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A probabilistic extension to Conway's Game of Life.

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Abstract The "Game of life" model was created in 1970 by the mathematician John Horton Conway using cellular automata. Since then, different extensions of these cellular automata have been used in many applications.

In this work, we introduce probabilistic cellular automata which include non-deterministic rules for transitions between successive generations of the automaton together with probabilistic decisions about life and death of the cells in the next generation of the automaton. Different directions of the neighbours of each cell are treated with the possibility of applying distinct probabilities. This way, more realistic situations can be modelled and the obtained results are also non-deterministic.

In this paper, we include a brief state of the art, the description of the model and some examples obtained with an implementation of the model made in Java.

Keywords Probabilistic Cellular Automata · Game of Life

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1 Introduction

1.1 The origins

The "Game of life" [14] model was created in 1970 by the mathematician John Horton Conway, using cellular automata. The original model consists of an infinite orthogonal grid composed by square cells. Each cell can be dead or alive. Each cell is adjacent to eight other cells. The time is considered discrete. The minimum quantity of time is a step. The state of each cell on the grid only depends of the state of the eight adjacent cells in the previous step. The rules are:

- Any dead cell with exactly three live neighbours becomes a living cell in the next step. (Reproduction)
- Any living cell with two or three live neighbours remains alive in the next step. In other case, the cell becomes a dead cell. (Underpopulation when they are less than two and overpopulation when they are more than three).

Paul Rendell constructed a Turing Machine using the Game of Life [17]. Since the Game of Life can be constructed by a Turing Machine, the Game of Life and a Universal Turing Machine are theoretically equivalents. The problem of deciding if a given pattern can be achieved using the Game of Live starting with an initial pattern is, in this sense, equivalent to the halt problem in a Turing Machine, and therefore undecidable.

1.2 Extensions

Many different extensions of the cellular automata introduced by Conway have been developed. We are going to show a brief state of the art of these extensions.

A first criterion for classifying the extensions of the game of life is the kind of grid set instead of the grid of squares of the classic Game of Life. In this sense, in the literature we can find grids of triangles [7], pentangles and hexagons [8]. Another extension is the use of a three-dimensional grid instead of the classic two-dimensional original [6,2].

Another criterion is the changes in the parameters of the rules. In this sense, a notation used, for example in [9] is the following: $E_1, E_2, \ldots / F_1, F_2, \ldots$ where E_i are the number of living neighbours cells in which a living cell remains alive, and F_i are the numbers of living neighbours cells in which a dead cell becomes alive. For example, the classic Conway's Game of Life would be denoted by $2, 3/3$.

Another criterion is the possible change in the states. One example of this type is the Quantum Game of Life in [10].

More on the line of this paper, some fuzzy extensions have been considered. For example in [16] a fuzzy extension for one-dimension and two-dimensions automata are introduced and compared with other possible fuzzy logics. This

general consideration can be used in the Conway's Game of Life. Unlike the approach of this paper, this fuzzy extension can not assign different probabilities for the positions of the neighbours. Stochastic extensions are considered in [18], in which the authors include stochastic element (parametrized by a "temperature").

In the literature, the concept of Probabilistic Cellular Automata has been previously used, for example in [1,12].

A discussion of the new features of the method introduced in this paper has been included in 2.2.

1.3 Applications

Cellular automata have been used in many applications, such as car traffic control [13] or baggage traffic in an airport [4]. These extensions introduce ideas not only from cellular automata models but also from neural networks theory.

In [5] applications to earth-quakes cosmology, turbulence, biology and economics are mentioned. In [18], the authors include among their motivations, the categories: physical (microdynamics of a non equilibrium system), biological (reproduction and evolution), pattern formation and its relation to physical and biological formalisms.

Applications of probabilistic cellular automata in the scope of fluctuating lattice Boltzmann methods have been considered in previos works. For example, for simulating solid-fluid suspensions [15] where a Lattice Boltzman model for the fluid is used in the simulation. Also a brownian motion is included in the model adding fluctuating components to the fluid stress tensor. A statistical study of the fluctuating Lattice Boltzman equation can be found in [11].

Another application where a diffusive probabilistic celular automata is used can be seen in [12] where binary strings are classified in a non-deterministic way.

1.4 Overiew of the paper

Together with this section 1 (Introduction), a description of the model with some detailed examples of different situations in a step of a cell is shown in section 2. In section 3 some examples using the implementation in Java, developed by the authors, of a complete system are included. So me graphics for these examples are shown in this section. Finally, a section 4 of Conclusions and Future Related Work is included.

This paper is an extension of the oral presentation given in the international conference ESCO 2018 [3].

2 The PCAEGOL model

2.1 Description of the PCAEGOL model

In this work, we introduce a probabilistic cellular automata which is an extension of the Game of Live. It includes non-deterministic rules for transitions between successive generations of the automaton together with probabilistic decisions about life and death of the cells in next generation of the automaton. This way, more realistic situations can be modelled and the obtained results are also non-deterministic. We refer to this model as PCAEGOL, acronym for "Probabilistic Cellular Automata, Extension of the Game Of Life".

The basic idea of the PCAEGOL model is that the states of the neighbour cells of a given cell are not exactly known or even errors might occur in these states (with certain probability). Moreover, the resulting state of the cell is not deterministic and with certain probability a resultant living cell can die or a resultant dead cell can be considered live. In addition, the rules about when a cell becomes a live or a dead cell can be changed. The probabilities assigned to each neighbour cell is allowed to be different for each one. As far as the authors know, no other extension of the Conway's Game of Life considers this possibility.

In the next table we show a cell C0 with its eight neighbours: C1 to C8.

		C1 (i-1,j-1)	C2 (i-1,j)	C3 (i-1,j+1)		
		C4 (i,j-1)	C0 (i,j)	C5 (i,j+1)		
		C6 (i+1,j-1)	C7 (i+1,j)	C8 (i+1,j+1)		

We consider the vectors $pl =< pl0, pl1, pl2, pl3, pl4, pl5, pl6, pl7, pl8 >$ and $pd =< pd0, pd1, pd2, pd3, pd4, pd5, pd6, pd7, pd8 >$, where

pli = Probability of the cell Ci of being considered alive when it is really alive. $i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

pdi =Probability of the cell Ci of being considered alive when it is really death. $i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

PCAEGOL is an extension of the Game of life since we can consider $pl =< 1, 1, 1, 1, 1, 1, 1, 1, 1 >$ (by default) and $pd =< 0, 0, 0, 0, 0, 0, 0, 0, 0 >$ (by default).

For example, if $pl3 = 0.9$ then 90 % of the times cell C3 is considered a living cell when counting the live neighbours but 10 % of the times it is

considered a dead cell. That means that 10 % of the times an "error" occurs, and the initial system does not work as expected. On the contrary, pd0=0.05 means that if the result of applying the rules is that the central cell C0 is dead, 5 % of the times it is considered as a living cell.

Normally, *pli* will be a number close to 1 while *pdi* will be close to 0. When *pli* is not exactly 1 or *pdi* is not exactly 0 means that an "error" has occurred. These errors make the model non deterministic and can be used to model real situations in which one direction is more probable than others for living cells, for example, when simulating the growth of tissues.

Furthermore, an extension of the rules of the Game of Life has been considered as follows:

- MAXO = Maximum number of neighbouring living cells not to die of overpopulation.
- MINU = Minimum number of neighbouring living cells not to die of underpopulation.
- MAXB = Maximum number of neighbouring living cells to be born in the next generation.
- MINB = Minimum number of neighbouring living cells to be born in the next generation.

Parameters MAXO, MINU, MAXB and MINB correspond to
MINU, MINU+1, ..., MAXO / MINB, MINB+1, ..., MAXB

in the traditional notation.

Parameters *pl*, *pd*, MAXO, MINU, MAXB and MINB can be changed in order to simulate some characteristics. For example, if we want to simulate a high density of living cells, the MAXO can be changed to a value greater than 3. In this case, cells do not die of overpopulation until MAXO neighbours are alive. Furthermore, if we want to simulate a great quantity of births we can decrease MINB or increase MAXB, for example: MINB = 1 and MAXB = 4 means that a dead cell will be born if the number of live neighbours is between one and four. In this case, even with only one live neighbour the cell can be born. Both cases, high density and high births rate are characteristics of cells in cancerous tissues.

As an example, we consider the following situation with three different cases:

		C1 Dead	C2 Alive	C3 Alive		
		C4 Alive	C0 Alive	C5 Dead		
		C6 Dead	C7 Alive	C8 Dead		

Case 1:

If we consider $pl =< 1, 1, 1, 1, 1, 1, 1, 1, 1 >$, $pd =< 0, 0, 0, 0, 0, 0, 0, 0, 0 >$, MAXO=3, MINU =2, MAXB = 3 and MINB = 3, then we have the classic Game of Life and for the next step, the result is:

C0 dies of overpopulation, since it has fourth live neighbours.

Case 2:

If we consider $pl =< 0.99, 0.95, 0.95, 0.95, 0.9, 0.9, 0.85, 0.85, 0.85 >$, $pd =< 0.01, 0.05, 0.1, 0.15, 0.05, 0.15, 0.05, 0.1, 0.15 >$, MAXO=3, MINU =2, MAXB = 3 and MINB = 3, then:

In order to decide when a cell is considered dead or alive we need to generate a sequence of nine random numbers from a uniform distribution $[0, 1]$ (RN).

The obtained sequence using a random number generator was:

$$RN = \{RN0, RN1, \ldots, RN8\} = \{0.4667595, 0.963136, 0.1046177, 0.1179487, 0.1056324, 0.9278201, 0.1179487, 0.223967, 0.06180911\}$$

1. C1 is initially dead so the probability of being consider alive is $pd1 = 0.01$. Now, we use RN1, the fist random number of RN, in order to decide if C1 is finally considered dead or alive. Since RN1 = 0.963136 is greater than 0.01, the cell C1 is considered dead.
2. C2 is initially alive so the probability of being consider alive is $pl2 = 0.95$. We use RN2 to decide if C2 is finally considered dead or alive. Since RN2 = 0.1046177 is smaller than 0.95, the cell C2 is considered alive.
3. C3 is initially alive so the probability of being consider alive is $pl3 = 0.95$. We use RN3 to decide if C3 is finally considered dead or alive. Since RN3 = 0.1179487 is smaller than 0.95, the cell C3 is considered alive.
4. C4 is initially alive so the probability of being consider alive is $pl4 = 0.9$. We use RN4 to decide if C4 is finally considered dead or alive. Since RN4 = 0.1056324 is smaller than 0.9, the cell C4 is considered alive.
5. C5 is initially dead so the probability of being consider alive is $pd5 = 0.15$. We use RN5 to decide if C5 is finally considered dead or alive. Since RN5 = 0.9278201 is greater than 0.15, the cell C5 is considered dead.

6. C6 is initially dead so the probability of being consider alive is $pd6 = 0.05$. We use RN6 to decide if C6 is finally considered dead or alive. Since RN6 = 0.1179487 is greater than 0.05, the cell C6 is considered dead.
7. C7 is initially alive, so the probability of being consider alive is $pl7 = 0.85$. We use RN7 to decide if C7 is finally considered dead or alive. Since RN7 = 0.223967 is smaller than 0.85, the cell C6 is considered alive.
8. C8 is initially dead, so the probability of being consider alive is $pd8 = 0.15$. We use RN8 to decide if C8 is finally considered dead or alive. Since RN8 = 0.06180911 is smaller than 0.15, the cell C6 is considered alive.

The number of living cells among the eight neighbours are five (more than in the classic model), which is greater than MAXO = 3 so the cell C0 is initially considered dead (overpopulation). Since pd0=0.01, we use RN0 to decide if C0 is finally considered dead or alive. Since RN0 = 0.4667595 is greater than 0.01 the cell C0 is considered dead.

Case 3:

If we consider $pl =< 0.99, 0.95, 0.95, 0.95, 0.9, 0.9, 0.85, 0.85, 0.85 >$,
$pd =< 0.01, 0.05, 0.1, 0.15, 0.05, 0.15, 0.05, 0.1, 0.15 >$,
MAXO=6, MINU =2, MAXB = 3, MINB=3, then:

For the same random numbers as the previous case we would obtain five live neighbours, which is smaller than MAXO = 6 and greater than MINU = 2, then the cell is initially considered alive for the next step. Since RN0 = 0.4667595 is smaller than $pl0 = 0.99$, the cell is finally considered alive in the next step.

2.2 New features of PCAEGOL

PCAEGOL method is very flexible since it allows the following two characteristics:

- The possibility of changing the limits of the rules in the classic Game of Life.
- The possibility of assigning probabilities for living and for dying not only for the current cell but also for every adjacent cell.

This flexibility allows us to consider not only "errors" in the state of the current cell but also "errors" in the assigned state of the adjacent cells.

In previous papers, such as [1], randomness is allowed in the transition rule. Since our approach is focused on assigning probability values to every adjacent cell, and even to the current cell, some directional properties can be considered through the assignation of the probabilities. For example, one directional characteristic could be the prioritization of the growth in one or several directions. This possibility of prioritizing one particular direction can be observed in the examples of the following section 3.

3 Examples

An implementation of this probabilistic cellular automaton has been developed using the programming language Java. Authors are interested in the simulation of tissues, specially for the case of cancerous tissues. This is an initial prototype, but the goal of the authors is improving the model in order to make it more flexible. Due to its nature, we believe this prototype has a great potencial as a useful tool for oncologists that in the future could estimate the parameters needed for a realistic simulation.

The following examples have been obtained using the implementation made in Java.

3.1 Example 1

The first example has the following parameters: The grid is 500×500 cells.
$pl =< 0.002, 0.8, 0.95, 0.75, 0.75, 0.95, 0.8, 0.002 >$,
$pd =< 0.0, 0.0015, 0.03, 0.001, 0.001, 0.03, 0.0015, 0.0 >$,
MAXO=4, MINU =2, MAXB = 3 and MINB=3.

Note that C3 and C6 have a very high probability in $pl : 0.95$, C2 and C7 have a high probability in $pl : 0.8$, C4 and C5: 0.75 and very low probability in $pl : 0.002$ for C1 and C8. That is, it is more probable that neighbours cells in the sinister diagonal (from bottom left to top right) are considered alive and, adjacent cells in the dexter diagonal (from top left to bottom right) have lower probability of being considered alive. All the probabilities in pd are very small, so the probability of being considered alive when they are really dead is very low. Parameters for MAXO, MINU, MAXB and MINB can be expressed with the notation 2,3,4/3.

From a random initial configuration of the 500×500 cells (with 0.5 probability of being alive). Step 100 and step 200 are shown in Fig. 1. Note that the sinister diagonal emerges from a random initial configuration in both figures.

3.2 Example 2

The second example has the following parameters: The grid is 500×500 cells.
$pl =< 0.96, 0.776, 0.05, 0.776, 0.776, 0.05, 0.776, 0.96 >$,
$pd =< 0.001, 0.001, 0.001, 0.001, 0.001, 0.001, 0.001, 0.001 >$,
MAXO=4, MINU =2, MAXB = 3 and MINB=3.

Note that C1 and C8 has a very high probability in $pl : 0.96$, C2, C4, C5 and C7 has a high probability in $pl : 0.776$, and very low probability in $pl : 0.001$ for C3 and C6 . That is, it is more probable that neighbours cells in the dexter diagonal (from top left to bottom right) are considered alive and, adjacent cells in the sinister diagonal (from bottom left to top right) has lower probability of been considered alive. All the probabilities in pd are very small: 0.001 so the probability of being considered alive when they are really dead is

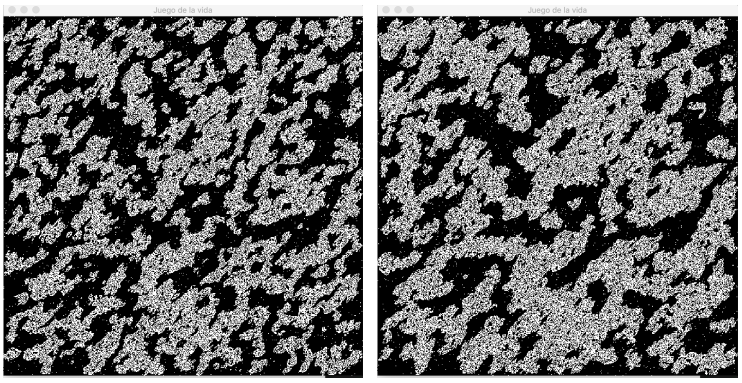

Fig. 1 Example 1 steps 100 and 200. Black and white points correspond to dead and living cells respectively.

very low. Parameters for MAXO, MINU, MAXB and MINB can be expressed with the notation 2,3,4/3.

In Fig. 2 the initial random configuration of the 500×500 cells (with 0.5 probability of been alive) is shown. In Fig. 3 steps 50, 100, 150 and 200 are shown. Note that the dexter diagonal emerges from a random initial configuration in the four figures.

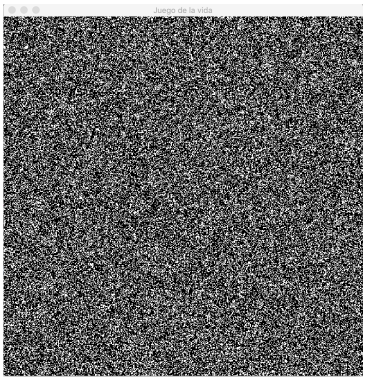

Fig. 2 Example 2. Step = 0. Black and white points correspond to dead and living cells respectively.

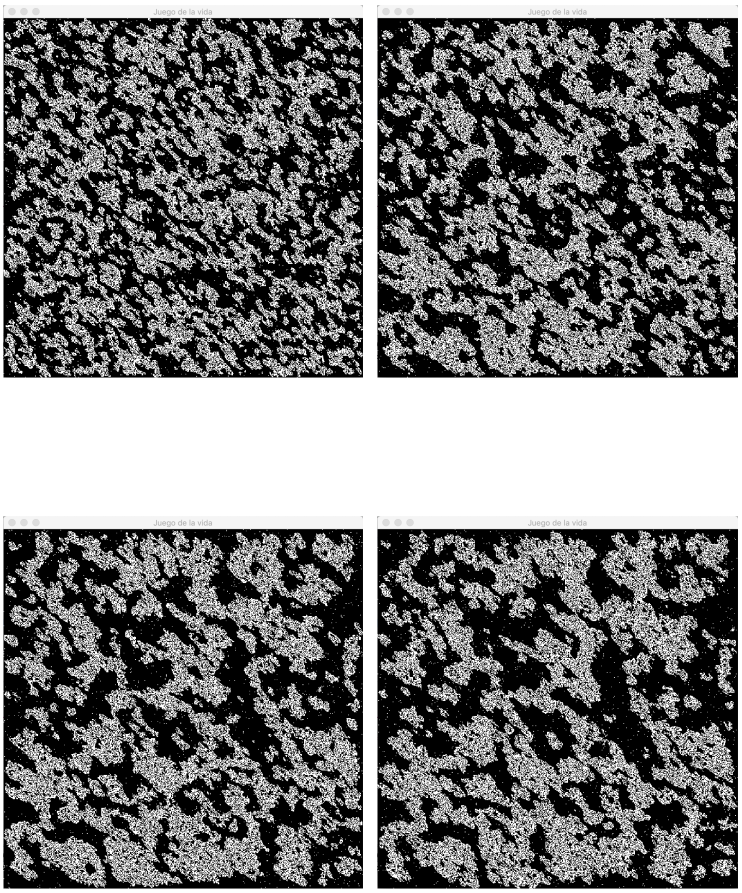

Fig. 3 Example 2. Steps = 50, 100, 150, 200. Black and white points correspond to dead and living cells respectively.

4 Conclusions and Future Related Work

4.1 Conclusions

In this work, the following conclusions can be underlined:

– A brief state of the art of extensions of the Game of Life (GoL) has been shown. Some extensions of GoL and applications of them have been referenced in the paper.

- A new extension of the Game of Life using probabilistic cellular automata called PCAEGOL has been described. Some examples for better understanding the description of the method have been included. The new features of the PCAEGOL model have been discussed.
- An implementation of PCAEGOL using Java has been developed. The implementation has been used for showing some examples.
- The examples of use of PCAEGOL show directional characteristics that, as far as we know, is a novelty since they are not considered in other probabilistic automata.

4.2 Future Related Work

The following extensions are being studied for future works:

- Extending this work to a 3D grid to simulate the evolution of the tissue in three dimensions. The basic idea is to consider two matrixes of $27 = 3^3$ probabilities instead of the two matrixes of $9 = 3^2$ probabilities used int this paper.
- Introducing a third state: alive but cancerous cells. This new state would allow assigning different probabilities to a cancerous cell than a normal living cell, simulating the different behaviour of these types of cells. In addition, different limits values for these cancerous cells can be assigned so that the differences of both types of cells can be potentiated. For example, allowing more adjacent living cells for cancerous cells than for normal cells.
- Introducing probabilistic functions instead of fixed probabilities. We consider that using probabilistic functions instead of fixed probabilities will improve the directional-spatial characteristics of the model.

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