A GENERALIZED HILBERT OPERATOR ACTING ON CONFORMALLY INVARIANT SPACES

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Abstract. If μ is a positive Borel measure on the interval $[0,1)$ we let $\mathcal{H}_\mu$ be the Hankel matrix $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where, for $n = 0,1,2,\dots$, μ_n denotes the moment of orden n of μ. This matrix induces formally the operator

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)z^n$$

on the space of all analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, in the unit disc $\mathbb{D}$. This is a natural generalization of the classical Hilbert operator. The action of the operators H_μ on Hardy spaces has been recently studied. This paper is devoted to study the operators H_μ acting on certain conformally invariant spaces of analytic functions on the disc such as the Bloch space, $BMOA$, the analytic Besov spaces, and the Q_s spaces.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$ and let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence in compact subsets. We also let H^p $(0 < p \leq \infty)$ be the classical Hardy spaces. We refer to [18] for the notation and results regarding Hardy spaces.

If μ is a finite positive Borel measure on $[0,1)$ and $n = 0,1,2,\dots$, we let μ_n denote the moment of order n of μ, that is, $\mu_n = \int_{[0,1)} t^n\,d\mu(t)$, and we let $\mathcal{H}_\mu$ be the Hankel matrix $(\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$. The matrix $\mathcal{H}_\mu$ induces formally an operator, which will be also called $\mathcal{H}_\mu$, on spaces of analytic functions by its action on the Taylor coefficients: $a_n \mapsto \sum_{k=0}^{\infty}\mu_{n,k}a_k, \quad n = 0,1,2,\dots$.
To be precise, if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(\mathbb{D})$ we define

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)z^n,$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

If μ is the Lebesgue measure on $[0,1)$ the matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix $\mathcal{H} = ((n+k+1)^{-1})_{n,k\geq 0}$, which induces the classical Hilbert operator $\mathcal{H}$ which has extensively studied recently (see [1, 13, 14, 16, 24]).

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Galanopoulos and Peláez [19] described the measures μ so that the generalized Hilbert operator $\mathcal{H}_\mu$ becomes well defined and bounded on H^1. Chatzifountas, Girela and Peláez [12] extended this work describing those measures μ for which $\mathcal{H}_\mu$ is a bounded operator from H^p into H^q, $0<p,q<\infty$.

Obtaining an integral representation of H_μ plays a basic role in these works. If μ is as above, we shall write throughout the paper

$$I_\mu(f)(z)=\int_{[0,1)}\frac{f(t)}{1-tz}\,d\mu(t), \tag{1.1}$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. It turns out that the operators H_μ and I_μ are closely related. In fact, in [19] and [12] the measures μ for which the operator I_μ is well defined in H^p $(0<p<\infty)$ are characterized and it is proved that for such measures we have $\mathcal{H}_\mu(f)=I_\mu(f)$ for all $f\in H^p$. These measures are Carleson-type measures.

If $I\subset\partial\mathbb{D}$ is an arc, $|I|$ will denote the length of I. The *Carleson square* $S(I)$ is defined as $S(I)=\{re^{it}:\, e^{it}\in I,\quad 1-\frac{|I|}{2\pi}\le r<1\}$.

If $s>0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu\left(S(I)\right)\le C|I|^s,\quad \text{for any interval } I\subset\partial\mathbb{D}.$$

If μ satisfies $\lim_{|I|\to 0}\frac{\mu\left(S(I)\right)}{|I|^s}=0$, then we say that μ is a *vanishing s-Carleson measure*.

A 1-Carleson measure, respectively, a vanishing 1-Carleson measure, will be simply called a Carleson measure, respectively, a vanishing Carleson measure.

We recall that Carleson [11] proved that $H^p\subset L^p(d\mu)$ $(0<p<\infty)$, if and only if μ is a Carleson measure. This result was extended by Duren [17] (see also [18, Theorem 9. 4]) who proved that for $0<p\le q<\infty$, $H^p\subset L^q(d\mu)$ if and only if μ is a q/p-Carleson measure.

Following [32], if μ is a positive Borel measure on $\mathbb{D}$, $0\le\alpha<\infty$, and $0<s<\infty$ we say that μ is an α-logarithmic s-Carleson measure if there exists a positive constant C such that

$$\frac{\mu\left(S(I)\right)\left(\log\frac{2\pi}{|I|}\right)^{\alpha}}{|I|^s}\le C,\quad \text{for any interval } I\subset\partial\mathbb{D}.$$

If $\mu\left(S(I)\right)\left(\log\frac{2\pi}{|I|}\right)^{\alpha}=\mathrm{o}\left(|I|^s\right)$, as $|I|\to 0$, we say that μ is a vanishing α-logarithmic s-Carleson measure.

A positive Borel measure μ on $[0,1)$ can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(A)=\mu\left(A\cap[0,1)\right),\quad \text{for any Borel subset } A \text{ of } \mathbb{D}.$$

In this way a positive Borel measure μ on $[0,1)$ is an s-Carleson measure if and only if there exists a positive constant C such that

$$\mu\left([t,1)\right)\le C(1-t)^s,\quad 0\le t<1,$$

and we have similar statements for vanishing s-Carleson measures and for α-logarithmic s-Carleson and vanishing α-logarithmic s-Carleson measures.

Our main aim in this paper is studying the operators $\mathcal{H}_\mu$ acting on conformally invariant spaces.

It is a standard fact that the set of all disc automorphisms (*i.e.*, of all one-to-one analytic maps f of $\mathbb{D}$ onto itself), denoted $\mathrm{Aut}(\mathbb{D})$, coincides with the set of all Möbius transformations of $\mathbb{D}$ onto itself:

$$\mathrm{Aut}(\mathbb{D}) = \{\lambda\varphi_a : |a| < 1, |\lambda| = 1\},$$

where $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$.

A space X of analytic functions in $\mathbb{D}$, defined via a semi-norm ρ, is said to be *conformally invariant* or *Möbius invariant* if whenever $f \in X$, then also $f \circ \varphi \in X$ for any $\varphi \in \mathrm{Aut}(\mathbb{D})$ and, moreover, $\rho(f \circ \varphi) \le C\rho(f)$ for some positive constant C and all $f \in X$. A great deal of information on conformally invariant spaces can be found in [5, 15, 30].

Let us start considering the Bloch space and $BMOA$. The *Bloch space* $\mathcal{B}$ consists of all analytic functions f in $\mathbb{D}$ with bounded invariant derivative:

$$f \in \mathcal{B} \quad \Leftrightarrow \quad \|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)\, |f'(z)| < \infty .$$

The *little Bloch space* $\mathcal{B}_0$ is the closure of the polynomials in the above norm of $\mathcal{B}$ and consists of all functions f analytic in $\mathbb{D}$ for which

$$\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0 .$$

A classical source for the Bloch space is [3]; see also [34]. Rubel and Timoney [30] proved that $\mathcal{B}$ is the biggest "natural" conformally invariant space.

The space $BMOA$ consists of those functions f in H^1 whose boundary values have bounded mean oscillation on the unit circle $\partial\mathbb{D}$ as defined by F. John and L. Nirenberg. There are many characterizations of $BMOA$ functions. Let us mention the following:

If f is an analytic function in $\mathbb{D}$, then $f \in BMOA$ if and only if

$$\|f\|_{BMOA} \stackrel{\text{def}}{=} |f(0)| + \|f\|_\star < \infty,$$

where

$$\|f\|_\star \stackrel{\text{def}}{=} \sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^2}.$$

It is clear that the seminorm $\|\cdot\|_\star$ is conformally invariant. If

$$\lim_{|a|\to 1} \|f \circ \varphi_a - f(a)\|_{H^2} = 0$$

we say that f belongs to the space $VMOA$. We mention [9, 21] as general references for the spaces $BMOA$ and $VMOA$. Let us recall that

$$H^\infty \subsetneq BMOA \subsetneq \bigcap_{0<p<\infty} H^p \quad \text{and } BMOA \subsetneq \mathcal{B}.$$

Other important Möbius invariant spaces are the analytic Besov spaces B^p $(1 < p < \infty)$ and the Q_s-spaces $(s > 0)$. These spaces will be considered in Section 3.

We close this section noticing that, as usual, we shall be using the convention that $C = C(p, \alpha, q, \beta, \dots)$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta \dots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Moreover, for two real-valued functions E_1, E_2 we write $E_1 \lesssim E_2$, or $E_1 \gtrsim E_2$, if there exists a positive constant C independent of the arguments such that $E_1 \leq CE_2$, respectively $E_1 \geq CE_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously then we say that E_1 and E_2 are equivalent and we write $E_1 \asymp E_2$.

2. The operator $\mathcal{H}_\mu$ acting on $BMOA$ and the Bloch space

We start characterizing those μ for which the operator I_μ is well defined in $BMOA$ and in the Bloch space. It turns out that they coincide.

Theorem 2.1. *Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent:*

(i) $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$.

(ii) *For any given $f \in \mathcal{B}$, the integral in (1.1) converges for all $z \in \mathbb{D}$ and the resulting function $I_\mu(f)$ is analytic in $\mathbb{D}$.*

(iii) *For any given $f \in BMOA$, the integral in (1.1) converges for all $z \in \mathbb{D}$ and the resulting function $I_\mu(f)$ is analytic in $\mathbb{D}$.*

Proof.

(i) $\Rightarrow$ (ii). It is well known that there exists a positive constant C such that

$$|f(z)| \leq C\|f\|_{\mathcal{B}} \log \frac{2}{1-|z|}, \quad (z \in \mathbb{D}), \text{ for every } f \in \mathcal{B}, \tag{2.1}$$

(see [3, p. 13]). Assume (i) and set $A = \int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t)$. Using (2.1) we see that

$$\int_{[0,1)} |f(t)|\, d\mu(t) \leq C\|f\|_{\mathcal{B}} \int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) = AC\|f\|_{\mathcal{B}}, \quad f \in \mathcal{B}. \tag{2.2}$$

This implies that

$$\int_{[0,1)} \frac{|f(t)|}{|1-tz|}\, d\mu(t) \leq \frac{AC\|f\|_{\mathcal{B}}}{1-|z|}, \quad (z \in \mathbb{D}), \quad f \in \mathcal{B}. \tag{2.3}$$

Using (2.2), (2.3), and Fubini's theorem we see that if $f \in \mathcal{B}$ then:

- For every $n \in \mathbb{N}$, the integral $\int_{[0,1)} t^n f(t)\, d\mu(t)$ converges absolutely and

$$\sup_{n \geq 0} \left| \int_{[0,1)} t^n f(t)\, d\mu(t) \right| < \infty.$$

- The integral $\int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t)$ converges absolutely, and

$$\int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t) \;=\; \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n f(t)\, d\mu(t) \right) z^n, \quad z \in \mathbb{D}.$$

Thus, if $f \in \mathcal{B}$ then $I_\mu(f)$ is a well defined analytic function in $\mathbb{D}$ and

$$I_\mu(f)(z) \;=\; \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n f(t)\, d\mu(t) \right) z^n, \quad z \in \mathbb{D}.$$

The implication (ii) $\Rightarrow$ (iii) is clear because $BMOA \subset \mathcal{B}$.

(iii) $\Rightarrow$ (i). Suppose (iii). Since the function $F(z) = \log \frac{2}{1-z}$ belongs to $BMOA$, $I_\mu(F)(z)$ is well defined for every $z \in \mathbb{D}$. In particular

$$I_\mu(F)(0) = \int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t)$$

is a complex number. Since μ is a positive measure and $\log \frac{2}{1-t} > 0$ for all $t \in [0,1)$, (i) follows. □

Our next aim is characterizing the measures μ so that I_μ is bounded in $BMOA$ or $\mathcal{B}$ and seeing whether or not I_μ and H_μ coincide for such measures. We have the following results.

Theorem 2.2. *Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)} \log \frac{2}{1-t} d\mu(t) < \infty$. Then the following three conditions are equivalent:*

(i) *The measure ν defined by $d\nu(t) = \log \frac{2}{1-t}\, d\mu(t)$ is a Carleson measure.*
(ii) *The operator I_μ is bounded from $\mathcal{B}$ into $BMOA$.*
(iii) *The operator I_μ is bounded from $BMOA$ into itself.*

Theorem 2.3. *Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$. If the measure ν defined by $d\nu(t) = \log \frac{2}{1-t}\, d\mu(t)$ is a Carleson measure, then $\mathcal{H}_\mu$ is well defined on the Bloch space and*

$$\mathcal{H}_\mu(f) \;=\; I_\mu(f), \quad \text{for all } f \in \mathcal{B}.$$

Theorem 2.2 and Theorem 2.3 together yield the following.

Theorem 2.4. *Let μ be a positive Borel measure on $[0,1)$ such that the measure ν defined by $d\nu(t) = \log \frac{2}{1-t}\, d\mu(t)$ is a Carleson measure. Then the operator $\mathcal{H}_\mu$ is bounded from $\mathcal{B}$ into $BMOA$.*

Proof of Theorem 2.2. Since $\int_{[0,1)} \log \frac{2}{1-t} d\mu(t) < \infty$, (2.1) implies that

$$\int_{[0,1)} |f(t)|\, d\mu(t) < \infty, \quad \text{for all } f \in \mathcal{B}$$

and this implies that

$$\int_0^{2\pi}\int_{[0,1)}\left|\frac{f(t)g(e^{i\theta})}{1-re^{i\theta}t}\right|d\mu(t)d\theta<\infty,\quad 0\le r<1,\ f\in\mathcal{B},\ g\in H^1.$$

Using this, Fubini's theorem and Cauchy's integral representation of H^1-functions [18, Theorem 3. 6], we deduce that whenever $f\in\mathcal{B}$ and $g\in H^1$ we have

$$\int_0^{2\pi} I_\mu(f)(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta \;=\; \int_0^{2\pi}\left(\int_{[0,1)}\frac{f(t)d\mu(t)}{1-re^{i\theta}t}\right)\overline{g(e^{i\theta})}\,d\theta \tag{2.4}$$

$$=\int_{[0,1)} f(t)\left(\int_0^{2\pi}\frac{\overline{g(e^{i\theta})}d\theta}{1-re^{i\theta}t}\right)d\mu(t)=\int_{[0,1)} f(t)\overline{g(rt)}\,d\mu(t),\quad 0\le r<1.$$

(i) $\Rightarrow$ (ii). Assume that ν is a Carleson measure and take $f\in\mathcal{B}$ and $g\in H^1$. Using (2.4) and (2.1), we obtain

$$\left|\int_0^{2\pi} I_\mu(f)(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta\right| \;=\; \left|\int_{[0,1)} f(t)\overline{g(rt)}\,d\mu(t)\right|$$

$$\lesssim \|f\|_{\mathcal{B}}\int_{[0,1)}|g(rt)|\log\frac{2}{1-t}\,d\mu(t) \;=\; \|f\|_{\mathcal{B}}\int_{[0,1)}|g(rt)|\;d\nu(t).$$

Since ν is a Carleson measure

$$\int_{[0,1)}|g(rt)|\;d\nu(t)\;\lesssim\;\|g_r\|_{H^1}\;\le\;\|g\|_{H^1}.$$

Here, as usual, g_r is the function defined by $g_r(z)=g(rz)$ $(z\in\mathbb{D})$.

Thus, we have proved that

$$\left|\int_0^{2\pi} I_\mu(f)(re^{i\theta})\overline{g(e^{i\theta})}\,d\theta\right|\;\lesssim\;\|f\|_{\mathcal{B}}\|g\|_{H^1},\quad f\in\mathcal{B},\ g\in H^1.$$

Using Fefferman's duality Theorem (see [21, Theorem 7. 1]) we deduce that if $f\in\mathcal{B}$ then $I_\mu(f)\in BMOA$ and

$$\|I_\mu(f)\|_{BMOA}\lesssim\|f\|_{\mathcal{B}}.$$

The implication (ii) $\Rightarrow$ (iii) is trivial because $BMOA\subset\mathcal{B}$.

(iii) $\Rightarrow$ (i). Assume (iii). Then there exists a positive constant A such that $\|I_\mu(f)\|_{BMOA}\le A\|f\|_{BMOA}$, for all $f\in BMOA$. Set

$$F(z)=\log\frac{2}{1-z},\quad z\in\mathbb{D}.$$

It is well known that $F\in BMOA$. Then $I_\mu(F)\in BMOA$ and

$$\|I_\mu(F)\|_{BMOA}\le A\|F\|_{BMOA}.$$

Then using again Fefferman's duality theorem we obtain that

$$\left|\int_0^{2\pi} I_\mu(F)(re^{i\theta})\,\overline{g(e^{i\theta})}\,d\theta\right|\lesssim\|g\|_{H^1},\quad g\in H^1.$$

Using (2.4) and the definition of F, this implies

$$\left| \int_{[0,1)]} \overline{g(rt)} \log \frac{2}{1-t} \, d\mu(t) \right| \lesssim \|g\|_{H^1}, \quad g \in H^1. \tag{2.5}$$

Take $g \in H^1$. Using Proposition 2 of [12] we know that there exists a function $G \in H^1$ with $\|G\|_{H^1} = \|g\|_{H^1}$ and such that

$$|g(s)| \leq G(s), \quad \text{for all } s \in [0,1).$$

Using these properties and (2.5) for G, we obtain

$$\begin{aligned} \int_{[0,1)} |g(rt)| \log \frac{2}{1-t} \, d\mu(t) &\leq \int_{[0,1)} G(rt) \log \frac{2}{1-t} \, d\mu(t) \\ &\leq C\|G_r\|_{H^1} \leq C\|G\|_{H^1} = C\|g\|_{H^1} \end{aligned}$$

for a certain constant $C > 0$, independent of g. Letting r tend to 1, it follows that

$$\int_{[0,1)} |g(t)| \log \frac{2}{1-t} \, d\mu(t) \lesssim \|g\|_{H^1}, \quad g \in H^1.$$

This is equivalent to saying that ν is a Carleson measure. □

It is worth noticing that for μ and ν as in Theorem 2.1, ν being a Carleson measure is equivalent to μ being an 1-logarithmic 1-Carleson measure. Actually, we have the following more general result.

Proposition 2.5. *Let μ be a positive Borel measure on $[0,1)$, $s > 0$, and $\alpha \geq 0$. Let ν be the Borel measure on $[0,1)$ defined by*

$$d\nu(t) = \left(\log \frac{2}{1-t} \right)^{\alpha} d\mu(t).$$

Then, the following two conditions are equivalent.

(a) *ν is an s-Carleson measure.*
(b) *μ is an α-logarithmic s-Carleson measure.*

Proof.
(a) ⇒ (b). Assume (a). Then there exists a positive constant C such that

$$\int_{[t,1)} \left(\log \frac{2}{1-u} \right)^{\alpha} d\mu(u) \leq C(1-t)^s, \quad t \in [0,1).$$

Using this and the fact that the function $u \mapsto \log \frac{2}{1-u}$ is increasing in $[0,1)$, we obtain

$$\left(\log \frac{2}{1-t} \right)^{\alpha} \int_{[t,1)} d\mu(u) \leq \int_{[t,1)} \left(\log \frac{2}{1-u} \right)^{\alpha} d\mu(u) \leq C(1-t)^s, \quad t \in [0,1).$$

This shows that μ is an α-logarithmic s-Carleson measure.

(b) ⇒ (a). Assume (b). Then there exists a positive constant C such that

$$\left(\log \frac{2}{1-t} \right)^{\alpha} \mu([t,1)) \leq C(1-t)^s, \quad 0 \leq t < 1. \tag{2.6}$$

For $0 \le u < 1$, set $F(u) = \mu([0,u)) - \mu([0,1)) = -\mu([u,1))$. Integrating by parts and using (2.6), we obtain

$$\begin{aligned}
\nu([t,1)) &= \int_{[t,1)} \left(\log\frac{2}{1-u}\right)^{\alpha} d\mu(u) \\
&= \left(\log\frac{2}{1-t}\right)^{\alpha} \mu([t,1)) - \lim_{u\to 1^-} \left(\log\frac{2}{1-u}\right)^{\alpha} \mu([u,1)) \\
&\quad + \alpha \int_{[t,1)} \mu([u,1)) \left(\log\frac{2}{1-u}\right)^{\alpha-1} \frac{du}{1-u} \\
&= \left(\log\frac{2}{1-t}\right)^{\alpha} \mu([t,1)) + \alpha \int_{[t,1)} \mu([u,1)) \left(\log\frac{2}{1-u}\right)^{\alpha-1} \frac{du}{1-u} \\
&\le C(1-t)^s + C\alpha \int_t^1 \frac{(1-u)^{s-1}}{\log\frac{2}{1-u}}\, du \\
&\lesssim (1-t)^s, \quad 0 \le t < 1.
\end{aligned}$$

Thus, ν is an s-Carleson measure. □

The following lemma will be needed in the proof of Theorem 2.3.

Lemma 2.6. *Let μ be a positive Borel measure in $[0,1)$ such that the measure ν defined by $d\nu(t) = \log\frac{1}{1-t} d\mu(t)$ is a Carleson measure. Then the sequence of moments $\{\mu_n\}$ satisfies*

$$\mu_n = \mathrm{O}\left(\frac{1}{n\log n}\right), \quad \text{as } n \to \infty.$$

Actually, we shall prove the following more general result.

Lemma 2.7. *Suppose that $0 \le \alpha \le \beta$, $s \ge 1$, and let μ be a positive Borel measure on $[0,1)$ which is a β-logarithmic s-Carleson measure. Then*

$$\int_{[0,1)} t^k \left(\log\frac{2}{1-t}\right)^{\alpha} d\mu(t) = \mathrm{O}\left(\frac{(\log k)^{\alpha-\beta}}{k^s}\right), \quad \text{as } k \to \infty.$$

Using Proposition 2.5, Lemma 2.6 follows taking $\alpha = 0$, $\beta = 1$, and $s = 1$ in Lemma 2.7.

Proof of Lemma 2.7. Arguing as in the proof of the implication (b)⇒(a) of Proposition 2.5, integrating by parts and using the fact that μ is a β-logarithmic 1-Carleson measure, we obtain

$$\begin{aligned}
&\int_{[0,1)} t^k \left(\log\frac{2}{1-t}\right)^{\alpha} d\mu(t) \qquad (2.7) \\
&= k\int_0^1 \mu([t,1)) t^{k-1} \left(\log\frac{2}{1-t}\right)^{\alpha} dt + \alpha \int_0^1 \mu([t,1)) t^k \left(\log\frac{2}{1-t}\right)^{\alpha-1} \frac{dt}{1-t} \\
&\lesssim k\int_0^1 (1-t)^s t^{k-1} \left(\log\frac{2}{1-t}\right)^{\alpha-\beta} dt + \alpha \int_0^1 (1-t)^{s-1} t^k \left(\log\frac{2}{1-t}\right)^{\alpha-\beta-1} dt.
\end{aligned}$$

Now, we notice that the weight functions

$$\omega_1(t) = (1-t)^s \left(\log \frac{2}{1-t}\right)^{\alpha-\beta} \quad \text{and} \quad \omega_2(t) = (1-t)^{s-1} \left(\log \frac{2}{1-t}\right)^{\alpha-\beta-1}$$

are regular in the sense of [29] (see [29, p. 6] and [2, Example 2]). Then, using Lemma 1.3 of [29] and the fact that the ω_j's are also decreasing, we obtain

$$\begin{aligned}\int_0^1 (1-t)^s t^{k-1} \left(\log \frac{2}{1-t}\right)^{\alpha-\beta} dt &\lesssim \int_{1-\frac{1}{k}}^1 (1-t)^s t^{k-1} \left(\log \frac{2}{1-t}\right)^{\alpha-\beta} dt \\ &\lesssim \frac{(\log k)^{\alpha-\beta}}{k^{s+1}}\end{aligned}$$

and

$$\begin{aligned}\int_0^1 (1-t)^{s-1} t^{k} \left(\log \frac{2}{1-t}\right)^{\alpha-\beta-1} dt &\lesssim \int_{1-\frac{1}{n}}^1 (1-t)^{s-1} t^{k} \left(\log \frac{2}{1-t}\right)^{\alpha-\beta-1} dt \\ &\lesssim \frac{(\log k)^{\alpha-\beta-1}}{k^{s}}.\end{aligned}$$

Using these two estimates in (2.7) yields

$$\int_{[0,1)} t^k \left(\log \frac{2}{1-t}\right)^{\alpha} d\mu(t) \lesssim \frac{(\log k)^{\alpha-\beta}}{k^s}$$

finishing the proof. □

We shall also use the characterization of the coefficient multipliers from $\mathcal{B}$ into ℓ^1 obtained by Anderson and Shields in [4].

Theorem A. *A sequence* $\{\lambda_n\}_{n=0}^{\infty}$ *of complex numbers is a coefficient multiplier from* $\mathcal{B}$ *into* ℓ^1 *if and only if*

$$\sum_{n=1}^{\infty} \left(\sum_{k=2^n+1}^{2^{n+1}} |\lambda_k|^2 \right)^{1/2} < \infty.$$

Bearing in mind Definition 1 of [4], Theorem A reduces to the case $p = 1$ in Corollary 1 in p. 259 of [4].

We recall that if X is a space of analytic functions in $\mathbb{D}$ and Y is a space of complex sequences, a sequence $\{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{C}$ is said to be a multiplier of X into Y if whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n \in X$ one has that the sequence $\{\lambda_n a_n\}_{n=0}^{\infty}$ belongs to Y. Thus:

By saying that $\{\lambda_n\}_{n=0}^{\infty}$ is a coefficient multiplier from $\mathcal{B}$ into ℓ^1 we mean that

$$\text{If } f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B} \text{ then } \sum_{n=0}^{\infty} |\lambda_n a_n| < \infty.$$

Actually, using the closed graph theorem, we can assert the following:

A complex sequence $\{\lambda_n\}_{n=0}^{\infty}$ is a multiplier from $\mathcal{B}$ to ℓ^1 if and only if there exists a positive constant C such that whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$, we have that $\sum_{n=0}^{\infty} |\lambda_n a_n| \le C\|f\|_{\mathcal{B}}$.

Proof of Theorem 2.3. Suppose that ν is a Carleson measure. Then, using Lemma 2.6, we see that there exists $C>0$ such that

$$|\mu_n| \le \frac{C}{n\log n}, \quad n\ge 2. \tag{2.8}$$

It is clear that

$$k^2\log^2 k \ge 2^{2n}n^2(\log 2)^2, \quad \text{if } 2^n+1\le k\le 2^{n+1} \text{ for all } \ n.$$

Then it follows that

$$\sum_{n=1}^{\infty}\left(\sum_{k=2^n+1}^{2^{n+1}}\frac{1}{k^2\log^2 k}\right)^{1/2} \lesssim \sum_{n=1}^{\infty}\left(\frac{2^n}{n^2 2^{2n}}\right)^{1/2} = \sum_{n=1}^{\infty}\frac{1}{n2^{n/2}} < \infty.$$

Using this, (2.8) and Theorem A, we obtain:

$$\text{The sequence of moments } \{\mu_n\}_{n=0}^{\infty} \text{ is a multiplier from } \mathcal{B} \text{ to } \ell^1. \tag{2.9}$$

Take now $f\in\mathcal{B}$, $f(z)=\sum_{n=0}^{\infty}a_nz^n$ $(z\in\mathbb{D})$. Using the simple fact that the sequence $\{\mu_n\}_{n=0}^{\infty}$ is a decreasing sequence of positive numbers and (2.9), we see that there exists $C>0$ such that

$$\sum_{k=0}^{\infty}|\mu_{n+k}\,a_k| \le \sum_{k=0}^{\infty}|\mu_k\,a_k| \le C\|f\|_{\mathcal{B}}, \quad n\,=\,0,\,1,\,2,\ldots. \tag{2.10}$$

This implies that $\mathcal{H}_\mu(f)(z)$ is well defined for all $z\in\mathbb{D}$ and that, in fact, $\mathcal{H}_\mu(f)$ is an analytic function in $\mathbb{D}$. Furthermore, since (2.10) also implies that we can interchange the order of summation in the expression defining $\mathcal{H}_\mu(f)(z)$, we have

$$\begin{aligned}\mathcal{H}_\mu(f)(z) &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n+k}a_k\right)z^n = \sum_{k=0}^{\infty}a_k\left(\sum_{n=0}^{\infty}\mu_{n+k}z^n\right)\\ &= \sum_{k=0}^{\infty}a_k\left(\sum_{n=0}^{\infty}\int_{[0,1)}t^{n+k}z^n\,d\mu(t)\right) = \sum_{k=0}^{\infty}\int_{[0,1)}\frac{a_kt^k}{1-tz}\,d\mu(t)\\ &= \int_{[0,1)}\frac{f(t)}{1-tz}\,d\mu(t) = I_\mu(f)(z), \quad z\in\mathbb{D}.\end{aligned}$$

□

We have the following result regarding compactness.

Theorem 2.8. *Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)}\log\frac{2}{1-t}\,d\mu(t)<\infty$. If the measure ν defined by $d\nu(t)=\log\frac{2}{1-t}\,d\mu(t)$ is a vanishing Carleson measure then:*

(i) *The operator I_μ is a compact operator from $\mathcal{B}$ into $BMOA$.*
(ii) *The operator I_μ is a compact operator from $BMOA$ into itself.*

Before embarking on the proof of Theorem 2.8 it is convenient to recall some facts about Carleson measures and to fix some notation.

If μ is a Carleson measure on $\mathbb{D}$, we define the Carleson-norm of μ, denoted $\mathcal{N}(\mu)$, as

$$\mathcal{N}(\mu) = \sup_{I \text{ subarc of } \partial\mathbb{D}} \frac{\mu(S(I))}{|I|}.$$

We let also $\mathcal{E}(\mu)$ denote the norm of the inclusion operator $i : H^1 \to L^1(d\mu)$. It turns out that these quantities are equivalent: There exist two positive constants A_1, A_2 such that

$$A_1 \mathcal{N}(\mu) \le \mathcal{E}(\mu) \le A_2 \mathcal{N}(\mu), \quad \text{for every Carleson measure } \mu \text{ on } \mathbb{D}.$$

For a Carleson measure μ on $\mathbb{D}$ and $0 < r < 1$, we let μ_r be the measure on $\mathbb{D}$ defined by

$$d\mu_r(z) = \chi_{\{r<|z|<1\}} d\mu(z).$$

We have that μ is a vanishing Carleson measure if and only if

$$\mathcal{N}(\mu_r) \to 0, \quad \text{as } r \to 1.$$

Proof of Theorem 2.8. Since $BMOA$ is continuously contained in the Bloch spaces, it suffices to prove (i).

Suppose that ν is a vanishing Carleson measure. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Bloch functions with $\sup_{n\ge1} \|f_n\|_{\mathcal{B}} < \infty$ and such that $\{f_n\} \to 0$, uniformly on compact subsets of $\mathbb{D}$. We have to prove that $I_\mu(f_n) \to 0$ in $BMOA$.

The condition $\sup_{n\ge1} \|f_n\|_{\mathcal{B}} < \infty$ implies that there exists a positive constant M such that

$$|f_n(z)| \le M \log \frac{2}{1-|z|}, \quad z \in \mathbb{D}, \quad n \ge 1. \tag{2.11}$$

Recall that for $0 < r < 1$, ν_r is the measure defined by

$$d\nu_r(t) = \chi_{\{r<t<1\}} \, d\nu(t).$$

Since ν is a vanishing Carleson measure, we have that $\mathcal{N}(\nu_r) \to 0$, as $r \to 1$, or, equivalently,

$$\mathcal{E}(\nu_r) \to 0, \quad \text{as } t \to 1. \tag{2.12}$$

Take $g \in H^1$ and $r \in [0, 1)$. Using (2.11) we have

$$\begin{aligned}
\int_{[0,1)} |f_n(t)||g(t)| \, d\mu(t) &= \int_{[0,r)} |f_n(t)||g(t)| \, d\mu(t) + \int_{[r,1)} |f_n(t)||g(t)| \, d\mu(t) \\
&\le \int_{[0,r)} |f_n(t)||g(t)| \, d\mu(t) + M \int_{[r,1)} \log \frac{2}{1-t} |g(t)| \, d\mu(t) \\
&= \int_{[0,r)} |f_n(t)||g(t)| \, d\mu(t) + M \int_{[0,1)} |g(t)| \, d\nu_r(t) \\
&\le \int_{[0,r)} |f_n(t)||g(t)| \, d\mu(t) + M\mathcal{E}(\nu_r)\|g\|_{H^1}.
\end{aligned}$$

Using (2.12) and the fact that $\{f_n\} \to 0$, uniformly on compact subsets of $\mathbb{D}$, it follows that

$$\lim_{n\to\infty} \int_{[0,1)} |f_n(t)||g(t)|\, d\mu(t) = 0, \quad \text{for all } g \in H^1.$$

Bearing in mind (2.4), this yields

$$\lim_{n\to\infty} \left(\lim_{r\to 1} \left| \int_0^{2\pi} I_\mu(f_n)(re^{i\theta}) \overline{g(e^{i\theta})}\, d\theta \right| \right) = 0, \quad \text{for all } g \in H^1.$$

By the duality relation $(H^1)^\star = BMOA$, this is equivalent to saying that $I_\mu(f_n) \to 0$ in $BMOA$. □

3. The operator $\mathcal{H}_\mu$ acting on Q_s spaces and Besov spaces

If $0 \le s < \infty$, we say that $f \in Q_s$ if f is analytic in $\mathbb{D}$ and

$$\|f\|_{Q_s} \stackrel{\text{def}}{=} \left(|f(0)|^2 + \rho_{Q_s}(f)^2\right)^{1/2} < \infty,$$

where

$$\rho_{Q_s}(f) \stackrel{\text{def}}{=} \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g(z,a)^s\, dA(z) \right)^{1/2}.$$

Here, $g(z,a)$ is the Green's function in $\mathbb{D}$, given by $g(z,a) = \log\left|\frac{1-\bar{a}z}{z-a}\right|$, while $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized area measure on $\mathbb{D}$. All Q_s spaces $(0 \le s < \infty)$ are conformally invariant with respect to the semi-norm ρ_{Q_s} (see e.g., [31, p. @1] or [15, p. 47]).

These spaces were introduced by Aulaskari and Lappan in [6] while looking for new characterizations of Bloch functions. They proved that for $s > 1$, Q_s is the Bloch space. Using one of the many characterizations of the space $BMOA$ (see, e. g., [9, Theorem 5] or [21, Theorem 6. 2]) we see that $Q_1 = BMOA$. In the limit case $s = 0$, Q_s is the classical Dirichlet space $\mathcal{D}$ of those analytic functions f in $\mathbb{D}$ satisfying $\int_{\mathbb{D}} |f'(z)|^2\, dA(z) < \infty$.

It is well known that $\mathcal{D} \subset VMOA$. Aulaskari, Xiao and Zhao proved in [8] that

$$\mathcal{D} \subsetneq Q_{s_1} \subsetneq Q_{s_2} \subsetneq BMOA, \qquad 0 < s_1 < s_2 < 1.$$

We mention the book [31] as an excelent reference for the theory of Q_s-spaces.

It is well known that the function $F(z) = \log\frac{2}{1-z}$ belong to Q_s, for all $s > 0$, (in fact, it is proved in [7] that the univalent functions in all Q_s-spaces $(0 < s < \infty)$ are the same). Using this we easily see that Theorem 2.1 and Theorem 2.4 can be improved as follows.

Theorem 3.1. *Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent:*

(i) $\int_{[0,1)} \log\frac{2}{1-t}\, d\mu(t) < \infty$.

(ii) *For any given $s \in (0,\infty)$ and any $f \in Q_s$, the integral in (1.1) converges for all $z \in \mathbb{D}$ and the resulting function $I_\mu(f)$ is analytic in $\mathbb{D}$.*

We remark that condition (ii) with $s \geq 1$ includes the points (ii) and (iii) of Theorem 2.1.

Theorem 3.2. *Let μ be a positive Borel measure on $[0,1)$ with $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$. Then the following two conditions are equivalent:*

(i) *The measure ν defined by $d\nu(t) = \log \frac{2}{1-t}\, d\mu(t)$ is a Carleson measure.*

(ii) *For any given $s \in (0,\infty)$, the operator I_μ is bounded from Q_s into $BMOA$.*

We remark that (ii) with $s > 1$ reduces to condition (ii) of Theorem 2.2, while (ii) with $s = 1$ reduces to condition (iii) of Theorem 2.2.

These results cannot be extended to the limit case $s = 0$. Indeed, the function $F(z) = \log \frac{2}{1-z}$ does not belong to the Dirichlet space $\mathcal{D}$.

The Dirichlet space is one among the analytic Besov spaces.

For $1 < p < \infty$, the *analytic Besov space* B^p is defined as the set of all functions f analytic in $\mathbb{D}$ such that

$$\|f\|_{B^p} \stackrel{\text{def}}{=} (|f(0)|^p + \rho_p(f)^p)^{1/p} < \infty,$$

where

$$\rho_p(f) = \left(\int_{\mathbb{D}} (1-|z|^2)^{p-2} |f'(z)|^p \, dA(z) \right)^{1/p}.$$

All B^p spaces $(1 < p < \infty)$ are conformally invariant with respect to the semi-norm ρ_p (see [5, p. 112] or [15, p. 46]). We have that $\mathcal{D} = B^2$. A lot of information on Besov spaces can be found in [5, 15, 23, 33, 34]. Let us recall that

$$B^p \subsetneq B^q \subsetneq VMOA, \quad 1 < p < q < \infty.$$

From now on, if $1 < p < \infty$ we let p' denote the exponent conjugate to p, that is, p' is defined by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in B^p$ $(1 < p < \infty)$ then, see [23] or [33],

$$|f(z)| = \mathrm{o}\left(\left(\log \frac{1}{1-|z|} \right)^{1/p'} \right), \quad \text{as } |z| \to 1, \tag{3.1}$$

and there exists a positive constant $C >, 0$ such that

$$|f(z)| \leq C \|f\|_{B^p} \left(\log \frac{2}{1-|z|} \right)^{1/p'}, \quad z \in \mathbb{D}, \quad f \in B^p. \tag{3.2}$$

Clearly, (3.1) or (3.2) imply that the function $F(z) = \log \frac{2}{1-z}$ does not belong to B^p $(1 < p < \infty)$, a fact that we have already mentioned for $p = 2$. Our substitutes of Theorem 2.1 and Theorem 2.2 for Besov spaces are the following.

Theorem 3.3. *Let $1 < p < \infty$ and let μ be a positive Borel measure on $[0,1)$. We have:*

(i) *If* $\int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{1/p'} d\mu(t) < \infty$*, then for any given* $f \in B^p$*, the integral in (1.1) converges for all* $z \in \mathbb{D}$ *and the resulting function* $I_\mu(f)$ *is analytic in* $\mathbb{D}$*.*

(ii) *If for any given* $f \in B^p$*, the integral in (1.1) converges for all* $z \in \mathbb{D}$ *and the resulting function* $I_\mu(f)$ *is analytic in* $\mathbb{D}$*, then* $\int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{\gamma} d\mu(t) < \infty$ *for all* $\gamma < \frac{1}{p'}$*.*

Theorem 3.4. *Suppose that* $1 < p < \infty$ *and let* μ *be a positive Borel measure on* $[0, 1)$*. Let* ν *be the measure defined by*

$$d\nu(t) = \left(\log \frac{2}{1-t}\right)^{1/p'} d\mu(t).$$

(i) *If* ν *is a Carleson measure, then the operator* I_μ *is bounded from* B^p *into* $BMOA$*.*

(ii) *If* ν *is a vanishing Carleson measure then the operator* I_μ *is compact from* B^p *into* $BMOA$*.*

These results follow using the growth condition (3.2), the fact that if $\gamma < \frac{1}{p'}$ then the function $f(z) = \left(\log \frac{2}{1-z}\right)^{\gamma}$ belongs to B^p (see [23, Theorem 1]), and with arguments similar to those used in the proofs of Theorem 2.1, Theorem 2.2, and Theorem 2.8. We omit the details.

Let us work next with the operator $\mathcal{H}_\mu$ directly. In order to study its action on the Besov spaces we need some results on the Taylor coefficients of functions in B^p. The following result was proved by Holland and Walsh in [23, Theorem 2].

Theorem B. (i) *Suppose that* $1 < p \leq 2$*. Then there exists a positive constant* C_p *such that if* $f \in B^p$ *and* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$ *then*

$$\sum_{k=1}^{\infty} k^{p-1} |a_k|^p \leq C_p \, \rho_p(f)^p.$$

(ii) *If* $2 \leq p < \infty$ *then there exists* $C_p > 0$ *such that if* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$ *with* $\sum_{k=1}^{\infty} k^{p-1}|a_k|^p < \infty$ *then* $f \in B^p$ *and*

$$\rho_p(f)^p \leq C_p \sum_{k=1}^{\infty} k^{p-1}|a_k|^p.$$

If $p \neq 2$ *the converses to (i) and (ii) are false.*

Theorem B is the analogue for Besov spaces of results of Hardy and Littlewood for Hardy spaces (Theorem 6.2 and Theorem 6.3 of [18]).

In spite of the fact that the converse to (ii) is not true, the membership of f in B^p $(p > 2)$ implies some summability conditions on the Taylor coefficients $\{a_k\}$ of f. Indeed, Pavlović has proved the following result in [28, Theorem 2.3].

Theorem C. *Suppose that $2 < p < \infty$. Then there exists a positive constant C_p such that if $f \in B^p$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$ then*

$$\sum_{k=1}^{\infty} k|a_k|^p \le C_p\, \rho_p(f)^p.$$

These results allow us to obtain conditions on μ which are sufficient to ensure that $\mathcal{H}_\mu$ is well defined on the Besov spaces.

Theorem 3.5. *Let μ be a finite positive Borel measure on $[0,1)$.*

(i) *If $1 < p \le 2$ and $\sum_{k=1}^{\infty} \frac{\mu_k^{p'}}{k} < \infty$, then the operator $\mathcal{H}_\mu$ is well defined in B^p.*

(ii) *If $2 < p < \infty$ and $\sum_{k=1}^{\infty} \frac{\mu_k^{p'}}{k^{p'/p}} < \infty$, then the operator $\mathcal{H}_\mu$ is well defined in B^p.*

Proof. Suppose that $1 < p < \infty$ and $f \in B^p$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$. Since the sequence of moments $\{\mu_n\}_{n=0}^{\infty}$ is clearly decreasing we have

$$\sum_{k=1}^{\infty} |\mu_{n+k}||a_k| \le \sum_{k=1}^{\infty} |\mu_k||a_k|, \quad \text{for all } n \ge 0.$$

Consequently, we have:

(i) If $1 < p \le 2$ and $f \in B^p$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$, then

$$\sum_{k=1}^{\infty} |\mu_{n+k} a_k| \le \sum_{k=1}^{\infty} |\mu_k||a_k| = \sum_{k=1}^{\infty} k^{1-\frac{1}{p}} |a_k| \frac{\mu_k}{k^{1/p'}}, \quad n \ge 0.$$

Then using Hölder inequality and Theorem B (i), we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} |\mu_{n+k} a_k| &\le \left(\sum_{k=1}^{\infty} k^{p-1}|a_k|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} \frac{|\mu_k|^{p'}}{k}\right)^{1/p'} \\ &\le C\,\rho_p(f) \left(\sum_{k=1}^{\infty} \frac{|\mu_k|^{p'}}{k}\right)^{1/p'}, \quad n \ge 0.\end{aligned}$$

Then it is clear that the condition $\sum_{k=1}^{\infty} \frac{|\mu_k|^{p'}}{k} < \infty$ implies that the power series appearing in the definition of $\mathcal{H}_\mu(f)$ defines an analytic function in $\mathbb{D}$.

(ii) If $2 < p < \infty$ and $f \in B^p$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$, then

$$\sum_{k=1}^{\infty} |\mu_{n+k} a_k| \le \sum_{k=1}^{\infty} |\mu_k||a_k| = \sum_{k=1}^{\infty} k^{\frac{1}{p}} |a_k| \frac{\mu_k}{k^{1/p}}, \quad n \ge 0.$$

Then using Hölder inequality and Theorem B (ii), we obtain

$$\sum_{k=1}^{\infty}|\mu_{n+k}a_k| \le \left(\sum_{k=1}^{\infty}k|a_k|^p\right)^{1/p}\left(\sum_{k=1}^{\infty}\frac{|\mu_k|^{p'}}{k^{p'/p}}\right)^{1/p'}$$
$$\le C\,\rho_p(f)\left(\sum_{k=1}^{\infty}\frac{|\mu_k|^{p'}}{k^{p'/p}}\right)^{1/p'}, \quad n\ge 0.$$

Then we see that the condition $\sum_{k=1}^{\infty}\frac{|\mu_k|^{p'}}{k^{p'/p}} < \infty$ implies that the power series appearing in the definition of $\mathcal{H}_\mu(f)$ defines an analytic function in $\mathbb{D}$.

□

Let us turn to study when is the operator $\mathcal{H}_\mu$ bounded from B^p into itself. Let us mention that Bao and Wulan [10] considered an operator which is closely related to the operator $\mathcal{H}_\mu$ acting on the Dirichlet spaces $\mathcal{D}_\alpha$ ($\alpha\in\mathbb{R}$) which are defined as follows:

For $\alpha\in\mathbb{R}$, the space $\mathcal{D}_\alpha$ consists of those functions $f(z)=\sum_{n=0}^{\infty}a_n z^n$ analytic in $\mathbb{D}$ for which

$$\|f\|_{\mathcal{D}_\alpha}\stackrel{\text{def}}{=}\left(\sum_{n=0}^{\infty}(n+1)^{1-\alpha}\,|a_n|^2\right)^{1/2} < \infty.$$

Let us remark that $\mathcal{D}_0$ is the Dirichlet spaces $\mathcal{D}=B^2$, while $\mathcal{D}_1=H^2$.

Bao and Wulan proved that if μ is a positive Borel measure on $[0,1)$ and $0<\alpha<2$, then the operator $\mathcal{H}_\mu$ is bounded from $\mathcal{D}_\alpha$ into itself if and only if μ is a Carleson measure. Let us remark that this does not include the case $\alpha=0$. In fact, the following results are proved in [10].

Theorem D.

(i) *There exists a positive Borel measure μ on $[0,1)$ which is a Carleson measure but such that $\mathcal{H}_\mu(B^2)\not\subset B^2$.*

(ii) *Let μ be a positive Borel measure on $[0,1)$ such that the operator $\mathcal{H}_\mu$ is a bounded operator from B^2 into itself. Then μ is a Carleson measure.*

We can improve these results and, even more, we shall obtain extensions of these improvements to all B^p spaces ($1<p<\infty$). More precisely we are going to prove the following results.

Theorem 3.6. *Suppose that $1<p<\infty$ and $0<\beta\le\frac{1}{p}$. Then there exists a positive Borel measure μ on $[0,1)$ which is a β-logarithmic 1-Carleson measure but such that the operator $\mathcal{H}_\mu$ does not apply B^p into itself.*

Next we prove that μ being a β-logarithmic 1-Carleson measure for a certain β is a necessary condition for $\mathcal{H}_\mu$ being a bounded operator from B^p into itself.

Theorem 3.7. *Suppose that $1<p<\infty$ and let μ be a positive Borel measure on $[0,1)$ such that the operator $\mathcal{H}_\mu$ is bounded from B^p to itself. Then μ is a γ-logarithmic 1-Carleson measure for any $\gamma<1-\frac{1}{p}$.*

Finally, we obtain a sufficient condition for the boundedness of $\mathcal{H}_\mu$ from B^p into itself.

Theorem 3.8. *Suppose that $1 < p < \infty$, $\gamma > 1$, and let μ be a positive Borel measure on $[0, 1)$ which is a γ-logarithmic 1-Carleson measure. Then the operator $\mathcal{H}_\mu$ is a bounded operator from B^p into itself.*

We shall need a number of results on Besov spaces, as well as some lemmas, to prove these three theorems. First of all we notice that the Besov spaces can be characterized in terms of "dyadic blocks". In order to state this in a precise way we need to introduce some notation.

For a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $\mathbb{D}$, define the polynomials $\Delta_j f$ as follows:

$$\Delta_j f(z) = \sum_{k=2^j}^{2^{j+1}-1} a_k z^k, \quad \text{for } j \geq 1,$$
$$\Delta_0 f(z) = a_0 + a_1 z.$$

Mateljević and Pavlović proved in [25, Theorem 2. 1] (see also [27, Theorem C]) the following result.

Theorem E. *Let $1 < p < \infty$ and $\alpha > -1$. For a function f analytic in $\mathbb{D}$ we define*

$$Q_1(f) \stackrel{\text{def}}{=} \int_{\mathbb{D}} |f(z)|^p (1 - |z|)^\alpha \, dA(z), \quad Q_2(f) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \|\Delta_n f\|_{H^p}^p.$$

Then, $Q_1(f) \asymp Q_2(f)$.

Theorem E readily implies the following result.

Corollary 3.9. *Suppose that $1 < p < \infty$ and f is an analytic function in $\mathbb{D}$. Then*

$$f \in B^p \;\Leftrightarrow\; \sum_{n=0}^{\infty} 2^{-n(p-1)} \|\Delta_n f'\|_{H^p}^p < \infty.$$

Furthermore,

$$\rho_p(f)^p \asymp \sum_{n=0}^{\infty} 2^{-n(p-1)} \|\Delta_n f'\|_{H^p}^p.$$

Using Corollary 3.9 we can prove that the converses of (i) and (ii) in Theorem B hold if the sequence of Taylor coefficients $\{a_n\}$ decreases to 0. This is the analogue for Besov spaces of the result proved in [22] by Hardy and Littlewood for Hardy spaces (see also [27], [26, 7. 5. 9] and [35, Chapter XII, Lemma 6. 6]).

Theorem 3.10. *Suppose that $1 < p < \infty$ and let $\{a_n\}_{n=0}^{\infty}$ be a decreasing sequence of non-negative numbers with $\{a_n\} \to 0$, as $n \to \infty$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. Then*

$$f \in B^p \;\Leftrightarrow\; \sum_{n=1}^{\infty} n^{p-1} a_n^p < \infty.$$

Furthermore, $\rho_p(f)^p \asymp \sum_{n=1}^{\infty} n^{p-1} a_n^p$.

Proof. For every n, we have

$$z\left(\Delta_n f'\right)(z) = \sum_{k=2^n+1}^{2^{n+1}} k a_k z^k.$$

Since the sequence $\lambda = \{k\}_{k=0}^{\infty}$ is an increasing sequence of non-negative numbers, using Lemma A of [27] we see that

$$\|z\left(\Delta_n f'\right)\|_{H^p}^p \asymp 2^{np}\|\Delta_n f\|_{H^p}^p. \tag{3.3}$$

Now, set $h(z) = \sum_{n=0}^{\infty} z^n$ $(z \in \mathbb{D})$. Since the sequence $\tilde{\lambda} = \{a_n\}_{n=0}^{\infty}$ is a decreasing sequence of non-negative numbers, using the second part of Lemma A of [27], we see that

$$a_{2^n}^p\|\Delta_n h\|_{H^p}^p \lesssim \|\Delta_n f\|_{H^p}^p \lesssim a_{2^{n-1}}^p\|\Delta_n h\|_{H^p}^p. \tag{3.4}$$

Notice that $h(z) = \frac{1}{1-z}$ $(z \in \mathbb{D})$. Then it is well known that $M_p(r,h) \asymp (1-r)^{\frac{1}{p}-1}$ (recall that $1 < p < \infty$). Following the notation of [25], this can be written as $h \in H\left(p, \infty, 1-\frac{1}{p}\right)$. Then using Theorem 2. 1 of [25] (see also [26, p. 120]), we deduce that $\|\Delta_n\|_{H^p}^p \asymp 2^{n(p-1)}$. Using this and (3.4), it follows that

$$2^{n(p-1)}a_{2^n}^p \lesssim \|\Delta_n f\|_{H^p}^p \lesssim 2^{n(p-1)}a_{2^{n-1}}^p. \tag{3.5}$$

Using Corollary 3.9, (3.3), and (3.5), we see that

$$\rho_p(f)^p \asymp \sum_{n=0}^{\infty} 2^{-n(p-1)}\|z\,\Delta_n f'\|_{H^p}^p \asymp \sum_{n=0}^{\infty} 2^n\|\Delta_n f\|_{H^p}^p \asymp \sum_{n=0}^{\infty} 2^{np}a_{2^n}^p.$$

Now, the fact that $\{a_n\}$ is decreasing implies that $\sum_{n=0}^{\infty} 2^{np}a_{2^n}^p \asymp \sum_{n=1}^{\infty} n^{p-1}a_n^p$ and, then it follows that $\rho_p(f)^p \asymp \sum_{n=1}^{\infty} n^{p-1}a_n^p$. □

Remark 3.11. If f is an analytic function in $\mathbb{D}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, and $1 < p < \infty$ then any of the two conditions $f \in B^p$ and $\sum_{n=1}^{\infty} n^{p-1}|a_n|^p < \infty$ implies that $\{a_n\} \to 0$. Consequently, the condition $\{a_n\} \to 0$ can be omitted in the hypotheses of Theorem 3.10.

Suppose that $\beta \geq 0$, $s \geq 1$, $1 < p < \infty$, and μ is a positive Borel measure on $[0,1)$ which is a β-logarithmic s-Carleson measure. Using Lemma 2.7 and Theorem 3.5, it follows that $\mathcal{H}_\mu$ is well defined on B^p. Also, it is easy to see that $\int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{1/p'} d\mu(t) < \infty$, a fact that, using Theorem 3.3 (i), shows that I_μ is also well defined in B^p. Using then standard arguments it follows that I_μ and $\mathcal{H}_\mu$ coincide in B^p. Let us state this as a lemma.

Lemma 3.12. *Suppose that $\beta \geq 0$, $s \geq 1$, $1 < p < \infty$, and μ is a positive Borel measure on $[0,1)$ which is a β-logarithmic s-Carleson measure. Then the operators $\mathcal{H}_\mu$ and I_μ are well defined in B^p and $\mathcal{H}_\mu(f) = I_\mu(f)$, for all $f \in B^p$.*

Proof of Theorem 3.6. Let μ be the Borel measure on $[0,1)$ defined by

$$d\mu(t) = \left(\log \frac{2}{1-t}\right)^{-\beta} dt.$$

Since the function $x \mapsto \left(\log \frac{2}{1-x}\right)^{-\beta}$ is decreasing in $[0,1)$, we have

$$\mu\left([t,1)\right) = \int_t^1 \left(\log \frac{2}{1-x}\right)^{-\beta} dx \le (1-t)\left(\log \frac{2}{1-t}\right)^{-\beta}, \quad 0 \le t < 1.$$

Hence, μ is a β-logarithmic 1-Carleson measure. Then, taking $\alpha = 0$ in Lemma 2.7, we see that

$$\mu_k = \mathrm{O}\left(\frac{1}{k(\log k)^\beta}\right).$$

On the other hand,

$$\mu_k \ge \int_0^{1-\frac{1}{k}} t^k \left(\log \frac{2}{1-t}\right)^{-\beta} dt \gtrsim \frac{1}{(\log k)^\beta} \int_0^{1-\frac{1}{k}} t^k \, dt \gtrsim \frac{1}{k(\log k)^\beta}.$$

Thus, we have seen that μ is a β-logarithmic 1-Carleson measure which satisfies

$$\mu_n \asymp \frac{1}{n(\log n)^\beta}. \tag{3.6}$$

Take $p \in (1,\infty)$ and $\alpha > \frac{1}{p}$ and set

$$a_n = \frac{1}{(n+1)\left(\log(n+2)\right)^\alpha}, \quad n = 0,\, 1,\, 2,\, \ldots,$$

and

$$g(z) = \sum_{n=0}^\infty a_n z^n, \quad z \in \mathbb{D}.$$

Notice that $\{a_n\} \downarrow 0$ and that $\sum_{n=0}^\infty n^{p-1}|a_n|^p < \infty$. Hence, $g \in B^p$.

We are going to prove that $\mathcal{H}_\mu(g) \notin B^p$. This implies that $\mathcal{H}_\mu(B^p) \not\subset B^p$, proving the theorem.

We have $\mathcal{H}_\mu(g)(z) = \sum_{n=0}^\infty \left(\sum_{k=0}^\infty \mu_{n+k} a_k\right) z^n$. Notice that $a_k \ge 0$ for all k and that the sequence of moments $\{\mu_n\}$ is a decreasing sequence of non-negative numbers. Then it follows that the sequence $\{\sum_{k=0}^\infty \mu_{n+k} a_k\}_{n=0}^\infty$ of the Taylor coefficients of $\mathcal{H}_\mu(g)$ is decreasing. Consequently, we have that

$$\mathcal{H}_\mu(g) \in B^p \Leftrightarrow \sum_{n=1}^\infty n^{p-1} \left|\sum_{k=0}^\infty \mu_{n+k} a_k\right|^p < \infty. \tag{3.7}$$

Using the definition of the sequence $\{a_k\}$, (3.6) and the simple inequalities $\frac{k}{n+k} \ge \frac{1}{n+1}$ and $\log(n+k) \le (\log n)(\log k)$ which hold whenever $k,n \ge 10$, say, we obtain

$$\begin{aligned}
&\sum_{n=1}^\infty n^{p-1} \left|\sum_{k=0}^\infty \mu_{n+k} a_k\right|^p \ge \sum_{n=10}^\infty n^{p-1} \left|\sum_{k=10}^\infty \mu_{n+k} a_k\right|^p \\
&\gtrsim \sum_{n=10}^\infty n^{p-1} \left(\sum_{k=10}^\infty \left[\frac{1}{(n+k)\left(\log(n+k)\right)^\beta} \frac{1}{k\left(\log k\right)^\alpha}\right]\right)^p \\
&\gtrsim \sum_{n=10}^\infty \frac{1}{n(\log n)^{p\beta}} \left(\sum_{k=10}^\infty \frac{1}{k^2\, \left(\log k\right)^{\alpha+\beta}}\right)^p = \infty.
\end{aligned}$$

Bearing in mind (3.7), this implies that $H_\mu(g) \notin B^p$ as desired. □

Proof of Theorem 3.7. Suppose that $1 < p < \infty$ and $\gamma < 1 - \frac{1}{p}$. Let μ be a positive Borel measure on $[0,1)$ such that the operator $\mathcal{H}_\mu$ is a bounded operator from B^p into itself. Set $\alpha = 1 - \gamma$,

$$a_k = \frac{1}{k(\log k)^\alpha}, \quad k \geq 2,$$

and

$$f(z) = \sum_{k=2}^{\infty} a_k\, z^k, \quad z \in \mathbb{D}.$$

Since $\alpha > \frac{1}{p}$, using Theorem 3.10 we see that $f \in B^p$. By our assumption $H_\mu(f) \in B^P$, that is, $\|\mathcal{H}_\mu(f)\|_{B^p} < \infty$. We have

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=2}^{\infty} \mu_{n+k} a_k \right) z^n.$$

Since $a_k \geq 0$ for all k and $\{\mu_n\}$ is a decreasing sequence of non-negative numbers, it follows that the sequence $\{\sum_{k=2}^{\infty} \mu_{n+k} a_k\}_{n=0}^{\infty}$ is a decreasing sequence of non-negative numbers. Then, using Theorem 3.10 we obtain

$$\begin{aligned}
\|\mathcal{H}_\mu(f)\|_{B^p}^p &\gtrsim \sum_{n=1}^{\infty} n^{p-1} \left(\sum_{k=2}^{\infty} \mu_{n+k} a_k \right)^p \\
&\gtrsim \sum_{n=1}^{\infty} n^{p-1} \left(\sum_{k=2}^{\infty} \frac{1}{k(\log k)^\alpha} \int_{[0,1)} x^{n+k}\, d\mu(x) \right)^p \\
&\geq \sum_{n=1}^{\infty} n^{p-1} \left(\sum_{k=2}^{\infty} \frac{1}{k(\log k)^\alpha} \int_{[t,1)} x^{n+k}\, d\mu(x) \right)^p \\
&\geq \sum_{n=1}^{\infty} n^{p-1} \left(\sum_{k=2}^{\infty} \frac{t^{n+k}}{k(\log k)^\alpha} \right)^p \mu\left([t,1)\right)^p \\
&= \sum_{n=1}^{\infty} n^{p-1} t^{np} \left(\sum_{k=2}^{\infty} \frac{t^k}{k(\log k)^\alpha} \right)^p \mu\left([t,1)\right)^p, \quad \text{for all } t \in (0,1).
\end{aligned}$$

Now, it is well known that $\sum_{k=2}^{\infty} \frac{t^k}{k(\log k)^\alpha} \asymp \left(\log \frac{2}{1-t}\right)^{1-\alpha} = \left(\log \frac{2}{1-t}\right)^\gamma$ (see [35, Vol. I, p. 192]). Then it follows that

$$\begin{aligned}
\|\mathcal{H}_\mu(f)\|_{B^p}^p &\gtrsim \left(\log \frac{2}{1-t}\right)^{\gamma p} \left(\sum_{n=1}^{\infty} n^{p-1} t^{np} \right) \mu\left([t,1)\right)^p \\
&\asymp \left(\log \frac{2}{1-t}\right)^{\gamma p} \frac{1}{(1-t)^p} \mu\left([t,1)\right)^p.
\end{aligned}$$

Since $\|\mathcal{H}_\mu(f)\|_{B^p} < \infty$, this shows that μ is a γ-logarithmic 1-Carleson measure. □

The following lemma will be used to prove Theorem 3.8. It is an adaptation of [20, Lemma 7] to our setting. The proof is very similar to that of the latter but we include it for the sake of completeness.

Lemma 3.13. *Let p, γ, and μ be as in Theorem 3.8. Then, there exists a constant $C = C(p,\gamma,\mu) > 0$ such that if $f \in B^p$, $g(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathcal{H}ol(\mathbb{D})$, and we set*

$$h(z) = \sum_{k=0}^{\infty} c_k \left(\int_0^1 t^{k+1} f(t)\, d\mu(t) \right) z^k,$$

then

$$\|\Delta_n h\|_{H^p} \le C \left(\int_0^1 t^{2^{n-2}+1} |f(t)|\, d\mu(t) \right) \|\Delta_n g\|_{H^p}, \quad n \ge 3.$$

Proof. For each $n = 1, 2, \dots,$ define

$$\Upsilon_n(s) = \int_0^1 t^{2^n s+1} f(t)\, d\mu(t), \quad s \ge 0.$$

Clearly, Υ_n is a $C^\infty(0,\infty)$-function and

$$|\Upsilon_n(s)| \le \int_0^1 t^{2^{n-2}+1} |f(t)|\, d\mu(t), \quad s \ge \frac{1}{2}. \tag{3.8}$$

Furthermore, since $\sup_{0<x<1} \left(\log \frac{1}{x}\right)^2 x^{1/2} = C(2) < \infty$, we have

$$\begin{aligned} |\Upsilon_n''(s)| &\le \int_0^1 \left[\left(\log \frac{1}{t^{2^n}} \right)^2 t^{2^{n-1}} \right] t^{2^n s+1-2^{n-1}} |f(t)|\, d\mu(t) \\ &\le C(2) \int_0^1 t^{2^n s+1-2^{n-1}} |f(t)|\, d\mu(t) \le C(2) \int_0^1 t^{2^{n-2}+1} |f(t)|\, d\mu(t), \quad s \ge \tfrac{3}{4}. \end{aligned} \tag{3.9}$$

Then, using (3.8) and (3.9), for each $n = 1, 2, \dots,$ we can take a function $\Phi_n \in C^\infty(\mathbb{R})$ with $\mathrm{supp}(\Phi_n) \in \left(\frac{3}{4}, 4\right)$, and such that

$$\Phi_n(s) = \Upsilon_n(s), \quad s \in [1,2],$$

and

$$A_{\Phi_n} = \max_{s\in\mathbb{R}} |\Phi_n(s)| + \max_{s\in\mathbb{R}} |\Phi_n''(s)| \le C \int_0^1 t^{2^{n-2}+1} |f(t)|\, d\mu(t).$$

Following the notation used in [20, p. 236], we can then write

$$\begin{aligned} \Delta_n h(z) &= \sum_{k=2^n}^{2^{n+1}-1} c_k \left(\int_0^1 t^{k+1} f(t)\, d\mu(t) \right) z^k \\ &= \sum_{k=2^n}^{2^{n+1}-1} c_k \Phi_n\left(\frac{k}{2^n} \right) z^k = W_{2^n}^{\Phi_n} * \Delta_n g(z). \end{aligned}$$

So by using part (iii) of Theorem B of [20], we have

$$\|\Delta_n h\|_{H^p} = \|W_{2^n}^{\Phi_n} * \Delta_n g\|_{H^p} \le C_p A_{\Phi_n} \|\Delta_n g\|_{H^p}$$
$$\le C \left(\int_0^1 t^{2^{n-2}+1} |f(t)|\, d\mu(t) \right) \|\Delta_n g\|_{H^p}.$$

□

Proof of Theorem 3.8. By the closed graph theorem it suffices to show that $\mathcal{H}_\mu(B^p) \subset B^p$.

Take $f \in B^p$. Since μ is a γ-logarithmic 1-Carleson measure, using Lemma 3.12 we see that

$$\mathcal{H}_\mu(f)(z) = I_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n f(t)\, d\mu(t) \right) z^n, \quad z \in \mathbb{D}.$$

Also, using Corollary 3.9, we see that

$$\mathcal{H}_\mu(f) \in B^p \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} 2^{-n(p-1)} \|\Delta_n \left(\mathcal{H}_\mu(f)' \right)\|_{H^p}^p < \infty. \tag{3.10}$$

Now, we have

$$\Delta_n \left(\mathcal{H}_\mu(f)' \right)(z) = \sum_{k=2^n}^{2^{n+1}-1} (k+1) \left(\int_{[0,1)} t^{k+1} f(t)\, d\mu(t) \right) z^k.$$

Using Lemma 3.13 we obtain that

$$\|\Delta_n \left(\mathcal{H}_\mu(f)' \right)\|_{H^p} \lesssim \left(\int_{[0,1)} t^{2^{n-2}+1} |f(t)|\, d\mu(t) \right) \|\Delta_n F\|_{H^p}$$

with $F(z) = \sum_{k=0}^{\infty} (k+1) z^k$ $(z \in \mathbb{D})$. Now, we have that $M_p(r, F) = \mathrm{O}\left(\frac{1}{(1-r)^{2-\frac{1}{p}}} \right)$ and then it follows that $\|\Delta_n F\|_{H^p} = \mathrm{O}\left(2^{n(2-\frac{1}{p})} \right)$ (see, e. g., [25]). Using this and the estimate $|f(t)| \lesssim \left(\log \frac{2}{1-t} \right)^{1/p'}$, we obtain

$$\|\Delta_n \left(\mathcal{H}_\mu(f)' \right)\|_{H^p} \lesssim 2^{n(2-\frac{1}{p})} \left(\int_{[0,1)} t^{2^{n-2}+1} \left(\log \frac{2}{1-t} \right)^{1/p'} d\mu(t) \right),$$

which using the fact that μ is a γ-logarithmic 1-Carleson measure and Lemma 2.7 implies

$$\|\Delta_n \left(\mathcal{H}_\mu(f)' \right)\|_{H^p} \lesssim 2^{n(2-\frac{1}{p})} 2^{-n} n^{\frac{1}{p'}-\gamma} = 2^{n/p'} n^{\frac{1}{p'}-\gamma}.$$

This, together with the fact that $\gamma > 1$, implies that

$$\sum_{n=1}^{\infty} 2^{-n(p-1)} \|\Delta_n \left(\mathcal{H}_\mu(f)' \right)\|_{H^p}^p \lesssim \sum_{n=1}^{\infty} 2^{-n(p-1)} 2^{np/p'} n^{p(1-\gamma)-1}$$
$$= \sum_{n=1}^{\infty} n^{p(1-\gamma)-1} < \infty.$$

Bearing in mind (3.10), this shows that $\mathcal{H}_\mu(f) \in B^p$ and finishes the proof. □

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