Appl. Math. Mech. -Engl. Ed., **0**(0), 1–9 (0)
DOI 0

Applied Mathematics and Mechanics
(English Edition)

Contact problem for regular hexagon weakened with full-strength hole *

N. Odishelidze[1], F. Criado-Aldeanueva[2], J.M. Sanchez[3]

(1. Department of Computer Sciences, Tbilisi State University, Tbilisi. Georgia
2. Department of Applied Physics II, Polytechnic School, Malaga University, Spain
3. Department of Statistics and Operative Research, Malaga University, Spain.)

Abstract This paper addresses a problem of plane elasticity theory for a doubly connected body with an external boundary of regular hexagon shape and with a 6-fold symmetric hole at the center. It is assumed that all the six sides of the hexagon are subjected to uniform normal displacements via smooth rigid stamps, while uniformly distributed normal stress is applied to the internal hole boundary. By using the methods of complex analysis, the analytical image of Kolosov-Muskhelishvili's complex potentials and the shape of the hole contour are determined from the condition that the circumferential normal stress is constant along the hole contour. Numerical results are also given and shown in relevant graphs.

Key words plate elasticity theory, complex variable theory, stress state, regular polygons.

Chinese Library Classification number1, number2, number3
2010 Mathematics Subject Classification 74A10, 74S70, 74G70

1 Introduction

The boundary value problems of the plane theory of elasticity and plate bending with partially unknown boundaries (the problems of finding a *full-strength contour*) are closely related to very important problems of optimal projecting. Their solution ensures optimal distribution of stresses on the hole boundary by means of appropriate inspection of its form and location.

In the case of a finite plate, when constant normal and tangential stresses are prescribed on the hole borders and certain boundary conditions are given on the outer boundary, the problem in a general case reduces to a nonlinear problem of the theory of analytic functions.

In case the outer boundary of a plate is a convex broken line, the problem under consideration reduces to the linear problem of the theory of analytic functions.

The problem is formulated as follows: Find holes such that the tangential normal stress on the hole boundaries takes constant value. Such holes are called *full-strength ones*.

The boundary value problems of the theory of plane elasticity and plate bending related to infinite plates weakened by full-strength unknown holes in which normal stresses act on their boundaries and forces are applied at infinity have been analyzed in [1–5]. Also, the boundary value problems of a finite doubly-connected domain in which a part of its boundary is the unknown full strength and the other part is a polygonal line have been solved in [6, 7]. Finally, the cycle-symmetric and axis-symmetric problems of the theory of plane elasticity and bending with partially unknown boundaries have been studied in [8–14].

---

* Corresponding author F. Criado Aldeanueva, Department of Applied Physics, Polytechnic School, Malaga University (29071), Spain. E-mail: fcriado@uma.es

In the present work we investigate the problem of plane elasticity for a regular hexagon which is weakened by a full-strength hole, when tangential stresses on the boundary are equal to zero and normal displacements are piecewise constant. The linear segments are endowed with the boundary conditions of the third problem.

One the effective methods of solution of such kind problems turned out to be the methods of the theory of analytic functions.

The solution is written in quadratures and the unknown full-strength hole of the plate is constructed.

2 Problem statement and solution technique

Let an isotropic elastic body on the plane $z = x + iy$ occupy a doubly connected domain S, whose external boundary is a regular hexagon boundary and whose internal boundary is the required full-strength hole.

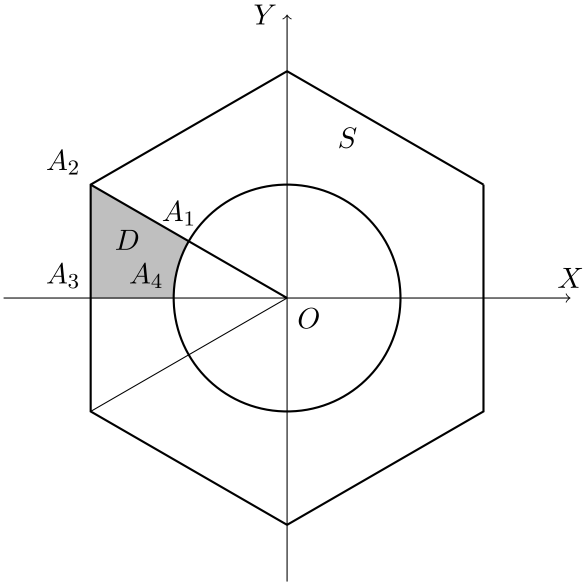


Fig. 1 Graph of the posed problem

The required full-strength hole includes the origin of coordinates. Hexagon's two vertices are laid at the axis Oy, and middle points of its two opposite sides are laid at the axis Ox (Fig. 1). It is assumed that all the six sides of the hexagon are subjected to uniform normal displacements under the action of concentrated normally compressive forces P, while uniformly distributed normal stress Q is applied to the internal hole boundary. With the above conditions, the normal displacement on each side of the hexagon is constant, i.e. $\nu_n = \nu =$ const. and the tangential stresses on the external and internal boundaries are zero, i.e. $\tau_{ns} = 0$.

Problem formulation : Find the stress state of the body and the shape of the unknown hole under the condition that the circumferential normal stress σ_s along the hole contour would take the constant value, i.e. $\sigma_s = K =$ const. Such a hole is referred to as full-strength.

Since the problem is 6-fold symmetric, then to investigate the stated problem, it is sufficient to consider the curvilinear quadrangle $A_1A_2A_3A_4$ which is denoted by D. The normal displacements and the tangential stresses are equal to zero $\nu_n = \tau_{ns} = 0$ at each segment $[A_1, A_2]$, $[A_3, A_4]$.

Introduce the following notations: $\Gamma_1 = A_1A_2$, $\Gamma_2 = A_2A_3$, $\Gamma_3 = A_3A_4$, $\gamma {=} A_4A_1$, $\Gamma = \cup_{j=1}^{3}\Gamma_j$. $P_1 = \int_{\Gamma_1} \sigma_n \, ds$, $P_2 = \int_{\Gamma_2} \sigma_n \, ds$, $P_3 = \int_{\Gamma_3} \sigma_n \, ds$, σ_n is the normal stress and A_3 is the middle point of hexagon side. So $P_2 = \int_{\Gamma_2} \sigma_n \, ds = -P/2$. Since the D is in the equilibrium state, then we have:

$$P_2 = P_1 \cos\frac{\pi}{3} + Q, \qquad P_3 = P_1 \sin\frac{\pi}{3} + Q\cos\frac{\pi}{3}.$$

Hence one obtains

$$P_1 = -2Q - P, \qquad P_3 = -\frac{\sqrt{3}}{2} + \frac{1-2\sqrt{3}}{2}Q.$$

The boundary conditions are

$$\nu_n = \begin{cases} 0 & t \in \Gamma_3 \cup \Gamma_1 \\ \nu & t \in \Gamma_3 \end{cases} \tag{1}$$

$$\begin{gathered} \tau_{ns} = 0, \qquad t \in \Gamma \cup \gamma, \\ \sigma_n = Q, \quad \sigma_s = K, \qquad t \in \gamma, \end{gathered} \tag{2}$$

$$\int_{\Gamma_1} \sigma_n \, ds = -2Q - P, \quad \int_{\Gamma_2} \sigma_n \, ds = -\frac{P}{2}, \quad \int_{\Gamma_3} \sigma_n \, ds = -\frac{\sqrt{3}}{2}P + \frac{1-2\sqrt{3}}{2}Q \tag{3}$$

On the basis of the well-known Kolosov-Muskelishvili's formulas [15], the problem reduces to finding the functions ψ, ϕ which are holomophic in the domain D with the following conditions

$$\Re e^{-i\alpha(t)} \left(\chi\varphi(t) - t\,\overline{\varphi'(t)} - \overline{\psi(t)}\right) = 2\mu\nu_n(t), \qquad t \in \Gamma, \tag{4}$$

$$\Re e^{-i\alpha(t)} \left(\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)}\right) = C(t), \qquad t \in \Gamma, \tag{5}$$

$$\varphi(t) + t\,\overline{\varphi'(t)} + \overline{\psi(t)} = -Qt, \qquad t \in \gamma, \tag{6}$$

$$\Re\varphi'(t) = \frac{\sigma_n + \sigma_s}{4} = \frac{-Q+K}{4}, \qquad t \in \gamma, \tag{7}$$

where χ, μ are elasticity constants, $C(t)$ is a piecewise–constant function, $\alpha(t)$ is the angle formed between the external normal n to contour and the abscissa axis Ox.

$$\alpha(t) = \alpha_k, \qquad t \in \Gamma_k, \quad k = 1,2,3, \qquad \alpha_1 = \frac{\pi}{3}, \quad \alpha_2 = \pi, \quad \alpha_3 = \frac{3\pi}{2} \tag{8}$$

$$C(t) = \Re\left(e^{-i\alpha(t)} i \left(\int_{A_1}^{t} \sigma_n(s_0) e^{i\alpha(s_0)} \, ds_0\right)\right) \tag{9}$$

Taking into account (3) and (8), the (9) has following form:

$$C(t) = \begin{cases} -\frac{\sqrt{3}(2Q+P)}{2}, & t \in \Gamma_2 \\ 0, & t \in \Gamma_1 \cup \Gamma_3 \end{cases} \tag{10}$$

Let, $t \in A_k A_{k+1}$, $k = 1,2$ and $t \in A_4 A_1$, then $t - A_k = i|t - A_k| e^{i\alpha_k}$. Hence we obtain

$$\Re t e^{-i\alpha(t)} = \Re e^{-i\alpha(t)} A(t) \tag{11}$$

where $A(t)$ is a piecewise-constant function, $A(t) = A_k$, $t \in A_k A_{k+1}$, $k = 1,2$ and $t \in A_4 A_1$.

Taking into account (8) one obtains:

$$\Re e^{-i\alpha(t)} A(t) = \begin{cases} \frac{\sqrt{3}}{2}a, & t \in \Gamma_2, \\ 0, & t \in \Gamma_1 \cup \Gamma_3. \end{cases}$$

where a is length of the regular hexagon side.

The functions $\varphi(z)$, $\overline{z}\varphi'(z) + \psi(z)$ will be assumed to be continuous everywhere on the boundary of the domain D, while functions $\varphi'(z)$, $\psi(z)$ are continuously extendable on the boundary D, with the exclusion of the point A_2 in the neighborhood of which is admitted the following estimate:

$$|\varphi'(z)| < M|z - A_2|^{-\delta}, \quad |\psi(z)| < M|z - A_2|^{-\delta}, \qquad 0 \le \delta < 1 \tag{12}$$

Summarizing (4) and (5) and differentiating with respect to arc abscissa s and taking into account that $\alpha(t)$, $C(t)$, $\nu_n(t)$ are the piecewise–constant functions on Γ, we obtain

$$\Im\varphi'(t) = 0, \qquad t \in \Gamma. \tag{13}$$

Equalities (7) and (13) are the Keldysh-Sedov problem for domain S:

$$\begin{aligned} \Re\left(\varphi'(t) - \frac{K-Q}{4}\right) &= 0, \qquad t \in A_4A_1 \\ \Im\left(\varphi'(t) - \frac{K-Q}{4}\right) &= 0, \qquad t \in \Gamma. \end{aligned} \tag{14}$$

Problem (14) has a unique solution [16]

$$\varphi'(z) = \frac{K-Q}{4} \tag{15}$$

Hence we obtain

$$\varphi(z) = \frac{K-Q}{4}z. \tag{16}$$

Here we neglect the constant summands.

Substituting the values $\varphi(t)$, $C(t)$ into the boundary conditions (5)-(6) and taking into account (12) one gets the following problem

$$\Re\left[e^{-i\alpha(t)}\left(\frac{K-Q}{2}t + \overline{\psi(t)}\right)\right] = C(t) = \begin{cases} -\frac{\sqrt{3}(2Q+P)}{2}, & t \in \Gamma_2 \\ 0, & t \in \Gamma_1 \cup \Gamma_3 \end{cases} \tag{17}$$

$$\Re t e^{-i\alpha(t)} = \Re e^{-i\alpha(t)}A(t), \qquad t \in \Gamma, \tag{18}$$

$$\left(\frac{K+Q}{2}t + \overline{\psi(t)}\right) = 0, \qquad t \in \gamma \tag{19}$$

Let the function $z = \omega(\zeta)$, $\zeta = \xi + i\eta$ map semicircle, $|\zeta| < 1$, $\Im\zeta > 0$, conformally onto the domain D. It is assumed that the vertices A_k of the hexagon line correspond to the point a_k of the semicircle $|\zeta| = 1$, $\Im\zeta > 0$, $a_k = \omega^{-1}(A_k)$, $k = 1, 2, 3$. It is further assumed, that $a_1 = 1$, $a_4 = -1$, $a_3 = i$. Here we can fix three points and the remaining ones are to be defined. Then the diameter $-1 \le \xi \le 1$ is mapped onto arc A_4A_1 and the semi-circumference γ_0: $|\gamma_0| = 1$, $\Im\zeta > 0$ is mapped onto the broken line Γ. The point $a_2 = e^{i\theta_2}$, $0 < \theta_2 < \frac{\pi}{2}$ is to be determined.

Because of (17)-(19) the boundary conditions become

$$\Re e^{-i\alpha(\sigma)}\overline{\psi(\sigma)} = -\frac{(K-Q)}{2}\Re e^{-i\alpha(\sigma)}A(\sigma) + C(\sigma), \qquad \sigma \in \gamma_0, \tag{20}$$

$$\Re e^{-i\alpha(\sigma)}\omega(\sigma) = \Re e^{-i\alpha(\sigma)}A(\sigma), \qquad \sigma \in \gamma_0, \tag{21}$$

$$\begin{gathered} \frac{K+Q}{2}\omega(\sigma) + \overline{\psi_0(\sigma)} = 0, \qquad \sigma \in (-1, 1) \\ \psi_0(\zeta) = \psi(\omega(\zeta)). \end{gathered} \tag{22}$$

Since $\alpha(t)$, $A(t)$, $C(t)$ are the piecewise-constant functions $\alpha(\omega(\sigma))$, $A(\omega(\sigma))$, $C(\omega(\sigma))$ are denoted by $\alpha(\sigma)$, $A(\sigma)$, $C(\sigma)$.

Let us introduce a piecewise-holomorphic function $W(\zeta)$ in the circle $|\zeta| < 1$.

$$W(\zeta) = \begin{cases} \frac{K+Q}{2}\omega(\zeta), & |\zeta| < 1, \quad \Im\zeta > 0 \\ -\psi_0(\overline{\zeta}), & |\zeta| < 1, \quad \Im\zeta < 0 \end{cases} \tag{23}$$

Taking into account (23)

$$W^+(\xi) = \frac{K+Q}{2}\omega(\xi), \qquad W^-(\xi) = -\overline{\psi_0(\xi)}, \qquad -1 < \xi < 1, \tag{24}$$

$$W^+(\sigma) = \frac{K+Q}{2}\omega(\sigma), \qquad \sigma \in \gamma_0, \qquad W^-(\sigma) = -\overline{\psi_0(\overline{\sigma})}, \qquad \sigma \in \gamma_0^*. \tag{25}$$

where the signs (+) and (−) mark, as usually, the limit values the function takes at the upper and lower edges and γ_0^* is mirror image of γ_0 about the Ox-axis. By virtue of (22) and (24), we obtain

$$W^+(\xi) - W^-(\xi) = 0, \qquad -1 < \xi < 1.$$

which means that $W(\zeta)$ is a holomorphic function on the circle $|\zeta| < 1$.

Considering (25), we get

$$\Re e^{-i\alpha(\sigma)} W(\sigma) = \frac{K+Q}{2} \Re e^{-i\alpha(\sigma)} A(\sigma), \qquad \sigma \in \gamma_0, \tag{26}$$

$$\Re e^{-i\alpha(\sigma)} W(\sigma) = -C(\sigma) + \frac{K-Q}{2} \Re e^{-i\alpha(\sigma)} A(\sigma), \qquad \sigma \in \gamma_0^* \tag{27}$$

Conditions (26) and (27) can be rewritten in the following form

$$\Re e^{-i\alpha(\sigma)} W(\sigma) = f(\sigma), \qquad \sigma \in \gamma' \tag{28}$$

where

$$\begin{gathered}
\gamma' = \gamma_0 \cup \gamma_0^*, \quad \alpha(\sigma) = \alpha(\overline{\sigma}), \quad \sigma \in \gamma_0^*, \quad \alpha_1 = \frac{\pi}{3}, \quad \alpha_2 = \pi, \quad \alpha_3 = \frac{3\pi}{2}, \\
f(\sigma) = \frac{K+Q}{2} \Re e^{-i\alpha(\sigma)} A(\sigma) = \begin{cases} \frac{(K+Q)a\sqrt{3}}{4}, & \sigma \in (a_2, a_3) \\ 0, & \sigma \in (a_1, a_2) \cup (a_3, a_4) \end{cases} \\
f(\sigma) = \frac{K-Q}{2} \Re e^{-i\alpha(\sigma)} A(\sigma) - C(\sigma) = \begin{cases} \frac{(K-Q)a\sqrt{3}}{4} + \frac{\sqrt{3}}{2}(2Q+P), & \sigma \in (\overline{a_3}, \overline{a_2}) \\ 0, & \sigma \in (a_4, \overline{a_3}) \cup (\overline{a_2}, a_1) \end{cases}
\end{gathered} \tag{29}$$

Thus, the problem in question has been reduced to the Riemann-Hilbert problem with piecewise-constant coefficients. The solution of this problem was obtained in [17] (by reducing it to a linear-conjugation problem). Here we reduce the problem to the Dirichlet problem for a circle and represent its solution in terms of the Schwarz integral, which is computationally convenient.

Function $e^{2i\alpha(\sigma)}$ is given by

$$e^{2i\alpha(\sigma)} = \frac{X(\sigma)}{\overline{X(\sigma)}}, \qquad |\sigma| = 1, \tag{30}$$

where $X(\zeta)$ is given as

$$X(\zeta) = \frac{\sqrt{(\zeta - \overline{a_3})(\zeta - a_3)}(\zeta - a_2)^{1/3}(\zeta - \overline{a_2})^{2/3}\sqrt[6]{a_2}}{\zeta}, \qquad |\zeta| < 1,$$

By virtue of (30) condition (28) becomes

$$\frac{W(\sigma)}{X(\sigma)} + \frac{\overline{W(\sigma)}}{\overline{X(\sigma)}} = \frac{2fe^{i\alpha(\sigma)}}{X(\sigma)}. \tag{31}$$

Condition (31) represents the boundary condition of the Dirichlet problem, whose solution is presented by Schwarz formula

$$\frac{W(\zeta)}{X(\zeta)} = \frac{1}{2\pi i} \int_{\gamma'} \frac{f(\sigma)e^{i\alpha(\sigma)}(\sigma+\zeta)\, d\sigma}{X(\sigma)\sigma(\sigma-\zeta)} \tag{32}$$

Since function $X(\zeta)$ has a simple pole at the point $\zeta = 0$, the function $W(\zeta)/X(\zeta)$ will have the first-order zero at this point $\zeta = 0$. Hence from (32) we get

$$\int_{\gamma'} \frac{f(\sigma)e^{i\alpha(\sigma)}\, d\sigma}{X(\sigma)\sigma} = 0 \tag{33}$$

Taking into account (29) and (8), (33) has the form:

$$-\frac{(K+Q)a\sqrt{3}}{4} \int_{a_2}^{a_3} \frac{d\sigma}{X(\sigma)\sigma} - \left(\frac{(K-Q)a\sqrt{3}}{4} + \frac{\sqrt{3}}{2}(2Q+P) \right) \int_{\overline{a_3}}^{\overline{a_2}} \frac{d\sigma}{X(\sigma)\sigma} = 0 \tag{34}$$

Thus we obtain the equation with respect to two unknown parameters a_2, K.

We could choose some value of K and then determine the parameter a_2. In this case, the problem becomes more complicated. For the purpose of computations, it is more convenient to calculate K for each fixed point $a_2 = e^{i\theta_2}$, $0 < \theta_2 < \pi/2$ and for the given P and Q. From (34) one obtains:

$$K = -\frac{Qa(A-B) + 2B(2Q+P)}{a(A+B)}, \tag{35}$$

where $A = \int_{a_2}^{a_3} \frac{d\sigma}{X(\sigma)\sigma)}$, $B = \int_{\overline{a_3}}^{\overline{a_2}} \frac{d\sigma}{X(\sigma)\sigma}$.

Remark A solution of the problem exists if

$$A + B \neq 0$$

By virtue of (33), the formula (32) has the form:

$$W(\zeta) = \frac{\zeta X(\zeta)}{\pi i} \int_{\gamma'} \frac{f(\sigma)e^{i\alpha(\sigma)}\, d\sigma}{X(\sigma)\sigma(\sigma-\zeta)} \tag{36}$$

i.e.:

$$W(\zeta) = \frac{\zeta X(\zeta)}{\pi i} \left(-\frac{(K+Q)a\sqrt{3}}{4} \int_{a_2}^{a_3} \frac{d\sigma}{X(\sigma)\sigma} - \left(\frac{(K-Q)a\sqrt{3}}{4} + \frac{\sqrt{3}}{2}(2Q+P) \right) \right) \int_{\overline{a_3}}^{\overline{a_2}} \frac{d\sigma}{X(\sigma)\sigma}.$$

By virtue of (23), equation of the contour $z = \omega(\xi)$ is presented by

$$\omega(\xi) = \frac{2W(\xi)}{K+Q}, \qquad -1 < \xi < 1. \tag{37}$$

3 Construction of the hexagon's full-strength hole

We will consider now some concrete cases: let the length of a regular hexagon side be $a = 3$. Having defined K by formula (35) for every fixed point $a_2 = e^{i\theta_2}$, $0 < \theta_2 < \pi/2$ and given P, Q, we define the function $W(\xi)$ by formula (36), and then using formula (37) we define the function $\omega(\xi)$. As a result of conformal mapping $z = \omega(\xi)$, $-1 < \xi < 1$, we obtain the part of an unknown full-strength contour.

To construct the hexagon's full-strength hole, at first, the arc $\gamma = A_4A_1$ of the required full-strength contour is constructed by (37): $w = \omega(\xi)$, $-1 < \xi < 1$. Then by virtue of this function, the next part of required full-strength contour is constructed by $r = \Re(\omega(\xi)) - i\Im(\omega(\xi))$. Thus one sixth part of the required full-strength hole is constructed (Fig. 1). Since the problem is 6-fold symmetric, the other parts of required full-strength hole can be obtained by rotating the graphs w, r through an angle of $\pi/3$.

As an illustration, some graphics of full-strength holes are presented in Fig 2 for different sets of parameters.

Remark :

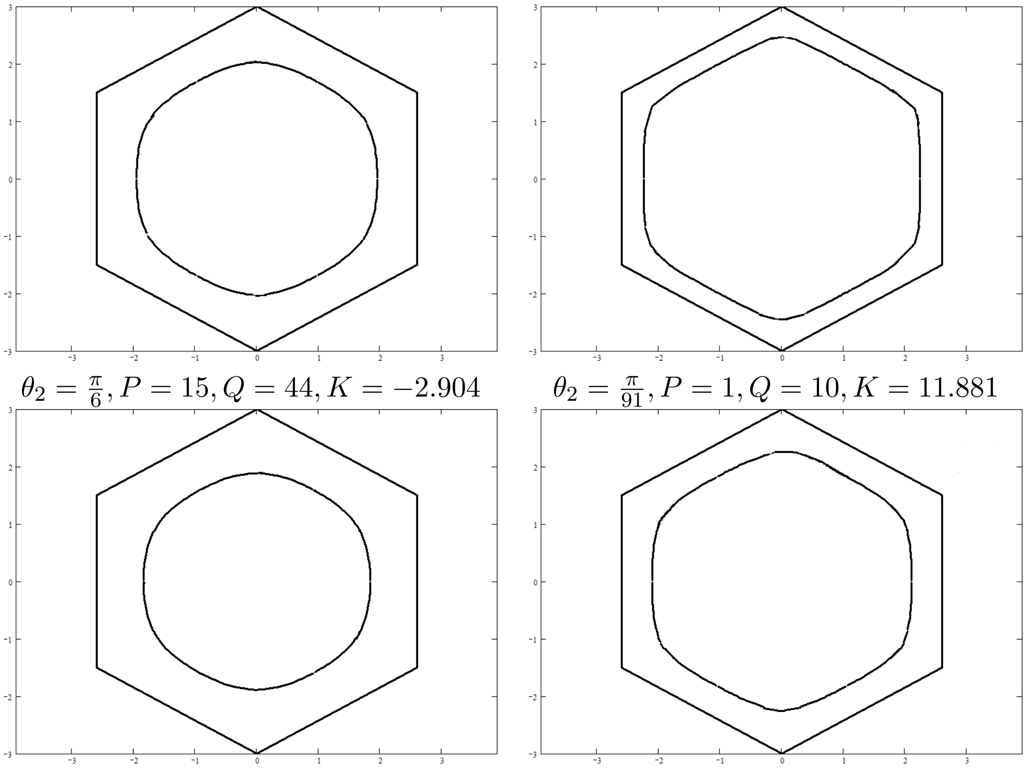
As a general case, the stated problem for a regular polygon is considered. This problem will be solved similarly as above. Let the considered polygon be with n vertices (n is an even number). Then the boundary conditions of the posed problem for a regular polygon are presented as follows:

$$\nu_n = \begin{cases} 0, & t \in \Gamma_3 \cup \Gamma_1, \\ \nu, & t \in \Gamma_3, \end{cases} \tag{38}$$

$$\tau_{ns} = 0, \qquad t \in \Gamma \cup \gamma, \tag{39}$$

$$\sigma_n = Q, \qquad \sigma_s = K, \qquad t \in \gamma \tag{40}$$

$$P_1 = \int_{\Gamma_1} \sigma_n\, ds = \frac{-2Q-P}{2\sin\frac{\pi}{n}}, \qquad P_2 = \int_{\Gamma_2} \sigma_n\, ds = \frac{-P}{2}, \qquad P_3 = \int_{\Gamma_3} \sigma_n\, ds = \frac{-P-2Q}{2}\cot\frac{\pi}{n} + Q\sin\frac{\pi}{n} \tag{41}$$



$\theta_2 = \frac{\pi}{6}, P = 15, Q = 44, K = -2.904$ $\theta_2 = \frac{\pi}{91}, P = 1, Q = 10, K = 11.881$

$\theta_2 = \frac{\pi}{4}, P = 1, Q = 10, K = 1.232$ $\theta_2 = \frac{\pi}{16}, P = 15, Q = 51, K = 13.45$

Fig. 2 Graphs of a full-strength Boundary

Then, the problem (4)-(7) will be considered for

$$\alpha(t)=\alpha_k,\quad t\in\Gamma_k,\quad k=1,2,3,\qquad \alpha_1=\frac{\pi}{2}-\frac{\pi}{n},\quad \alpha_2=\pi,\quad \alpha_3=\frac{3\pi}{2} \tag{42}$$

$$C(t)=\begin{cases}\frac{-P-2Q}{2}\cot\frac{\pi}{n}, & t\in\Gamma_2\\ 0 & t\in\Gamma_1\cup\Gamma_3\end{cases} \tag{43}$$

Taking into account (42), one obtains:

$$\Re e^{-i\alpha(t)}A(t)=\begin{cases}\frac{a}{2}\cot\frac{\pi}{n}, & t\in\Gamma_2,\\ 0, & t\in\Gamma_1\cup\Gamma_3\end{cases}$$

where a is the length of the regular poligon side . Following the above mentioned method we obtain the problem (28):

$$\Re e^{-i\alpha(\sigma)}W(\sigma)=f(\sigma),\qquad \sigma\in\gamma'$$

with

$$f(\sigma)=\frac{K+Q}{2}\Re e^{-i\alpha(\sigma)}A(\sigma)=\begin{cases}\frac{(K+Q)a}{4}\cot\frac{\pi}{n}, & \sigma\in(a_2,a_3)\\ 0, & \sigma\in(a_1,a_2)\cup(a_3,a_4)\end{cases} \tag{44}$$

$$f(\sigma)=\frac{K-Q}{2}\Re e^{-i\alpha(\sigma)}A(\sigma)-C(\sigma)=\begin{cases}\frac{(K-Q)a}{4}\cot\frac{\pi}{n}+\frac{(P+2Q)}{2}\cot\frac{\pi}{n}, & \sigma\in(\overline{a_3},\overline{a_2})\\ 0, & \sigma\in(a_4,\overline{a_3})\cup(\overline{a_2},a_1)\end{cases} \tag{45}$$

After some transformations we obtain the solution of the problem:

$$W(\zeta)=\frac{\zeta X(\zeta)}{\pi i}\int_{\gamma'}\frac{f(\sigma)e^{i\alpha(\sigma)}\,d\sigma}{X(\sigma)\sigma(\sigma-\zeta)} \tag{46}$$

i.e

$$W(\zeta)=\frac{\zeta X(\zeta)}{\pi i}\Bigg(-\frac{(K+Q)a}{4}\cot\frac{\pi}{n}\int_{a_2}^{a_3}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\zeta)} \\ -\left(\frac{(K-Q)a}{4}\cot\frac{\pi}{n}+\frac{(P+2Q)}{2}\cot\frac{\pi}{n}\right)\int_{\overline{a_3}}^{\overline{a_2}}\frac{d\sigma}{X(\sigma)\sigma(\sigma-\zeta)}\Bigg)$$

where

$$K=\frac{Qa(B-A)+2B(2Q+P)}{a(A+B)} \tag{47}$$

where

$$A=\int_{a_2}^{a_3}\frac{d\sigma}{X(\sigma)\sigma},\qquad B=\int_{\overline{a_3}}^{\overline{a_2}}\frac{d\sigma}{X(\sigma)\sigma}$$

Equation of the contour $z=\omega(\xi)$ is presented by

$$\omega(\xi)=\frac{2W(\xi)}{K+Q},\qquad -1<\xi<1. \tag{48}$$

and the required full-strength hole of a regular polygon can be obtained by rotating the graphs w, r ($w=\omega(\xi)$, $-1<\xi<1$, $r=\Re(\omega(\xi))-i\Im(\omega(\xi))$) through an angle of $2\pi/n$.

4 Conclusions

The applications of elastic plates that are weakened by holes are of relevant interest in diverse mechanical constructions (ships, buildings, aircrafts, etc).

It is known that, in some cases, external action may cause stress concentration, especially along the borders of holes, leading to plastic deformations or even structural fractures. So the problem is whether it is possible to find all such holes that minimum stresses along the borders take the minimum value (full-strength contour)

The most effective methods to study such kind of problems involve the theory of analytical functions of complex variable. This problem is linked to mechanics and geometry because the shape of a hole in a plate is required and the conformal mapping function (a generalization of the Christoffel-Schwarz formulas) is used to determine it. For this reason, formulas of Kolosov-Muskhelishvili have been used to investigate the problem. By means of the conformal mapping, the problem has been reduced to one of Riemann-Hilbert and its solution has been written in quadratures. The unknown full-strength hole of the plate has been constructed and some concrete cases have been analyzed in detail.

References

[1] Banichuk, N. V. Optimality conditions in the problem of seeking the hole shapes in elastic bodies. *J. Appl. Math. Mech.*, **41** (5), 920–925 (1977)

[2] Banichuk, N. V. *Optimization of forms of elastic bodies*, Nauka, Moscow (1980)

[3] Cherepanov, G. P. Inverse problems of the plane theory of elasticity. *Prikl. Mat. Meh.*, **38** (6), 963–979 (1974)

[4] Neuber, H. The optimization of stresses concentracion. In *Continuum mechanics and related problems of analysis*, Nauka, Moscow, 375–380 (1972)

[5] Vigdergauz, S. B. On a case of the inverse problem of the two-dimensional theory of elasticity. *J. Appl. Math. Mech.*, **41** (5), 902–908 (1977)

[6] Bantsuri, R. D. Some inverse problems of plane elasticity and of bending of thin plates. In *Continuum mechanics and related problems of analysis (Tbilisi, 1991)*, Metsniereba, Tbilisi, 100–107 (1993)

[7] Bantsuri, R. D. and Isakhanov, R. S. Some inverse problems in elasticity theory. *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR*, **87**, 3–20 (1987)

[8] Bantsuri, R. On one mixed problem of the plane theory of elasticity with a partially unknown boundary. *Proc. A. Razmadze Math. Inst.*, **140**, 9–16 (2006)

[9] Bantsuri, R. Solution of the mixed problem of plate bending for a multi-connected domain with partially unknown boundary in the presence of cyclic symmetry. *Proc. A. Razmadze Math. Inst.*, **145**, 9–22 (2007)

[10] Bantsuri, R. and Mzhavanadze, Sh. The mixed problem of the theory of elasticity for a rectangle weakened by unknown equiv-strong holes. *Proc. A. Razmadze Math. Inst.*, **145**, 23–33 (2007)

[11] Odishelidze, N. T. and Criado-Aldeanueva, F. F. On a mixed problem in the plane theory of elasticity for a domain with a partially unknown boundary. *Prikl. Mekh.*, **42** (3), 110–118 (2006)

[12] Odishelidze, N. and Criado-Aldeanueva, F. Some axially symmetric problems of the theory of plane elasticity with partially unknown boundaries. *Acta Mech.*, **199** (1-4), 227–240 (2008)

[13] Odishelidze, N. T. Solution of the mixed problem of the plane theory of elasticity for a multiply connected domain with partially unknown boundary in the presence of axial symmetry. *Proc. A. Razmadze Math. Inst.*, **146**, 97–112 (2008)

[14] Odishelidze, N. and Criado-Aldeanueva, F. A mixed problem of plate bending for a doubly connected domains with partially unknown boundary in the presence of cycle symmetry. *Science China Physics, Mechanics and Astronomy*, **53** (10), 1884–1894 (2010)

[15] Muskhelishvili, N. I. *Some basic problems of the mathematical theory of elasticity.* Noordhoff International Publishing, Leiden (1977)

[16] Keldysh, M.V. and Sedov, L.D. The effective solution of some boundary problems for harmonic functions. *Dokl. Akad. Nauk SSSR*, **16** (1), 7–10 (1937)

[17] Muskhelishvili, N. I. *Singular integral equations*, Dover Publications Inc., New York (1992)