{21}a'_{33}$

$(J_{(\bar{0},\bar{2}),(\bar{0},\bar{3}),(\bar{1},\bar{3})})$: $a_{23}a'_{13} = -a'_{33}a'_{22} = a'_{23}a'_{31}$

$(J_{(\bar{0},\bar{2}),(\bar{1},\bar{0}),(\bar{1},\bar{1})})$: $a'_{20}a''_{12} = a_{12}a''_{01} = a'_{21}a''_{03}$

$(J_{(\bar{0},\bar{2}),(\bar{1},\bar{0}),(\bar{1},\bar{3})})$: $a'_{20}a''_{23} = -a_{23}a''_{03} = a'_{23}a''_{01}$

$(J_{(\bar{0},\bar{2}),(\bar{1},\bar{1}),(\bar{1},\bar{2})})$: $a'_{21}a''_{23} = -a_{23}a''_{12} = a'_{22}a''_{01}$

$(J_{(\bar{0},\bar{2}),(\bar{1},\bar{2}),(\bar{1},\bar{2})})$: $a'_{22}a''_{02} = b'_2$

$(J_{(\bar{0},\bar{2}),(\bar{1},\bar{3}),(\bar{1},\bar{3})})$: $a'_{23}b'_1 = a''_{33}b_2$

$(J_{(\bar{1},\bar{0}),(\bar{1},\bar{0}),(\bar{1},\bar{2})})$: $b'_0 = -a'_{20}a''_{02}$

$(J_{(\bar{1},\bar{0}),(\bar{1},\bar{1}),(\bar{1},\bar{1})})$: $a'_{11}a''_{01} = -a'_{20}a''_{11}$

$(J_{(\bar{1},\bar{0}),(\bar{1},\bar{1}),(\bar{1},\bar{3})})$: $-a'_{13}a''_{01} = b'_1 = a'_{31}a''_{03}$

$(J_{(\bar{1},\bar{0}),(\bar{1},\bar{2}),(\bar{1},\bar{3})})$: $-a'_{23}a''_{02} = a'_{10}a''_{23} = a'_{32}a''_{03}$

$(J_{(\bar{1},\bar{1}),(\bar{1},\bar{1}),(\bar{1},\bar{2})})$: $a'_{22}a''_{11} = -a'_{31}a''_{12}$

$(J_{(\bar{1},\bar{1}),(\bar{1},\bar{2}),(\bar{1},\bar{2})})$: $a'_{32}a''_{12} = -b'_2$

$(J_{(\bar{1},\bar{1}),(\bar{1},\bar{3}),(\bar{1},\bar{3})})$: $b'_1 = a'_{21}a''_{33}$

$(J_{(\bar{1},\bar{2}),(\bar{1},\bar{2}),(\bar{1},\bar{3})})$: $a'_{13}a''_{23} = -a'_{22}a''_{33}$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{1}),(\bar{0},\bar{3})})$: $a_{11}a_{23} = -b_1$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{1}),(\bar{1},\bar{1})})$: $a_{11}a'_{21} = -a'_{11}a'_{12}$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{1}),(\bar{1},\bar{3})})$: $a_{11}a'_{23} = -a'_{10}a'_{13}$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{2}),(\bar{0},\bar{3})})$: $a_{12}a_{33} = a_{11}a_{23} = -b_1$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{2}),(\bar{1},\bar{1})})$: $a_{12}a'_{31} = -a'_{13}a'_{21} = a'_{11}a'_{22}$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{2}),(\bar{1},\bar{3})})$: $a_{12}a'_{33} = -a'_{11}a'_{23} = a'_{13}a'_{20}$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{3}),(\bar{1},\bar{0})})$: $b_1 = a'_{10}a'_{31} = a'_{13}a'_{30}$

$(J_{(\bar{0},\bar{1}),(\bar{0},\bar{3}),(\bar{1},\bar{2})})$: $b_1 = a'_{11}a'_{32} = a'_{12}a'_{33}$

$(J_{(\bar{0},\bar{1}),(\bar{1},\bar{0}),(\bar{1},\bar{0})})$: $b'_0 = a'_{10}a''_{01}$

$(J_{(\bar{0},\bar{1}),(\bar{1},\bar{0}),(\bar{1},\bar{2})})$: $-a'_{10}a''_{12} = a_{12}a''_{02} = a'_{12}a''_{03}$

$(J_{(\bar{0},\bar{1}),(\bar{1},\bar{1}),(\bar{1},\bar{1})})$: $a'_{11}a''_{12} = -a_{12}a''_{11}$

$(J_{(\bar{0},\bar{1}),(\bar{1},\bar{1}),(\bar{1},\bar{3})})$: $b'_1 = a'_{11}a''_{23} = -a'_{13}a''_{01}$

$(J_{(\bar{0},\bar{1}),(\bar{1},\bar{2}),(\bar{1},\bar{3})})$: $-a'_{12}a''_{33} = a''_{23}a_{11} = a'_{13}a''_{02}$

$(J_{(\bar{0},\bar{2}),(\bar{0},\bar{2}),(\bar{0},\bar{3})})$: $a_{12}a_{23} = -b_2$

$(J_{(\bar{0},\bar{2}),(\bar{0},\bar{2}),(\bar{1},\bar{1})})$: $a'_{21}a'_{23} = b_2$

$(J_{(\bar{0},\bar{2}),(\bar{0},\bar{2}),(\bar{1},\bar{3})})$: $a'_{21}a'_{23} = b_2$

$(J_{(\bar{0},\bar{2}),(\bar{0},\bar{3}),(\bar{1},\bar{0})})$: $a'_{10}a_{23} = a'_{23}a'_{30} = -a'_{20}a'_{32}$

$(J_{(\bar{0},\bar{2}),(\bar{0},\bar{3}),(\bar{1},\bar{2})})$: $a_{23}a'_{12} = a'_{32}a'_{21} = -a'_{22}a'_{30}$

$(J_{(\bar{0},\bar{2}),(\bar{1},\bar{0}),(\bar{1},\bar{0})})$: $a'_{20}a''_{02} = -b'_0$

$(J_{(\bar{0},\bar{2}),(\bar{1},\bar{0}),(\bar{1},\bar{2})})$: $a'_{20}b'_2 = a''_{02}b_2 = -a'_{22}b'_0$

$(J_{(\bar{0},\bar{2}),(\bar{1},\bar{1}),(\bar{1},\bar{1})})$: $a'_{21}b'_1 = -a''_{11}b_2$

$(J_{(\bar{0},\bar{2}),(\bar{1},\bar{1}),(\bar{1},\bar{3})})$: $a'_{21}a''_{33} = b'_1 = -a'_{23}a''_{11}$

$(J_{(\bar{0},\bar{2}),(\bar{1},\bar{2}),(\bar{1},\bar{3})})$: $a'_{22}a''_{03} = a_{12}a''_{23} = a'_{23}a''_{12}$

$(J_{(\bar{1},\bar{0}),(\bar{1},\bar{0}),(\bar{1},\bar{1})})$: $b'_0 = a''_{01}a'_{10}$

$(J_{(\bar{1},\bar{0}),(\bar{1},\bar{0}),(\bar{1},\bar{3})})$: $b'_0 = -a'_{30}a''_{03}$

$(J_{(\bar{1},\bar{0}),(\bar{1},\bar{1}),(\bar{1},\bar{2})})$: $a'_{12}a''_{01} = a'_{30}a''_{12} = a'_{21}a''_{02}$

$(J_{(\bar{1},\bar{0}),(\bar{1},\bar{2}),(\bar{1},\bar{2})})$: $a'_{22}a''_{02} = b'_2$

$(J_{(\bar{1},\bar{0}),(\bar{1},\bar{3}),(\bar{1},\bar{3})})$: $a'_{33}a''_{03} = -a'_{20}a''_{33}$

$(J_{(\bar{1},\bar{1}),(\bar{1},\bar{1}),(\bar{1},\bar{3})})$: $a'_{23}a''_{11} = -b'_1$

$(J_{(\bar{1},\bar{1}),(\bar{1},\bar{2}),(\bar{1},\bar{3})})$: $-a'_{33}a''_{12} = a'_{11}a''_{23} = b'_1$

$(J_{(\bar{1},\bar{2}),(\bar{1},\bar{2}),(\bar{1},\bar{3})})$: $b'_2 = a'_{12}a''_{23}$

A very important fact is that, for any arbitrary choice of $a_{11}, a_{12}, a'_{10}, a'_{11}, a'_{12}, a'_{13}, a''_{01} \in \mathbb{F}^\times$, (71) forces the remaining 21 scalars to be

$$b_1 = \frac{a'_{10}a'_{11}a'_{12}a'_{13}}{a_{11}a_{12}}, \quad b_2 = \frac{a'_{10}a'_{11}a'_{12}a'_{13}}{a_{11}^2}, \quad a_{23} = \frac{-a'_{10}a'_{11}a'_{12}a'_{13}}{a_{11}^2 a_{12}},$$

$$a'_{20} = \frac{a'_{10}a'_{11}}{a_{11}}, \quad a'_{21} = \frac{-a'_{11}a'_{12}}{a_{11}}, \quad a'_{22} = \frac{a'_{12}a'_{13}}{a_{11}}, \quad a'_{23} = \frac{-a'_{10}a'_{13}}{a_{11}},$$

$$\begin{aligned} &a_{33} = \frac{-a'_{10}a'_{11}a'_{12}a'_{13}}{a_{11}a_{12}^2}, \quad a'_{30} = \frac{a'_{10}a'_{11}a'_{12}}{a_{11}a_{12}}, \quad a'_{31} = \frac{a'_{11}a'_{12}a'_{13}}{a_{11}a_{12}}, \quad a'_{32} = \frac{a'_{10}a'_{12}a'_{13}}{a_{11}a_{12}}, \\ &a'_{33} = \frac{a'_{10}a'_{11}a'_{13}}{a_{11}a_{12}}, \quad b'_0 = a'_{10}a''_{01}, \quad a''_{02} = \frac{-a_{11}a''_{01}}{a'_{11}}, \quad a''_{03} = \frac{-a_{11}a_{12}a''_{01}}{a'_{11}a'_{12}}, \\ &a''_{11} = \frac{-a_{11}a''_{01}}{a'_{10}}, \quad a''_{12} = \frac{a_{11}a_{12}a''_{01}}{a'_{10}a'_{11}}, \quad b'_1 = -a'_{13}a''_{01}, \\ &b'_2 = \frac{-a'_{12}a'_{13}a''_{01}}{a'_{11}}, \quad a''_{23} = \frac{-a'_{13}a''_{01}}{a'_{11}}, \quad a''_{33} = \frac{a_{11}a'_{13}a''_{01}}{a'_{11}a'_{12}}; \end{aligned} \tag{85}$$

and conversely, this provides a solution of (71), whose set of solutions is therefore a 7-parametric family. Making the 7 free parameters equal to 1, and substituting in (85), we just obtain the concrete solution provided in (72).

Conversely, we assume that the scalars $a_{\alpha,\beta}, b_\alpha^{(i)} \in \mathbb{F}^\times$ satisfy all the equations in (71) (and hence they satisfy (85) too) and let us prove that then $\mathcal{L}$ is a Lie algebra. First, it is not difficult to check case by case that $[x, y] = -[y, x]$ if $x, y \in \mathcal{L}_\alpha$ for $\alpha \in \mathbb{Z}_2 \times \mathbb{Z}_4$, as in the above four models, so that the skew-symmetric extension is well defined. Alternatively, those computations can be skipped if we recall the existence of a solution making $\mathcal{L}$ a Lie algebra, since at least there must be a description of the exceptional Lie algebra $\mathfrak{e}_8$ with the products as in (70). That is, we once again follow the lines of Remark 3.2. Besides, we only have to check Jacobi identities $J(\mathcal{L}_\alpha, \mathcal{L}_\beta, \mathcal{L}_\gamma) = 0$ for all $\alpha \prec \beta \prec \gamma$, denoted again by $(J_{\alpha,\beta,\gamma})$. The strategy is much the same as for the model in Sect. 3.3: it is enough to check that the equations in (71) are equivalent to the list of identities provided in Table 2: one/two for each of the possibilities $(J_{\alpha,\beta,\gamma})$ for $\alpha \prec \beta \prec \gamma$.

All these identities can be easily derived from (85) by direct substitution. Although the process is fairly straightforward, it is also tedious: there are 77 cases to consider (112 equations, most of them redundant, as they ultimately reduce to the 21 equations displayed in (71)). It is therefore important to note that not all of these checks are strictly necessary. Indeed, it suffices to have the relations between $a_{\alpha,\beta}a_{\alpha+\beta,\gamma}$ and $a_{\beta,\gamma}a_{\beta+\gamma,\alpha}$ for all $\alpha, \beta, \gamma \in \mathbb{Z}_2 \times \mathbb{Z}_4$ (it is not important whether they are equal, opposite, multiples, etc.), as the existence of a solution is already known in advance.

Finally, taking into account Remark 3.4 (since Equation (70) ensures that conditions *i)* and *ii)* are satisfied in our graded algebra), uniqueness follows, and therefore simplicity as well. □

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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