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On the possibility of quantitative definition of knowledge

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In this paper the possibility of quantitative definition of knowledge, information and uncertainty (ignorance) is investigated in such a way, that initial knowledge and useful information in sum would give the final knowledge.

Earlier these fundamental notions in the case of probabilistic uncertainty were considered by MacDonald (1954); Schouten (1955); Gamba (1954).

Here, the problem above mentioned is considered in the case of combined uncertainty: probabilistic and possibilistic.

Keywords: measure of information; entropies; possibility and probability measures; splitting procedure.

1. Introduction

In order to elaborate a theory of communication which could be useful for designing a great variety of real information system it is necessary to consider the superposition of two kind of uncertainty: probabilistic and possibilistic. In such case Zadeh (1968) and de Luca and Termini (1972) entropies are used. In Criado and Gachechiladze (1997) the genetic connection of these entropies with Shannon's entropy is considered.

The realization of this superposition in statistical distributions is achieved by the probability measure splitting procedure (Torralba et al. 2002).

2. Preliminary remarks

Let X be a set of events (x_1, x_2, \dots, x_m) having probabilities $\{p_i\}_{i=1, \dots, n}$ and $\mu : X \rightarrow [0, 1]$ some fuzzy set of X .

The occurrence of a single event in which two kinds of uncertainties are accumulated brings about an information, I , which depends on both variables p and

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μ :

$$\tilde{I} = I(\mu_{\tilde{X}}(x), p(x)) \quad (1)$$

Analogously, for dual fuzzy point:

$$\tilde{I}^D = I(\mu_{\tilde{X}^D}(x), p(x)) = I(1 - \mu_{\tilde{X}}(x), p(x)) \quad (2)$$

Below will be show that information supplied by non-fuzzy event x with probability $p(x)$, $I(p(x))$, is

$$I(p(x)) = I(\mu_{\tilde{X}}(x), p(x)) + I(\mu_{\tilde{X}^D}(x), p(x)), \quad (3)$$

In order to find the I function, its two obvious properties are considered.

- (1) First, consider two finite sets X and Y . Let \tilde{X} and \tilde{Y} be fuzzy sets of X and Y correspondingly with membership functions $\mu_{\tilde{X}}$ and $\mu_{\tilde{Y}}$. If for some x and y , $\mu_{\tilde{X}}(x) = \mu_{\tilde{Y}}(y) = \mu$, then the membership function at the point $(x, y) \in X \times Y$ has the same value (Gachechiladze 1988), and if non-fuzzy events x and y with probabilities p and q are independent, we will have the equality

$$I(\mu, pq) = I(\mu, p) + I(\mu, q) \quad (4)$$

for every $0 \leq p, q \leq 1, 0 \leq \mu \leq 1$.

- (2) Second, the information supplied by a fuzzy event having a certain probability measure p must be directly proportional to a membership function of this event. More exactly, the changing of the membership function by the factor λ entails an equal changing of the information supplied by this event. This lead to

$$I(\lambda\mu, p) = \lambda I(\mu, p) \quad (5)$$

for any $0 \leq p \leq 1, 0 \leq \mu \leq 1$ and non-negative number λ .

Theorem 1. Let $\tilde{I} = I(\mu, p)$ be a function satisfying conditions (4) and (5). Thus, \tilde{I} has the form

$$\tilde{I} = I(\mu, p) = -k\mu \log p, \quad (6)$$

where k is an arbitrary positive constant.

Proof. Relation (5) implies that

$$I(\mu, p) = \mu I(1, p), \quad (7)$$

and (4) establishes that, for particular value $\mu = 1$, one has

$$I(1, p, q) = I(1, p) + I(1, q). \quad (8)$$

Assuming that

$$I(1, p) = F(\log p), \quad (9)$$

from (8) we obtain the following relation

$$F(x + y) = F(x) + F(y), \quad (10)$$

where $x = \log p$, $y = \log q$.

The unique solution of the functional equation (10) is

$$F(x) = kx, \quad (11)$$

where k is an arbitrary constant. From (9) and (7) we have (6).

Let us now return to the set of fuzzy points $\widetilde{x}_1, \dots, \widetilde{x}_n$. We have established that the information supplied by fuzzy event \widetilde{x}_i is

$$\widetilde{I}_i = I(\mu_i, p_i) = -k\mu_i \log p_i \quad (12)$$

As \widetilde{x}_i 's probability measure is p_i , the information \widetilde{I}_i occurs with the same probability. The average of the information corresponding to the events $\widetilde{x}_1, \dots, \widetilde{x}_n$ will be

$$Z(\widetilde{X}) = \sum_{i=1}^n p_i \widetilde{I}_i = -k \sum_{i=1}^n \mu_i p_i \log p_i, \quad (13)$$

which is the Zadeh's entropy of fuzzy random events $\widetilde{x}_1, \dots, \widetilde{x}_n$ (Zadeh 1968).

Analogously,

$$Z(\widetilde{X}^D) = \sum_{i=1}^n p_i \widetilde{I}_i^D = -k \sum_{i=1}^n \mu_i^D p_i \log p_i. \quad (14)$$

It is evident that

$$Z(\widetilde{X}) + Z(\widetilde{X}^D) = H(X). \quad (15)$$

where $H(X)$ is the Shannon's entropy. \square

2.1. Splitting of Shannon's entropy.

Let us perform a splitting of the set X point by point. In this case Shannon's entropy of probability distribution $(\{p_i\}_1^n, H(\{p_i\}_1^n))$, turns into $H(\{\mu_i p_i\}_1^n, \{\mu_i^D p_i\}_1^n)$, where μ_i is the membership function of a fuzzy point $\widetilde{x}_i \in \widetilde{X}$ ($i = 1, \dots, n$) and \widetilde{X} is a fuzzy set of the universal set X . If we avail the branching property of function $H(\cdot)$ (Feinstein 1958) then

$$\begin{aligned} H(\{\mu_i p_i\}_1^n, \{\mu_i^D p_i\}_1^n) &= H(\mu_1 p_1, \mu_1^D p_1; \dots; \mu_n p_n, \mu_n^D p_n) = \\ &= Z(\widetilde{X}/\{p_i\}_1^n) + Z(\widetilde{X}^D/\{p_i\}_1^n) + L(\widetilde{X}/\{p_i\}_1^n) + L(\widetilde{X}^D/\{p_i\}_1^n), \end{aligned} \quad (16)$$

where¹

$$Z\left(\tilde{X}/\{p_i\}_1^n\right) = -\sum_{i=1}^n \mu_i p_i \log p_i \quad (17)$$

$$Z\left(\tilde{X}^D/\{p_i\}_1^n\right) = -\sum_{i=1}^n (1 - \mu_i) p_i \log p_i \quad (18)$$

and

$$Z\left(\tilde{X}/\{p_i\}_1^n\right) + Z\left(\tilde{X}^D/\{p_i\}_1^n\right) = H\left(\{p_i\}_1^n\right). \quad (19)$$

$Z(\tilde{X}/\{p_i\}_1^n)$ is Zadeh's entropy (Zadeh 1968), i.e. an entropy of fuzzy set \tilde{X} with respect to probability distribution $\{p_i\}_1^n$.

$$L\left(\tilde{X}/\{p_i\}_1^n\right) = -\sum_{i=1}^n p_i \mu_i \log \mu_i, \quad (20)$$

$$L\left(\tilde{X}^D/\{p_i\}_1^n\right) = -\sum_{i=1}^n (1 - \mu_i) p_i \log(1 - \mu_i). \quad (21)$$

Function $L(\tilde{X}/\{p_i\}_1^n)$ is actually a Kullback directed divergence $I(\{\mu_i p_i\}_1^n : \{p_i\}_1^n)$, (Kullback 1997) and

$$L\left(\tilde{X}/\{p_i\}_1^n\right) + L\left(\tilde{X}^D/\{p_i\}_1^n\right) = -\sum_{i=1}^n p_i (\mu_i \log \mu_i + (1 - \mu_i) \log(1 - \mu_i)) \quad (22)$$

is a weighted entropy of de Luca and Termini (1972).

We see that (16) is nothing else than Hirota's measure of uncertainty (Hirota 1982). Notice that if we avail the branching property in some other way then we can rewrite (16) in the following form:

$$\begin{aligned} H\left(\{\mu_i p_i\}_1^n, \{\mu_i^D p_i\}_1^n\right) &= H\left(\mu_1 p_1, \dots, \mu_n p_n; \mu_1^D p_1, \dots, \mu_n^D p_n\right) = \\ &= H\left(P(\tilde{X}), P(\tilde{X}^D)\right) + P(\tilde{X}) H\left(\frac{\{\mu_i p_i\}_1^n}{P(\tilde{X})}\right) + P(\tilde{X}^D) H\left(\frac{\{\mu_i^D p_i\}_1^n}{P(\tilde{X}^D)}\right), \end{aligned} \quad (23)$$

where

$$P(\tilde{X}) = \sum_{i=1}^n \mu_i p_i, \quad P(\tilde{X}^D) = \sum_{i=1}^n \mu_i^D p_i. \quad (24)$$

In addition functions $Z(\tilde{X}/\{p_i\}_1^n)$ and $L(\tilde{X}/\{p_i\}_1^n)$ are connected by Jumarie's entropy (Jumarie 1975):

$$E_J\left(\{\mu_i\}_1^n, \{p_i\}_1^n\right) = Z(\tilde{X}/\{p_i\}_1^n) + Z(\tilde{X}^D/\{p_i\}_1^n) + L(\tilde{X}/\{p_i\}_1^n) \quad (25)$$

¹ $\log(\cdot) \triangleq \log_2(\cdot)$

2.2. Zadeh's and De Luca and Termini's β -entropies

Formulas (17) and (19) represent particular cases of Zadeh's and De Luca and Termini's β -entropies respectively. The characteristic equations of $Z(\tilde{X}/\{p_i\}_1^n)$ and $L(\tilde{X}/\{p_i\}_1^n)$ are particular cases of a more general equation whose solutions are β -entropies; they are connected with the usual β -entropy of probability (sub)distribution as $(Z(\tilde{X}/\{p_i\}_1^n) + Z(\tilde{X}^D/\{p_i\}_1^n))$ and $(L(\tilde{X}/\{p_i\}_1^n) + L(\tilde{X}^D/\{p_i\}_1^n))$ with Shannon's entropy of split distribution (Klir and Mariano 1987). β -entropy of probability (sub)distribution $\{\mu_i p_i\}_1^n$, $\sum_{i=1}^n \mu_i p_i \leq 1$, is defined as

$$\begin{aligned} H^\beta(\{\mu_i p_i\}_1^n) &= - \sum_{i=1}^n (\mu_i p_i)^{\beta+1} \log(\mu_i p_i) = \\ &= Z^\beta(\tilde{X}/\{\mu_i p_i\}_1^n) + L^\beta(\tilde{X}/\{\mu_i p_i\}_1^n), \end{aligned} \quad (26)$$

where

$$Z^\beta(\tilde{X}/\{\mu_i p_i\}_1^n) = - \sum_{i=1}^n (\mu_i p_i)^{\beta+1} \log p_i \quad (27)$$

is the β -entropy of Zadeh and

$$L^\beta(\tilde{X}/\{\mu_i p_i\}_1^n) = - \sum_{i=1}^n (\mu_i p_i)^{\beta+1} \log \mu_i \quad (28)$$

is the weighted β -entropy of De Luca and Termini. Consequently,

$$Z(\tilde{X}/\{p_i\}_1^n) = Z^0(\tilde{X}/\{\mu_i p_i\}_1^n) \quad (29)$$

and

$$L(\tilde{X}/\{p_i\}_1^n) = L^0(\tilde{X}/\{\mu_i p_i\}_1^n) \quad (30)$$

Definitions analogous to (26)-(30) can be given for dual subsets.

2.3. Characterization of $Z^\beta(\tilde{X}/\{\mu_i p_i\}_1^n)$.

Let X be a set of events $\{x_1, \dots, x_n\}$ having probabilities $\{p_1, \dots, p_n\}$ and $\mu : X \rightarrow [0; 1]$ some fuzzy subset of X . Realization of one event in which two kinds of uncertainty are accumulated gives an information I_Z^β depending on two variables p and μ :

$$I_Z^\beta = I_Z^\beta(\mu_{\tilde{X}}(x), p(x)). \quad (31)$$

Similarly for a dual point:

$$I_Z^\beta = I_Z^\beta(\mu_{\tilde{X}^D}(x), p(x)). \quad (32)$$

Let us try to find a function I_Z^β that has following properties:

$$I_Z^\beta(\mu, pq) = q^\beta I_Z^\beta(\mu, p) + p^\beta I_Z^\beta(\mu, q) \quad (33)$$

and

$$I_Z^\beta(\lambda\mu, p) = \lambda^{\beta+1} I_Z^\beta(\mu, p), \quad (34)$$

for any $0 \leq p \leq 1$, $0 < \mu \leq 1$ and non-negative number λ .

Theorem 2. Let $I_Z^\beta(\mu, p)$ be a function satisfying conditions (33) and (34). Then

$$I_Z^\beta(\mu, p) = k\mu(\mu p)^\beta \log p, \quad (35)$$

where k is an arbitrary constant.

Proof. Condition (34) means that

$$I_Z^\beta(\mu, p) = \mu^{\beta+1} I_Z^\beta(1, p). \quad (36)$$

For $\mu = 1$ from (33) follows

$$I_Z^\beta(1, pq) = q^\beta I_Z^\beta(1, p) + p^\beta I_Z^\beta(1, q), \quad (37)$$

or

$$(pq)^{-\beta} I_Z^\beta(1, pq) = p^{-\beta} I_Z^\beta(1, p) + q^{-\beta} I_Z^\beta(1, q). \quad (38)$$

Assuming that

$$p^{-\beta} I_Z^\beta(1, p) = F(\log p).$$

From (38) one can obtain

$$F(x + y) = F(x) + F(y), \quad (39)$$

where $x = \log p$, $y = \log q$.

The unique continuous solution of this functional equation is

$$F(x) = k \log p,$$

where k is an arbitrary constant. So

$$I_Z^\beta(1, p) = p^\beta F(\log p) = kp^\beta \log p.$$

Finally

$$I_Z^\beta(\mu, p) = \mu^{\beta+1} I_Z^\beta(1, p) = k\mu^{\beta+1} p^\beta \log p.$$

formula (35) is proved. □

Let us return a set \tilde{X} which is regarded as the whole set of fuzzy points $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$. We have established that information gained from realization of fuzzy event \tilde{x}_i is

$$I_Z^\beta(\mu_i p_i) = k \mu_i (\mu_i p_i)^\beta \log p_i. \quad (40)$$

The mathematical expectation of this quantity with respect to probability distribution $\{p_i\}_1^n$ leads to (27).

It is worth to dwell on properties (33) and (34) when $\beta = 0$. In this case

$$I_Z^0(\mu, pq) = I_Z^0(\mu, p) + I_Z^0(\mu, q) \quad (41)$$

and

$$I_Z^0(\mu, p) = \mu I_Z^0(1, p). \quad (42)$$

We can consider $I_Z^0(1, p)$ as a measure of probabilistic uncertainty contained in the event x . When event x is split the probabilistic uncertainty becomes proportional to a membership function of fuzzy event \tilde{x} . $I_Z^0(1, p)$ as though it were split like a set (Klir and Mariano 1987) or a probability measure: $p(x) = \mu p(x) + (1 - \mu)p(x)$ (Torralba et al. 2002). (41) reflects additivity of an unsplit probability measure for independent events taking into account (42).

2.4. Characterization of $L^\beta(\tilde{X}/\{\mu_i p_i\}_1^n)$.

Characterization of this quantity is analogous to that of the above case, because comparing (27) and (28) one can see that they may be obtained one from another by replacement $p \leftrightarrow \mu$. Evidently an interpretation of postulates will be essentially different. As in the case of Zadeh's entropy one can assert that

$$I_L^\beta = I_L^\beta(\mu_{\tilde{X}}, p). \quad (43)$$

Suppose that the function I_L^β has the following properties:

$$I_L^\beta(\mu_1 \mu_2, p) = \mu_2^{\beta+1} I_L^\beta(\mu_1, p) + \mu_1^{\beta+1} I_L^\beta(\mu_2, p) \quad (44)$$

and

$$I_L^\beta(\mu, \gamma p) = \gamma^\beta I_L^\beta(\mu, p) \quad (45)$$

for any $0 \leq p \leq 1$, $0 < \mu \leq 1$ and non-negative γ .

Theorem 3. Let I_L^β be a function satisfying conditions (43) and (44), then

$$I_L^\beta(\mu, p) = k(\mu p)^\beta \mu \log \mu, \quad (46)$$

where k is an arbitrary constant.

Proof. (45) means that

$$I_L^\beta(\mu, p) = p^\beta I_L^\beta(\mu, 1). \quad (47)$$

for $p = 1$ from (44) one can obtain

$$I_L^\beta(\mu_1\mu_2, 1) = \mu_2^{\beta+1}I_L^\beta(\mu_1, 1) + \mu_1^{\beta+1}I_L^\beta(\mu_2, 1),$$

or

$$(\mu_1\mu_2)^{-\beta-1}I_L^\beta(\mu_1\mu_2, 1) = \mu_1^{-\beta-1}I_L^\beta(\mu_1, 1) + \mu_2^{-\beta-1}I_L^\beta(\mu_2, 1). \tag{48}$$

Assuming

$$\mu^{-\beta-1}I_L^\beta(\mu, 1) = F(\log \mu)$$

we come to (39) and consequently

$$I_L^\beta(\mu, 1) = k\mu^{\beta+1} \log \mu \tag{49}$$

Then, according to (47),

$$I_L^\beta(\mu, p) = p^\beta I_L^\beta(\mu, 1) = kp^\beta \mu^{\beta+1} \log \mu$$

(46) is obtained. □

Fitting with each fuzzy event the quantity (49) and considering a mathematical expectation with respect to probability distribution $\{p_i\}_1^n$, we obtain (28).

For $\beta = 0$, (44) and (45) gives:

$$I_L^0(\mu_1\mu_2, p) = \mu_2 I_L^0(\mu_1, p) + \mu_1 I_L^0(\mu_2, p) \tag{50}$$

and

$$I_L^0(\mu, p) = I_L^0(\mu, 1) = I_L(\mu). \tag{51}$$

$I_L(\mu)$ can be considered as a measure of possibilistic uncertainty contained in a fuzzy event \tilde{x} .

We see that it is independent from classical chance.

Relation (50) represents a law for calculation of possibilistic uncertainty in the case where the fuzzy event is considered as a result of two sequential splittings with membership functions μ_1 and μ_2 and reflect independence of final result from succession of splittings.

It seems natural that a measure of possibilistic uncertainty for fuzzy subset \tilde{X} may be represented by the following sum

$$L(\{\mu_i\}_1^n) = \sum_{i=1}^n I_L(\mu_i). \tag{52}$$

Notice that De Luca and Termini's entropy can be represented by this function as

$$E_{DT}(\{\mu_i\}_1^n, \{\mu_i^D\}_1^n) = L(\{\mu_i\}_1^n) + L(\{\mu_i^D\}_1^n). \tag{53}$$

Comparing properties (33) and (34); (41) and (42) with (44) and (45); (50) and (51) in spite of the complete symmetry of (27) and (28) relatively to $p \leftrightarrow \mu$, one can observe the difference in the structure of equations which in the presence of specific conditions probabilistic and possibilistic uncertainties superposition reflects the fundamental difference between classical chance and fuzziness.

3. Ignorance, knowledge and information

1) The combined uncertainty of measured quantity is defined by formula (16):

$$\begin{aligned} H(\tilde{X}) &= H\left(\{p_i \mu_i\}_1^n, \{p_i \mu_i^D\}_1^n\right) = - \sum_{x_i \in X} (\tilde{p}_i \log \tilde{p}_i + \tilde{p}_i^D \log \tilde{p}_i^D) = \\ &= H(\{p_i\}_1^n) + \sum_{x_i \in X} p_i (G(\mu_i) + G(\mu_i^D)) \end{aligned} \quad (54)$$

where

$$\begin{aligned} p_i &= \tilde{p}_i + \tilde{p}_i^D, & \tilde{p}_i &= \mu_i p_i, & \tilde{p}_i^D &= \mu_i^D p_i, \\ G(\mu_i) &= -\mu_i \log \mu_i, & G(\mu_i^D) &= -\mu_i^D \log \mu_i^D \end{aligned} \quad (55)$$

μ_i is the membership function of fuzzy subset \tilde{X} of universal set X , μ_i^D is the dual membership function.

2) Combined uncertainty of an experiment (valuation):

$$\begin{aligned} H(\tilde{X}') &= H\left(\{\nu_i q_i\}_1^n, \{\nu_i^D q_i\}_1^n\right) = - \sum_{x'_i \in X'} (\tilde{q}_i \log \tilde{q}_i + \tilde{q}_i^D \log \tilde{q}_i^D) = \\ &= H(\{q_i\}_1^n) + \sum_{x'_i \in X'} q_i (G(\nu_i) + G(\nu_i^D)), \end{aligned} \quad (56)$$

where

$$\begin{aligned} q_i &= \tilde{q}_i + \tilde{q}_i^D, & \tilde{q}_i &= \nu_i q_i, & \tilde{q}_i^D &= \nu_i^D q_i, \\ G(\nu_i) &= -\nu_i \log \nu_i, & G(\nu_i^D) &= -\nu_i^D \log \nu_i^D \end{aligned}$$

ν_i is the membership function of a fuzzy point $\tilde{x}'_i \in \tilde{X}'$ of universal set X' , ν_i^D is the dual membership function and $\{q_i\}_{i=1}^n$ are the probabilities of events $\{x'_i\}_{i=1}^n = \tilde{X}'$.

3) Joint combined uncertainty of a measured (valuated) quantity and an experi-

ment (valuation) (split entropy on the Cartesian product $X \times X'$):

$$\begin{aligned}
H(\tilde{X}, \tilde{X}^D, \tilde{X}', \tilde{X}'^D) &= \\
= H(\mu_1 \wedge \nu_1 p(x_1, x'_1), (\mu_1 \wedge \nu_1)^D p(x_1, x'_1); \dots; \mu_n \wedge \nu_n p(x_n, x'_n), (\mu_n \wedge \nu_n)^D p(x_n, x'_n)) &= \\
= Z\left(\frac{\tilde{X} \times \tilde{X}'}{\{p(x_i, x'_i)\}_{11}^{nn}}\right) + Z\left(\frac{(\tilde{X} \times \tilde{X}')^D}{\{p(x_i, x'_i)\}_{11}^{nn}}\right) + \\
+ L\left(\frac{\tilde{X} \times \tilde{X}'}{\{p(x_i, x'_i)\}_{11}^{nn}}\right) + L\left(\frac{(\tilde{X} \times \tilde{X}')^D}{\{p(x_i, x'_i)\}_{11}^{nn}}\right) \quad (57)
\end{aligned}$$

where

$$\begin{aligned}
Z\left(\frac{\tilde{X} \times \tilde{X}'}{\{p(x_i, x'_i)\}_{11}^{nn}}\right) &= - \sum_{x_i \in X} \sum_{x'_i \in X'} p(x_i, x'_i) ((\mu(x_i) \wedge \nu(x'_i)) \log p(x_i, x'_i)), \\
Z\left(\frac{(\tilde{X} \times \tilde{X}')^D}{\{p(x_i, x'_i)\}_{11}^{nn}}\right) &= - \sum_{x_i \in X} \sum_{x'_i \in X'} p(x_i, x'_i) ((\mu(x_i) \wedge \nu(x'_i))^D \log p(x_i, x'_i)) \\
L\left(\frac{\tilde{X} \times \tilde{X}'}{\{p(x_i, x'_i)\}_{11}^{nn}}\right) &= - \sum_{x_i \in X} \sum_{x'_i \in X'} p(x_i, x'_i) ((\mu(x_i) \wedge \nu(x'_i)) \log (\mu(x_i) \wedge \nu(x'_i))) \\
L\left(\frac{(\tilde{X} \times \tilde{X}')^D}{\{p(x_i, x'_i)\}_{11}^{nn}}\right) &= - \sum_{x_i \in X} \sum_{x'_i \in X'} p(x_i, x'_i) \left((\mu(x_i) \wedge \nu(x'_i))^D \log (\mu(x_i) \wedge \nu(x'_i))^D \right) \quad (58)
\end{aligned}$$

4) Average conditional combined uncertainty of an experiment when the measured quantity is given:

$$\begin{aligned}
H(\tilde{X}' | \tilde{X}) &= \sum_{x_i \in X} p_i H(\tilde{X}' | \tilde{x}_i) = \\
&= \left(Z\left(\frac{\tilde{X} \times \tilde{X}'}{\{p(x_i, x'_i)\}_{11}^{nn}}\right) - Z\left(\frac{\tilde{X}}{\{p_i\}_1^n}\right) \right) + \\
&\quad + \left(L\left(\frac{\tilde{X} \times \tilde{X}'}{\{p(x_i, x'_i)\}_{11}^{nn}}\right) - L\left(\frac{\tilde{X}}{\{p_i\}_1^n}\right) \right), \quad (59)
\end{aligned}$$

where

$$\begin{aligned}
Z\left(\frac{\tilde{X} \times \tilde{X}'}{\{p(x_i, x'_i)\}_{11}^{nn}}\right) - Z\left(\frac{\tilde{X}}{\{p_i\}_1^n}\right) &= \\
= \sum_{x_i \in X} \sum_{x'_i \in X'} p(x_i, x'_i) [I_Z(p(x_i, x'_i), \mu(x_i) \wedge \nu(x'_i)) - I_Z(p_i, \mu_i)] \\
L\left(\frac{\tilde{X} \times \tilde{X}'}{\{p(x_i, x'_i)\}_{11}^{nn}}\right) - L\left(\frac{\tilde{X}}{\{p_i\}_1^n}\right) &= \\
= \sum_{x_i \in X} \sum_{x'_i \in X'} p(x_i, x'_i) [I_L(\mu(x_i) \wedge \nu(x'_i)) - I_L(\mu(x_i))]^2
\end{aligned}$$

²In all expressions operation $\wedge \equiv \min$. It is necessary to note that instead of Cartesian product, corre-

5) Average conditional uncertainty of a measured quantity when the outcome of the experiment is given:

$$\begin{aligned}
H(\tilde{X}' | \tilde{x}) &= \sum_{x'_i \in X'} q_i H(\tilde{X} | \tilde{x}'_i) = \\
&= \left[Z \left((\tilde{X} \times \tilde{X}') / \{p(x_i, x'_i)\}_{11}^{nn} \right) - Z \left(\tilde{X}' / \{q_i\}_1^n \right) \right] + \\
&\quad + \left[L \left((\tilde{X} \times \tilde{X}') / \{p(x_i, x'_i)\}_{11}^{nn} \right) - L \left(\tilde{X}' / \{q_i\}_1^n \right) \right] = \\
&= \sum_{x_i \in X} \sum_{x'_i \in X'} p(x_i, x'_i) [I_Z(p(x_i, x'_i), \mu(x_i) \wedge \nu(x'_i)) - I_Z(q(x_i), \nu(x'_i)) + \\
&\quad + I_L(\mu(x_i) \wedge \nu(x'_i)) - I_L(\nu(x'_i))]. \quad (60)
\end{aligned}$$

For all these uncertainties the well known relation is valid:

$$H(\tilde{X}) + H(\tilde{X}') - H(\tilde{X}, \tilde{X}') = H(\tilde{X}) - H(\tilde{X} | \tilde{X}') = H(\tilde{X}') - H(\tilde{X}' | \tilde{X}). \quad (61)$$

Now, for values of measured quantity x'_i introduce the following notions:

- (1) Initial ignorance $J(\tilde{X}) = H(\tilde{X})$, which is equal to initial uncertainty of measured quantity.
- (2) Final ignorance $J(\tilde{X} | \tilde{X}') = H(\tilde{X} | \tilde{X}')$, which is equal to final uncertainty of measured quantity, when the outcome of an experiment is given.
- (3) Diminution of ignorance = increase of knowledge =

$$= -\Delta J(\tilde{X} | \tilde{X}') = +\Delta K(\tilde{X} | \tilde{X}') = J(\tilde{X}) - J(\tilde{X} | \tilde{X}') = I(\tilde{X}') - \Gamma(\tilde{X}' | \tilde{X}),$$

where $J(\tilde{X} | \tilde{X}') = H(\tilde{X} | \tilde{X}')$.

- (4) Full information $I(\tilde{X}') = H(\tilde{X}')$ is equal to full uncertainty of the experiment. Numerically quantity I is equal to the activity, expressed in binary choices, which is necessary for getting this information; in other words, it is equal to an effort necessary for accomplish the experiment. So the increase of knowledge is connected with some effort. ΔK can be considered as a measure of correlation between values of measured quantity and outcomes of an experiment. ΔK also can be considered as a rate of information transmission, when \tilde{X} and \tilde{X}' are transmitter and receiver correspondingly, or as measure of redundancy when \tilde{X} and \tilde{X}' are both sets of symbols of some fuzzy code.
- (5) Useless information $\Gamma(\tilde{X} | \tilde{X}') = H(\tilde{X} | \tilde{X}')$ is equal to the uncertainty of the outcome of an experiment, when the value of a measured quantity is given. This part Γ of full information I was called useless, because it gives nothing for useful information, i.e. for knowledge increasing.

sponding to simultaneous splitting, one can avail Cartesian product corresponding to sequential splitting. It is easy to obtain expressions (57)-(60) corresponding to sequential splitting. For example, (58) can be rewritten in the following form:

$$Z \left((\tilde{X} \times \tilde{X}') / \{p(x_i, x'_i)\}_{11}^{nn} \right) = - \sum_{x_i \in X} \sum_{x'_i \in X'} p(x_i, x'_i) (\mu(x_i) \nu(x'_i)) \log p(x_i, x'_i)$$

and so on.

Consider the case, when values of probability distribution are only 1 or 0. In this case

$$\begin{aligned}
H(\tilde{X}) &\rightarrow I_L(\mu(x)) + I_L(\mu^D(x)), \\
H(\tilde{X}') &\rightarrow I_L(\nu(x')) + I_L(\nu^D(x')), \\
H(\tilde{X} \times \tilde{X}') &\rightarrow I_L(\mu(x) \wedge \nu(x')) + I_L((\mu(x) \wedge \nu(x'))^D), \\
H(\tilde{X}' | \tilde{X}) &\rightarrow I_L(\mu(x) \wedge \nu(x')) - I_L(\mu(x)) \\
H(\tilde{X} | \tilde{X}') &\rightarrow I_L(\mu(x) \wedge \nu(x')) - I_L(\nu(x'))
\end{aligned} \tag{62}$$

In accordance with this formulae one can write:

- (1) Initial knowledge

$$J(\tilde{X}) \rightarrow I_L(\mu(x)) + I_L(\mu^D(x)),$$

- (2) Final ignorance

$$J(\tilde{X} | \tilde{X}') \rightarrow I_L(\mu(x) \wedge \nu(x')) - I_L(\nu(x'))$$

- (3) Diminution of ignorance (increase of knowledge)

$$\begin{aligned}
-\Delta J(\tilde{X}, \tilde{X}') &= +\Delta K(\tilde{X}, \tilde{X}') \rightarrow \\
&\rightarrow I_L(\mu(x)) + I_L(\mu^D(x)) + I_L(\nu(x')) - I_L(\mu(x) \wedge \nu(x')) = \\
&= I_L(\nu(x')) + I_L(\nu^D(x')) + I_L(\mu(x)) - I_L(\mu(x) \wedge \nu(x')). \tag{63}
\end{aligned}$$

- (4) Full information

$$I(\tilde{X}') \rightarrow I_L(\nu(x')) + I_L(\nu^D(x'))$$

- (5) Useless information

$$\Gamma(\tilde{X}' | \tilde{X}) \rightarrow I_L(\mu(x) \wedge \nu(x')) - I_L(\mu(x))$$

Notice, that the minimum of ignorance $J = 0$ corresponds to crisp (membership function = 1) and deterministic (probability values = 0 or 1) event. Maximum ignorance $J = \log(2n)$ is realized when fuzzy subset coincide with its dual, as it is clear from qualitative consideration, and all probabilities are equal.

4. Conclusions

In this work, the entropy is regarded as a measure of a quantity of information taking into account two kind of uncertainty: probabilistic and possibilistic. Some characterizations of this functional are studied and the relation Zadeh's entropy function and De Luca and Termini's weighted entropy function with Shannon's entropy (the only function that is generally accepted for a meaningful measure of

uncertainty) are shown. Finally, ignorance, knowledge and information provided by experiments is quantified by using previous results.

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