Spin(7) structures, spinors and nilmanifolds

Ph. D. Dissertation

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Tesis doctoral, Programa de Doctorado en Matemáticas
Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias
Universidad de Málaga, 2021

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EDITA: Publicaciones y Divulgación Científica. Universidad de Málaga

Esta tesis ha sido financiada por la Ayuda de Formación de Profesorado Universitario FPU16/03475.

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Agradecimientos

En primer lugar, quiero dar las gracias a mi familia, que ha sido mi soporte vital:

A mi hermana Andrea, por alegrar mi día a día, compartir conmigo parte de su gran corazón, y apoyarme siempre de manera incondicional.

A mis padres, Félix y Fina, por transmitirme el valor de la educación e inculcarme un hábito de trabajo. Este privilegio junto con el cariño y las facilidades que me han dado mientras estudiaba, han sido fundamentales para llegar aquí.

A Aurelia, por cuidarnos y mimarnos a Andrea y a mí como si fuésemos familia.

En especial, quiero dedicar esta tesis a mi abuela Fina, quien, con su amabilidad, buen humor, cariño, inteligencia y fuerza, me acompañó a lo largo de casi todo este camino. Aparte de apoyo, ha sido una referencia para mí, y me siento afortunada y agradecida por todo el tiempo que hemos compartido.

Estos cuatro años han sido intensos y la compañía de mis amigxs me ha ayudado a sobrellevarlos. Quiero dar las gracias a quienes me han ayudado a desconectar cuando estaba agobiada, a quienes han celebrado conmigo la alegría de resolver un problema, y a quienes me han apoyado cuando estaba desmotivada o frustrada. En particular:

A Carmen, Laura, Delia, Antonio y Noelia, por permanecer a mi lado todo este tiempo.

A mis amigas de la UIMP, por ser fuente de inspiración y por el entusiasmo que me transmitís en cada uno de nuestros reencuentros. A Marina por partida doble; gracias por el apoyo y los ratos que compartimos durante nuestro primer año en Madrid.

A mis amigxs de matemáticas (fundamentalmente), por reservarme tiempo siempre que he estado en Málaga y por las tardes *merendando cerveza*.

A Blanca, por su generosidad, por su ayuda cuando llegué a Madrid, por añadir humor a nuestro día a día y por animarme a empezar actividades nuevas, como bailar o aprender alemán. En definitiva, ha sido una suerte compartir con ella la mayor parte de esta experiencia.

A todxs mis compañerxs de piso en Calle Vascos, por transformar aquel piso de paredes blancas y vacías en una pequeña comunidad en la que siempre he estado muy cómoda.

A José, Alberto y Miguel Ángel, con quienes he compartido de un modo u otro las *alegrías y miserias* del doctorado.

A Elena, por sus enseñanzas de swing y la confianza que hemos construido al margen del baile.

A Carlos, por los bailes, las noches en el cine Doré y los dobles de después, y sobre todo, por el apoyo que siempre me ha ofrecido.

A Víctor, por las experiencias compartidas, el cariño, la confianza y ayuda en todo lo que he necesitado.

Quiero también agradecer a las personas que me han acompañado y de las que he aprendido estos años dentro de la universidad:

A mis profesorxs del grado, quienes a través de sus clases alimentaron mi curiosidad por las matemáticas y me impulsaron a comenzar la carrera académica.

A mis profesores del máster, gracias a los cuales aprendí por encima de las que yo creía que eran mis posibilidades. Gracias en especial a Fran, por el tiempo que nos dedicó para el trabajo de fin de Máster y las ideas que compartió con nosotros; fue un trabajo muy productivo.

A mis compañerxs de doctorado y profesorxs del departamento en la UCM por construir un ambiente estimulante a través de seminarios y cursos de doctorado, en los que he disfrutado aprendiendo e intercambiando ideas. Mención especial a Edu, por las innumerables discusiones acerca de *la filosofía de los problemas* y a Ángel, por compartir parte de su sabiduría conmigo y aconsejarme siempre que lo he necesitado.

A Juan, por introducirme en el tema de los orbifolds simplécticos, y por las largas conversaciones acerca del caso de dimensión 4, que han dado lugar al Capítulo 3 de esta tesis. Sin duda, hicimos un buen equipo.

A los profesorxs del departamento de Álgebra, Geometría y Topología, por su acogida a mi vuelta. A Elena, por los buenos ratos en el despacho durante este año tan raro.

To everyone who discussed with me about maths at conferences, workshops and research stays.

I also want to acknowledge Ilka Agricola and Anna Fino for their hospitality during my research stays in Marburg and Torino.

Por supuesto quiero agradecer el trabajo de mis directores, Giovanni y Vicente. Ellos han guiado mi investigación y han contribuido a ampliar mi perspectiva a través de los problemas y lecturas que me han propuesto, en ocasiones bastante distintas entre sí. También me han animado cuando me sentía insegura de la calidad de mi trabajo, y han fomentado mi autonomía en la medida de lo posible; esto hace que me encuentre preparada para avanzar. Quiero subrayar mi agradecimiento a Giovanni, por escuchar cada una de mis preocupaciones y aconsejarme siempre al respecto.

Finalmente quiero agradecer a las personas que luchan por la diversidad en la ciencia, a las que favorecen que el contenido de la investigación sea accesible para todxs, y a las que garantizan la calidad de la enseñanza pública. En especial, quiero dar las gracias a mis compañerxs Patricia, Elena y Gianni, de la asociación WOMAT, por contribuir a enriquecer mi perspectiva, confiar en mí, y, sobre todo, por el trabajo (voluntario) que hacen para combatir la desigualdad de género en las matemáticas.

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Introduction

This thesis consists of a series of papers that I wrote during my Ph.D. program, some of them in collaboration with other authors. These papers deal with various problems from the area of geometric structures, namely the study of Spin(7) structures and the construction of examples of symplectic structures and closed G_2 structures. In the last problem, we pay special attention to the topological property of formality. The techniques we need for this are mainly spinor theory, left-invariant geometric structures on nilmanifolds, and resolution of orbifolds. The purpose of this introduction is to present the state of the art on these topics and to outline the main ideas and results of this thesis.

From the point of view of Riemannian geometry, holonomy theory motivates the study of non-integrable geometric structures. The *holonomy group* $\mathrm{Hol}(g)$ of a Riemannian manifold (M, g) is an invariant that measures how vectors on T_pM change under parallel transport along loops with basepoint p. Shortly after its definition, one goal was to determine the possible holonomy groups of simply connected irreducible complete manifolds. The assumption that M is simply connected guarantees that $\mathrm{Hol}(g)$ is a connected Lie subgroup of $\mathrm{SO}(n)$; in this case, the hypotheses that (M, g) is irreducible and complete avoids the situation in which $\mathrm{Hol}(g)$ is a product. In fact, the de Rham decomposition theorem [39] shows that if (M, g) is simply connected and complete, then it is a Riemannian product $(M_1, g_1) \times \ldots \times (M_\ell, g_\ell)$ where the action of $\mathrm{Hol}(g_i)$ on $T_{p_i}M_i$ is irreducible. Cartan computed holonomy groups of symmetric manifolds using Lie group theory in [26] and [27]. Later, Berger treated the case of non-symmetric manifolds in his celebrated paper [17] and obtained the following result:

Theorem 1. *Let (M, g) be a simply connected irreducible complete non-symmetric Riemannian n-dimensional manifold. Exactly one of the following cases hold:*

$\mathrm{Hol}(g) = \mathrm{SO}(n)$,

$\mathrm{Hol}(g) = \mathrm{U}(m) \subset \mathrm{SO}(2m)$ *with* $n = 2m$ *and* $m \geq 2$,

$\mathrm{Hol}(g) = \mathrm{SU}(m) \subset \mathrm{SO}(2m)$ *with* $n = 2m$ *and* $m \geq 2$,

$\mathrm{Hol}(g) = \mathrm{Sp}(k) \subset \mathrm{SO}(4k)$ *with* $n = 4k$ *and* $k \geq 2$,

$\mathrm{Hol}(g) = \mathrm{Sp}(k) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4k)$ *with* $n = 4k$ *and* $k \geq 2$,

$\mathrm{Hol}(g) = \mathrm{G}_2 \subset \mathrm{SO}(7)$ *with* $n = 7$,

$\mathrm{Hol}(g) = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$ *with* $n = 8$.

The groups $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(k)$, and $\mathrm{Sp}(k)\cdot\mathrm{Sp}(1)$ are known as *special holonomy groups*, and the groups G_2 and $\mathrm{Spin}(7)$ are the so-called *exceptional holonomy groups*. A nice consequence of Berger's theorem is that holonomy groups are related to real division algebras. The holonomy groups $\mathrm{U}(m)$ and $\mathrm{SU}(m)$ are associated with the so-called *Kähler* and *Calabi-Yau* manifolds; these are complex manifolds from the point of view of differential geometry. The groups $\mathrm{Sp}(k)$ and $\mathrm{Sp}(k)\cdot\mathrm{Sp}(1)$ are related to quaternions and correspond to *hyperKähler* and *quaternionic-Kähler* manifolds. The groups G_2 and $\mathrm{Spin}(7)$ are simply connected and they are related to the octonions. More precisely, the multiplicative structure on $\mathbb{R}^8=\mathbb{O}$ determines a *triple cross product* $\times$ on $\mathbb{R}^8$, namely, an alternating map $\mathbb{R}^8\times\mathbb{R}^8\times\mathbb{R}^8\to\mathbb{R}^8$ such that the product $u\times v\times w$ has length $\|u\wedge v\wedge w\|$ and it is perpendicular to the vectors u, v and w. The contraction with the scalar product, $\Omega_0(u,v,w,z):=\langle u\times v\times w,z\rangle$, gives a 4-form that in terms of the standard orthonormal frame $(e_0,\dots,e_7)$ is:

$$\begin{aligned}\Omega_0 = {}& e^{0123}-e^{0145}-e^{0167}-e^{0246}+e^{0257}-e^{0347}-e^{0356}\\ &+e^{4567}-e^{2367}-e^{2345}-e^{1357}+e^{1346}-e^{1256}-e^{1247}.\end{aligned}$$

Denote $\mathbb{R}^8=\mathbb{R}(e_0)\times\mathbb{R}^7$; there is a cross product $\times'$ on $\mathbb{R}^7$ determined by $u\times' v=e_0\times u\times v$ if $u,v\in\mathbb{R}^7$, or equivalently a 3-form $\varphi_0=i(e_0)\Omega_0$. $\mathrm{Spin}(7)$ is the subgroup of $\mathrm{SO}(8)$ that preserves the triple cross product on $\mathbb{R}^8$, namely $\mathrm{Spin}(7)=\mathrm{Stab}(\Omega_0)$, and G_2 is the subgroup of $\mathrm{SO}(7)$ that preserves the cross product $\times'$, namely $\mathrm{G}_2=\mathrm{Stab}(\varphi_0)$. Of course, $\mathrm{G}_2\subset\mathrm{Spin}(7)$.

Berger's proof was algebraic, and at the time of publication of [17] there were no examples of complete metrics with holonomy G_2 and $\mathrm{Spin}(7)$. These were provided by Bryant and Salamon [24] in 1989. Another problem arising from this list was the construction of compact manifolds with holonomy $\mathrm{SU}(m)$, $\mathrm{Sp}(k)$, $\mathrm{Sp}(k)\cdot\mathrm{Sp}(1)$, G_2, and $\mathrm{Spin}(7)$. The construction of such metrics involves deep analytic theorems. For example, the proof of the existence of metrics with holonomy $\mathrm{SU}(m)$ and $\mathrm{Sp}(k)$ uses Yau's theorem. This result solves the Calabi conjecture and implies that a compact Kähler manifold with trivial canonical bundle admits a Calabi-Yau metric. Compact examples with holonomy G_2 and $\mathrm{Spin}(7)$ were the last to appear in 1996; later in this introduction we discuss their construction, developed by Joyce in the series of papers [71], [72], and [73].

The condition that a Riemannian manifold (M,g) has holonomy contained in a group G splits into a topological and an analytic obstruction. This is due to the *holonomy principle*, which relates $\mathrm{Hol}(g)$ to parallel tensors on M:

Proposition 2. *[74, Lemma 2.5.2] Let (M,g) be a Riemannian manifold, let $p\in M$ and let $\mathrm{Hol}(g)$ be the holonomy group with basepoint p. Then,*

1. *If T is a parallel tensor on M, then $\mathrm{Hol}(g)\subset\mathrm{Stab}(\mathrm{T}_p)$.*
2. *If S is a tensor on $\mathbb{R}^n$ such that $\mathrm{Hol}(g)\subset\mathrm{Stab}(\mathrm{S})$, there exists a parallel tensor T on M such that $\mathrm{T}_p=\mathrm{S}$.*

The difficulty of finding examples with special and exceptional holonomy, together with this result, motivated the study of the notion of a *geometric structure* associated to a Lie group $\mathrm{G}\subset\mathrm{SO}(n)$. A G structure on an oriented Riemannian manifold (M,g) consists of a reduction of the oriented orthonormal frame bundle of M from $\mathrm{SO}(n)$ to G. This notion is equivalent to the existence of tensors $\{\mathrm{T}_i\}$ whose common stabilizer is the group G. For this reason we denote by $(M^n,g,\{\mathrm{T}_i\})$ a G structure on a n-dimensional Riemannian manifold. Let us focus for a moment on the cases $\mathrm{U}(m)$, $\mathrm{SU}(m)$, G_2 and $\mathrm{Spin}(7)$:

1. (M^{2m},g,J) is a $\mathrm{U}(m)$ structure or an almost Hermitian structure if J is an almost complex structure compatible with g. More precisely, for every $p\in M^{2m}$ there is an isometry $f_p\colon(T_pM^{2m},g_p)\to(\mathbb{C}^m,\langle\cdot,\cdot\rangle)$ such that $f_p\circ J_p\circ f_p^{-1}(v)=\mathbf{i}v$ for $v\in\mathbb{C}^m$. In this case, we define the 2-form $\omega(v,w)=g(Jv,w)$.

2. (M^{2m}, g, J, Θ) is a $\mathrm{SU}(m)$ structure if (M^{2m}, g, J) is a $\mathrm{U}(m)$ structure and the maps $\{f_p\}_{p\in M}$ also satisfy $f_p^*(dz_1\wedge\cdots\wedge dz_m) = \Theta_p$. Of course, $(z_1, \ldots, z_m)$ are the coordinates on $\mathbb{C}^m$.

3. (M^7, g, φ) is a G_2 structure if φ is a 3-form such that for every $p \in M^7$ there is an isometry $f_p\colon (T_pM^7, g_p) \to (\mathbb{R}^7, \langle\cdot,\cdot\rangle)$ such that $f_p^*\varphi_0 = \varphi_p$.

4. (M^8, g, Ω) is a $\mathrm{Spin}(7)$ structure if Ω is 4-form such that for every $p \in M^8$ there is an isometry $f_p\colon (T_pM^8, g_p) \to (\mathbb{R}^8, \langle\cdot,\cdot\rangle)$ such that $f_p^*\Omega_0 = \Omega_p$.

The notion of G structure also allows us to study geometric situations that are not characterized by holonomy properties. This is the case of $\mathrm{U}(m)$ and $\mathrm{SU}(m)$ structures on $(2m+1)$-dimensional manifolds. The first are also called *almost contact metric* structures and these are related to contact geometry.

Interesting geometric properties arise when one requires that the tensors defining the G structure satisfy partial differential equations; these are often less restrictive than the condition that the holonomy is contained in G. Examples include *almost Kähler* and *Hermitian structures*, which are symplectic and complex manifolds from the point of view of differential geometry. A $\mathrm{U}(m)$ structure (M, g, J) is almost Kähler if $d\omega = 0$ and Hermitian if the Nijenhuis tensor N_J vanishes. This fact motivated Gray and Hervella to start a classification program for G-structures; in [59] they treated the case of almost Hermitian structures. The *intrinsic torsion* Γ is the object that allows us to classify G structures. This is a section of a bundle $\mathcal{W}$ over M with fibre $\mathbb{R}^n \otimes \mathfrak{g}^\perp$; here $\mathfrak{g}$ denotes the Lie algebra of $\mathrm{G} \subset \mathrm{SO}(n)$ viewed as a subspace of $\Lambda^2\mathbb{R}^n = \mathfrak{so}(n)$, where we take its orthogonal complement. The G module $\mathbb{R}^n \otimes \mathfrak{g}^\perp$ decomposes into irreducible invariant subspaces, which in turn determine a splitting $\mathcal{W} = \oplus_{i\in I}\mathcal{W}_i$. Non-integrable classes are defined by the condition $\Gamma \in \oplus_{i\in J}\mathcal{W}_i$ for some $J \subset I$, $J \neq \emptyset$; the torsion-free case corresponds to $\Gamma = 0$ and is equivalent to the condition $\mathrm{Hol}(g) \subset \mathrm{G}$.

It is customary to describe different classes in terms of the covariant derivative or the exterior derivative of the tensors defining the structure. Let us focus for a moment on the case of G_2 structures obtained by Fernandez and Gray in [48] and later reformulated by Bryant in [23]. Classes of G_2 structures are determined by $d\varphi$ and $d\star\varphi$; more precisely, there are *torsion forms* $\tau_k \in \Omega^k(M)$ such that

$$\begin{aligned} d\varphi &= \tau_0 \star \varphi + 3\,\tau_1 \wedge \varphi + \star\tau_3, \\ d \star \varphi &= 4\,\tau_1 \wedge \star\varphi + \tau_2 \wedge \varphi, \end{aligned}$$

and in addition τ_2 and τ_3 satisfy the conditions: $\tau_2 \wedge \star\varphi = 0$, $\tau_3 \wedge \star\varphi = 0$ and $\tau_3 \wedge \varphi = 0$. These equations follow from the decomposition of the spaces $\Lambda^4(\mathbb{R}^7)^*$ and $\Lambda^5(\mathbb{R}^7)^*$ into G_2 invariant irreducible parts. The 1-form τ_1 is the so-called *Lee form* of the structure. *Pure classes* of G_2 structures correspond to the case where all but one torsion form vanish; the most studied are *nearly parallel* G_2 structures, characterized by the condition $d\varphi = \tau_0 \star \varphi$, *closed* G_2 structures, defined by the condition $d\varphi = 0$, and *locally conformally parallel* G_2 structures, described by the conditions $d\varphi = 3\,\tau_1 \wedge \varphi$ and $d \star \varphi = 4\,\tau_1 \wedge \star\varphi$. The class of *coclosed* G_2 structures is determined by $d \star \varphi = 0$. In [37] the authors prove that coclosed G_2 structures exist on any compact manifold with a G_2 structure using Gromov's h-principle [60]. Explicit examples of these are hypersurfaces of 8-dimensional manifolds with a parallel $\mathrm{Spin}(7)$ structure. If the hypersurface is totally umbilic, like the sphere $S^7 \subset \mathbb{R}^8$, the G_2 structure is nearly parallel.

Manifolds with certain holonomy groups or geometric structures fit into the theory of spinor geometry; it was Wang who first explored this connection [112]. More precisely, Wang's theorem states that a complete simply connected irreducible Riemannian manifold which is not flat has a parallel spinor if and only if its holonomy group is simply connected, that is, if $\mathrm{Hol}(g)$ is one of $\mathrm{SU}(m)$, $\mathrm{Sp}(k)$, G_2, $\mathrm{Spin}(7)$. In terms of geometric structures, if the structure group G is simply connected, then the manifold is spin and is endowed with a certain number of nowhere-vanishing spinors.

Dirac began studying spinors when he tried to construct a relativistic wave operator $\not{D}$; this essentially consisted in finding a square root for the Laplacian on $\mathbb{R}^n$. His calculations led him to introduce the *Clifford algebra* Cl_n of $\mathbb{R}^n$: this is the $\mathbb{R}$-algebra with unit generated by $\mathbb{R}^n$ and the quotient relations $v \cdot v = -|v|^2 \cdot 1$. The operator $\not{D}$ is the so-called Dirac operator; for a nice approach to this see the introduction of [54]. One of the greatest achievements of spinor theory is the Atiyah-Singer index theorem, which relates the index of the Dirac operator to a topological invariant: the $\hat{A}$-genus. Moreover, spin geometry plays an important role in various geometric problems: it provides nowhere-vanishing vector fields on spheres, shows the existence of metrics with positive scalar curvature as well as the integrality of certain characteristic classes.

The universal covering $\mathrm{Ad}\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is constructed from the Clifford algebra: $\mathrm{Spin}(n)$ is a multiplicative subgroup of $\mathrm{Cl}_n \setminus \{0\}$,

$$\mathrm{Spin}(n) = \{v_1 \cdots v_{2k} \text{ s.t. } 2k \leq n,\, |v_j| = 1\},$$

and the covering map corresponds to the conjugation $\mathrm{Ad}(g)(x) = gxg^{-1}$. The spinor formalism in $\mathbb{R}^n$ consists of an irreducible Cl_n module Δ_n arising from an isomorphism $\rho\colon \mathrm{Cl}_n \to \mathbf{k}(m)$ or $\rho\colon \mathrm{Cl}_n \to \mathbf{k}(m) \oplus \mathbf{k}(m)$; here $\mathbf{k}(m)$ denotes the algebra of m-dimensional matrices over the (skew) field $\mathbf{k} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. For a while there were problems in extending the spinor formalism from $\mathbb{R}^n$ to orientable manifolds; the notion of a *spin structure* overcame these difficulties. Orientable anifolds admitting a spin structure are called *spin manifolds* and are characterized by the vanishing of the second Stiefel-Whitney class.

Let (M, g) be a Riemannian oriented n-dimensional manifold and denote by $\mathrm{P}_{\mathrm{SO}}(M)$ its principal $\mathrm{SO}(n)$ bundle. A spin structure consists of a principal $\mathrm{Spin}(n)$ bundle $\mathrm{P}_{\mathrm{Spin}}(M)$ and a covering map $\mathrm{p}\colon \mathrm{P}_{\mathrm{Spin}}(M) \to \mathrm{P}_{\mathrm{SO}}(M)$ that is compatible with $\mathrm{Ad}\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$, i.e., $\mathrm{p}(\gamma y) = \mathrm{Ad}(\gamma)\mathrm{p}(y)$ for $\gamma \in \mathrm{Spin}(n)$ and $y \in \mathrm{P}_{\mathrm{Spin}}(M)$. If (M, g) is spin, its *spinor bundle* is defined as

$$\Sigma(M) = \mathrm{P}_{\mathrm{Spin}}(M) \times_{\rho'} \Delta_n,$$

where $\rho'\colon \mathrm{Spin}(n) \to \mathrm{End}(\Delta_n)$ comes from an irreducible representation $\mathrm{Cl}_n \to \mathrm{End}(\Delta_n)$. The peculiarity of its sections, the so-called *spinors*, is that they can be multiplied by vectors and forms; the existence of this multiplication is a consequence of the fact that ρ' extends to a map $\mathrm{Cl}_n \to \mathrm{End}(\Delta_n)$. Moreover, the Levi-Civita connection lifts to the spinor bundle, and this allows one to define partial differential equations for spinors without introducing additional information. This is the case for the *harmonic* condition, characterized by being in the kernel of the *Dirac operator*. This is a self-adjoint first order operator; its expression in terms of a local orthonormal frame $(e_1, \ldots, e_n)$ is the following:

$$\not{D}\eta = \sum_{i=1}^{n} e_i \nabla_{e_i} \eta.$$

Friedrich proved in [53] that the first eigenvalue λ of the Dirac operator is related to the scalar curvature by the inequality $\lambda^2 \geq \frac{n}{4(n-1)} \min_{p \in M}\{\mathrm{scal}_p\}$. He also proved that both sides are equal if there is a *Killing spinor*, defined by the equation $\nabla_X \eta = \mu X \cdot \eta$. These were introduced before in the context of general relativity, but Killing spinors first appeared in [53] in the area of Riemannian geometry. The relationship between harmonic spinors and geometric structures is explored later in this introduction, as it is part of the work developed in Chapter 2. Killing spinors determine nearly parallel G_2 structures and *nearly Kähler* SU(3) structures, characterized by the conditions $d\omega = 3\mathfrak{Re}(\Theta)$ and $d\mathfrak{Im}(\Theta) = -2\omega^2$. Examples of Riemannian manifolds carrying these structures are the spheres S^7 and S^6, equipped with their standard metrics.

Certain types of geometric structures affect curvature properties. The Riemannian curvature $\mathcal{R}$ satisfies $\mathcal{R} \in \mathrm{Sym}^2(\mathfrak{hol}(g))$ pointwise; this gives additional constraints on the Ricci tensor in the case of special and exceptional holonomy. The Ricci tensor of a Kähler manifold is determined by the so-called *Ricci* 1*-form*, if $\mathrm{Hol}(g)$ is simply connected, then the manifold is Ricci-flat, and if $\mathrm{Hol}(g) = \mathrm{Sp}(k) \cdot \mathrm{Sp}(1)$, then the metric is Einstein. In fact, known examples of compact odd-dimensional simply connected manifolds endowed with a Ricci flat metric are exactly compact 7-manifolds with holonomy G_2. In the case of G structures, the Ricci tensor is determined by the torsion forms, as expressed in [23] and [69] for the case of G_2 and Spin(7) structures. The most illustrative example is that G structures determined by Killing spinors are associated with Einstein metrics; this property is a consequence of the formula obtained in [54, p.64], which relates the Ricci tensor of the metric to the covariant derivative of a spinor. Ricci flatness for manifolds with a simply connected holonomy group can also be proved by combining this formula with Wang's theorem.

The interplay between special and exceptional holonomy and cohomological properties is well known in the case of compact Kähler manifolds: these are formal and their de Rham cohomology algebra admits a Hodge decomposition and satisfies the hard Lefschetz property. The Weitzenböck formula for the Laplacian allows us to generalize the Hodge decomposition to compact Riemannian manifolds (M, g) with holonomy contained in a group $G \subset \mathrm{SO}(n)$ from Berger's list. The space of harmonic forms $\mathcal{H}^k(M, \mathbb{R})$ admits a decomposition into a direct sum of subspaces determined by the irreducible components of the representation of G in $\Lambda^k(\mathbb{R}^n)^*$; this allows the definition of the *refined Betti numbers*. More precisely, let $\Lambda^k(\mathbb{R}^n)^* = \oplus_{i \in I} \Lambda^k_i$ be the direct sum decomposition into G irreducible subspaces. There exists a decomposition $\Omega^k(M) = \oplus_{i\in I} \Omega^k_i(M)$ and the Weitzenböck formula guarantees that the Laplacian preserves each $\Omega^k_i(M)$. From this it follows that

$$\mathcal{H}^k(M) = \oplus_{i \in I} \mathcal{H}^k_i(M).$$

Moreover, if two representations Λ^k_i and Λ^l_j are isomorphic, then $\mathcal{H}^k_i(M) \cong \mathcal{H}^l_j(M)$. The refined Betti numbers are $b^k_i = \dim(\mathcal{H}^k_i(M))$. Further obstructions arise if the holonomy equals G; for example, manifolds with holonomy G_2 and Spin(7) have $b_1 = 0$.

The formality property for compact Kähler manifolds was proved by Deligne, Griffiths, Morgan, and Sullivan in [40] and is a consequence of the $\partial\overline{\partial}$-Lemma. The notion of formality comes from the *rational homotopy theory* founded by Sullivan in [107]. This theory is devoted to the study of the torsion-free part of higher homotopy groups $\pi_k(M) \otimes \mathbb{Q}$, $k \geq 2$, and introduces algebraic objects such as *commutative differential graded algebras over* $\mathbb{Q}$ (CDGAs for short) and their *minimal models*. The minimal model of a CDGA $(\mathcal{A}, d)$ is a minimal CDGA (see Definition 4.14) $(\mathcal{M}, d)$ and a homomorphism $\Psi\colon (\mathcal{M}, d) \to (\mathcal{A}, d)$ that induces an isomorphism between their cohomology algebras.

Let M be a connected simplicial complex of finite type and let $(\mathcal{A}_{PL}(M), d)$ be the CDGA of rational polynomial forms. A rational polynomial k-form on M consists of a collection $\{\omega_\sigma\}_{\sigma\subset M}$ of k-forms on the simplices $\sigma \subset M$ whose coefficients are polynomials over $\mathbb{Q}$. These k-forms are compatible, i.e., if $\sigma \subset \partial\sigma'$ then $\omega_\sigma = \omega_{\sigma'}|_\sigma$. The PL de Rham theorem guarantees that the cohomology of $(\mathcal{A}_{PL}(M), d)$ is $H^*(M, \mathbb{Q})$. The invariant introduced by Sullivan is the *minimal model* of M, defined as the minimal model of the CDGA $(\mathcal{A}_{PL}(M), d)$. This always exists and is unique up to isomorphism. The explicit relation between rational homotopy groups and minimal models was found in [107, Theorem 10.1]:

Theorem 3. *Let M be a connected nilpotent simplicial complex of finite type and let $(\mathcal{M}, d)$ be its minimal model. If $k \geq 2$ the rational homotopy group $\pi_k(M) \otimes \mathbb{Q}$ is dual to the space of degree k generators of $\mathcal{M}$.*

The hypothesis that M is *nilpotent* requires that $\pi_1(M)$ is nilpotent and acts on $\pi_k(M)$ as a nilpotent homomorphism. If the minimal model of $(\mathcal{A}_{PL}(M), d)$ is equal to the minimal model of $(H^*(M, \mathbb{Q}), d = 0)$ we say that M is *formal*. Computing the minimal model is a formal procedure, and this explains the name of the property: rational homotopy groups of formal spaces are formally obtained from rational cohomology groups.

If M is a smooth manifold, the real minimal model is constructed from the de Rham complex $(\Omega^*(M), d)$. In practice, computing the minimal model may be quite involved; the notion of *s-formality* has become customary to decide whether a manifold is formal or not. Briefly, this property depends on the generators of the minimal model whose degree is less than or equal to s. Poincaré duality property allows to prove in [49] that a compact oriented manifold of dimension $2n$ or $2n-1$ is formal if and only if it is $(n-1)$-formal. Moreover, non-vanishing *Massey products* are often used to show that a manifold is non-formal. For their precise definition and their relation to formality, see [100, Section 1.6].

The result in [40] implies that compact manifolds with holonomy $\mathrm{SU}(m)$ and $\mathrm{Sp}(k)$ are formal. Compact manifolds with holonomy contained in $\mathrm{Sp}(k)\cdot\mathrm{Sp}(1)$ that have positive scalar curvature are also formal [4]; the proof takes advantage of the formality of compact Kähler manifolds. It remains to determine whether compact manifolds with exceptional holonomy are formal or not. There are partial results collected in [29], [38] and [76]. The results in [29] and [76] are based on an idea of Verbitsky in [111], where he defines a differential operator $\mathcal{L}_\omega$ on a Kähler manifold (M, g, J) to give an alternative proof of the formality of Kähler manifolds. This operator is well-defined on Riemannian manifolds endowed with a parallel k-form; the study of operators $\mathcal{L}_\varphi$, $\mathcal{L}_{\star\varphi}$ or $\mathcal{L}_\Omega$ defined by φ, $\star\varphi$ or Ω for the case where the holonomy is contained in G_2 or $\mathrm{Spin}(7)$ has proved fruitful but does not answer the question of the formality of such manifolds. Moreover, the paper [38] focuses on the case of 7-manifolds; among other results, the authors show that a non-formal compact manifold with holonomy G_2 should have $b_2 \geq 4$.

The search for compact examples of manifolds with certain types of geometric structures often begins with *nilmanifolds and solvmanifolds*. These spaces arise as compact quotients of a Lie group G by a lattice Γ; the Lie group is nilpotent in the first case and solvable in the second. Nilmanifolds and solvmanifolds are special from a topological point of view. These are aspherical spaces and satisfy that $\pi_1(\Gamma\backslash G) = \Gamma$; nilmanifolds have first Betti number $b_1 \geq 2$ and solvmanifolds have $b_1 \geq 1$. Nomizu's theorem [98] states that the real minimal model of a nilmanifold $\Gamma\backslash G$ is determined by its Chevalley-Eilenberg CDGA $(\Lambda\mathfrak{g}^*, d)$, where the differential d is determined by $d\alpha(X, Y) = \alpha[X, Y]$ if $\alpha \in \mathfrak{g}^*$. Of course, we denote by $\mathfrak{g}$ the Lie algebra of G . Hattori's theorem [64] states that the Chevalley-Eilenberg CDGA is a model for a subclass of solvmanifolds, but it may not be minimal. This subclass is that of *completely solvable solvmanifolds* and consists of those in which the adjoint endomorphisms $\mathrm{ad}(X)\colon \mathfrak{g} \to \mathfrak{g}$, $X \in \mathfrak{g}$, have only real eigenvalues. Moreover, non-abelian nilmanifolds are non-formal [62], while solvmanifolds can be both formal and non-formal.

The geometric structures that we employ are induced by left-invariant geometric structures on the Lie group. Metrics underlying them have special curvature properties; according to [91], these are either flat or have strictly negative scalar curvature. In addition, non-flat metrics are not Einstein. Partial differential equations determining that the geometric structure belongs to a specific class are transformed into a system of equations involving the structure equations of the Lie algebra. Of course, this approach simplifies the problem and it is the reason why we frequently refer to geometric structures on nilpotent or solvable Lie algebras. Nilpotent Lie algebras of dimensions less or equal to 7 are classified, see [14] and [58]; with the classification at hand, several papers are devoted to determining which nilpotent Lie algebras admit a G structure in some specific class.

The behavior of such geometric structures is wide but limited. An illustrative example is the case of the Kodaira-Thurston manifold, a 4-dimensional nilmanifold which was the first known symplectic manifold that cannot be endowed with a Kähler structure. Of course, since non-abelian nilmanifolds are non-formal, these are not Kähler. In addition, non-abelian completely solvable solvmanifolds do not admit metrics with holonomy contained in G_2 or $\mathrm{Spin}(7)$. According to the Cheeger-Gromoll Splitting Theorem, if that were the case for a completely solvable solvmanifold $(\Gamma\backslash G, g)$, its universal covering would be $\mathbb{R}^k \times N$ with $k = b_1(\Gamma\backslash G)$ and N simply connected and compact. The universal covering of $\Gamma\backslash G$ is G, which is diffeomorphic to $\mathbb{R}^7$ or $\mathbb{R}^8$. Therefore, $b_1(\Lambda\mathfrak{g}^*, d) = 7, 8$ and G is necessarily abelian. Similarly, some types of non-integrable geometric structures do not occur in nilmanifolds and solvmanifolds. This is the case for those that induce positive scalar curvature metrics, such as nearly Kähler $\mathrm{SU}(3)$ structures and nearly parallel G_2 structures. The same holds for a subclass of locally conformally parallel (LCP for short) G_2 and $\mathrm{Spin}(7)$ structures on both nilmanifolds and solvmanifolds. The latter is described by the equation $d\Omega = \theta \wedge \Omega$; the 1-form θ is also called the Lee form of the structure. If the Lee form is coclosed, the scalar curvature of the metric associated with the LCP structure is positive. Moreover, manifolds admitting an LCP structure with nowhere-vanishing Lee form satisfy a structure theorem [70]. They are mapping tori of a manifold N whose universal cover is compact. In the case of G_2, N has a nearly Kähler $\mathrm{SU}(3)$ structure, and in the case of $\mathrm{Spin}(7)$, N is endowed with a nearly parallel G_2 structure. It follows from the characterization that both nilmanifolds and solvmanifolds do not admit left-invariant LCP structures.

Topological properties of nilmanifolds and solvmanifolds being restrictive, orbifold resolution is a way to construct compact examples of G structures in manifolds with different topological properties. This is the case for the simply connected symplectic manifolds in [11] and [50]. Finite group actions on nilmanifolds are not hard to construct. If the action preserves a left-invariant G structure, the orbit space of the nilmanifold by the group determines an orbifold with a G structure. Its desingularization, if possible, yields a manifold with such G structure and different topological properties. This procedure is discussed in more detail later in the introduction.

We now proceed to the discussion of the main results of each chapter. We divide it into two parts: the purpose of the first part is to study $\mathrm{Spin}(7)$ structures from the point of view of spinor theory, the second part is devoted to the resolution of symplectic and G_2

orbifolds. The works that support the publication of this thesis as a compendium of papers are [85, 86, 87]. The papers [85] and [87] correspond to the second part of the thesis and are respectively contained in Chapters 4 and 3. The paper [86] is included in the first part and corresponds to Chapter 1. The preprint [12] is currently being revised for publication and complements the work developed in the article [86]. Therefore, its presentation is relevant to the state of the art of this thesis. To make the exposition clearer, we present its content in Chapter 2 instead of doing it in this introduction.

A spinorial approach to Spin(7) manifolds and geometric structures defined by spinors

Since Fernández classified non-integrable Spin(7) structures [43], only a few papers have been devoted to their study. One reason is that there are still many open problems concerning G_2 structures. In addition, the classification of Spin(7) structures is small: there are only 4 classes of Spin(7) structures, compared with the 16 classes of G_2 structures and $U(m)$ structures. A special feature of Spin(7) geometry is that the Spin(7) form is closed if and only if it is parallel, and the classes are determined by $d\Omega$. The space $\Lambda^5(\mathbb{R}^8)^*$ decomposes as a direct sum of two Spin(7)-invariant subspaces and thus non-integrable pure Spin(7) classes are:

1. Locally conformally parallel, if $d\Omega = \theta \wedge \Omega$ for a closed 1-form θ.
2. Balanced, if $(\star d\Omega) \wedge \Omega = 0$.

In Chapter 1 we use the spinor approach to rewrite the classification of Spin(7) structures in terms of the covariant derivative of the spinor defining the structure. This motivates a method for constructing balanced examples in Chapter 2 and suggests the introduction of a new class of geometric structures in low dimensions: *spin-harmonic structures*.

Spinorial classification of Spin(7) manifolds

Chapter 1 of this thesis is devoted to the study of Spin(7) structures from the point of view of spinor geometry. This work continues the formalism developed in [1] for the case of SU(3) and G_2 structures and complements the paper [69], which also uses spinors as a tool to study Spin(7) structures. Moreover, this approach serves to recover the results in [83] and [84] about G_2 structures on hypersurfaces of manifolds with a Spin(7) structure and Spin(7) structures on S^1-principal bundles over G_2 manifolds. This approach turns out to be useful for the construction of balanced and locally conformally balanced Spin(7) structures on quasi abelian Lie algebras.

The first part of this work consists in rewriting the classification of Spin(7) structures in terms of spinors. To establish the set-up we first recall that, Cl_8 being isomorphic to $\mathbb{R}(16)$, the spinor representation is $\Delta_8 = \mathbb{R}^{16}$. This space decomposes into two 8-dimensional subspaces, namely the positive and negative spaces $\Delta_\pm$, which are the eigenspaces of the endomorphism determined by multiplication by the volume element, namely $e_0 \cdots e_8 \in Cl_8$, and hence Spin(8)-invariant. The stabilizer of a nonzero spinor lying in the positive or the negative subspace under the action of Spin(8) is isomorphic to Spin(7); the images of these subgroups by the adjoint map $\mathrm{Ad}\colon \mathrm{Spin}(8) \to \mathrm{SO}(8)$ are not conjugate in SO(8), but they are conjugate in O(8).

Let (M, g) be a spin Riemannian 8-dimensional manifold; the decomposition $\Delta_8 = \Delta_+ \oplus \Delta_-$ gives a splitting of the spinor bundle, $\Sigma(M) = \Sigma^+(M) \oplus \Sigma^-(M)$. As stated in Proposition

1.8, a unit-length spinor η in $\Sigma^+(M)$ yields a Spin(7) structure by means of the expression:

$$\Omega(W,X,Y,Z) = \frac{1}{2}((-WXYZ + WZYX)\eta, \eta).$$

Moreover, Proposition 1.13 proves that the covariant derivative of η contains the same information as the intrinsic torsion of the structure Γ. The precise relation between them allows us to prove:

Theorem A (Theorem 1.21). *The* Spin(7) *structure determined by a spinor* η *is,*

1. *Parallel if* $\nabla\eta = 0$.
2. *Balanced if* $\not{D}\eta = 0$.
3. *Locally conformally parallel if there exists* $V \in \mathfrak{X}(M)$ *such that* $\nabla_X\eta = \frac{2}{7}(X^* \wedge V^*)\eta$. *In this case,* $\not{D}\eta = V\eta$.

The Dirac operator plays a central role in the classification because it determines the *Lee form* of the structure. This is defined as $\theta = -\frac{1}{7} \star (\star(d\Omega) \wedge \Omega)$; in terms of the spinorial description $\theta = \frac{8}{7}V^*$ as Proposition 1.23 states. The geometric condition that the Spin(7) structure is balanced yields a harmonic spinor; the spinor is thus a solution to a partial differential equation that is interesting from the analytical point of view.

The way we obtain spinor equations differs from the approach in [1]. If ϕ is the spinor that determines a G structure then $\nabla_X\phi = \frac{1}{2}\Gamma(X)\phi$; here Γ denotes the intrinsic torsion of the G structure, and $\mathrm{G} \in \{\mathrm{SU}(3), \mathrm{G}_2, \mathrm{Spin}(7)\}$. Let (N, g, J, Θ) be a SU(3) structure, there are $\gamma \in \Omega^1(N)$ and $\mathcal{S}_N \in \mathrm{End}(TN)$ such that $\Gamma = i(\mathcal{S}_N)\mathfrak{Re}\,(\Theta) - \frac{2}{3}\gamma \otimes \omega$, with $(i(\mathcal{S}_N)\mathfrak{Re}\,(\Theta))(X,Y,Z) = \mathfrak{Re}\,(\Theta)(\mathcal{S}_N(X), Y, Z)$. Let (Q, g, φ) be a G_2 structure, there is $\mathcal{S}_Q \in \mathrm{End}(TQ)$ such that $\Gamma = -\frac{2}{3}i(\mathcal{S}_Q)\varphi$. These equalities hold because $\mathfrak{su}(3)^\perp = \langle\omega\rangle \oplus i(\mathbb{R}^6)\mathfrak{Re}\,(\Theta)$, and $\mathfrak{g}_2^\perp = i(\mathbb{R}^7)\varphi$. Let ϕ_N and ϕ_Q be the spinors that determine the geometric structures on N and Q. According to [1, Lemmas 2.2 and 2.3] we have:

$$\begin{aligned} \nabla_X\phi_N =& \frac{1}{2}\Gamma(X)\phi_N = \mathcal{S}_N(X)\phi_N + \gamma(X)\mathrm{j}(\phi_N), \\ \nabla_X\phi =& \frac{1}{2}\Gamma(X)\phi_Q = \mathcal{S}_Q(X)\phi. \end{aligned}$$

where j is a complex structure on $\Sigma(N)$ that anticommutes with the Clifford product by a vector field (see subsection 2.2.2). Observe that $\mathbb{R}^8$ is not contained in $\mathfrak{spin}(7)^\perp$ as a subrepresentation (see subsection 1.2.3); in addition, $\nabla_X\eta \in \Sigma^+(M)$ and $\mathcal{S}(X)\eta \in \Sigma^-(M)$. For this reason, we work with the equation $\nabla_X\eta = \frac{1}{2}\Gamma(X)\eta$.

In this work, we introduce the notion of a G_2 *distribution*: a 7-dimensional cooriented distribution with a G_2 structure in a Spin(7) manifold. This is a systematic approach that serves to unify various geometric situations involving G_2 and Spin(7) structures, namely G_2 hypersurfaces of Spin(7) manifolds, warped products of a G_2 manifold with $\mathbb{R}$, and S^1-principal bundles with base a G_2 manifold; some of these have already been studied by Martín-Cabrera in [83] and [84]. For example, if Q is a hypersurface of a Spin(7) manifold (M, g, Ω), there is an induced G_2 structure $\varphi = i(N)\Omega$, where N is a unit normal vector field. The type of the G_2 structure φ depends on the class of the Spin(7) structure Ω and the Riemannian properties of the embedding, as Theorem 1.39 shows. The key idea of this part of the chapter, which is also exploited in Chapter 2, is the following: the spinor that determines the Spin(7) structure of the ambient manifold also induces the G_2 structure of the distribution. That is, a single object encodes all the geometric information.

The formalism of G_2 distributions enables us to deal with left-invariant Spin(7) structures on quasi abelian Lie groups. A motivation for focusing on such Lie groups is that the study of G_2 structures on quasi-abelian Lie groups has been fruitful. In [51] the author determined quasi-abelian Lie algebras that admit a coclosed G_2 structure. In [52], these examples served him to construct cohomegneity-one manifolds with holonomy SU(4) by solving the Hitchin flow equation.

Quasi abelian Lie groups are semidirect products $\mathbb{R} \ltimes_\epsilon \mathbb{R}^7$, where $\varepsilon = \exp(\mathrm{ad}(\mathcal{E}))$ with $\mathcal{E} \in \mathbb{R}(7)$. Of course, these are solvable Lie groups. A left-invariant Spin(7) structure on $\mathbb{R} \ltimes_\epsilon \mathbb{R}^7$ restricts to a parallel G_2 structure on the hypersurfaces $\{t\} \times \mathbb{R}^7$. The type of the Spin(7) structure depends exactly on the endomorphism $\mathcal{E}$, as proved in Theorem 1.49. Pure classes of Spin(7) structures are obtained by imposing a certain restriction on the complex eigenvalues of the skew-symmetric part of $\mathcal{E}$. Moreover, the trace of $\mathcal{E}$ determines the component of the Lee form that is parallel to dt. From this investigation, we obtain compact examples by finding a lattice; this only occurs when $\mathcal{E}$ is traceless. As we explained before, solvmanifolds do not admit invariant locally conformally parallel Spin(7) structures, so we search for balanced Spin(7) structures. In the section 1.8 we give the first example of a balanced Spin(7) manifold with $b_1 = 2$, which is not a product $S^1 \times N^7$.

Our results allow us to tackle classification problems of Spin(7) structures in quasi abelian nilpotent Lie algebras of which there are 14 up to isomorphism. We determine those that admit a balanced structure or *a strict locally conformally balanced* Spin(7) structure. The latter are defined in the context of supergravity theory and satisfy the condition that the Lee form is closed and non-zero. Our analysis concludes the following:

Theorem B (Theorem 1.4)**.** *Let L_3 be the Lie algebra of the 3-dimensional Heisenberg group, let L_4 be the unique indecomposable 4-dimensional nilpotent Lie algebra, and let A_k be the k-dimensional abelian Lie algebra.*

1. *Every* Spin(7) *structure on the abelian Lie algebra A_8 is parallel.*
2. *The Lie algebras $\mathfrak{g} = A_5 \oplus L_3$ or $\mathfrak{g} = A_3 \oplus L_4$ admit strict locally conformally balanced structures but they do not admit balanced structures.*
3. *The remaining quasi abelian nilpotent Lie algebras admit a balanced structure and a strict locally conformally balanced structure.*

Spin-harmonic structures and nilmanifolds

The goal of Chapter 2 is to construct balanced Spin(7) structures on 8-dimensional nilmanifolds. We make use of the spinor equations obtained in Chapter 1. Our approach leads us to introduce a new class of geometric structures on low-dimensional manifolds: *spin-harmonic structures.*

The first compact manifold endowed with a balanced Spin(7) structure was obtained in [46] and consists of a product of a 5-dimensional nilmanifold with a 3-torus. Later, thanks to the work in [83] and [84], more compact examples were provided, such as the products $N \times S^1$ with (N, g, φ) a G_2 structure which is closed or *purely coclosed.* The last is defined by the conditions $d \star \varphi = 0$ and $d\varphi \wedge \varphi = 0$. In this chapter we restrict ourselves to Riemannian nilmanifolds $(N^6 \times T^2, g_6 + g_2)$, where (N^6, g_6) is a 6-dimensional nilmanifold and (T^2, g_2) is the flat torus; the Spin(7) structure is also assumed to be invariant in the T^2 direction. The reason for the simplification is that 8-dimensional nilpotent Lie algebras are not classified and the list of 7-dimensional nilmanifolds is quite extensive. We separately analyze the case where $N^6 = N^5 \times S^1$ and $g_6 = g_5 + g_1$, with g_1 the flat metric on S^1; our study allows us to recover the Spin(7) structure in [46].

The Spin(7) structure on $N^6 \times T^2$ induces an SU(3) structure on N^6, or an SU(2) structure on N^5 if $N^6 = N^5 \times S^1$. According to [35], the forms $(\alpha, \omega_1, \omega_2, \omega_3) \in \Omega^1(N^5) \times \Omega^2(N^5)^3$ determine an SU(2) structure if

1. $\omega_i \wedge \omega_j = 0$ for $i \neq j$, $\omega_1^2 = \omega_2^2 = \omega_3^2$ and $\alpha \wedge \omega_1^2 \neq 0$,
2. If $i(X)\omega_1 = i(Y)\omega_2$, then $\omega_3(X, Y) \geq 0$.

Equations for the SU(3) and the SU(2) structure are derived from the balanced condition for the Spin(7) structure. These do not correspond to any class according to [1, Theorem 3.7] and Corollary 2.39. Being the equations quite complicated, we use the spinorial approach developed in Chapter 1. This consists in finding harmonic spinors on $N^k \times T^{8-k}$ for $k \in \{5, 6\}$. For this purpose, we divide our strategy into three steps: a dimensional reduction, a choice of spin structure on the nilmanifold, and a formula for the Dirac operator in terms of the structure equations.

The dimensional reduction consists in relating the harmonic spinor on $N^k \times T^{8-k}$ to a spinor on N^k. The spinor bundle of N^k turns out to be the pullback of the spinor bundle $\Sigma^+(N^k \times T^{8-k})$ by the inclusion; this is deduced from the fact that $\mathrm{Cl}_5 = \mathbb{C}(4)$ and $\mathrm{Cl}_6 = \mathbb{R}(8)$. As a consequence of our assumptions for the Spin(7) structure, there is a unique way to define a spinor $\eta' \in \Sigma(N^k)$ starting from a positive spinor $\eta \in \Sigma^+(N^k \times T^{8-k})$; the spinor η is harmonic if and only if η' is harmonic. Motivated by the dimensional reduction, we define the notion of a *spin-harmonic structure* as the geometric structure determined by a unit-length harmonic spinor; equations in terms of the forms defining the structure were obtained in [1] for G_2 and SU(3) structures; in section 2.4 we compute them for SU(2) structures.

We restrict our attention to the spinor which determines a left-invariant geometric structure: we endow the nilmanifold with its trivial spinor bundle and choose constant spinors. That is, geometric properties are determined by the Lie algebra and do not depend on the lattice. Finally, we obtain a formula for the Dirac operator of such a spinor in terms of the structure equations of the Lie algebra:

Proposition C (Proposition 2.41)**.** *Suppose that $(e_1, \ldots, e_n)$ is an orthonormal frame and let ϕ be a left-invariant spinor on a solvable Lie algebra. Then*

$$4\not{D}\phi = -\sum_{i=1}^{n} (e^i \wedge de^i + i(e_i)de^i)\phi.$$

In the case of 6-dimensional nilmanifolds, we solve the equation $\not{D}\phi = 0$ directly. The strategy for finding left-invariant spin-harmonic structures on 5-dimensional nilmanifolds is to compute the square of the Dirac operator $\not{D}^2$. This approach allows us to determine all the left-invariant metrics that admit harmonic spinors. In this case, according to Proposition 2.50, if ϕ is a left-invariant spinor then:

$$\not{D}^2\phi = \mu\phi + v\mathrm{j}_1\phi.$$

Here $\mu > 0$ and $v \in \mathfrak{X}(N^5)$ are determined by the metric and the structure equations of the Lie algebra. Of course, v is left-invariant. In addition, the map j_1 is a complex structure on the space of spinors; it exists because $\mathrm{Cl}_5 \cong \mathbb{C}(4)$. From this formula, one derives that metrics that admit harmonic spinors are characterized by the equation $\|v\| = \mu$; moreover, the space of harmonic spinors is 4-dimensional. In this case, the vector v has a geometric interpretation: the form obtained by the musical isomorphism v^* is proportional to α. The following theorem summarizes our findings:

Theorem D (Theorems 2.53, 2.58, Subsection 2.6.3 and Proposition 2.59)**.** *Let N^k be a k-dimensional nilmanifold and let $\mathfrak{n}$ be the Lie algebra of its universal covering. Suppose in addition that $\mathfrak{n}$ is non-abelian.*

1. *If $k = 5$ and N admits left-invariant spin-harmonic structure, then $\mathfrak{n} = \mathrm{L}_{5,j}$, $j = 1, 2, 3, 4, 6$.*
2. *If $k = 6$ and N does not admit a left-invariant spin-harmonic structure then $\mathfrak{n}$ equals $\mathrm{L}_3 \oplus \mathrm{A}_3$ or $\mathrm{L}_4 \oplus \mathrm{A}_2$.*
3. *The Lie algebras $L_3 \oplus A_5$ and $L_4 \oplus A_4$ do not admit balanced* Spin(7) *structures.*

The results from Chapter 2 suggest that there are many balanced Spin(7) structures. This phenomenon is related to the result of Hitchin in [67], which states that every 8-dimensional spin manifold admits a harmonic spinor. However, this spinor need not determine a balanced Spin(7) structure, since it could have zeros. Moreover, the equation $\not{D}\eta = 0$ is overdetermined; both facts should lead us to investigate whether there is an h-principle in the sense of Gromov for such a structure.

Orbifolds with geometric structures and its resolution

Introduced by Satake in [106] as V-manifolds, orbifolds have been studied from different points of view and have proved useful in numerous geometric contexts. Orbifolds are locally modeled on $\mathbb{R}^n/\Gamma$, where Γ is a finite subgroup of $\mathrm{SO}(n)$, so they have *singularities.* In the local model, these are the orbits of points that are fixed by a non-identity element of Γ. Several objects coming from differential geometry are also useful in the context of orbifolds: metrics, forms, bundles, and operators.

In this thesis, we present a method for resolving orbifolds with symplectic structures or closed G_2 structures. We aim to obtain manifolds with such geometric structures satisfying some specific topological properties. The orbifolds we start with are usually the quotient of a manifold by a finite group of diffeomorphisms preserving the geometric structure. Some topological properties of the resolution, such as the fundamental group or cohomology groups, can be derived from those of the orbifold and the singular locus; see, e.g., Proposition 4.38.

Celebrated examples produced by orbifold desingularization are Joyce's compact manifolds with holonomy G_2 and Spin(7). His strategy is based on both orbifold resolution techniques and analytic existence theorems. These orbifolds are obtained as quotients of 7 or 8-dimensional flat tori under the action of a group preserving the geometric structure. Under mild assumptions on the singular locus, one can use techniques of algebraic geometry to resolve the orbifold and endow it with a 1-parameter family of geometric structures whose torsion tends to 0. Theorems 11.6.1 and 13.6.1 in [74] guarantee that the perturbation of the family is torsion-free. In both cases, the action of the group is constructed in such a way that the fundamental group of the orbifold is finite; in the case of Spin(7), the resolution has $\hat{A}(M) = 1$. These topological properties ensure that the holonomy of the manifold is G_2 or Spin(7).

Resolution of 4-dimensional symplectic orbifolds

Chapter 3 is devoted to the proof of the existence of a resolution for compact 4-dimensional symplectic orbifolds, using techniques from algebraic geometry as in the papers [11], [50] and [93].

From the point of view of differential geometry, classical theorems of symplectic geometry are adapted to the context of orbifolds; see [93] for a precise approach. An example is the existence of Darboux charts, which are of the form (U, ω_0) with $U \subset \mathbb{C}^m/\Gamma$; the isotropy group Γ is a subgroup of $\mathrm{U}(m)$ and ω_0 is the standard symplectic form. Other examples include the construction of a compatible almost complex structure and the normal bundle around a singularity. A notable achievement of the resolution of symplectic orbifold techniques is a counterexample to the *Thurston-Weinstein conjecture* in dimension 8 [50]. This conjecture states that a simply connected symplectic manifold of dimension greater than or equal to 8 is formal, and was first proved false in [5] in dimension ≥ 10. Another remarkable example is the construction of a 6-dimensional simply connected non-Kähler manifold which is both complex and symplectic [11].

The resolution procedure in these examples is ad-hoc and involves techniques derived from the resolution of algebraic singularities. These techniques have already been used to desingularize symplectic orbifolds in [28], where the authors prove that such a resolution exists if the singularities are *isolated points*; we briefly discuss the strategy. In this situation, the unique fixed point of a non-identity element of the isotropy is 0. Therefore, its resolution consists in replacing a neighbourhood of the singular point in the orbifold by a neighbourhood of the exceptional divisor of a projective resolution of the quotient singularity $\mathbb{C}^m/\Gamma$, constructed as in the classical works of Hironaka [65] and [66]. The symplectic form is obtained by interpolating the Kähler form of the resolution with ω_0 by the *inflation process* introduced by Thurston in [108].

It has not been proved that every symplectic orbifold admits a symplectic resolution. As we outline in the introduction to Chapter 3, there are special cases for which desingularization is possible. In Chapter 3 we prove:

Theorem E (Theorem 3.26)**.** *Let (X, ω) be a compact symplectic 4-orbifold. There exists a symplectic manifold $(\tilde{X}, \tilde{\omega})$ and a smooth map $\pi : (\tilde{X}, \tilde{\omega}) \to (X, \omega)$ which is a symplectomorphism outside an arbitrarily small neighbourhood of the singular set of X.*

This theorem was previously proved by Chen in [30], using techniques from symplectic geometry, such as symplectic fillings of contact manifolds and symplectic reduction. Our method is different and follows the ideas of [28] and its generalization [93]. The paper [93] deals with orbifolds with *homogeneous isotropy*, i.e., those whose singular components do not intersect each other. In this case, the desingularization takes place in the normal bundle, which has a complex singularity in the fiber; to make the resolution of different fibers compatible one with each other, the authors require the algebraic resolution of [41] instead of Hironaka's classical theorems. The special feature of the resolution in [41] is that it is equivariant under the action of groups.

Symplectic 4-dimensional orbifolds have the advantage that the configuration of the singularities is simpler than in higher-dimensional symplectic orbifolds. This follows from the fact that non-identity elements in $\mathrm{U}(2)$ fix the origin or a complex line. Apart from isolated singularities, we define the singular subsets Σ^* and Σ^1 by means of a Darboux chart (U, ω_0) with $U \subset \mathbb{C}^2/\Gamma$:

1. $x \in \Sigma^*$ if there exists a complex line $L \subset \mathbb{C}^2$ such that for every $1 \neq \gamma \in \Gamma$ we have $\mathrm{Fix}(\gamma) = L$.

2. $x \in \Sigma^1$ if there exist at least two complex lines $L_1, L_2 \subset \mathbb{C}^2$ and $\gamma_1, \gamma_2 \in \Gamma$ so that $L_1 = \mathrm{Fix}(\gamma_1)$ and $L_2 = \mathrm{Fix}(\gamma_2)$.

The connected components of Σ^* are surfaces and Σ^1 contains intersections of the closure of non-closed components of Σ^*. The challenging part of the resolution is precisely to make compatible the resolutions of different singular surfaces whose closures intersect at a point in Σ^1. Points in Σ^* have neighbourhoods contained in $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_m)$, which is topologically a manifold. There are several ways to find a resolution of this local model, but we choose to endow the quotient with the structure of a complex manifold and change the symplectic form by a perturbation. To make this construction global, one has to construct the normal bundle of the singularity and introduce a connection. Moreover, it is possible to change the local model of $x \in \Sigma^1$ to obtain a local model in which x is an isolated singularity. This is proved by first arguing that $\mathbb{C}^2/\Gamma = (\mathbb{C}^2/\Gamma')/(\Gamma/\Gamma')$, where Γ' is the normal subgroup formed by elements fixing some complex line. Then one proves that $\mathbb{C}^2/\Gamma'$ is a smooth complex manifold using results from invariant group theory and finally one observes that Γ/Γ' acts freely on $(\mathbb{C}^2 - \{0\})/\Gamma'$.

As a consequence of this discussion, the strategy for resolving a symplectic 4-dimensional orbifold without isolated singularities has 4 steps. First, we define a manifold atlas in $X - \Sigma^1$ and a closed 2-form ω' which is zero in a small neighbourhood of Σ^1 and symplectic outside of it. The Riemann extension theorem allows us to extend this atlas as an orbifold atlas on X such that the singularities are isolated. Then, an orbifold symplectic form is constructed from ω' and we finally proceed as in [28] to resolve the isolated singularities.

A compact non-formal closed G_2 manifold with $b_1 = 1$

The purpose of Chapter 4 is to construct a compact non-formal manifold with $b_1 = 1$ equipped with a closed G_2 structure but admitting no metric with holonomy contained in G_2. This is the first example with such properties. The construction follows the ideas of the paper [47] and it requires the development of a resolution theorem for closed G_2 orbifolds inspired by the paper [75].

The geographical problem concerning topological properties of compact manifolds that admit a closed G_2 structure but cannot be endowed with a torsion-free G_2 structure is far from being understood. As pointed out in the introduction of Chapter 4, prior to the publication of this paper, the known such examples with $b_1 = 1$ were formal [47], [81]. At that time, there was no reason to believe that we could not find a non-formal example with $b_1 = 1$; in fact, the examples in [34] are nilmanifolds and thus, non-formal with $b_1 \geq 2$. It is worth mentioning that the construction of an example with $b_1 = 0$ remains open. As we stated earlier, the main theorem in Chapter 4 is the following:

Theorem F (Propositions 4.44, 4.46)**.** *There exists a compact non-formal closed* G_2 *manifold* M *with* $b_1 = 1$ *which cannot be endowed with a torsion-free* G_2 *structure.*

The construction is based on a resolution procedure already used in [47]. We define a closed G_2 orbifold X as the orbit space of a $\mathbb{Z}_2$ action on a nilmanifold N preserving the closed G_2 structure that was obtained in [34]. The resolution M of X is non-formal because X is non-formal; the reason is that the $\mathbb{Z}_2$ action preserves a non-zero Massey product on N. The non-zero Massey product on X lifts to M by pullback. Moreover, to guarantee that

$b_1(M) = 1$, we construct the action so that $b_1(X) = 1$ because the first Betti number is not changed by the resolution procedure (see Proposition 4.38). Remarkably, the singular locus of the orbifold consists of 16 disjoint copies of the 3-dimensional Heisenberg manifold; to the best of our knowledge this is the first time in which such a configuration occurs.

To desingularize this orbifold, we develop a method for resolving closed G_2 orbifolds, inspired by the work of Joyce and Karigiannis [75], where they resolve the case of an orbifold X that is the quotient of a manifold N with a torsion-free G_2 structure by the action of $\mathbb{Z}_2$; the resolution has a torsion-free G_2 structure. This and the foundational work of Joyce [71], [72], are the only cases of resolution of orbifolds with holonomy contained in G_2 that have been studied so far. For the resolution in [75], they require the additional hypothesis that the singular locus L of the action, which is a 3-dimensional manifold, has a nowhere-vanishing harmonic 1-form. The strategy they follow is similar to that used by Joyce in his foundational work, and is described in the introduction to Chapter 4; let us go into some details for a moment.

The normal bundle to L in N has a complex structure determined by the nowhere-vanishing 1-form; hence the normal bundle to L in X has fiber $\mathbb{C}^2/\mathbb{Z}_2$, whose algebraic resolution is the *Eguchi-Hanson* space (see subsection 4.2.2). Moreover, the hypothesis that the 1-form is closed guarantees that the G_2 structures that they define are closed; the hypothesis that it is co-closed helps to ensure that their torsion is small. Influenced by this work, we prove the theorem:

Theorem G (Theorem 4.32)**.** *Let* (M, φ, g) *be a closed* G_2 *structure on a compact manifold. Suppose that* $\mathrm{j}\colon M \to M$ *is an involution such that* $\mathrm{j}^*\varphi = \varphi$ *and consider the orbifold* $X = M/\mathrm{j}$. *Let* $L = \mathrm{Fix}(\mathrm{j})$ *be the singular locus of* X *and suppose that there is a nowhere-vanishing closed* 1*-form* $\theta \in \Omega^1(L)$. *Then there exists a compact* G_2 *manifold endowed with a closed* G_2 *structure* $(\widetilde{X}, \widetilde{\varphi}, \widetilde{g})$ *and a map* $\rho\colon \widetilde{X} \to X$ *such that:*

1. *The map* $\rho\colon \widetilde{X} - \rho^{-1}(L) \to X - L$ *is a diffeomorphism.*
2. *There exists a small neighbourhood* U *of* L *such that* $\rho^*(\varphi) = \widetilde{\varphi}$ *on* $\widetilde{X} - \rho^{-1}(U)$.

Compared with the work in [75], the lack of need to estimate the torsion is reflected in both the statement and the proof. On the one hand, we require that the nowhere-vanishing 1-form is closed; this condition means that each connected component of the singular locus is a mapping torus over a 2-dimensional manifold. On the other hand, although we use the same strategy to prove the existence of the resolution, some technical parts are somewhat simplified or avoided.

Finally, both arguments we use to prove that the manifold M constructed in Theorem F does not admit a metric with holonomy contained in G_2 (see Proposition 4.46 and Remark 4.47) are based on formality. The manifold M does not satisfy the *almost formality obstruction* obtained in [29], which we briefly recall. The de Rham algebra of a manifold with holonomy contained in G_2 is quasi-isomorphic to a differential algebra with all the differentials 0 except in degree 3. The algebra is constructed from the differential operator $\mathcal{L}_\varphi$. This implies that the Massey products are zero except possibly those $\langle [\alpha], [\beta], [\gamma] \rangle$ such that $|\alpha| + |\beta| = 4$ and $|\beta| + |\gamma| = 4$; here $|\alpha|$ denotes the degree of α and so on. The manifold M is not almost-formal because it has a non-zero Massey product $\langle [\alpha], [\beta], [\gamma] \rangle$ such that $|\alpha| = |\gamma| = 1$ and $|\beta| = 2$.

CHAPTER 1

Spinorial classification of Spin(7) manifolds

Lucía Martín-Merchán

Abstract

We describe the different classes of Spin(7) structures in terms of spinorial equations. We relate them to the spinorial description of G_2 structures in some geometric situations. Our approach enables us to analyze Spin(7) structures on quasi abelian Lie algebras.

MSC classification [2010]: Primary 53C27; Secondary 53C10, 22E25.

Key words: Spin(7)-structures, Spinor, Intrinsic torsion, Characteristic connection, G_2 distributions, Quasi abelian Lie algebras.

1.1 Introduction

Berger's list [17] (1955) of possible holonomy groups of simply connected, irreducible and non-symmetric Riemannian manifolds contains the so-called exceptional holonomy groups, G_2 and Spin(7), which occur in dimensions 7 and 8 respectively. Non-complete metrics with exceptional holonomy were given by Bryant in [22], complete metrics were obtained by Bryant and Salamon in [24], but compact examples were not constructed until 1996, when Joyce published [71], [72] and [73].

The remaining groups of Berger's list different from $\mathrm{SO}(n)$, called special holonomy groups, are $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(k)$ and $\mathrm{Sp}(k) \cdot \mathrm{Sp}(1)$. If the holonomy of a Riemannian manifold is contained in a group G, the manifold admits a G structure, that is, a reduction to G of its frame bundle. Therefore, holonomy is homotopically obstructed by the existence of G structures. Examples of manifolds endowed with G structures for some of the holonomy groups in the Berger list are not only easier to obtain than manifolds with holonomy in G, but also relevant in M-theory, especially if they admit a characteristic connection [56], that is, a metric connection with totally skew-symmetric torsion whose holonomy is contained in G. It is worth mentioning that Ivanov proved in [69] that each manifold with a Spin(7) structure admits a unique characteristic connection. Moreover, Friedrich proved in [55] that Spin(7) is the unique compact simple Lie group G such that every G structure admit a unique characteristic connection.

The Lie group G_2 is compact, simple and simply connected. It consists of the endomorpisms of $\mathbb{R}^7$ which preserve the cross product from the imaginary part of the octonions [104]. Hence, a G_2 structure on a manifold Q determines a 3-form Ψ, a metric and an orientation. In [48], Fernández and Gray classified G_2 structures into 16 different classes in terms of $\nabla\Psi$. Related to this, the analysis of the intrinsic torsion in [32] allowed to obtain equations involving $d\Psi$ and $d(\star\Psi)$ for each of the 16 classes, determined by the G_2 irreducible components of Λ^4T^*Q and Λ^5T^*Q. In particular, the holonomy of Q is contained in G_2 if and only if $d\Psi = 0$ and $d(\star\Psi) = 0$. The Lie group Spin(7) is also compact, simple and simply connected. It is the group of endomorphisms of $\mathbb{R}^8$ which preserve the triple cross product from the octonions [104]. Thus, a Spin(7) structure on a manifold M determines 4-form Ω, a metric and an orientation. In [43], Fernández classified Spin(7) structures into 4 classes in terms of differential equations for $d\Omega$, which are determined by the Spin(7) irreducible components of Λ^5T^*M. Parallel structures satisfy $d\Omega = 0$, locally conformally parallel structures satisfy $d\Omega = \theta \wedge \Omega$ for a closed 1-form θ and balanced structures satisfy $\star(d\Omega) \wedge \Omega = 0$. A generic Spin(7) structure, which does not satisfy any of the previous conditions, is called mixed.

The relationship between G_2 and Spin(7) structures was firstly explored by Martín-Cabrera in [84]. Each oriented hypersurface of a manifold equipped with a Spin(7) structure naturally inherits a G_2 structure whose type is determined by the Spin(7) structure of the ambient manifold and some extrinsic information of the submanifold, such as the Weingarten operator. Following the same viewpoint, Martín-Cabrera constructed Spin(7) structures on S^1-principal bundles over G_2 manifolds in [83]. Both approaches allowed him to construct manifolds with G_2 and Spin(7) structures of different pure types.

It turns out that manifolds admitting SU(3), G_2 and Spin(7) structures are spin and their spinor bundle has a unit-length section η which determines the structure. In [1], spinorial formalism was used to deal with different aspects of SU(3) and G_2 structures, such as the classification of both types of structures, SU(3) structures on hypersurfaces of G_2 manifolds and specific types of Killing spinors. A clear advantage of this viewpoint is that a unique object, the spinor, encodes the whole geometry of the structure. For instance, a G_2 structure on a Riemannian manifold (Q, g) with associated 3-form Ψ is determined by a suitable spinor η according to the formula $\Psi(X, Y, Z) = -(\eta, XYZ\eta)$, where $(\cdot, \cdot)$ denotes the scalar product in the spinor bundle and juxtaposition of vectors indicates the Clifford product. Any oriented hypersurface Q' with unit normal vector field N inherits an SU(3) structure implicitly defined by $\Psi = N^* \wedge \omega + \mathrm{Re}(\Theta)$, where $N^*(X) = g(N, X)$ for $X \in TQ$. But both the 2-form ω and the $(3, 0)$-form $\mathrm{Re}(\Theta)$ turn out to be determined by the same spinor η.

In this paper we follow the ideas of [1] to describe the geometry of Spin(7) structures from a spinorial viewpoint, starting from the classification of these structures, continuing to analyze the relationship between G_2 and Spin(7) structures and finishing with the study of Spin(7) structures on quasi abelian Lie algebras.

Our first result, Theorem 1.21 in section 1.3, describes each type of Spin(7) structure in terms of differential equations involving the spinor η that determines the structure (see section 1.2 for details). Parallel Spin(7) structures have already been studied from a spinorial point of view and correspond to the equation $\nabla\eta = 0$. In order to state the spinorial equations for the remaining classes consider D the Dirac operator on the spinor bundle.

Theorem 1.1. *A* Spin(7) *structure determined by* η *is:*

1. *Balanced if* $D\eta = 0$.
2. *Locally conformally parallel if there exists* $V \in \mathfrak{X}(M)$ *such that* $\nabla_X\eta = \frac{2}{7}(X^* \wedge V^*)\eta$. *In this case,* $D\eta = V\eta$.

Moreover, in Proposition 1.23 we determine the torsion forms of the structure and we obtain that the Lee form is $\Theta = \frac{7}{8}V^*$ where $D\eta = V\eta$.

Our techniques also allow us to identify the intrinsic torsion of the structure and to obtain the formula for the unique characteristic connection of each Spin(7) structure, given by Ivanov in [69, Theorem 1.1]. Along the way, in section 1.6 we also show that the spinorial equation for balanced structures also follows from [69, Theorem 9.1].

We also introduce the concept of G_2 distributions, a general setting to relate G_2 and Spin(7) structures.

Definition 1.2. Let (M, g) be an oriented 8-dimensional Riemannian manifold and let $\mathcal{D}$ be a cooriented distribution of codimension 1. We say that $\mathcal{D}$ has a G_2 structure if the principal SO(7) bundle $P_{SO}(\mathcal{D})$ is spin and the spinor bundle $\Sigma(\mathcal{D})$ admits a unit-length section.

This construction allows us to obtain the results obtained in [83] and [84] about G_2 structures on hypersurfaces of Spin(7) manifolds and S^1-principal bundles over G_2 manifolds. Related to this, we also study warped products of manifolds admitting a G_2 structure with $\mathbb{R}$.

The formalism of G_2 distributions enables us to study Spin(7) structures on quasi abelian Lie algebras, that is, Lie algebras with a codimension 1 abelian ideal. To state the result, which is Theorem 1.49, we suppose that the Lie algebra is $\mathfrak{g} = \langle e_0, \ldots, e_7 \rangle$ and the abelian ideal is $\mathbb{R}^7 = \langle e_1, \ldots, e_7 \rangle$; we also assume that $\mathfrak{g}$ is endowed with the canonical metric and the canonical volume form.

Theorem 1.3. *Denote by $\mathcal{E} = \mathrm{ad}(e_0)|_{\mathbb{R}^7}$ and let $\mathcal{E}_{13}$ and $\mathcal{E}_{24}$ be the symmetric and skew-symmetric parts of the endomorphism. Then, $\mathfrak{g}$ admits a* Spin(7) *structure of type:*

1. *Parallel, if and only if $\mathcal{E}_{13} = 0$ and the eigenvalues of $\mathcal{E}_{24}$ are $0, \pm\lambda_1 i, \pm\lambda_2 i, \pm(\lambda_1+\lambda_2)i$, for some $0 \leq \lambda_1 \leq \lambda_2$.*

2. *Locally conformally parallel and non-parallel if and only if $\mathcal{E}_{13} = h\,\mathrm{Id}$ with $h \neq 0$ and the eigenvalues of $\mathcal{E}_{24}$ are $0, \pm\lambda_1 i, \pm\lambda_2 i, \pm(\lambda_1 + \lambda_2)i$, for some $0 \leq \lambda_1 \leq \lambda_2$.*

3. *Balanced if and only if $\mathfrak{g}$ is unimodular and the eigenvalues of $\mathcal{E}_{24}$ are $0, \pm\lambda_1 i, \pm\lambda_2 i$, $\pm(\lambda_1 + \lambda_2)i$, for some $0 \leq \lambda_1 \leq \lambda_2$.*

Moreover, if $\mathcal{E}_{24} \neq 0$ then it admits a Spin(7) *structure of mixed type.*

This result allows us to provide an example of a nilmanifold admitting both a left-invariant balanced structure and a left-invariant mixed structure. This nilmanifold has $b_1 = 2$ but it is not a product $S^1 \times Q$. We also compute an example of a left-invariant strict locally conformally balanced structure, that is, a mixed structure whose Lee form is closed and non-exact. We also obtain a compact manifold admitting a parallel Spin(7) structure as a quotient of a quasi abelian simply connected Lie group. The Lie group is not abelian, but it is endowed with a flat metric. In particular, the solvmanifold is a $\mathbb{Z}_2$ quotient of a torus T^7.

In addition, we determine nilpotent quasi abelian Lie algebras that admit balanced and locally conformally balanced structures:

Theorem 1.4. *Let L_3 be the Lie algebra of the 3-dimensional Heisenberg group, L_4 the unique irreducible 4-dimensional nilpotent Lie algebra and A_j the j-dimensional abelian Lie algebra.*

1. *Every* Spin(7) *structure on the abelian Lie algebra A_8 is parallel.*

2. *The Lie algebras $\mathfrak{g} = A_5 \oplus L_3$ or $\mathfrak{g} = A_3 \oplus L_4$ admit strict locally conformally balanced structures. However, they do not admit balanced structures.*

3. *The rest of the quasi abelian nilpotent Lie algebras admit a balanced structure and a strict locally conformally balanced structure.*

This paper is organized as follows. Section 1.2 contains a review of algebraic aspects of Spin(7) geometry. Section 1.3 identifies the instrinsic torsion of the Levi-Civita connection with the covariant derivative of the spinor that determines the structure, section 1.4 provides the spinorial classification of Spin(7) structures, section 1.5 is devoted to obtain the torsion forms of Spin(7) structures in terms of spinors and section 1.6 provides an alternative proof of the existence of the characteristic connection. Section 1.7 provides a complete analysis of G_2 structures on distributions and then focuses on the particular cases described above. Section 1.8 deals with invariant structures on quasi abelian Lie algebras and provides compact examples. Finally section 1.9 is devoted to the study of quasi abelian nilpotent structures and its Spin(7) structures.

Acknowledgements

I would like to thank my thesis directors, Giovanni Bazzoni and Vicente Muñoz, for useful conversations, advices and encouragement. I also want to thank Mario García Fernández and Ilka Agricola for useful comments.

The author is supported by a grant from Ministerio de Educación, Cultura y Deporte, Spain (FPU16/03475).

1.2 Preliminaries

In this section we introduce some aspects of Clifford algebras, 8-dimensional spin manifolds and Spin(7) representations, which can be found in [54], [79] and [104] as well as the notations that we use in the sequel.

1.2.1 The real representation of Cl_8

The Clifford algebra Cl_8 is isomorphic to the algebra of endomorphisms of $\mathbb{R}^{16}$. We denote such an isomorphism by $\rho\colon \mathrm{Cl}_8 \to \mathrm{End}(\mathbb{R}^{16})$; this is indeed the unique irreducible repesentation of Cl_8 up to equivalence [79, Chapter 1, Theorem 4.3]. There is also an inner product on $\mathbb{R}^{16}$, which we denote by $(\cdot,\cdot)$, such that the Clifford multiplication by a vector of $\mathbb{R}^8$ is a skew-symmetric transformation [79, Chapter 1, Theorem 5.3].

Fix an orientation of $\mathbb{R}^8$ and let ν_8 be a unit-length positively oriented volume form of $\mathbb{R}^8$. Consider the Spin(8) equivariant endomorphism:

$$\nu_8\cdot\colon \mathbb{R}^{16} \to \mathbb{R}^{16}, \quad \phi \longmapsto \nu_8\phi.$$

Since $\nu_8^2 = 1$, there is a splitting $\mathbb{R}^{16} = \Delta^+ \oplus \Delta^-$ where $\Delta^\pm$ is the eigenspace associated to ± 1. In addition, this endomorphism anticommutes with the Clifford multiplication by a vector.

It is well known that Spin(8) contains three distinct conjugacy classes of the group Spin(7) [79, Chapter 4, Proposition 10.4]. The first one is obtained from the adjoint action $\mathrm{Ad}\colon \mathrm{Spin}(8) \to \mathrm{SO}(8)$ as the stabilizer of any non-zero $v \in \mathbb{R}^8$. The remaining ones, which we denote by $\mathrm{Spin}(7)^\pm$, are constructed from ρ as the stabilizer of a non-zero spinor $\phi_\pm \in \Delta^\pm$. The adjoint action embeds $\mathrm{Spin}(7)^\pm$ into SO(8) because $-1 \notin \mathrm{Stab}(\phi_\pm)$. Note also that the conjugacy classes $\mathrm{Spin}(7)^\pm$ depend on the choice of an orientation of $\mathbb{R}^8$ and these are conjugated in Pin(8).

Remark 1.5. We can construct ρ from the representation of the complex Clifford algebra and the real structure constructed in [54, Chapter 1]. The construction that allows to obtain an irreducible representation of Cl_6 is similar but there is a difference that we outline. Let $\mathbb{C}\mathrm{l}_{2k}$

be the Clifford algebra of $(\mathbb{C}^{2k}, \sum_{i=1}^{2k} z_i^2)$, according to [54, p.13] there is a 2^k-dimensional complex space Δ_{2k} and an isomorphism $\kappa_{2k}\colon \mathbb{C}\mathrm{l}_{2k} \to \mathrm{End}(\Delta_{2k})$. The multiplication by the complex volume form $\nu_{2k}^{\mathbb{C}} = i^k \nu_{2k}$ splits Δ_{2k} into two eigenspaces $\Delta_{2k}^{\pm}$ associated to the eigenvalue ± 1 which are irreducible under the action of $\mathrm{Spin}(2k)$.

1. There is a $\mathrm{Spin}(8)$ equivariant real structure φ_8 on Δ_8 which commutes with $\nu_8^{\mathbb{C}}$ (see [54, p.32]). Thus, a real representation is $(\Delta_8^+)_+ \oplus (\Delta_8^-)_-$, where $(\Delta_8^+)_\pm$ and $(\Delta_8^-)_\pm$ are the eigenspaces associated to the eigenvalue ± 1 of φ_8 on Δ_8^+ and Δ_8^-.

2. There is a $\mathrm{Spin}(6)$ equivariant real structure φ_6 on Δ_6 that anticommutes with $\nu_6^{\mathbb{C}}$. Thus the real representation of Cl_6 is $(\Delta_6)_+ = \{\phi + \varphi_6(\phi)\colon \phi \in \Delta_6^+\}$, the eigenspace associated to $+1$ of φ_6. In addition, if $\eta = \phi + \varphi_6(\phi) \neq 0$ is a real spinor, then $\mathrm{Stab}_{\mathrm{Spin}(6)}(\eta) = \mathrm{Stab}_{\mathrm{Spin}(6)}(\phi) = \mathrm{Stab}_{\mathrm{Spin}(6)}(\varphi_6(\phi)) = \mathrm{SU}(3)$.

The hermitian metric h on Δ_8 constructed in [54, p.24] makes the Clifford multiplication a skew-symmetric transformation. In particular, h is $\mathrm{Spin}(8)$ invariant. The fact that $\Delta_8^{\pm}$ are irreducible $\mathrm{Spin}(8)$ modules guarantees that $b(\phi, \eta) = h(\varphi_8(\phi), \eta)$ is a symmetric bilinear form on $\Delta_8^{\pm}$ and therefore the restrictions of h to the real and the imaginary part of $\Delta_8^{\pm}$ are real-valued. The subspaces Δ_8^+ and Δ_8^- are orthogonal with respect to h because the multiplication by $\nu_{\mathbb{C}}$ preserves h. Therefore the real part of h is a scalar product on $(\Delta_8^+)_+ \oplus (\Delta_8^-)_-$ with the same properties as $(\cdot,\cdot)$.

1.2.2 $\mathrm{Spin}(7)$ structures

Let (M, g) be an oriented Riemannian 8-manifold and let $\mathrm{P}_{\mathrm{SO}}(M)$ be the associated $\mathrm{SO}(8)$ frame bundle. If M is spin, i.e. if $w_2(M) = 0$, the $\mathrm{Spin}(8)$ principal bundle $\mathrm{P}_{\mathrm{Spin}}(M)$ over M is a double covering $\pi\colon \mathrm{P}_{\mathrm{Spin}}(M) \to \mathrm{P}_{\mathrm{SO}}(M)$ equivariant under the adjoint action $\mathrm{Ad}\colon \mathrm{Spin}(8) \to \mathrm{SO}(8)$. The associated spinor bundle is $\Sigma(M) = \mathrm{P}_{\mathrm{Spin}}(M) \times_\rho \mathbb{R}^{16}$ and it is equipped with a metric induced by $(\cdot,\cdot)$ which we denote by the same symbols. In addition, there is a splitting $\Sigma(M) = \Sigma(M)^+ \oplus \Sigma(M)^-$, where $\Sigma(M)^{\pm} = \mathrm{P}_{\mathrm{Spin}}(M) \times_\rho \Delta^{\pm}$.

Also note that $X(\Sigma(M)^{\pm}) \subset \Sigma(M)^{\mp}$ if $X \in \mathfrak{X}(M)$ and that for each nowhere-vanishing spinor $\eta\colon M \to \Sigma(M)^{\pm}$ the map:

$$TM \to \Sigma(M)^{\mp}, \quad X \longmapsto X\eta, \tag{1.1}$$

is an isomorphism.

The Clifford multiplication with a vector field provides an action of $\Lambda^k T^*M$ defined as follows.

1. The product with a covector is defined by $X^*\phi = X\phi$, where we used the canonical identification between the tangent and the cotangent bundle: $X^* = g(X, \cdot)$.

2. If the product is defined on $\Lambda^\ell T^*M$ when $\ell \leq k$, we define

$$(X^* \wedge \beta)\phi = X(\beta\phi) + (i(X)\beta)\phi,$$

where $i(X)\beta$ denotes the contraction, $\beta \in \Lambda^k T^*M$ and $X \in TM$. This product is extended lineary to $\Lambda^{k+1}T^*M$.

For instance:

$$(X^* \wedge Y^*)\phi = (XY + g(X, Y))\phi, \tag{1.2}$$
$$(X^* \wedge Y^* \wedge Z^*)\phi = (XYZ + g(X, Y)Z - g(X, Z)Y + g(Y, Z)X)\phi. \tag{1.3}$$

Observe also that $\Sigma^{\pm}(M) = \{\phi_p : \nu\phi_p = \pm\phi_p\}$ where ν is the positively oriented unit-length volume form of (M, g).

The action $\mathrm{Spin}(8) \times \mathbb{R}^{16} \to \mathbb{R}^{16}$ lifts to an action $\mathrm{P}_{\mathrm{Spin}}(M) \times \Sigma(M) \to \Sigma(M)$, so that the existence of a unit-length spinor $\eta \in \Gamma(\Sigma(M)^{\pm})$ determines an identification between $\mathrm{Spin}(7)^{\pm}$ and the stabilizer of η_p at each $p \in M$. This defines a Spin(7) principal subundle $\mathrm{Stab}(\eta) \subset \mathrm{P}_{\mathrm{Spin}}(M)$ and therefore, $\mathrm{Ad}(\mathrm{Stab}(\eta))$ is a Spin(7) reduction of $\mathrm{P}_{\mathrm{SO}}(M)$. In this paper we focus on Spin(7) structures determined by positive spinors. This condition is not restrictive due to the following result which is not difficult to check.

Lemma 1.6. *Let (M, g) be a connected oriented spin manifold and let $\Sigma(M)$ be its spinor bundle. Let $\overline{\Sigma(M)}$ be the spinor bundle associated to the opposite orientation on M. There is an isomorphism of $\mathrm{Cl}(M)$ modules $\mathcal{R} : \Sigma(M) \to \overline{\Sigma(M)}$. Therefore, $\mathcal{R}(\Sigma(M)^{\pm}) = \overline{\Sigma(M)}^{\mp}$.*

For the convenience of the reader, we shall relate this spinorial approach with the point of view of positive triple cross products [104, Definitions 6.1, 6.12].

Lemma 1.7. *Let (M, g) be a Riemannian oriented spin manifold that admits a unit-length spinor $\eta : M \to \Sigma(M)^{\pm}$. Then there is a well defined map:*

$$TM \times TM \times TM \to TM, \quad (X, Y, Z) \longmapsto X \times Y \times Z \ \textit{s.t,}\ (X \times Y \times Z)\eta = (X^* \wedge Y^* \wedge Z^*)\eta,$$

which is in turn a positive triple cross product. The associated 4-form $\Omega(W, X, Y, Z) = g(W, X \times Y \times Z)$ satisfies that $\star\Omega = \pm\Omega$.

Moreover [104, Theorem 10.3] states that there is a 1 to 1 correspondence between 4-forms Ω that define a positive triple cross product with $\Omega \wedge \Omega > 0$, and sections of the projectivization of $\Sigma(M)^{+}$.

According to the previous discussion we summarize our basic assumptions in a Proposition. In the sequel given a frame $(e_0, \ldots, e_7)$ and a spinor ϕ we use short-hand notation e^i for $g(e_i, \cdot)$, e^{ijkl} for $e^i \wedge e^j \wedge e^k \wedge e^l$ and $e_{ijk}\phi$ for $e_i e_j e_k \phi$.

Proposition 1.8. *Let (M, g) be an oriented spin manifold and suppose that there exists a positive unit-length spinor. Consider the triple cross product on M defined as in Lemma 1.7.*

1. *The associated 4-form is self-dual and is determined by*

$$\Omega(W, X, Y, Z) = \frac{1}{2}((-WXYZ + WZYX)\eta, \eta).$$

2. *Given local orthonormal vector fields e_0, e_1, e_2, e_4 such that e_4 is perpendicular to $e_3 = e_0 \times e_1 \times e_2$ there exists a positively oriented orthonormal frame $(e_0, \ldots, e_7)$ such that:*

$$\begin{aligned} \Omega = \quad & e^{0123} - e^{0145} - e^{0167} - e^{0246} + e^{0257} - e^{0347} - e^{0356} \\ & + e^{4567} - e^{2367} - e^{2345} - e^{1357} + e^{1346} - e^{1256} - e^{1247}. \end{aligned} \tag{1.4}$$

   A local frame with this property is called a Cayley frame.

Proof. Taking into account Lemma 1.7 and equation (1.3) the associated 4-form of the triple cross product, which is self-dual, is:

$$\begin{aligned} \Omega(W, X, Y, Z) &= ((X \times Y \times Z)\eta, W\eta) = ((XYZ + g(X, Y)Z - g(X, Z)Y + g(Y, Z)X)\eta, W\eta) \\ &= \frac{1}{2}((-WXYZ + WZYX)\eta, \eta). \end{aligned}$$

The second statement can be found in [104, Theorem 7.12]. A Cayley frame $(e_0, \ldots, e_7)$ satisfies $(e_0 \cdots e_7)\eta = \eta$; it is thus positively oriented. □

1.2.3 Spin(7) representations

Let us denote the standard basis of $\mathbb{R}^8$ by $(e_0, \dots, e_7)$, and the standard Spin(7) structure of $\mathbb{R}^8$ by Ω_0, given by (1.4). We also denote $\Lambda^k = \Lambda^k(\mathbb{R}^8)^*$.

The representation of $\mathrm{Spin}(7) = \mathrm{Stab}(\Omega_0) \subset \mathrm{SO}(8)$ on Λ^k induces an orthogonal decomposition of this space into irreducible Spin(7) invariant subspaces. The expression Λ^k_l denotes such an l-dimensional subspace of Λ^k. The Hodge star operator $\star$ gives Spin(7) equivariant isomorphisms between Λ^k and Λ^{8-k} determining that $\Lambda^k_l = \star\Lambda^{8-k}_l$ if $k \leq 4$. We briefly describe the splitting; a complete proof can be found in [43] and [104, Theorem 9.8]. The decomposition goes as follows:

$$\begin{aligned}\Lambda^2 =& \Lambda^2_7 \oplus \Lambda^2_{21},\\ \Lambda^3 =& \Lambda^3_8 \oplus \Lambda^3_{48},\\ \Lambda^4 =& \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}.\end{aligned}$$

The first one comes from the orthogonal splitting $\Lambda^2 = \mathfrak{so}(8) = \mathfrak{spin}(7) \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{spin}(7)^\perp$. An alternative description is obtained from the map:

$$\Lambda^2 \to \Lambda^2, \quad \beta \longmapsto \star(\beta \wedge \Omega_0),$$

which is Spin(7)-equivariant, symmetric and traceless. Therefore, Λ^2 splits into eigenspaces which must coincide with the previous ones due to the irreducibility. One can check that the eigenvalues are 3 on Λ^2_7 and -1 on Λ^2_{21}. Moreover, the set

$$\{\alpha_j = \frac{1}{2}(e^{0j} + i(e_j)i(e_0)\Omega_0)\}_{j=1}^7 \tag{1.5}$$

is an orthonormal basis of Λ^2_7 and the projection $\mathrm{p}^2_7 \colon \Lambda^2 \to \Lambda^2_7$ is consequently determined by the equation:

$$\mathrm{p}^2_7(u^* \wedge v^*) = \frac{1}{4}(u^* \wedge v^* + i(v)i(u)\Omega_0). \tag{1.6}$$

The subspaces involved in the splitting of Λ^3 are:

$$\Lambda^3_8 = i(\mathbb{R}^8)\Omega_0, \quad \Lambda^3_{48} = \ker(\cdot \wedge \Omega_0 \colon \Lambda^3 \to \Lambda^7).$$

In order to describe the last one observe that the Hodge star operator splits Λ^4 into two 35-dimensional spaces: anti self-dual and self-dual forms. The space of anti self-dual forms is Λ^4_{35} and the space of self-dual forms is $\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}$. Obviously, $\Lambda^4_1 = \langle \Omega_0 \rangle$ and the space Λ^4_7 is the image of the map,

$$j \colon \mathfrak{m} \to \Lambda^4, \quad j(\beta) = \rho_*(\beta)\Omega_0,$$

with $\rho \colon \mathrm{SO}(8) \to \Lambda^4 T^*M$, $\rho(g) = (g^{-1})^*\Omega_0$. That is, j is the restriction to $\mathfrak{m}$ of the map determined by $j(u^* \wedge v^*) = u^* \wedge i(v)\Omega_0 - v^* \wedge i(u)\Omega_0$ and therefore, $\Lambda^7_4 = \{u^* \wedge i(v)\Omega_0 - v^* \wedge i(u)\Omega_0,\ u, v \in \mathbb{R}^8\}$. The subspace Λ^4_{27} is the orthogonal complement of $\Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{35}$.

We now describe the irreducible decomposition of $\Lambda^1 \otimes \mathfrak{m}$ which is related with the intrinsic torsion of the Levi-Civita connection (see Section 1.3).

Proposition 1.9. *Let $(e_0, \dots, e_7)$ be a Cayley basis and let $\mathrm{p}^2_7 \colon \Lambda^2 \to \mathfrak{m}$ be the orthogonal projection. Consider the* Spin(7)*-equivariant maps:*

$$\Theta \colon \Lambda^3 \to \Lambda^1 \otimes \mathfrak{m}, \quad \beta \longmapsto \Theta(\beta) = \sum_{j=0}^{7} e_j \otimes \mathrm{p}^2_7(i(e_j)\beta),$$

$$\Xi \colon \Lambda^1 \otimes \mathfrak{m} \to \Lambda^3, \quad \alpha \otimes \beta \longmapsto \alpha \wedge \beta = 3\,\mathrm{alt}(\alpha \otimes \beta),$$

where $\mathrm{alt}(\mathrm{T})(v_1, \dots, v_n) = \frac{1}{n!}\sum_{\sigma \in S_n}(-1)^{\mathrm{sgn}\,\sigma}\mathrm{T}(v_{\sigma(1)}, \dots, v_{\sigma(n)})$. *The eigenvalues of $\Xi \circ \Theta$ are $\frac{9}{4}$ and $\frac{1}{2}$. They are associated to the eigenspaces Λ^3_8 and Λ^3_{48} respectively.*

Proof. The map $\Xi \circ \Theta$ is symmetric and Spin(7)-equivariant, so that its eigenspaces must be Λ^3_8 and Λ^3_{48}. A direct computation in the cases $i(e_0)\Omega_0 \in \Lambda^3_8$ and $e^{123} + e^{145} \in \Lambda^3_{48}$ shows that the eigenvalues are $\frac{9}{4}$ on Λ^3_8 and $\frac{1}{2}$ on Λ^3_{48}. □

We formulate an alternative description of $\Lambda^1 \otimes \mathfrak{m}$ which is proved in the same manner.

Proposition 1.10. *Let $(e_0, \ldots, e_7)$ be an orthonormal frame, and let $\mathrm{p}^2_7\colon \Lambda^2 \to \mathfrak{m}$ be the orthogonal projection.Consider the* O(8) *equivariant maps,*

$$\iota\colon \mathbb{R}^8 \to \Lambda^1 \otimes \mathfrak{m}, \quad \iota(v) = \sum_{i=0}^{7} e^i \otimes \mathrm{p}^2_7(e^i \wedge v^*),$$
$$\kappa\colon \Lambda^1 \otimes \mathfrak{m} \to \mathbb{R}^8, \quad \kappa(\Gamma) = \sum_{i=0}^{7} (i(e_i)\Gamma(e_i))^\sharp,$$

which do not depend on the orthonormal basis chosen. Then $\iota(\mathbb{R}^8) = \Theta(\Lambda^3_8)$ and $\ker(\kappa) = \Theta(\Lambda^3_{48})$. Moreover, $\kappa \circ \iota(v) = \frac{7}{4}v$ for any $v \in \mathbb{R}^8$.

The study of the space $\Lambda^1 \otimes \Lambda^4_7$ is done similarly; this turns out to be isomorphic to $\Lambda^1 \otimes \mathfrak{m}$. For instance, it is not difficult to check that the map $\mathrm{alt}\colon \Lambda^1 \otimes \Lambda^4_7 \to \Lambda^5$ is a Spin(7) equivariant isomorphism.

A Spin(7) structure on the Riemannian manifold (M, g) determines a splitting of $\Lambda^k T^*M$ into subbundles $\Lambda^k_l T^*M = R \times_{\mathrm{Spin}(7)} \Lambda^k_l$ where R is the Spin(7) reduction R of the SO(8) principal bundle given by the Cayley frames. We also denote by $\Omega^k_l(M)$ the space of smooth sections of $\Lambda^k_l T^*M$. In addition, the maps j, Θ, Ξ, ι, κ induce bundle homomorphisms that we call by the same name. We also consider the subbundles of $T^*M \otimes \Lambda^2_7 T^*M$ defined by:

$$\chi_1 = \Theta(\Lambda^3_{48} T^*M), \qquad \chi_2 = \Theta(\Lambda^3_8 T^*M). \tag{1.7}$$

1.3 The intrinsic torsion

We compute the intrinsic torsion Γ of the Levi-Civita connection which is a section of the bundle $TM \otimes \Lambda^2_7 T^*M$. Recall that the Levi-Civita connection ∇ on TM induces a connection ω on $\mathrm{P}_{\mathrm{SO}}(M)$. A connection on the Spin(7) reduction R is $\omega' = \mathrm{p}(\omega)|_{TR}$, where p denotes the orthogonal projection to $\mathfrak{spin}(7)$. The connection that ω' induces on TM is denoted by ∇' and determines the intrinsic torsion by means of the expression:

$$\nabla_X Y = \nabla'_X Y + \Gamma(X)Y.$$

The skew-symmetric endomorphism $\Gamma(X)$ can be identified with a 2-form which lies in $\Omega^2_7(M)$ for each $X \in TM$. To compute it, define H as the subspace of Δ_+ which is orthogonal to η with respect to the scalar product $(\cdot, \cdot)$ defined in Section 1.2.1. Of course, H depends on the choice of the spinor η. We first prove that the vector bundles $\Lambda^2_7 T^*M$ and H are isomorphic.

Lemma 1.11. *There is a well defined* Spin(7)*-equivariant map*

$$\Lambda^2 T^*M \to H, \quad \alpha \longmapsto \alpha\eta,$$

whose kernel is $\Lambda^2_{21} T^*M$. Indeed, its restriction $c\colon \Lambda^2_7 T^*M \to H$ is an isomorphism whose inverse is given by $(c^{-1}\phi)(X, Y) = \frac{1}{4}(\phi, (XY + g(X, Y))\eta)$.

Proof. The spinor $\beta\eta$ is perpendicular to η if $\beta \in \Lambda^2 T^*M$. Therefore, the map is well-defined and it is Spin(7)-equivariant because $\mathrm{Spin}(7) = \mathrm{Stab}(\eta_p)$.

To prove that c is an isomorphism, we first claim that if $(e_0, \ldots, e_7)$ is a Cayley frame then $\alpha_j\eta = 4e^{0j}\eta$, where the 2-forms α_j are defined as in equation (1.5). Observe that we only need to check this formula for $j = 1$ because c is Spin(7)-equivariant and $\mathrm{G}_2 = \mathrm{Spin}(7) \cap \mathrm{Stab}(e_0)$ acts transitively on the 6-sphere generated by $(e_1, \ldots, e_7)$. In this case, $\alpha_1 = e^{01} + e^{23} - e^{45} - e^{67}$ and if $(i,j) \in \{(2,3),(5,4),(7,6)\}$, then $\Omega(e_0, e_1, e_i, e_j) = 1$; we now prove that this equality implies that $e^{01}\eta = e^{ij}\eta$. First observe that $e^{01}\eta$ and $e^{ij}\eta$ are unit-length positive spinors. In addition, according to Proposition 1.8 (1), $1 = -(e^{01ij}\eta, \eta) = (e^{01}\eta, e^{ij}\eta)$; therefore $e^{01}\eta = e^{ij}\eta$.

Moreover, taking into account that $\{e^{0i}\eta\}_{i=1}^7$ is an orthonormal basis of H we obtain:

$$c^{-1}(\phi) = \frac{1}{4}\sum_{i=1}^{7}(\phi, e^{0i}\eta)\alpha_i.$$

If $X = e_0$, $Y = e_1$ are orthonormal vectors, then $\alpha_j(e_0, e_1) = (e^{0j} - i(e_0)i(e_j)\Omega)(e_0, e_1) = \delta_{j1}$. Hence, $c^{-1}\phi(e_0, e_1) = \frac{1}{4}(\phi, e_0 e_1 \eta)$.

Finally, for dimensional reasons the Clifford product with η must vanish on $\Lambda^2_{21}T^*M$. □

Remark 1.12. These computations and others that we do in the sequel in terms of Cayley frames may be computed alternatively from a representation of Cl_8.

The previous result enables us to find a formula for the intrinsic torsion:

Proposition 1.13. *The intrinsic torsion is given by* $\Gamma(X) = 2c^{-1}\nabla_X\eta$.

Proof. We also denote by ∇ and ∇' the induced connections on the spinor bundle. According to [54, p. 60]:

$$\nabla_X\phi = \nabla'_X\phi + \frac{1}{2}\Gamma(X)\phi,$$

where $\Gamma(X)$ acts on ϕ as a 2-form. The holonomy of the connection ∇' is contained in Spin(7) and $\mathrm{Stab}(\eta_p) = \mathrm{Spin}(7)$; therefore $\nabla'\eta = 0$. Finally, if $X \in TM$ then $\nabla_X\eta \in H$ and $\Gamma(X) \in \Lambda^2_7 T^*M$ thus, Lemma 1.11 shows that $\Gamma(X) = 2c^{-1}\nabla_X\eta$. □

1.4 Classification of Spin(7) structures.

The classification of Spin(7) structures was obtained in [43, Theorem 5.3]. There it is proved that $\nabla\Omega \in \Gamma(TM^* \otimes \Lambda^4_7 T^*M)$ and that $\Lambda^1 \otimes \Lambda^4_7$ splits into two irreducible Spin(7) subspaces, described in terms of the isomorphism $\mathrm{Id}\otimes j\colon \Lambda^1 \otimes \mathfrak{m} \to \Lambda^1 \otimes \Lambda^4_7$ (see Section 1.2.3 for the definition of j). Those are of course $(\mathrm{Id}\otimes j)\circ\Theta(\Lambda^3_{48})$ and $(\mathrm{Id}\otimes j)\circ\Theta(\Lambda^3_8)$.

We also denote by $\mathrm{Id}\otimes j$ the induced map from $T^*M \otimes \Lambda^2_7 T^*M$ to $T^*M \otimes \Lambda^4_7 T^*M$ and we define $\mathcal{W}_1 = (\mathrm{Id}\otimes j)(\chi_1)$ and $\mathcal{W}_2 = (\mathrm{Id}\otimes j)(\chi_2)$, where χ_j are defined as in equation (1.7).

Moreover, it is straightforward to check that $\mathrm{Id}\otimes j(\Gamma) = \nabla\Omega$ and that $\mathrm{alt}(\nabla\Omega) = d\Omega$. These considerations allow us to describe the classification of Spin(7) structures in three different ways.

Definition 1.14. Let Γ be the intrinsic torsion of the Spin(7) structure determined by Ω. The type of the structure is given by the equivalent conditions:

	Γ	$\nabla\Omega$	$d\Omega$
Parallel	0	0	0
Balanced	χ_1	$\mathcal{W}_1$	$\star(d\Omega)\wedge\Omega = 0$
Locally conformally parallel	χ_2	$\mathcal{W}_2$	$\theta\wedge\Omega, \quad \theta \in \Omega^1(M)$

In other case, the structure is said to be of mixed type.

Definition 1.15. The Lee form of Ω is the unique $\theta \in \Omega^1(M)$ such that the orthogonal projection to $\Omega^5_8(M)$ of $d\Omega$ is $\theta \wedge \Omega$.

Remark 1.16. According to Proposition 1.10 locally conformally parallel Spin(7) structures are the class of Spin(7) structures with vectorial torsion in the sense of [2]. In [2, Proposition 2.2] there is a characterization of compact manifolds with vectorial torsion and formulas for the Ricci tensor.

Remark 1.17. If the structure is locally conformally parallel then $d\theta = 0$. Let O be a contractible open set. Take a primitive f of $-\frac{1}{4}\theta|_O$ and define the metric $g' = e^{2f} g|_O$. The associated Spin(7) structure is $\Omega' = e^{4f}\Omega|_O$ and it satisfies $d\Omega' = 0$. Therefore, $\Omega|_O$ is conformal to a parallel structure. This justifies the name.

We now focus on obtaining an alternative description in terms of spinors. For that purpose, decompose $\Gamma = \Gamma_1 + \Gamma_2$ according to the splitting $\chi_1 \oplus \chi_2$ and write $\Gamma_2(X) = \frac{4}{7}\mathrm{p}^2_7(X^* \wedge V^*)$. Taking into account Proposition 1.10 and equation (1.6) we obtain:

1. $\kappa(\Gamma_2) = V^*$,
2. $\Xi(\Gamma_2) = \frac{4}{7}\sum_{i=0}^7 e^i \wedge \mathrm{p}^2_7(e^i \wedge V^*) = \frac{1}{7}\sum_{i=0}^7 e^i \wedge i(e_i)i(V)\Omega = \frac{3}{7}i(V)\Omega.$

Remark 1.18. Let $Z(V) = \{p \in M \text{ s.t } V(p) = 0\}$. The open set $M - Z(V)$ has a G_2 structure defined by $i(V/\|V\|)\Omega$.

Remark 1.19. We added a factor $\frac{4}{7}$ in order to avoid a constant in Theorem 1.20.

We compute the action of the Dirac operator D on the spinor η that determines the Spin(7) structure.

Proposition 1.20. *Let Ω be a* Spin(7) *structure determined by a spinor η. Let $\Gamma = \Gamma_1 + \Gamma_2$ be its intrinsic torsion with $\Gamma_2(X) = \frac{4}{7}\mathrm{p}^2_7(X^* \wedge V^*)$. Then,*

1. *The map $\Lambda^3T^*M \to \Sigma(M)^-$, $\alpha \longmapsto \alpha\eta$ is* Spin(7) *equivariant and its kernel is $\Lambda^3_{48}T^*M$. Moreover, $(i(X)\Omega)\eta = 7X\eta$.*
2. *The action of the Dirac operator on η is given by $D\eta = V\eta$.*

Proof. The first statement is a consequence of Schur's Lemma. One can check the equality $i(X)\Omega\eta = 7X\eta$ by supposing that X is unit-length and using a Cayley frame such that $X = e_0$.

For the second we compute in terms of a Cayley local frame $(e_0, \ldots, e_7)$,

$$2D\eta = \sum_{i=0}^{7} e_i\Gamma(e_i)\eta = \sum_{i=0}^{7}(e^i \wedge \Gamma(e_i) - i(e_i)\Gamma(e_i))\eta = \Xi(\Gamma)\eta - \kappa(\Gamma)\eta = 2V\eta.$$

We used Proposition 1.13 to obtain the first equality. For the last, we used the formulas (1) and (2) above; we also took into account that $\kappa(\Gamma_1) = 0$ by Proposition 1.10, and that $\Xi(\Gamma_1)\eta = 0$ by the first part of the proposition. □

Theorem 1.21. *The* Spin(7) *structure determined by a spinor η is,*

1. *Parallel if $\nabla\eta = 0$*
2. *Balanced if $D\eta = 0$*
3. *Locally conformally parallel if there exists $V \in \mathfrak{X}(M)$ such that $\nabla_X\eta = \frac{2}{7}(X^* \wedge V^*)\eta$. In this case, $D\eta = V\eta$.*

Proof. The equation for balanced structures follows from Proposition 1.20 and the equation for locally conformally parallel structures follows from Lemma 1.11 and Proposition 1.13. □

1.5 Torsion forms of a Spin(7) structure

In this section we describe the torsion forms of a Spin(7) structure by means of the spinor that defines it. That is, we determine the projections of $\star d\Omega$ to the spaces $\Omega^3_8(M)$ and $\Omega^3_{48}(M)$. Note that the projection is given by $\mathrm{p}^3_8\colon \Omega^3(M) \to \Omega^3_8(M)$, $\mathrm{p}^3_8(\beta) = -\frac{1}{7}\star(\star(\beta\wedge\Omega)\wedge\Omega)$.

For that purpose, denote by D the Dirac operator on $\Sigma(M)$. Since $D\eta \in \Sigma(M)^-$, the isomorphism (1.1) ensures the existence of a unique vector field V such that

$$D\eta = V\eta. \tag{1.8}$$

Then, the 3-form $\gamma_8(X,Y,Z) = (D\eta, (X\times Y\times Z)\eta) = (i(V)\Omega)(X,Y,Z)$ obviously lies in $\Omega^3_8(M)$.

Proposition 1.22. *In terms of the previously defined notation,*

$$\star d\Omega = 2(\gamma_8 - 12\,\mathrm{alt}(c^{-1}\nabla\eta)).$$

Proof. Taking into account that ∇ is a metric connection on the spinor bundle and acts as a derivation for the Clifford product, we obtain:

$$\begin{aligned}(\nabla_T\Omega)(W,X,Y,Z) &= \frac{1}{2}\Big(((-WXYZ + WZYX)\nabla_T\eta,\eta) + ((-WXYZ+WZYX)\eta,\nabla_T\eta)\Big)\\ &= \frac{1}{2}((-ZYXW + XYZW - WXYZ + WZYX)\eta,\nabla_T\eta).\end{aligned}$$

Take orthonormal vectors X,Y,Z and an orthonormal oriented basis $(X_0,\dots X_7)$ such that $X_0 = X$, $X_1 = Y$ and $X_2 = Z$. Then, using the previous equality and the fact that the basis is orthonormal:

$$\begin{aligned}\delta\Omega(X,Y,Z) &= -\sum_{i=3}^{7}\nabla_{X_i}\Omega(X_i,X,Y,Z) = -2\sum_{i=3}^{7}(XYZ\eta, X_i\nabla_{X_i}\eta)\\ &= -2(D\eta,(X\times Y\times Z)\eta) + 2(XYZ\eta, X\nabla_X\eta + Y\nabla_Y\eta + Z\nabla_Z\eta)\\ &= -2((D\eta,(X\times Y\times Z)\eta) - (YZ\eta,\nabla_X\eta) + (XZ\eta,\nabla_Y\eta) - (XY\eta,\nabla_Z\eta))\\ &= -2((D\eta,(X\times Y\times Z)\eta) - 12\,\mathrm{alt}(c^{-1}\nabla\eta)(X,Y,Z)).\end{aligned}$$

The third equality follows from $\sum_{i=3}^{7} X_i\nabla_{X_i}\eta = D\eta - \sum_{i=1}^{3} X_i\nabla_{X_i}\eta$. Note that the coefficient 12 comes from the normalization of alt and the expression $c^{-1}(\nabla_X\eta)(X,Y) = \frac{1}{4}((XY + g(X,Y))\eta,\nabla_X\eta)$. ☐

We decompose $\star d\Omega$ according to the splitting $\Omega^3(M) = \Omega^3_8(M)\oplus\Omega^3_{48}(M)$:

Proposition 1.23. *The 3-form $\gamma_{48} = 3\gamma_8 - 84\,\mathrm{alt}(c^{-1}\nabla\eta)$ lies in $\Omega^3_{48}(M)$ and*

$$\star d\Omega = \frac{2}{7}\gamma_{48} + \frac{8}{7}\gamma_8.$$

Moreover, the Lee form is given by $\theta = \frac{8}{7}V^*$, where V is defined as in the equation (1.8).

Proof. Take a unit-length vector X and a Cayley frame $(e_0,e_1,\dots,e_7)$ such that $X = e_0$. Then:

$$\begin{aligned}(\gamma_8\wedge\Omega)(e_1,\dots,e_7) &= (D\eta,(e_{123} - e_{145} - e_{167} - e_{246} + e_{257} - e_{347} - e_{356})\eta)\\ &= 7(D\eta,e_0\eta) = 7V^*(X),\\ (12\,\mathrm{alt}(c^{-1}\nabla\eta)\wedge\Omega)(e_1,\dots,e_7) &= \mathfrak{S}(\nabla_{e_1}\eta,e_{23}\eta) - \mathfrak{S}(\nabla_{e_1}\eta,e_{45}\eta) - \mathfrak{S}(\nabla_{e_1}\eta,e_{67}\eta)\\ &\quad - \mathfrak{S}(\nabla_{e_2}\eta,e_{46}\eta) + \mathfrak{S}(\nabla_{e_2}\eta,e_{57}\eta) - \mathfrak{S}(\nabla_{e_3}\eta,e_{47}\eta)\\ &\quad - \mathfrak{S}(\nabla_{e_3}\eta,e_{56}\eta) = 3(D\eta,e_0\eta) = 3V^*(X).\end{aligned}$$

We took into account formula (1.4) to determine the non-zero terms $\gamma_8(e_i, e_j, e_k)\Omega(e_l, e_m, e_p, e_q)$ that appear in $\gamma_8 \wedge \Omega(e_1, \dots, e_7)$. In the second computation, we denoted by $\mathfrak{S}$ the cyclic sums in the indices involved. To arrange the last term observe that each index appears 3 times and:

$$\mathfrak{S}(\nabla_{e_1}\eta, e_{23}\eta) = (e_1\nabla_{e_1}\eta + e_2\nabla_{e_2}\eta + e_3\nabla_{e_3}\eta, e_{123}\eta) = (e_1\nabla_{e_1}\eta + e_2\nabla_{e_2}\eta + e_3\nabla_{e_3}\eta, e_0\eta),$$
$$-\mathfrak{S}(\nabla_{e_1}\eta, e_{45}\eta) = (e_1\nabla_{e_1}\eta + e_4\nabla_{e_4}\eta + e_5\nabla_{e_5}\eta, -e_{145}\eta) = (e_1\nabla_{e_1}\eta + e_4\nabla_{e_4}\eta + e_5\nabla_{e_5}\eta, e_0\eta),$$

and so on. We used that $e_{123}\eta = e_0\eta = -e_{145}\eta$ for the last equalities. This is deduced from the equality $e^{01}\eta = e^{23}\eta = -e^{45}\eta$, that we obtained in the proof of Proposition 1.11. Taking into account that Cayley frames are positively oriented, we obtain $\star(V^*) = \frac{1}{7}(\gamma_8 \wedge \Omega) = 4\,\mathrm{alt}(c^{-1}\nabla\eta)$, so that γ_{48} lies in $\Omega^3_{48}(M)$. Finally, taking into account the formula for $\star d\Omega$ in Proposition 1.22, we get $\star d\Omega = \frac{2}{7}\gamma_{48} + \frac{8}{7}\gamma_8$.

To compute the Lee form we used that the projection of $d\Omega$ to $\Omega^3_8(M)$ is $-\frac{8}{7}\star\gamma_8$ and the formula $i(X)\Omega = \star(X^* \wedge \Omega)$, which can be checked by considering a Cayley frame and $X = e_0$. □

1.6 The characteristic connection

The characteristic connection of a Spin(7) structure is a connection ∇^c with totally skew-symmetric torsion such that $\nabla^c\Omega = 0$. The computations above allow us to prove the existence and uniqueness of the characteristic connection for manifolds with a Spin(7) structure. This is a well known result which appears in [69, Theorem 1.1]. Our proof is based on the argument of Theorem 3.1 in [55] and uses the notation of section 1.2.3.

Proposition 1.24. *Given a* Spin(7) *structure, there exists a unique characteristic connection whose torsion* $\mathrm{T} \in \Omega^3(M)$ *is determined by the expression:*

$$\mathrm{T} = -\delta\Omega - \frac{7}{6}\star(\theta \wedge \Omega).$$

Proof. A connection with skew-symmetric torsion $\mathrm{T} \in \Omega^3(M)$ is given by $\nabla_X Y + \frac{1}{2}\mathrm{T}(X, Y, \cdot)^\sharp$, where $\mathrm{T}(X, Y, \cdot)^\sharp$ is the vector field such that $(\mathrm{T}(X, Y, \cdot)^\sharp)^* = \mathrm{T}(X, Y, \cdot)$. Thus, the lifting to the spinor bundle is $\nabla_X\phi + \frac{1}{4}i(X)\mathrm{T}\phi$.

Taking into account that the condition $\nabla^c\Omega = 0$ is equivalent to $\nabla^c\eta = 0$ and that the kernel of the Clifford product with η on Λ^2T^*M is $\Lambda^2_{21}T^*M$, we deduce that the set of characteristic connections is isomorphic to the set of 3-forms $\mathrm{T} \in \Omega^3(M)$ such that

$$-4c^{-1}\nabla_X\eta = i(X)\mathrm{T}\eta = \mathrm{p}^2_7(i(X)\mathrm{T})\eta, \quad \forall X \in \mathfrak{X}(M).$$

The last equality may be rewritten as $-4c^{-1}\nabla\eta = \Theta(\mathrm{T})\eta$. From the definition of γ_{48} given in Proposition 1.23 we obtain: $-4\Xi(c^{-1}\nabla\eta) = -12\,\mathrm{alt}(c^{-1}\nabla\eta) = \frac{1}{7}(\gamma_{48} - 3\gamma_8)$. Finally, taking into account the eigenvalues of $\Xi \circ \Theta$, we deduce:

$$\mathrm{T} = \frac{1}{7}(2\gamma_{48} - \frac{4}{3}\gamma_8) = \star d\Omega - \frac{4}{3}\gamma_8 = -\delta\Omega - \frac{7}{6}\star(\theta \wedge \Omega).$$

To obtain the second equality we used the formula for $d\Omega$ from Lemma 1.23. To check the last one, note that $\gamma_8 = i(V)\Omega = \star(V^* \wedge \Omega) = \frac{7}{8}\star(\theta \wedge \Omega)$. □

Remark 1.25. The Spin(7) structure is balanced if and only if $\mathrm{T} \in \Omega^3_{48}(M)$ and locally conformally parallel if and only if $\mathrm{T} \in \Omega^3_8(M)$.

Remark 1.26. The equation for balanced structures given in Theorem 1.21 is also deduced from [69, Theorem 9.1], which states that the Spin(7) structure determined by η on a Riemannian manifold (M, g) is balanced for the metric $e^{\frac{6}{7}f}g$ if and only if it satisfies the equations

$$\nabla^{\mathrm{T}}\eta = 0, \tag{1.9}$$

$$(df - \frac{1}{2}\mathrm{T})\eta = 0, \tag{1.10}$$

where ∇^{T} is the g-metric connection with totally skew-symmetric torsion T. That is, $\nabla^{\mathrm{T}}\phi = \nabla_X\phi + \frac{1}{4}i(X)\mathrm{T}\phi$ for $\phi \in \Sigma(M)$. This connection has an associated Dirac operator, which is related to D:

$$D^{\mathrm{T}}\phi = \sum_{i=0}^{7} e_i \nabla^T_{e_i}\phi = D\phi + \frac{1}{4}\sum_{i=0}^{7} e_i \wedge (i(e_i)\mathrm{T})\phi = D\phi + \frac{3}{4}\mathrm{T}\phi.$$

Assuming [69, Theorem 9.1], if we suppose that the structure is balanced for the metric g, equations (1.9) and (1.10) imply that $0 = D^{\mathrm{T}}\eta = D\eta + \frac{3}{4}\mathrm{T}\eta = D\eta$. Conversely if we suppose that $D\eta = 0$ and we choose T the torsion of the characteristic connection, then $\nabla^{\mathrm{T}}\eta = 0$ and $0 = D^{\mathrm{T}}\eta = D\eta + \frac{3}{4}\mathrm{T}\eta$, so that $\mathrm{T}\eta = 0$. According to Proposition 1.20, $\mathrm{T} \in \Omega^3_{48}(M)$ so that structure is balanced.

1.7 G_2 distributions

In this section we define the notion of a G_2 distribution on a Spin(7) manifold in terms of spinors and we study the torsion of the structure with respect to a suitable connection on the distribution. Then we relate the Spin(7) structure of the ambient manifold with the G_2 structure of the distribution. This approach enables us to study G_2 structures on submanifolds of Spin(7) manifolds, S^1-principal fibre bundles over G_2 manifolds and warped products of manifolds admitting a G_2 structure with $\mathbb{R}$. Our analysis is very similar to the description of G_2 structures from a spinorial viewpoint done in [1], which we briefly recall.

A 7-dimensional Riemannian manifold (Q, g) is equipped with a G_2 structure if it is spin. Its spinor bundle admits a unit-length section η because $\mathrm{rk}(\Sigma(Q)) = 8 > 7 = \dim(Q)$. A cross product is constructed from the spinor and is determined by a 3-form Ψ. Denote by ∇^Q both the Levi-Civita connection of the manifold and its lifting to the spinor bundle; an endomorphism $\mathcal{S}$ of TQ is defined by the condition:

$$\nabla^Q_X\eta = \mathcal{S}(X)\eta.$$

The intrinsic torsion is $-\frac{2}{3}i(\mathcal{S})\Psi$ [1, Proposition 4.4], so that pure types of G_2 structures are given by the G_2 irreducible components of $\mathrm{End}(TQ)$. It is known that $\mathrm{End}(\mathbb{R}^7) = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$, where χ_i are irreducible G_2 representations, defined by:

$$\chi_1 = \langle \mathrm{Id} \rangle, \quad \chi_2 = \mathfrak{g}_2, \quad \chi_3 = \mathrm{Sym}^2_0(\mathbb{R}^7), \quad \chi_4 = \{A\colon \mathbb{R}^7 \to \mathbb{R}^7\colon A(X) = X \times S, \quad S \in \mathbb{R}^7\},$$

where $\mathrm{Sym}^2_0(\mathbb{R}^7)$ denotes the set of symmetric and traceless endomorphisms. The dimensions of the previous spaces are 1, 14, 27 and 7 respectively.

Denote by R_Q a G_2 reduction of the SO(7) principal bundle $\mathrm{P}_{\mathrm{SO}}(Q)$ and define $\chi_i(Q) = R_Q \times_{\mathrm{G}_2} \chi_i$, then the pure classes of G_2 structures are determined by the condition $\mathcal{S} \in \chi_i(Q)$. For instance, nearly parallel G_2 structures satisfy $\mathcal{S} \in \chi_1(Q)$, almost parallel or calibrated are those with $\mathcal{S} \in \chi_2(Q)$, and locally conformally calibrated are such that $\mathcal{S} \in \chi_4(Q)$. Indeed in the nearly parallel case it holds that $\mathcal{S}(X) = \lambda_0 X$ for some $\lambda_0 \in \mathbb{R}$. Moreover mixed classes are also relevant, for instance cocalibrated structures correspond to $\mathcal{S} \in \chi_1(Q) \oplus \chi_3(Q)$.

Taking this into account, we define G$_2$ structures on distributions and characterise the existence of such structures.

Definition 1.27. Let (M, g) be an oriented 8-dimensional Riemannian manifold and let $\mathcal{D}$ be a cooriented distribution of codimension 1. We say that $\mathcal{D}$ has a G$_2$ structure if the principal SO(7) bundle $\mathrm{P}_{\mathrm{SO}}(\mathcal{D})$ is spin and the spinor bundle $\Sigma(\mathcal{D})$ admits a unit-length section.

Remark 1.28. Let $\bar{\rho}\colon \mathrm{Cl}_7 \to \mathrm{End}(\mathbb{R}^8)$ be irreducible representation of Cl_7 with $\bar{\rho}(e_1 \cdots e_7) = \mathrm{Id}$. If the bundle $\mathrm{P}_{\mathrm{SO}}(\mathcal{D})$ is spin, then $\Sigma(\mathcal{D}) = \mathrm{P}_{\mathrm{Spin}}(\mathcal{D}) \times_{\bar{\rho}} \mathbb{R}^8$. This is a vector bundle over M, with $\mathrm{rk}(\Sigma(\mathcal{D})) = 8$. Therefore, it is not automatic that the bundle $\Sigma(\mathcal{D}) \to M$ admits a unit-length section.

Lemma 1.29. *Let (M, g) be an oriented 8-dimensional Riemannian manifold and let $\mathcal{D}$ be a cooriented distribution of codimension 1. Take a unit-length vector field N perpendicular to $\mathcal{D}$ such that $TM = \langle N \rangle \oplus \mathcal{D}$ as oriented bundles. The manifold M is spin if and only if the bundle $\mathrm{P}_{\mathrm{SO}}(\mathcal{D})$ is spin. In this case, the spinor bundles are related by $\Sigma(\mathcal{D}) = \Sigma(M)^+$ and it holds*

$$X \cdot_{\mathcal{D}} \phi = NX\phi, \ \text{if } X \in \mathcal{D}, \quad \phi \in \Sigma(\mathcal{D}), \tag{1.11}$$

where we suppressed the symbol $\cdot_M$ to denote the Clifford product on M.

Therefore M has a Spin(7) *structure if and only if $\mathcal{D}$ has a* G$_2$ *structure.*

Proof. The bundle $\mathrm{P}_{\mathrm{SO}}(\mathcal{D})$ is a reduction of $\mathrm{P}_{\mathrm{SO}}(M)$ because of the inclusion:

$$i\colon \mathrm{P}_{\mathrm{SO}}(\mathcal{D}) \to \mathrm{P}_{\mathrm{SO}}(M), \quad (X_1, \dots, X_7) \to (N, X_1, \dots, X_7).$$

Suppose that $\mathrm{P}_{\mathrm{SO}}(\mathcal{D})$ is spin and denote the spin bundle by $\pi_{\mathcal{D}}\colon \mathrm{P}_{\mathrm{Spin}}(\mathcal{D}) \to \mathrm{P}_{\mathrm{SO}}(\mathcal{D})$. Then, we define the principal Spin(8) bundle $\mathrm{P}_{\mathrm{Spin}}(M) = \mathrm{P}_{\mathrm{Spin}}(\mathcal{D}) \times_{\mathrm{Spin}(7)} \mathrm{Spin}(8)$ and the map:

$$\pi_M\colon \mathrm{P}_{\mathrm{Spin}}(M) \to \mathrm{P}_{\mathrm{SO}}(M), \quad [\tilde{F}, \tilde{\varphi}] \to \mathrm{Ad}(\tilde{\varphi})(i(\pi_{\mathcal{D}}(\tilde{F}))),$$

which is a double covering and Ad-equivariant. Therefore, M is spin. Conversely, if M is spin then the pullback $i^*(\mathrm{P}_{\mathrm{Spin}}(M))$ is the spin bundle of $\mathrm{P}_{\mathrm{SO}}(\mathcal{D})$.

Moreover, the irreducible 8-dimensional representation of Cl_7 which maps the volume form to the identity is constructed from the composition

$$\mathrm{Cl}_7 \to \mathrm{Cl}_8^0 \xrightarrow{\rho} \mathrm{Gl}(\Delta^+),$$

where the first map is induced by the embedding $\mathbb{R}^7 \to \mathrm{Cl}_8^0$, $v \to e_0 v$, and $(e_0, \dots, e_7)$ denotes the canonical basis of $\mathbb{R}^8$.

Therefore, the spinor bundle $\Sigma(\mathcal{D})$ coincides with $\Sigma(M)^+$ and Clifford products of vectors and spinors are related by the formula (1.11).

□

From now on we assume that the manifold (M, g) has a Spin(7) structure Ω, constructed from a unit-length section η of the spinor bundle $\Sigma(M)^+$, as in Proposition 1.8. We now equip M with a distribution $\mathcal{D}$ as in Lemma 1.29. We denote by $\Omega^k(\mathcal{D})$ the space of smooth sections of $\Lambda^k \mathcal{D}^*$.

Remarks 1.30. In this situation we observe:

1. If $\beta \in \Omega^{2k}(\mathcal{D})$ and $\phi \in \Sigma(\mathcal{D})$ then $\beta \cdot_{\mathcal{D}} \phi = \beta\phi$.

2. There is an orthogonal decomposition $\Sigma(\mathcal{D}) = \langle\eta\rangle \oplus (\mathcal{D} \cdot_{\mathcal{D}} \eta)$.

3. The section η defines a cross product on $\mathcal{D}$ by means of:

$$(X \times Y)\eta = (X^* \wedge Y^*)\eta = (XY + g(X,Y))\eta,$$

which is determined by $\Psi_{\mathcal{D}}(X,Y,Z) = (X\eta, (Y \times Z)\eta) = -(\eta, XYZ\eta)$.

4. The cross product is determined by $\Psi_{\mathcal{D}} = i(N)\Omega$. Therefore, taking into account that $\star\Omega = \Omega$ we obtain $\Omega = N^* \wedge \Psi_{\mathcal{D}} + \star_{\mathcal{D}}\Psi_{\mathcal{D}}$.

We now equip $\mathcal{D}$ with a suitable connection $\nabla^{\mathcal{D}}$, the projection of covariant derivative of the ambient manifold ∇^M to $\mathcal{D}$. That is,

Definition 1.31. The covariant derivative $\nabla^{\mathcal{D}}$ of $\mathcal{D}$ is determined by the expression:

$$\nabla^M_X Y = \nabla^{\mathcal{D}}_X Y + g(\mathcal{T}(X), Y)N, \quad X, Y \in \mathcal{D},$$

where $\mathcal{T} \in \mathrm{End}(\mathcal{D})$ is given by: $2g(\mathcal{T}(X),Y) = -N(g(X,Y)) - g([X,N],Y) - g([Y,N],X) + g([X,Y],N)$.

The definition of $\mathcal{T}$ follows from the Koszul formulas. We decompose $\mathcal{T}$ into its symmetric and skew-symmetric parts, which we call $\mathcal{W}$ and $\mathcal{L}$ respectively. Of course,

$$g(\mathcal{W}(X),Y) = \frac{1}{2}(N(g(X,Y)) - g([X,N],Y) - g([Y,N],X)), \tag{1.12}$$

$$g(\mathcal{L}(X),Y) = \frac{1}{2}g([X,Y],N) = -\frac{1}{2}dN^*(X,Y). \tag{1.13}$$

The connection $\nabla^{\mathcal{D}}$ is a metric connection and the tensor $\mathcal{L} = -\frac{1}{2}dN^*$ measures the lack of integrability of the distribution.

We also denote by $\nabla^{\mathcal{D}}$ the lift of this connection to the spinor bundle $\Sigma(\mathcal{D})$. This is a metric connection with respect to $(\cdot,\cdot)$ and behaves as a derivation with respect to the Clifford product. Hence $\nabla^{\mathcal{D}}\eta \in \langle\eta\rangle^{\perp}$, and there is an endomorphism of $\mathcal{D}$ that we denote by $\mathcal{S}_{\mathcal{D}}$ such that $\nabla^{\mathcal{D}}_X\eta = \mathcal{S}_{\mathcal{D}}(X)\eta$. Let us define $\chi_i(\mathcal{D}) = R_{\mathcal{D}} \times \chi_i$, where $R_{\mathcal{D}}$ is the G_2 reduction of $P_{SO}(\mathcal{D})$ determined by $\Psi_{\mathcal{D}}$; there is a splitting of $\mathrm{End}(\mathcal{D})$ and we decompose $\mathcal{S}$ according to it:

$$\mathcal{S}_{\mathcal{D}}(X) = \lambda\,\mathrm{Id} + S_2 + S_3 + S_4,$$

where $\lambda \in C^\infty(M)$, $S_2 \in \chi_2(\mathcal{D})$, $S_3 \in \chi_3(\mathcal{D})$, $S_4 \in \chi_4(\mathcal{D})$. We let $S \in \mathfrak{X}(\mathcal{D})$ be such that $S_4(X) = X \times S$.

We relate these components with the Spin(7) structure defined on M:

Lemma 1.32. *The covariant derivative ∇^M at $\Sigma(M)^+$ in the direction of $\mathcal{D}$ is $\nabla^M_X\phi = \nabla^D_X\phi - \frac{1}{2}NT(X)\phi$. In particular, define $\mathcal{A} = \mathcal{S}_{\mathcal{D}} - \frac{1}{2}\mathcal{T}$; then,*

$$\nabla^M_X\eta = N\mathcal{A}(X)\eta. \tag{1.14}$$

Proof. Let $\phi \in \Sigma(M)^+$ be a spinor. Let $F = (X_0, X_1, \ldots, X_7)$ be a local orthonormal frame with $X_0 = N$ and $X_1, \ldots, X_7 \in \mathcal{D}$, denote by $\tilde{F}$ its lifting to $P_{\mathrm{Spin}}(M)$ and write $\phi(p) = [\tilde{F}, s(p)]$. According to [54, p. 60], if $X \in \mathcal{D}$ then,

$$\nabla^M_X\phi = [\tilde{F}, ds_p(X)] + \frac{1}{2}\sum_{0\leq i<j\leq 7} g(\nabla^M_X X_i, X_j)X_iX_j\phi.$$

Taking into account Definition 1.31 we obtain the equalities $g(\nabla^M_X X_i, X_j) = g(\nabla^{\mathcal{D}}_X X_i, X_j)$ for $1 \le i < j \le 7$, and $g(\nabla^M_X N, X_j) = -g(N, \nabla^M_X X_j) = -g(\mathcal{T}(X), X_j)$. Therefore,

$$\nabla^M_X \eta = [\tilde{F}, ds_p(X)] + \frac{1}{2} \sum_{1\le i<j\le 7} g(\nabla^{\mathcal{D}}_X X_i, X_j) X_i X_j \phi - \frac{1}{2}\sum_{j=1}^{7} g(\mathcal{T}(X), X_j) N X_j \phi.$$

The formula for the covariant derivative in [54, p. 60] allows to conclude that the first two summands correspond to $\nabla^{\mathcal{D}}_X \phi$ under identification provided in Lemma 1.29. In addition, the last summand is equal to $-\frac{1}{2} N T(X)\phi$. The second statement follows from the equality $\nabla^{\mathcal{D}}_X \eta = \mathcal{S}_{\mathcal{D}}(X)\eta$ for $X \in \mathcal{D}$. ☐

We decompose $\mathcal{L}$ and $\mathcal{W}$ according to the splitting of $\mathrm{End}(\mathcal{D})$ into irreducible parts and then we decompose $\mathcal{A}$:

1. $\mathcal{L} = L_2 + L_4$, where $L_2 \in \chi_2(\mathcal{D})$, $L_4 \in \chi_4(\mathcal{D})$ and let $L \in \mathfrak{X}(\mathcal{D})$ such that $L_4(X) = X \times L$.
2. $\mathcal{W} = h\,\mathrm{Id} + W_3$, where $h \in C^\infty(M)$, $W_3 \in \chi_3(\mathcal{D})$.
3. $\mathcal{A} = \mu\,\mathrm{Id} + A_2 + A_3 + A_4$, where $\mu = \lambda - \frac{h}{2}$, $A_2 = S_2 - \frac{1}{2}L_2$, $A_3 = S_3 - \frac{1}{2}W_3$, $A_4 = S_4 - \frac{1}{2}L_4$. We also denote $A = S - \frac{1}{2}L$.

We compute $\star d\Omega$ in terms of the previous endomorphisms and $\nabla^{\mathcal{D}}_N \eta$. Our first lemma is deduced from [1, Theorems 4.6, 4.8].

Lemma 1.33. *Let $(X_1, \dots, X_7)$ be an orthonormal local frame of $\mathcal{D}$. Then*

$$\sum_{i=1}^{7} X_i \mathcal{A}(X_i)\eta = -7\mu\eta - 6NA\eta.$$

Proof. We split the endomorphism $\mathcal{A}$ into its G_2 irreducible components and then compute each term separately. It is obvious that $\sum_{i=1}^7 X_i \mu X_i \eta = -7\mu\eta$. Moreover,

$$\sum_{i=1}^{7} X_i (X_i \times A)\eta = \sum_{i=1}^{7} X_i (X_i N A + g(X_i, A) N)\eta = -6NA.$$

Finally consider the G_2-equivariant map, $m\colon \mathcal{D} \otimes \mathcal{D} \to \Sigma(\mathcal{D})$, $m(X, Y) = XY\eta$. For dimensional reasons, its kernel must be $\chi_2(\mathcal{D}) \oplus \chi_3(\mathcal{D})$. If $k \in \{2, 3\}$, then:

$$\sum_{i=1}^{7} X_i A_k(X_i)\eta = m\left(\sum_{i=1}^{7} (A_k)_{ij} X_i X_j\right) = 0,$$

where we denote by $(A_k)_{ij}$ the entries of the matrix A_k with respect to the basis $(X_1, \dots, X_7)$. ☐

Remarks 1.34.

1. Since $\nabla^M_N \eta$ is perpendicular to η, there exists $U \in \mathfrak{X}(\mathcal{D})$ such that $\nabla^M_N \eta = -NU\eta$. In order to compute $\nabla^M_N \eta$ we may take $F = (X_0, X_1, \dots, X_7)$ a local orthonormal frame of M such that $N = X_0$, a lifting $\tilde{F} \in \mathrm{P}_{\mathrm{Spin}}(M)$ and write $\eta(p) = [\tilde{F}, s(p)]$. According to [54, p. 60],

$$\nabla^M_{X_0}\eta = [\tilde{F}, ds(X_0)] + \frac{1}{2}\sum_{0\le i<j\le 7} g(\nabla_{X_0} X_i, X_j) X_i X_j \eta \tag{1.15}$$

$$= [\tilde{F}, ds(X_0)] + \frac{1}{2}\left(X_0 \nabla_{X_0} X_0 + \sum_{1\le i<j\le 7} g(\nabla_{X_0} X_i, X_j) X_i X_j\right)\eta. \tag{1.16}$$

Therefore, U depends on the local information of the section and $\nabla_{X_0} X_i$.

2. From item (1) of this remark, equation (1.14) and Lemma 1.33, the Dirac operator of M is
$$D^M\eta = U\eta + \sum_{i=1}^{7} X_i N\mathcal{A}(X_i)\eta = (U - 6A + 7\mu N)\eta.$$

Lemma 1.35. *Define the forms $\beta_2 \in \Omega^2(\mathcal{D})$ and $\beta_3 \in \Omega^3(\mathcal{D})$ by:*

$$\beta_2(X,Y) = g(A_2(X),Y), \quad \beta_3(X,Y,Z) = \operatorname{alt}(i(A_3(\cdot))\Psi_{\mathcal{D}})(X,Y,Z).$$

Then

1. $N^* \wedge i(N)(12\operatorname{alt}(c^{-1}\nabla\eta)) = i(U-2A)(N^*\wedge\Psi_{\mathcal{D}}) - 2N^*\wedge\beta_2$,
2. $12\operatorname{alt}(c^{-1}\nabla\eta)|_{\mathcal{D}} = 3i\mu\Psi_{\mathcal{D}} - 3i(A)(\star_{\mathcal{D}}\Psi_{\mathcal{D}}) + 3\beta_3$.

Proof. The first equality is a consequence of the symmetric or the skew-symmetric property of each factor; if $X, Y \in \mathcal{D}$ then:

$$\begin{aligned} 12\operatorname{alt}(c^{-1}\nabla\eta)(N,X,Y) &= -(XY\eta, NU\eta) - (NY\eta, N\mathcal{A}(X)\eta) + (NX\eta, N\mathcal{A}(Y)\eta) \\ &= -i(U)\Psi_{\mathcal{D}}(X,Y) - 2(Y\eta, (A_2(X) + X\times A)\eta) \\ &= \left(i(U-2A)(N^*\wedge\Psi_{\mathcal{D}}) - 2N^*\wedge\beta_2\right)(N,X,Y). \end{aligned}$$

Observe that we used equation (1.14) to compute $\nabla_X\eta$ and $\nabla_Y\eta$. To check the second one, first note that according to Lemma 1.11 and equation (1.14), if $X, Y, Z \in \mathcal{D}$ then:

$$\begin{aligned} 4(c^{-1}\nabla\eta)(X,Y,Z) &= (N\mathcal{A}(X)\eta, YZ\eta) = -(YZ\mathcal{A}(X)\eta, N\eta) = \Omega(N,Y,Z,\mathcal{A}(X)) \\ &= \Psi_{\mathcal{D}}(\mathcal{A}(X),Y,Z). \end{aligned}$$

Observe that the third equality is deduced from Porposition 1.8 by taking into account that $0 = g(Y,\mathcal{A}(X))(Z\eta,N\eta) = g(Z,\mathcal{A}(X))(Y\eta,N\eta) = g(Y,Z)(\mathcal{A}(X)\eta,N\eta)$.

Thus, $12\operatorname{alt}(c^{-1}\nabla\eta)|_{\mathcal{D}} = 3\operatorname{alt}(i(\mathcal{A}(\cdot))\Psi_{\mathcal{D}})$. We compute each term in the decomposition of $\mathcal{A}$ separately. It is clear that $3\operatorname{alt}(i(\mu\operatorname{Id})\Psi_{\mathcal{D}})(X,Y,Z) = 3\mu\Psi_{\mathcal{D}}(X,Y,Z)$ and $3\operatorname{alt}(i(A_3(\cdot))\Psi_{\mathcal{D}}) = 3\beta_3$. Moreover, the equality $\operatorname{alt}(i(A_2(\cdot))\Psi_{\mathcal{D}}) = 0$ follows from the fact that $A_2 \in \chi_2(\mathcal{D})$. Finally, if X, Y and Z are orthonormal vectors in $T\mathcal{D}$, then:

$$i(A_4(X))\Psi_{\mathcal{D}}(Y,Z) = ((X\times A)\eta, (Y\times Z)\eta) = (XA\eta, YZ\eta) = -(A\eta, (X\times Y\times Z)\eta).$$

Therefore, $3\operatorname{alt}(i(A_4(\cdot))\Psi_{\mathcal{D}})(X,Y,Z) = -3(A\eta,(X\times Y\times Z)\eta) = i(A)(\star_{\mathcal{D}}\Psi_{\mathcal{D}})(X,Y,Z)$. □

From Lemmas 1.33 and 1.35 and the decomposition of $\star d\Omega$ obtained in Proposition 1.23 we conclude:

Proposition 1.36. *Let $U \in \mathfrak{X}(\mathcal{D})$ such that $\nabla^M_N\eta = -NU\eta$ and define the forms $\beta_2 \in \Omega^2(\mathcal{D})$ and $\beta_3 \in \Omega^3(\mathcal{D})$ by:*

$$\beta_2(X,Y) = g(A_2(X),Y), \quad \beta_3(X,Y,Z) = \operatorname{alt}(i(A_3(\cdot))\Psi_{\mathcal{D}})(X,Y,Z).$$

Then, the pure components of $\star d\Omega$ in terms of the G_2 structure are:

$$\begin{aligned} (\star d\Omega)_{48} &= \frac{2}{7}\left(-4i(A+U)N^*\wedge\Psi_{\mathcal{D}} + 3i(A+U)\star_{\mathcal{D}}\Psi_{\mathcal{D}}\right) + 4N^*\wedge\beta_2 - 6\beta_3, \\ (\star d\Omega)_8 &= \frac{8}{7}i(U - 6A + 7\mu N)(N^*\wedge\Psi_{\mathcal{D}} + \star_{\mathcal{D}}\Psi_{\mathcal{D}}). \end{aligned}$$

Proof. We first compute γ_8 and γ_{48}. First recall that $\gamma_8 = i(V)\Omega$ with $D\eta = V\eta$. In order to compute $D\eta$ we consider a local orthonormal frame $(X_0, X_1, \ldots, X_7)$, with $X_0 = N$. Then, according to Lemma 1.33 and Remark 1.38, we obtain:

$$D\eta = (U - 7\mu N + 6A)\eta.$$

Thus $\gamma_8 = i(U - 7\mu N + 6A)\Omega = -7\mu\Psi_{\mathcal{D}} + i(U + 6A)(N^* \wedge \Psi_{\mathcal{D}} + \star_{\mathcal{D}}\Psi_{\mathcal{D}})$. In addition, $\gamma_{48} = 3\gamma_8 - 84\,\mathrm{alt}(c^{-1}\nabla\eta)$; the previous computation and Lemma 1.35 allow us to obtain:

$$\gamma_{48} = (-4i(A + U)N^* \wedge \Psi + 3i(A + U) \star_{\mathcal{D}} \Psi_{\mathcal{D}}) + 4N^* \wedge \beta_2 - 6\beta_3.$$

Note that the terms $-21\mu\Psi_{\mathcal{D}}$ of $3\gamma_8$ and $21\mu\Psi_{\mathcal{D}}$ of $84\,\mathrm{alt}(c)^{-1}\nabla\eta|_{\mathcal{D}}$ cancel one to each other. The conclusion follows from Proposition 1.23. □

1.7.1 Hypersurfaces

Consider an 8-dimensional Spin(7) manifold (M, g), whose Spin(7) form is constructed from a unit-length section η of the spinor bundle $\Sigma(M)^+$, as in Proposition 1.8. Let Q be an oriented hypersurface and take a unit-length vector field N such that $TM = \langle N \rangle \oplus TQ$ as oriented vector bundles.

The tubular neighbourhood theorem guarantees the existence of a cooriented distribution $\mathcal{D}$ defined on a neighbourhood O of Q such that $\mathcal{D}|_Q = TQ$. The coorientation is determined by a unit-length extension of the normal vector field that we also denote by N. Both $\mathcal{D}$ and Q have G_2 structures determined by the spinor η; we relate them by using Proposition 1.36 in the manifold O.

According to Definition 1.31, the Levi-Civita connection of the hypersurface Q is $\nabla^{\mathcal{D}}|_Q$. Moreover, $\mathcal{L}|_Q = 0$ and $\mathcal{W}|_Q$ is the Weingarten operator. Therefore, the restriction of $\mathcal{S}_{\mathcal{D}}$ at Q is the endomorphism $\mathcal{S}$ of the submanifold Q. Decompose $\mathcal{S}|_Q$ and $\mathcal{W}|_Q$ with respect to the G_2 splitting of $\mathrm{End}(TQ)$:

1. $\mathcal{S} = \lambda\,\mathrm{Id} + S_2 + S_3 + S_4$
2. $\mathcal{W}|_Q = H\,\mathrm{Id} + W_3$,

where $\lambda \in C^\infty(M)$, $S_2 \in \chi_2(Q)$, $S_3, W_3 \in \chi_3(Q)$, $S_4 \in \chi_4(Q)$ and $H \in C^\infty(Q)$ is the mean curvature. We also denote by S the vector field on Q such that $S_4(X) = X \times S$.

Corollary 1.37. *Let $U \in \mathfrak{X}(Q)$ such that $\nabla^M_N\eta|_Q = -NU\eta$ and $\Psi_Q = i(N)\Omega$. Define the forms $\beta_2 \in \Omega^2(Q)$ and $\beta_3 \in \Omega^3(Q)$ by:*

$$\beta_2(X, Y) = g(S_2(X), Y), \quad \beta_3(X, Y, Z) = \mathrm{alt}(i((S_3 - \frac{1}{2}W_3)(\cdot))\Psi_{\mathcal{D}})(X, Y, Z).$$

Then, the pure components of $\star d\Omega$ in terms of the G_2 structure are:

$$(\star d\Omega)_{48} = \frac{2}{7}(-4i(S + U)N^* \wedge \Psi_Q + 3i(S + U) \star_Q \Psi_Q) + 4N^* \wedge i^*\beta_2 - 6\beta_3,$$
$$(\star d\Omega)_8 = \frac{8}{7}i\left(U - 6S + 7(\lambda - \frac{1}{2}H)N\right)(N^* \wedge \Psi_Q + \star_Q\Psi_Q).$$

Remark 1.38. Note that the condition $\nabla_N\eta|_Q = -NU\eta$ does not depend on the extension of the vectors. Moreover, we usually compute U taking into account equation (1.15). Note that it depends on the values of the spinor in the direction of N.

Therefore, the Spin(7) type of the ambient manifold provides relations between the G_2 type of the hypersurface, the vector U and the Weingarten operator. Before stating the result, we recall that a hypersurface is said to be totally geodesic if $\mathcal{W} = 0$, totally umbilic if $W_3 = 0$ and minimal if $H = 0$.

Theorem 1.39. *Let (M, g) be a Riemannian manifold endowed with a* Spin(7) *structure determined by a spinor η. Let Q be an oriented hypersurface with normal vector N and let $U \in \mathfrak{X}(Q)$ be such that $\nabla_N \eta|_Q = -NU\eta$.*

1. *If M has a parallel* Spin(7) *structure, then Q has a cocalibrated* G_2 *structure. Moreover,*
   - *1.1 $\mathcal{S} = 0$ if and only if Q is totally geodesic.*
   - *1.2 $\mathcal{S} \in \chi_1(Q)$ if and only if Q is totally umbilic.*
   - *1.3 $\mathcal{S} \in \chi_3(Q)$ if and only if Q is a minimal hypersurface.*
2. *If M has a locally conformally parallel* Spin(7) *structure, then $\mathcal{S} \in \chi_1(Q) \oplus \chi_3(Q) \oplus \chi_4(Q)$. Indeed,*
   - *2.1 $\mathcal{S} \in \chi_1(Q)$ if and only if $U = 0$ and Q is totally umbilic.*
   - *2.2 $\mathcal{S} \in \chi_1(Q) \oplus \chi_4(Q)$ if and only if Q is totally umbilic.*
3. *If M has a balanced* Spin(7) *structure, then:*
   - *3.1 $\mathcal{S} \in \chi_2(Q) \oplus \chi_3(Q)$ if and only if $U = 0$ and Q is a minimal hypersurface.*
   - *3.2 $\mathcal{S} \in \chi_1(Q) \oplus \chi_2(Q) \oplus \chi_3(Q)$ if and only if $U = 0$.*
   - *3.3 $\mathcal{S} \in \chi_2(Q) \oplus \chi_3(Q) \oplus \chi_4(Q)$ if and only if Q is a minimal hypersurface.*

Proof. The parallel case follows from the equalities $U = S = 0$, $S_2 = 0$, $2\lambda = H$ and $2S_3 = W_3$. The locally conformally parallel case follows from the equalities $U = -S$, $S_2 = 0$ and $2S_3 = W_3$, which imply that $S \in \chi_1(Q) \oplus \chi_3(Q) \oplus \chi_4(Q)$. Finally the balanced case follows from $U = 6S$ and $2\lambda = H$. □

1.7.2 Principal bundles over a G_2 manifold

Let Q be a G_2 manifold and let $\pi\colon M \to Q$ be a $G = \mathbb{R}$ or $G = S^1$ principal bundle over Q; identify its Lie algebra $\mathfrak{g}$ with $\mathbb{R}$.

Define the vertical field $N(p) = \frac{d}{dt}\big|_{t=0}(p\exp(t))$. A connection $\omega\colon TM \to \mathfrak{g}$ defines a horizontal distribution $\mathcal{H}$. Consider the metric on M such that:

1. The map $d\pi\colon \mathcal{H}_p \to T_{\pi(p)}Q$ is an isometry.
2. The vector $N(p)$ has unit-lenght and it is perpendicular to $\mathcal{H}_p$.

The projection $d\pi$ induces a map $p\colon \mathrm{P}_{\mathrm{SO}}(\mathcal{H}) \to \mathrm{P}_{\mathrm{SO}}(Q)$ so that the pullback to $\mathrm{P}_{\mathrm{Spin}}(Q)$ defines a spin structure $\mathrm{P}_{\mathrm{Spin}}(\mathcal{H})$ over $\mathrm{P}_{\mathrm{SO}}(\mathcal{H})$. The map $\tilde{p}\colon \mathrm{P}_{\mathrm{Spin}}(\mathcal{H}) \to \mathrm{P}_{\mathrm{Spin}}(Q)$, which is canonically defined, has the property that $\tilde{p}(\tilde{\varphi}\tilde{F}) = \tilde{\varphi}\tilde{p}(\tilde{F})$ if $\tilde{\varphi} \in \mathrm{Spin}(8)$, inducing a map between the spinor bundles, which we call $\bar{p}$. Note that this map yields isomorphisms $\Sigma(\mathcal{H})_p \to \Sigma(Q)_{\pi(p)}$. Moreover, let $X \in TQ$ and denote by X^h its horizontal lift, then $\bar{p}(X^h \cdot_{\mathcal{H}} \phi) = X\bar{p}(\phi)$. Therefore, from a section $\bar{\eta}\colon Q \to \Sigma(Q)$ we define a section $\eta\colon M \to \Sigma(\mathcal{H})$ by means of the expression $\bar{p}(\eta) = \bar{\eta}$. If we denote by Ψ_Q the G_2 form on Q, then $\Psi_{\mathcal{D}} = \pi^*\Psi_Q$.

Furthermore, one can check that $\nabla^{\mathcal{H}}_{X^h} Y^h = (\nabla^Q_X Y)^h$. Let $\mathcal{S} \in \mathrm{End}(Q)$ be such that $\nabla^Q_X \bar\eta = \mathcal{S}(X)\bar\eta$, then endomorphism $\mathcal{S}_{\mathcal{D}}$ of the distribution is the lift of $\mathcal{S}$, that is:

$$\nabla^{\mathcal{H}}_{X^h}\eta = \mathcal{S}(X)^h\eta.$$

Therefore the distribution $\mathcal{H}$ and the manifold Q have the same type of G_2 structure. In order to classify the Spin(7) structure on M, denote the curvature of the connection ω by:

$$\mathfrak{L}(X,Y) = [X^h, Y^h] - [X,Y]^h \in \langle N\rangle, \quad X, Y \in TQ.$$

Since $\mathfrak{L}(X,Y) \in \langle N\rangle$, we also denote by $\mathfrak{L}$ the 2-form $g(\mathfrak{L}(X,Y), N)$. As a skew-symmetric endomorphism, we decompose $\mathfrak{L} = \bar L_2 + \bar L_4$ where $\bar L_4(X) = X \times \bar L$ for some vector field $\bar L \in \mathfrak{X}(Q)$.

Corollary 1.40. *Suppose that $\nabla^Q_X\bar\eta = \mathcal{S}(X)\cdot_Q \bar\eta$ with $\mathcal{S}(X) = \lambda\,\mathrm{Id} + S_2 + S_3 + S_4$ where $\lambda \in C^\infty(Q)$, $S_2 \in \chi_2(Q)$, $S_3 \in \chi_3(Q)$, $S_4 \in \chi_4(Q)$ and let $S \in \mathfrak{X}(Q)$ be such that $S_4(X) = X \times S$. Define $\beta_2 \in \Omega^2(Q)$ and $\beta_3 \in \Omega^3(Q)$ by:*

$$\beta_2(X,Y) = g\left(S_2(X) - \frac{1}{2}\bar L_2(X), Y\right), \quad \beta_3(X,Y,Z) = \mathrm{alt}(i(S_3(\cdot))\Psi_Q)(X,Y,Z).$$

The pure components of $\star d\Omega$ in terms of the G_2 structure are:

$$(\star d\Omega)_{48} = \frac{2}{7}\left(-4i(S^h - \frac{1}{4}\bar L^h)N^* \wedge \pi^*\Psi_Q + 3i(S^h - \frac{1}{4}\bar L^h)\pi^*(\star_Q\Psi_Q)\right) + 4N^*\wedge\pi^*\beta_2 - 6\pi^*\beta_3,$$
$$(\star d\Omega)_8 = \frac{8}{7}i\left(\frac{15}{4}\bar L^h - 6S^h + 7\lambda N\right)(N^*\wedge\pi^*\Psi_Q + \pi^*(\star_Q\Psi_Q)).$$

Proof. It suffices to check the equalities $\mathcal{W} = 0$, $g(\mathcal{L}(X),Y) = \frac{1}{2}\pi^*\mathfrak{L}(X,Y)$, and $U = \frac{3}{4}\bar L^h$. From these we obtain $\mu = \lambda$, $A_2 = S_2^h - \frac{1}{2}L_2^h$, $A_3 = S_3^h$, and $A = S^h - \frac{1}{2}\bar L^h$; the conclusion then follows from Proposition 1.36.

Before proving the equalities we observe that $[X^h, N] = 0$ if $X \in \mathfrak{X}(Q)$ because ω is left-invariant. In addition, by the Koszul formulas:

$$\nabla_N N = 0,$$
$$g(\nabla_N X^h, Y^h) = -\frac{1}{2}g([X^h,Y^h],N) = -\frac{1}{2}g([X^h,Y^h] - [X,Y]^h, N) = -\frac{1}{2}\mathfrak{L}(X,Y),$$

for orthogonal vectors $X, Y \in \mathfrak{X}(Q)$.

The claim $\mathcal{W} = 0$ follows from equation (1.12) and the fact that $[X^h, N] = 0$ if $X \in \mathfrak{X}(Q)$. Taking into account formula (1.13), and the fact that $g([X^h,Y^h],N) = \mathfrak{L}(X,Y)$, we obtain $g(\mathcal{L}(X^h), Y^h) = -\frac{1}{2}\mathfrak{L}(X,Y)$.

We finally compute the vector U in terms of the formula (1.15). Let $F = (X_1, \ldots, X_7)$ be a local orthonormal frame of $\mathcal{H}$ which lifts some local frame of TQ. Take a lift $\tilde F \in \mathrm{P}_{\mathrm{Spin}}(\mathcal{H})$ and write $\eta(p) = [\tilde F, s(p)]$. We also denote $X_0 = N$.

By definition, if $\bar\eta(\pi(p)) = [\tilde p(\tilde F(p)), \bar s(\pi(p))]$ then $s(p) = \bar s(\pi(p))$ so that $ds_p(N) = 0$. Taking into account the computations above we obtain:

$$\nabla_N\eta = \frac{1}{2}\sum_{0\le i<j\le 7} g(\nabla_N X_i, X_j)X_iX_j\eta = -\frac{1}{4}\pi^*\mathfrak{L}\eta.$$

Define $\gamma_i(X,Y) = g(\bar L_i(X), Y)$, for $i \in \{2,4\}$, then:

$$-\frac{1}{4}\pi^*\mathfrak{L}\eta = -\frac{1}{4}\pi^*\gamma_4\eta = -\frac{3}{4}N\bar L^h\eta.$$

Here we used that $\pi^*\gamma_2\eta = 0$ because $\mathfrak{g}_2 \subset \mathfrak{spin}(7) = \Lambda^2_{21}$ and $\pi^*\gamma_4 = -i(N)i(\bar L^h)\Omega$. One can check that $\pi^*\gamma_4\eta = 3N\bar L^h\eta$ by using a Cayley basis with $N = e_0$ and $\bar L^h$ proportional to e_1; the computation is similar to the one we did in the proof of Lemma 1.11 to check the equality $e^{01}\eta = e^{ij}\eta$. Thus, $U = \frac{3}{4}\bar L^h$. □

1.7.3 Warped products

We analyze Spin(7) structures on warped products of a G_2 manifold with $\mathbb{R}$. Recall that a warped product of two Riemannian manifolds (X_1, g_1) and (X_2, g_2) is $(X_1 \times X_2, g_1 + e^{2f_1} g_2)$ where $f_1 : X_1 \to \mathbb{R}$ is a smooth function. Therefore, we distinguish two cases.

Warped product $(Q \times \mathbb{R}, e^{2f} g + dt^2)$

Consider a G_2 manifold (Q, g) and a smooth function $f : \mathbb{R} \to \mathbb{R}$. Define the Riemannian manifold $(M = Q \times \mathbb{R}, g_M = e^{2f} g + dt^2)$, and denote the projection to Q by $\pi : M \to Q$. This is the so-called spin cone.

The distribution $\mathcal{D} = TQ$ obviously admits a G_2 structure. The spinor bundle is $\Sigma(M)^+ = \Sigma(TQ \times \mathbb{R}) = \pi^*(\Sigma(Q))$. We denote the spinors at $(x, t) \in M$ by (ϕ, t) with $\phi \in \Sigma(Q)_x$; observe that $((\phi, t), (\phi, t)) = (\phi, \phi)$. Clifford products are related by $(X \cdot_Q \phi, t) = e^{-f} X \cdot_{\mathcal{D}} (\phi, t) = e^{-f} \frac{\partial}{\partial t} X(\phi, t)$ if $X \in TQ$. In the last expression, we suppressed the symbol $\cdot$ to denote the Clifford product on M.

A unit-length section η is defined from a section $\bar{\eta} : Q \to \Sigma(Q)$ by $\eta : M \to \Sigma(\mathcal{D})$, $\eta(x, t) = (\bar{\eta}(x), t)$. Denote by Ψ_Q the G_2 form on Q, then $\Psi_{\mathcal{D}} = e^{3f} \pi^* \Psi_Q$ and $\star_{\mathcal{D}}(\Psi_{\mathcal{D}}) = e^{4f} \star_Q (\Psi_Q)$. In addition, taking into account that $\nabla^{\mathcal{D}}_X Y = \nabla^Q_X Y$ if $X, Y \in \mathfrak{X}(Q)$, we obtain $\nabla^{\mathcal{D}}_X \eta = e^{-f} \mathcal{S}(X) \cdot_{\mathcal{D}} \eta$, where $\nabla^Q_X \bar{\eta} = \mathcal{S}(X) \bar{\eta}$. Thus $\mathcal{S}_{\mathcal{D}} = e^{-f} \mathcal{S}$.

Corollary 1.41. *Suppose that* $\nabla^Q_X \bar{\eta} = \mathcal{S}(X) \cdot_Q \bar{\eta}$ *with* $\mathcal{S}(X) = \lambda \operatorname{Id} + S_2 + S_3 + S_4$ *where* $\lambda \in C^\infty(Q)$, $S_2 \in \chi_2(Q)$, $S_3 \in \chi_3(Q)$, $S_4 \in \chi_4(Q)$. *Let* $S \in \mathfrak{X}(Q)$ *be such that* $S_4(X) = X \times S$. *Denote by* Ψ_Q *the* G_2*-form on* Q *and define* $\beta_2 \in \Omega^2(Q)$ *and* $\beta_3 \in \Omega^3(Q)$ *by:*

$$\beta_2(X, Y) = g\left(S_2(X), Y\right), \quad \beta_3(X, Y, Z) = \operatorname{alt}(i(S_3(\cdot))\Psi_Q)(X, Y, Z).$$

The pure components of $\star d\Omega$ *in terms of the* G_2 *structure are:*

$$(\star d\Omega)_{48} = \frac{2}{7}\left(-4e^{2f} i(S) dt \wedge \pi^*\Psi_Q + 3e^{3f} i(S)\pi^*(\star_Q \Psi_Q)\right) + 4e^f dt \wedge \pi^*\beta_2 - 6e^{2f}\pi^*\beta_3,$$
$$(\star d\Omega)_8 = \frac{8}{7} i\left(-6e^{-f} S + 7(\lambda e^{-f} + \frac{1}{2} f')\frac{\partial}{\partial t}\right)(e^{3f} dt \wedge \pi^*\Psi_Q + e^{4f}\pi^*(\star_Q \Psi_Q)).$$

Proof. The result follows from Proposition 1.36 once we check that $\mathcal{W} = -f' \operatorname{Id}$, $\mathcal{L} = 0$ and $U = 0$. Observe that both the expression $\mathcal{S}_{\mathcal{D}} = e^{-f} S$ and the equalities above ensure that $\mu = \lambda e^{-f} + \frac{1}{2} f'$, $A_2 = e^{-f} S_2$, $A_3 = e^{-f} S_3$ and $A = e^{-f} S$.

The distribution $\mathcal{D}$ is integrable, so that $\mathcal{L} = 0$. Take an orthonormal frame of TQ, $(X_1, \ldots, X_7)$ and note that $g_M(\mathcal{W}(X_i), X_j) = -f' e^{2f} \delta_{ij}$ according to formula (1.12), so that $\mathcal{W} = -f' \operatorname{Id}$. We now compute U taking into account equation (1.15). First, since η is constant in the vertical direction, the term $[\tilde{F}, ds_p(\frac{\partial}{\partial t})]$ vanishes. Moreover, from the Koszul formulas we deduce:

$$\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0 = \nabla_{\frac{\partial}{\partial t}} (e^{-f} X_i).$$

Therefore, $\nabla_{\frac{\partial}{\partial t}} \eta = 0$. □

Warped product $(Q \times \mathbb{R}, g + e^{2f} dt^2)$

Consider a G_2 manifold (Q, g) and a smooth function $f : Q \to \mathbb{R}$. Define the Riemannian manifold $(M = Q \times \mathbb{R}, g_M = g + e^{2f} dt^2)$, and denote the projection to Q by $\pi : M \to Q$.

The distribution $\mathcal{D} = TQ$ obviously admits a G_2 structure. The spinor bundle is $\Sigma(M)^+ = \Sigma(TQ \times \mathbb{R}) = \pi^*(\Sigma(Q))$. We denote the spinors at $(x, t) \in M$ by (ϕ, t) with $\phi \in \Sigma(Q)_x$; observe that $((\phi, t), (\phi, t)) = (\phi, \phi)$. Clifford products are related by $(X \cdot_Q \phi, t) = X \cdot_{\mathcal{D}} (\phi, t) =$

$e^{-f}\frac{\partial}{\partial t}X(\phi,t)$ if $X \in TQ$. We suppressed again the symbol $\cdot$ to denote the Clifford product on M.

A unit-length section η is defined from a section $\bar{\eta}\colon Q \to \Sigma(Q)$ by $\eta\colon M \to \Sigma(\mathcal{D})$, $\eta(x,t) = (\bar{\eta}(x),t)$. Denote by Ψ_Q the G_2 form on Q, then $\Psi_{\mathcal{D}} = \pi^*\Psi_Q$ and $\star_{\mathcal{D}}(\Psi_{\mathcal{D}}) = \star_Q(\Psi_Q)$. In addition, since $\nabla^{\mathcal{D}}_X Y = \nabla^Q_X Y$ when $X, Y \in \mathfrak{X}(Q)$, we obtain $\mathcal{S}_{\mathcal{D}} = \mathcal{S}$, with $\mathcal{S} \in \mathrm{End}(TQ)$ such that $\nabla^Q_X\bar{\eta} = \mathcal{S}(X)\bar{\eta}$.

Corollary 1.42. *Suppose that $\nabla^Q_X\bar{\eta} = \mathcal{S}(X)\cdot_Q \bar{\eta}$ with $\mathcal{S}(X) = \lambda\,\mathrm{Id} + S_2 + S_3 + S_4$ where $\lambda \in C^\infty(Q)$, $S_2 \in \chi_2(Q)$, $S_3 \in \chi_3(Q)$, $S_4 \in \chi_4(Q)$. Let $S \in \mathfrak{X}(Q)$ be such that $S_4(X) = X \times S$. Denote by Ψ_Q the G_2-form on Q and define $\beta_2 \in \Omega^2(Q)$ and $\beta_3 \in \Omega^3(Q)$ by:*

$$\beta_2(X,Y) = g\left(S_2(X),Y\right), \quad \beta_3(X,Y,Z) = \mathrm{alt}(i(S_3(\cdot))\Psi_Q)(X,Y,Z).$$

The pure components of $\star d\Omega$ in terms of the G_2 structure are:

$$\begin{aligned}(\star d\Omega)_{48} =& \frac{2}{7}\left(-4i\left(S + \frac{1}{2}\,\mathrm{grad}(f)\right)e^f dt \wedge \pi^*\Psi_Q + 3i\left(S + \frac{1}{2}\,\mathrm{grad}(f)\right)\pi^*(\star_Q\Psi_Q)\right)\\ &+ 4e^f dt \wedge \pi^*\beta_2 - 6\pi^*\beta_3,\\ (\star d\Omega)_8 =& \frac{8}{7}i\left(\frac{1}{2}\,\mathrm{grad}(f) - 6S + 7\lambda e^{-f}\frac{\partial}{\partial t}\right)\left(e^f dt \wedge \pi^*\Psi_Q + \pi^*(\star_Q\Psi_Q)\right).\end{aligned}$$

Proof. The result immediatly from Proposition 1.36 once we check that $\mathcal{W} = 0$, $\mathcal{L} = 0$ and $U = \frac{1}{2}\,\mathrm{grad}(f)$. Observe that both the expression $\mathcal{S}_{\mathcal{D}} = S$ and the equalities above ensure that $\mu = \lambda$, $A_2 = S_2$, $A_3 = S_3$ and $A = S$.

The distribution $\mathcal{D}$ is integrable, so that $\mathcal{L} = 0$. Take an orthonormal frame of TQ, $(X_1,\ldots,X_7)$ and note that $g_M(\mathcal{W}(X_i),X_j) = 0$ according to equation (1.12). We now compute U taking into account equation (1.15). First, since η is constant in the vertical direction, the term $[\tilde{F}, ds_p(e^{-f}\frac{\partial}{\partial t})]$ vanishes. Moreover, from the Koszul formulas we deduce:

$$\begin{aligned} g(\nabla_{e^{-f}\frac{\partial}{\partial t}}X_i, X_j) &= 0,\\ g\left(\nabla_{e^{-f}\frac{\partial}{\partial t}}e^{-f}\frac{\partial}{\partial t}, X_i\right) &= -X_i(f).\end{aligned}$$

Therefore, $\nabla_N\eta = -\frac{1}{2}e^{-f}\frac{\partial}{\partial t}\,\mathrm{grad}(f)\eta$. □

1.8 Spin(7) structures on quasi abelian Lie algebras

As an application of the previous section, we study Spin(7) structures on quasi abelian Lie algebras. The geometric setting is that of a simply connected Lie group with a left-invariant Spin(7) structure, endowed with an integrable distribution which inherits a G_2 structure. The integral submanifolds of the distribution are actually flat, so that the G_2 distribution is parallel and these submanifolds have non-trivial Weingarten operators. Finding a lattice in the Lie group, if that is possible, allows us to give compact examples.

First of all, let us recall the definition of a quasi abelian Lie algebra:

Definition 1.43. A Lie algebra $\mathfrak{g}$ is quasi abelian if it contains a codimension 1 abelian ideal $\mathfrak{h}$.

The information of $\mathfrak{g}$ is encoded in $\mathrm{ad}(x)$ for any vector x transversal to $\mathfrak{h}$. The following result shows that $\mathfrak{h}$ is unique in $\mathfrak{g}$ with exception of the Lie algebras $\mathbb{R}^n$ and $L_3 \oplus \mathbb{R}^{n-3}$, where L_3 is the Lie algebra of the 3-dimensional Heisenberg group, which is generated by the basis (x,y,z) with relations $[x,y] = z$ and $[x,z] = [y,z] = 0$.

Lemma 1.44. *Let $\mathfrak{g}$ be a n-dimensional quasi abelian Lie algebra with $n \geq 3$. If $\mathfrak{g}$ is not isomorphic to $\mathbb{R}^n$ or $L_3 \oplus \mathbb{R}^{n-3}$, then it has a unique codimension 1 abelian ideal. Moreover, codimension 1 abelian ideals in $L_3 \oplus \mathbb{R}^{n-3}$ are parametrized by $\mathbb{RP}^1$.*

Proof. Suppose that $\mathfrak{g}$ is not isomorphic to $\mathbb{R}^n$ and let $\mathfrak{h}$ be a codimension 1 abelian ideal with a transversal vector x. Let $\mathfrak{h}'$ be a codimension 1 abelian ideal different from $\mathfrak{h}$. If $u \in \mathfrak{h}$ is such that $x + u \in \mathfrak{h}'$ and $v \in \mathfrak{h} \cap \mathfrak{h}'$, then $0 = [x + u, v] = \mathrm{ad}(x)(v)$. Taking into account that $\mathfrak{h} \cap \mathfrak{h}'$ is $(n-2)$-dimensional and $\mathfrak{g}$ is not abelian we conclude that $\mathfrak{h} \cap \mathfrak{h}' = \ker(\mathrm{ad}(x)|_{\mathfrak{h}})$ and $\mathrm{ad}(x)(\mathfrak{h}) = \langle z \rangle$ for some $z \in \mathfrak{h}$. Let $y \in \mathfrak{h}$ be such that $[x, y] = z$ and observe that $z \in [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}'$, that is, $z \in \mathfrak{h} \cap \mathfrak{h}'$ and $[x, z] = 0$. Therefore, $\mathfrak{g}$ is isomorphic to $L_3 \oplus \mathbb{R}^{n-3}$.

In addition, from the discussion above we obtain that $\mathfrak{h}' = \langle v, z \rangle \oplus \mathbb{R}^{n-3}$ for some $v \in \langle x, y \rangle$. Conversely, all the subspaces of the previous form are actually codimension 1 abelian ideals. Therefore, they are parametrized by $\mathbb{RP}^1$.

□

A left-invariant Spin(7) structure on a Lie group is determined by the choice of a Spin(7) form Ω, which is in turn determined by a direction in the space of positive spinors Δ^+.

Define the set $\mathcal{QA}$ with elements $(\mathfrak{g}, \mathfrak{h}, g, \nu_g, \Omega)$ where $\mathfrak{g}$ is a non-trivial quasi abelian Lie algebra with a specific codimension 1 abelian ideal $\mathfrak{h}$, g is a metric on $\mathfrak{g}$, ν_g is a volume form on $\mathfrak{g}$ and Ω is a Spin(7) structure on $(\mathfrak{g}, g, \nu_g)$. We say that $\varphi' \colon (\mathfrak{g}, \mathfrak{h}, g, \nu_g, \Omega) \to (\mathfrak{g}', \mathfrak{h}', g', \nu_{g'}, \Omega')$ is an isomorphism if φ is an isomorphism of Lie algebras such that $\varphi'(\mathfrak{h}) = \mathfrak{h}'$, $(\varphi')^* g' = g$, $\varphi^* \nu_{g'} = \nu_g$ and $\varphi^* \Omega' = \Omega$.

Lemma 1.45. *The set $\overline{\mathcal{QA}}$ of isomorphisms classes of $\mathcal{QA}$ is given by:*

$$\overline{\mathcal{QA}} = \Big((\mathrm{End}(\mathbb{R}^7) - \{0\}) \times \mathbb{P}(\Delta^+) \Big) \,/\, \mathrm{O}(7),$$

where O(7) *acts via*

$$\varphi \cdot (\mathcal{E}, [\eta]) = (\det(\varphi)\varphi \circ \mathcal{E} \circ \varphi^{-1}, [\rho(\tilde{\varphi})\eta]), \tag{1.17}$$

where $\tilde{\varphi}$ is a lifting to Spin(8) *of the unique $\varphi' \in \mathrm{SO}(8)$ such that $\varphi'|_{\mathbb{R}^7} = \varphi$.*

Proof. A map $(\mathrm{End}(\mathbb{R}^7) - \{0\}) \times \mathbb{P}(\Delta^+) \to \mathcal{QA}$ is defined as follows. Take a pair $(\mathcal{E}, \bar{\eta})$ and define the Lie structure on $\mathbb{R}^8$ with oriented basis $(e_0, \ldots, e_7)$ such that $\mathbb{R}^7 = \langle e_1, \ldots, e_7 \rangle$ is a maximal abelian ideal and $\mathcal{E} = \mathrm{ad}(e_0)|_{\mathbb{R}^7}$. We endow this algebra with the canonical metric, the standard volume form and the spin structure determined by η.

It is obvious that a representative of each element of $\overline{\mathcal{QA}}$ can be chosen to lie in the image of our map. Moreover, if two structures given by $(\mathcal{E}, \bar{\eta})$ and $(\mathcal{E}', \bar{\eta}')$ are isomorphic via φ', then:

1. $\varphi'(e_0) = \pm e_0$ and $\varphi = \varphi'|_{\mathbb{R}^7} \in \mathrm{O}(7)$ because φ' preserves the metric and the orientation.

2. Denote by $\tilde{\varphi}$ any lifting of φ' to Spin(8). The equality $(\varphi')^*\Omega' = \Omega$ implies that $\mathrm{Stab}(\Omega) = (\varphi')^{-1} \circ \mathrm{Stab}(\Omega') \circ (\varphi')$. Thus $\mathrm{Stab}(\eta) = \tilde{\varphi}^{-1}\,\mathrm{Stab}(\eta')\tilde{\varphi}$. But $\mathrm{Stab}(\rho(\tilde{\varphi})^{-1}\eta') = \tilde{\varphi}^{-1}\,\mathrm{Stab}(\eta')\tilde{\varphi}$, so that $\eta = \pm\rho(\tilde{\varphi})^{-1}\eta'$.

3. $\varphi \circ \mathcal{E} = \det(\varphi)\mathcal{E}' \circ \varphi$, because φ' is an isomorphism of Lie algebras.

□

From now on we denote by $(\mathbb{R}^8, \mathcal{E}, [\eta])$ the element $(\mathfrak{g}, \mathfrak{h}, g, \nu, \Omega) \in \mathcal{QA}$ where $\mathfrak{g} = \mathbb{R}^8$ as vector spaces, $\mathfrak{h} = \mathbb{R}^7$ is the maximal abelian ideal, $\mathrm{ad}(e_0) = \mathcal{E}$, g is the canonical metric, ν is the canonical volume form and η is a spinor that determines the Spin(7) form Ω.

Remark 1.46. To obtain an analogue of Lemma 1.45, suppressing the condition $\varphi'(\mathfrak{h}) = \mathfrak{h}'$ in the definition of isomorphism, we need to treat separately the case of the Lie algebra $L_3 \oplus \mathbb{R}^5$. For this purpose, define $\mathcal{E}(x) = e_1^*(x)e_2$ and observe that Lemmas 1.44 and 1.45 allow us to suppose that any isomorphism of structures with underlying Lie algebra $L_3 \oplus \mathbb{R}^5$ is represented by $\varphi' \colon (\mathbb{R}^8, \lambda\mathcal{E}, [\eta]) \to (\mathbb{R}^8, \lambda'\mathcal{E}, [\eta'])$, for some $\lambda, \lambda' \neq 0$.

The set $\varphi'(\mathbb{R}^7)$ is a codimension 1 abelian ideal, hence Lemma 1.44 guarantees that $\varphi'(e_0) = \cos(\theta)e_0 + \sin(\theta)e_1$. Denote $\mathbb{R}^6 = \langle e_2, \ldots, e_7\rangle$ and let $v, v' \in \mathbb{R}^6$ be such that $\varphi'(v) = -\mu\sin(\theta)e_0 + \mu\cos(\theta)e_1 + v'$. Then, $0 = \varphi'[e_0, v] = [\cos(\theta)e_0 + \sin(\theta)e_1, -\mu\sin(\theta)e_0 + \mu\cos(\theta)e_1 + v'] = \mu\lambda' e_2$. Therefore $\mu = 0$, $\mathbb{R}^6$ is φ'-invariant and $\varphi'(e_1) = \mp\sin(\theta)e_0 \pm \cos(\theta)e_1$.

Denote by φ_1 the restriction of φ' to $\langle e_0, e_1\rangle$ and note that: $\lambda\varphi'(e_2) = \varphi'[e_0, e_1] = [\varphi'(e_0), \varphi'(e_1)] = \det(\varphi_1)\lambda' e_2$. Hence $\varphi'(e_2) = \det(\varphi_1)\frac{\lambda'}{\lambda}e_2$ and $|\lambda| = |\lambda'|$. Then, φ' is determined by φ_1 and $\varphi_2 = \varphi'|_{\mathbb{R}^5}$, where $\mathbb{R}^5 = \langle e_3, \ldots, e_7\rangle$, under the conditions $\frac{\lambda'}{\lambda}\det(\varphi_2) = 1$ and $\varphi'(e_2) = \det(\varphi_1)\frac{\lambda'}{\lambda}e_2$.

The condition over the spinor is obviously $\eta' = \pm\rho(\tilde{\varphi})\eta$, where $\tilde{\varphi}$ is any lifting of φ' to Spin(8).

In the following result we describe the action which appears in Lemma 1.45.

Lemma 1.47. *Under the action of* O(7) *on* $\mathrm{End}(\mathbb{R}^7)$,

$$\varphi \cdot \mathcal{E} = \det(\varphi)\varphi \circ \mathcal{E} \circ \varphi^{-1}, \tag{1.18}$$

the sets $\langle \mathrm{Id}\rangle$, $\mathrm{Sym}_0^2(\mathbb{R}^7)$ *and* $\Lambda^2\mathbb{R}^7$ *are parametrized respectively by:*

1. $[0, \infty)$,
2. $\{(\lambda_1, \ldots, \lambda_7) \colon \lambda_i \leq \lambda_{j+1}, \sum_{j=1}^7 \lambda_j = 0\}/\sim$, *where* $(\lambda_1, \ldots, \lambda_7) \sim (-\lambda_7, \ldots, -\lambda_1)$,
3. $\{(\lambda_1, \lambda_2, \lambda_3) \colon 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3\}$.

Proof. The first claim is obvious and the second follows from the fact that each symmetric matrix has an oriented orthonormal basis of ordered eigenvectors. Note also that $-\mathrm{Id} \cdot \mathrm{diag}(\lambda_1, \ldots, \lambda_7) = \mathrm{diag}(-\lambda_7, \ldots, -\lambda_1)$, hence $(\lambda_1, \ldots, \lambda_7)$ is related to $(-\lambda_7, \ldots, -\lambda_1)$.

If $\mathcal{E}$ is a skew-symmetric endomorphism of $\mathbb{R}^7$, there is a hermitian basis in $\mathbb{C}^7$ of eigenvectors and the eigenvalues are of the form $(-\lambda_3 i, -\lambda_2 i, -\lambda_1 i, 0, \lambda_1 i, \lambda_2 i, \lambda_3 i)$ with $0 \leq \lambda_j \leq \lambda_{j+1}$. In addition, the real parts of the eigenspaces associated to $-\lambda_j i$ and $\lambda_j i$ coincide. Thus, there is a positively oriented orthonormal basis $(v_1, w_1, v_2, w_2, v_3, w_3, u)$ of $\mathbb{R}^7$, such that $\mathcal{E}(v_j) = \lambda_j w_j$ and $\mathcal{E}(u) = 0$. Finally note that $(\lambda_1, \lambda_2, \lambda_3)$ are well-defined in the orbit. □

Now we compute the invariants that we defined for G_2 distributions on $\mathbb{R}^7$:

Proposition 1.48. *Consider* $(\mathbb{R}^8, \mathcal{E}, [\eta]) \in \mathcal{QA}$ *and decompose* $\mathcal{E}$ *according to the* G_2 *structure induced by* η, *that is* $\mathcal{E} = h\,\mathrm{Id} + E_2 + E_3 + E_4$, *where* $h \in \mathbb{R}$, $E_2 \in \chi_2$, $E_3 \in \chi_3$, $E_4 \in \chi_4$ *and* $E_4(X) = X \times E$ *for some* $E \in \mathbb{R}^7$. *Define* $\Psi, \beta_3 \in \Lambda^3 T^*\mathbb{R}^7$ *by* $\Psi = i(e_0)\Omega$ *and* $\beta_3(X, Y, Z) = \mathrm{alt}(i(E_3(\cdot))\Psi)$. *Then,*

$$(\star d\Omega)_{48} = \frac{2}{7}\left(6i(E)e^0 \wedge \Psi - \frac{9}{2}i(E) \star_{\mathbb{R}^7} \Psi\right) - 6\beta_3,$$
$$(\star d\Omega)_8 = -\left(\frac{12}{7}E + 4he_0\right)(e^0 \wedge \Psi + \star_{\mathbb{R}^7}\Psi).$$

Proof. The result follows immediately from Proposition 1.36 once we check that: $\mu = -\frac{1}{2}h$, $A_2 = 0$, $A_3 = -\frac{1}{2}E_3$, $A = 0$ and $U = -\frac{3}{2}E$.

To obtain this, first observe that $\nabla^{\mathfrak{h}}\eta = 0$ and $\mathcal{L} = 0$ because $\mathfrak{h}$ is an abelian ideal. From the formula of the Weingarten operator we obtain: $\mathcal{W} = h\,\mathrm{Id} + E_3$. To compute U we use again equation (1.15), obtaining that:

$$\nabla_N \eta = \frac{3}{2} e_0 E \eta,$$

here we used that $\nabla_{e_0} e_0 = 0$ because $\mathfrak{h}$ is an ideal and $\nabla_{e_0} e_j = (E_2 + E_4)(e_j)$ if $j > 0$. The last equality follows from Koszul formulas; these imply that $2g(\nabla_{e_0} e_j, e_0) = 0$, and $2g(\nabla_{e_0} e_j, e_k) = g(\mathcal{E}(e_j), e_k) - g(\mathcal{E}(e_k), e_j) = 2g((E_2 + E_4)(e_j), e_k)$ for $k > 0$; the last equality follows from the fact that $E_2 + E_4$ is the skew-symmetric part of $\mathcal{E}$.

□

In the next result we characterise the different types of Spin(7) structures on quasi abelian Lie algebras in terms of Lemma 1.47. For this purpose, we recall that a Lie algebra is unimodular if the volume form is not exact. In the case of the Lie algebra $(\mathbb{R}^8, \mathcal{E})$, this notion is equivalent to the fact that $\mathcal{E}$ is traceless.

Theorem 1.49. *Consider the Lie algebra $(\mathbb{R}^8, \mathcal{E})$ endowed with the standard metric and volume form. Denote by $\mathcal{E}_{13}$ and $\mathcal{E}_{24}$ the symmetric and skew-symmetric parts of the endomorphism $\mathcal{E} \neq 0$. Then, the Lie algebra admits a* Spin(7) *structure of type:*

1. *Parallel, if and only if $\mathcal{E}_{13} = 0$ and $\mathcal{E}_{24}$ is associated to $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$ with $0 \leq \lambda_1 \leq \lambda_2$, as in Lemma 1.47.*
2. *Locally conformally parallel and non-parallel if and only if $\mathcal{E}_{13} = h\,\mathrm{Id}$ with $h \neq 0$ and $\mathcal{E}_{24}$ is associated to $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$ with $0 \leq \lambda_1 \leq \lambda_2$, as in Lemma 1.47.*
3. *Balanced if and only if it is unimodular and $\mathcal{E}_{24}$ is associated to $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$ with $0 \leq \lambda_1 \leq \lambda_2$, as in Lemma 1.47.*

In addition, if $\mathcal{E}_{24} \neq 0$ then it admits a Spin(7) *structure of mixed type.*

Proof. We identify $\mathcal{E}_{24}$ with a 2-form γ. There is a positively oriented orthonormal basis $(X_1, \ldots, X_7)$ of $\mathbb{R}^7$ such that $\gamma = \lambda_1 X^{23} + \lambda_2 X^{45} + \lambda_3 X^{67}$, where $0 \leq \lambda_j \leq \lambda_{j+1}$. Here we denoted $X^{ij} = X_i^* \wedge X_j^*$.

Taking into account Proposition 1.48, the first three items are proved once we check that the existence of a spinor η with $\gamma\eta = 0$ is equivalent to the fact that $\mathcal{E}_{24}$ is associated to $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$ with $0 \leq \lambda_1 \leq \lambda_2$. This spinor exists if and only if $\rho_7(\lambda_1 X_2 X_3 + \lambda_2 X_4 X_5 + \lambda_3 X_6 X_7)$ is non-invertible for some 8-dimensional real irreducible representation $\rho_7 \colon \mathrm{Cl}_7 \to \mathrm{End}(\mathbb{R}^8)$ which maps the volume form ν_7 to the identity, because they are all equivalent [79, Proposition 5.9].

It is known that the two different irreducible representations of Cl_7 are constructed from the octonions $\mathbb{O}$ [79, p. 51]. More precisely, these are the extension to Cl_7 of the maps $\rho_\theta \colon \mathbb{R}^7 \to \mathrm{End}(\mathbb{R}^8)$, $\rho_\theta(v)(x) = \theta v x$, where $\theta = \pm 1$ and $\mathbb{R}^7$ is viewed as the imaginary part of the octonions. Define the isometry φ of $\mathbb{R}^7$ which maps X_i to e_i and note that the volume form is fixed by the extension of φ to the Clifford algebra. The extensions of ρ_θ and φ to Cl_7 are denoted in the same way. We check the previous condition using the representation $\rho_7 = \rho_\theta \circ \varphi \colon \mathrm{Cl}_7 \to \mathrm{End}(\mathbb{R}^8)$, taking θ such that $\rho_\theta(\nu_7) = \mathrm{Id}$. A direct computation shows that the determinant of $\rho_7(\lambda_1 X_2 X_3 + \lambda_2 X_4 X_5 + \lambda_3 X_6 X_7)$ is given by:

$$(\lambda_1 + \lambda_2 + \lambda_3)^2 (\lambda_1 + \lambda_2 - \lambda_3)^2 (\lambda_1 - \lambda_2 - \lambda_3)^2 (\lambda_1 - \lambda_2 + \lambda_3)^2.$$

Since $\lambda_1 \leq \lambda_2 \leq \lambda_3$, the endomorphism is non-invertible if and only if $\lambda_3 = \lambda_2 + \lambda_1$.

Finally, if $\mathcal{E}_{24} \neq 0$ then $\rho_7(\lambda_1 X_2 X_3 + \lambda_2 X_4 X_5 + \lambda_3 X_6 X_7) \neq 0$ so that there is a spinor such that $E \neq 0$; Proposition 1.48 guarantees that it induces a Spin(7) structure of mixed type. □

Recall that solvmanifolds are compact quotients $\Gamma\backslash G$, where G is a simply connected solvable Lie group and Γ is a discrete lattice. This forces the Lie algebra $\mathfrak{g}$ of G to be unimodular [91, Lemma 6.2]. Proposition 1.48 allows us to conclude the following:

Corollary 1.50. *There exists no quasi abelian solvmanifold with a left-invariant locally conformally parallel and non-parallel* Spin(7) *structure.*

Of course, a torus is solvmanifold which admits a parallel Spin(7) structure.

Corollary 1.51. *If* $(\mathbb{R}^8, \mathcal{E})$ *is a quasi abelian Lie algebra such that* $\mathcal{E}$ *is skew-symmetric, then it is flat. In particular, quasi abelian Lie algebras which admit a left-invariant parallel* Spin(7) *structure are flat.*

Proof. Let $(\mathbb{R}^8, \mathcal{E})$ be a quasi abelian Lie algebra and denote by $\mathcal{E}_{13}$ and $\mathcal{E}_{24}$ the symmetric and skew-symmetric parts of $\mathcal{E}$. It is straightforward to check that if $i, j > 0$ then:

$$\nabla_{e_0} e_0 = 0, \quad \nabla_{e_0} e_j = \mathcal{E}_{24}(e_j), \quad \nabla_{e_i} e_0 = -\mathcal{E}_{13}(e_i), \quad \nabla_{e_i} e_j = g(\mathcal{E}_{13}(e_i), e_j) e_0.$$

From this, one deduces that if $i, j, k > 0$, then the curvature tensor is given by:

$$\begin{aligned} R(e_0, e_j)e_0 =& -(\mathcal{E}_{24} \circ \mathcal{E}_{13} + \mathcal{E}_{13} \circ \mathcal{E}_{24})(e_j), \\ R(e_0, e_j)e_k =& -g(\mathcal{E}_{13}(e_k), (\mathcal{E} + \mathcal{E}_{24})(e_j))e_0, \\ R(e_i, e_j)e_0 =& \,0, \\ R(e_i, e_j)e_k =& \,g(\mathcal{E}_{13}(e_j), e_k)\mathcal{E}_{13}(e_i) - g(\mathcal{E}_{13}e_i, e_k)\mathcal{E}_{13}(e_j). \end{aligned}$$

Therefore, if $\mathcal{E}$ is skew-symmetric then the Lie group is flat. □

Remark 1.52. Corollary 1.50 also follows from the fact that locally conformally parallel Spin(7) structures with a co-closed Lee form are associated with positive scalar curvature metrics (see [69]) and left-invariant metrics on solvable Lie groups have non-positive scalar curvature [91]. Corollary 1.51 also follows from the fact that parallel Spin(7) structures are Ricci-flat and left-invariant metrics on solvable Lie groups with vanishing scalar curvature are flat [91].

Examples

Let $\mathfrak{g}$ be a quasi abelian Lie algebra determined by an endomorphism $\mathcal{E}$. Consider the unique simply connected Lie group G whose Lie algebra is $\mathfrak{g}$. The split exact sequence of Lie algebras $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ lifts to a split exact sequence of Lie groups $0 \to (\mathbb{R}^7, +) \to G \to (G/\mathbb{R}^7 = \mathbb{R}, +) \to 0$. This splitting and the conjugation ϵ on G by the elements of $(\mathbb{R}, +)$, provide an isomorphism $G \cong (\mathbb{R}, +) \ltimes_\epsilon (\mathbb{R}^7, +)$. Therefore $\frac{d}{dt}\big|_{t=s} d(\epsilon(t)) = s\mathcal{E}$, so that $d(\epsilon(t)) = \exp(t\mathcal{E}) = \epsilon(t)$, taking into account that the exponential map of $\mathbb{R}^7$ is the identity.

A nilmanifold with a balanced and a locally conformal balanced Spin(7) structure.

Define the endomorphism of $\mathbb{R}^7$

$$\mathcal{E} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and consider the quasi abelian Lie algebra $(\mathbb{R}^8, \mathcal{E})$. Note that this is a nilpotent Lie algebra with $(de^0, de^1, de^2, \dots, de^7) = (0, e^{02}, 2e^{03}, e^{04}, e^{05}, e^{06}, e^{07}, 0)$, where $d\beta(X,Y) = -\beta([X,Y])$.

The symmetric part of $\mathcal{E}$ is traceless and the eigenvalues of its skew-symmetric part are $(\lambda_1, \lambda_2, \lambda_1 + \lambda_2)$. Therefore, Theorem 1.49 guarantees that $(\mathbb{R}^8, \mathcal{E})$ admits both a balanced and a mixed Spin(7) structure. An alternative argument that avoids the computation of the eigenvalues of $\mathcal{E}$ is the following. Let Ω_0 be the standard Spin(7) structure on $\mathbb{R}^8$ and let η be the spinor that determines the structure. Let us identify the skew-symmetric part of $\mathcal{E}$ with the 2-form $\gamma = e^{23} + \frac{1}{2}(e^{12} + e^{45} + e^{56} + e^{67})$ as usual. The equalities $e_2e_3\eta = -e_4e_5\eta = -e_6e_7\eta$ and $e_1e_2\eta = -e_5e_6\eta$ imply that $\gamma\eta = 0$. Therefore, the 4-form associated to the structure is Ω_0.

On some nilpotent Lie algebras, the existence of a lattice is guaranteed by general theorems [80]. This case is simple and we compute it explicitly. The matrix of the endomorphism $\exp(t\mathcal{E})$ is:

$$\begin{pmatrix} 1 & -t & t^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -t & \frac{t^2}{2} & -\frac{t^3}{6} \\ 0 & 0 & 0 & 0 & 1 & -t & \frac{t^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we define $\Gamma = 6\mathbb{Z}e_0 \times_\epsilon (\mathbb{Z}e_1 \times \mathbb{Z}e_2 \times \cdots \times \mathbb{Z}e_7)$, then $\Gamma\backslash G$ is a compact manifold with $\pi_1(\Gamma\backslash G) = \Gamma$ which inherits both a balanced and a mixed Spin(7) left-invariant structure.

Moreover, we claim that $\Gamma\backslash G$ is not diffeomorphic to $Q \times S^1$ for any 7-dimensional submanifold Q. Since $b_1(\Gamma\backslash G) = 2$, it is sufficient to prove that if a nilmanifold $\Gamma'\backslash G'$ is diffeomorphic to $Q \times S^1$ then, $b_1(Q \times S^1) \geq 3$, or equivalently, $b_1(Q) \geq 2$. This claim turns out to be true because we can check that Q is homotopically equivalent to a nilmanifold. On the one hand, Q is an Eilenberg-McLance space $K(1, \pi_1(Q))$, because G' is contractible. On the other hand a group is isomorphic to a lattice of a nilpotent Lie group if and only if it is nilpotent, torsion-free and finitely generated [102, Theorem 2.18]. Since $\Gamma' = \pi_1(\Gamma'\backslash G') = \pi_1(Q) \times \mathbb{Z}$, both $\pi_1(Q)$ and Γ' satisfy the conditions listed above. Thus, there is a nilmanifold Q' such that $\pi_1(Q') = \pi_1(Q)$, which is an Eilenberg-MacLane space $K(1, \pi_1(Q))$. Therefore, Q' and Q have the same homotopy type and $b_1(Q) = b_1(Q') \geq 2$, because Q' is a nilmanifold.

This nilmanifold also has a strict locally conformally balanced Spin(7) structure (see Definition 1.53), a structure of mixed type with closed and non-exact Lee form. According to Theorem 1.48, if we show that there exists a spinor η and $\lambda \neq 0$ such that $\gamma\eta = -\lambda e^7\eta$, then the Lee form of the Spin(7) structure determined by η is μe^7 for some $\mu \in \mathbb{R}$ and $d(\mu e^7) = 0$. Take the octonionic representation ρ, which extends to Cl_7 the map $\rho\colon \mathbb{R}^7 \to \mathrm{End}(\mathbb{R}^8)$, $\rho(v)(x) = vx$ where $\mathbb{R}^7$ is viewed as the imaginary part of the octonions.

The previous condition is then equivalent to $(\rho(e_7)\rho(\gamma) - \lambda\,\mathrm{Id})\eta = 0$ for some $\eta \in \mathbb{R}^8$, that is, $\lambda \neq 0$ is a real eigenvalue of $\rho(e_7)\rho(\gamma)$. Computing this condition we obtain two eigenvalues $\lambda_\pm = \pm\sqrt{3}$. The unit-lenght eigenvectors associated to λ_+ are $\eta_+^1 = \frac{1}{\sqrt{15}}(0, -\sqrt{3}, 0, -\sqrt{3}, 0, 3, 0, 0)$ and $\eta_+^2 = \frac{1}{\sqrt{75}}(-\sqrt{3}, 0, 3\sqrt{3}, 0, -6, 0, 3, 0)$; these associated to λ_- are $\eta_-^1 = \frac{1}{\sqrt{15}}(0, -\sqrt{3}, 0, \sqrt{3}, 0, 3, 0, 0)$ and $\eta_-^2 = \frac{1}{\sqrt{75}}(\sqrt{3}, 0, -3\sqrt{3}, 0, -6, 0, 3, 0)$.

The 4-form associated to η_+^1 is $\Omega_0 = e^0 \wedge \Psi + \star\Psi$, where $\star$ is the Hodge star of the canonical metric on $\mathbb{R}^7$ and:

$$\begin{aligned}\Psi =& e^{12} \wedge (-\frac{1}{5}e^3 - 2\frac{\sqrt{3}}{5}e^6 + 2\frac{\sqrt{3}}{5}e^7) - 2\frac{\sqrt{3}}{5}e^{13} \wedge (e^4 + e^6) - \frac{1}{5}e^{14} \wedge (3e^5 + 2e^7) \\ & - \frac{2}{5}e^{156} + \frac{3}{5}e^{167} - 2\frac{\sqrt{3}}{5}e^{23}(e^5 + e^7) + e^{246} + \frac{1}{5}e^{257}\frac{1}{5}e^{34}(-2e^5 + 3e^7) - \frac{3}{5}e^{356} \\ & - \frac{2}{5}e^{367} - 2\frac{\sqrt{3}}{5}e^{457} + 2\frac{\sqrt{3}}{5}e^{567}.\end{aligned}$$

A compact manifold with a parallel and a mixed Spin(7) **structure.**

Take the same spinor and basis of $\mathbb{R}^7$ as the previous example. Consider the skew-symmetric endomorphism such that $\mathcal{E}(e_2) = e_3$, $\mathcal{E}(e_4) = e_5$ and $\mathcal{E}(X) = 0$ on $\langle e_2, e_3, e_4, e_5\rangle^\perp$. The rank of this matrix is two and it is associated to $(0, 1, 1)$. Therefore, Theorem 1.49 guarantees that $(\mathbb{R}^8, \mathcal{E})$ admits both a parallel and a mixed Spin(7) structure. The matrix of the endomorphism $\exp(t\mathcal{E}_2)$ in the previous basis is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(t) & \sin(t) & 0 & 0 & 0 & 0 \\ 0 & -\sin(t) & \cos(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(t) & -\sin(t) & 0 & 0 \\ 0 & 0 & 0 & \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If $t \in \pi\mathbb{Z}$, the previous matrix has integers coefficients so that $\gamma = \pi\mathbb{Z}e_0 \times_\epsilon (\mathbb{Z}e_1 \times \mathbb{Z}e_2 \times \cdots \times \mathbb{Z}e_7)$ is a subgroup. Moreover, $\Gamma\backslash G$ is a compact manifold with $\pi_1(\Gamma\backslash G) = \Gamma$ and inherits from G both a parallel left-invariant Spin(7) structure and a mixed left-invariant one.

According to Remark 1.51, this manifold is flat. It is the mapping torus of $\exp(\pi\mathcal{E})\colon X \to X$, where X is a 7-torus. Indeed, since $\exp(\pi\mathcal{E})^2 = \mathrm{Id}$, the 8-torus is a 2-fold connected covering of $\Gamma\backslash G$.

1.9 Balanced and locally conformally balanced structures on quasi abelian Lie algebras

In this section we focus on Spin(7) structures on quasi abelian nilpotent Lie algebras. As Corollary 1.50 states, a locally conformally parallel structure on a quasi abelian nilpotent Lie algebra is parallel. In fact, if a quasi abelian nilpotent Lie algebra admits a parallel structure, then it is flat; this implies that the Lie algebra is abelian. Therefore, we search for quasi abelian nilpotent Lie algebras which admit a balanced structure. In addition, we introduce a special type of mixed structure, which we call locally conformally balanced and we analyze its existence on quasi abelian nilpotent Lie algebras.

A Spin(7) structure on a Riemannian manifold is locally conformally balanced if on each contractible neighbourhood there is a conformal change of the metric whose associated Spin(7) structure is balanced, that is:

Definition 1.53. A Spin(7) structure is locally conformally balanced if its Lee form is closed. In addition, if the Lee form is not exact, we say that it is strict locally conformally balanced.

Of course, balanced and locally conformally calibrated structures are locally conformally balanced. The interesting case is when the structure is mixed and the Lee form is not exact.

Remark 1.54. Our spinorial approach enables us to characterise locally conformally balanced structures. Let $V \in TM$ such that $D\eta = V\eta$. We compute the Dirac operator of V as an element of $\mathrm{Cl}(M)$, that is, $DV = \sum_{i=1}^{7} X_i \nabla_{X_i} V$ for an orthonormal local basis $(X_0, \ldots, X_7)$:

$$\begin{aligned} DV = \sum_{i,j=0}^{7} g(\nabla_{X_i} V, X_j) X_i X_j = \sum_{i<j} \Big(g(\nabla_{X_i} V, X_j) - g(\nabla_{X_j} V, X_i) \Big) X_i X_j - \sum_{i=0}^{7} g(\nabla_{X_i} V, X_i) \\ = 2 \sum_{i<j} dV^*(X_i, X_j) X_i X_j + \mathrm{div}(V). \end{aligned}$$

The Lee form is $\frac{8}{7} V^*$; therefore the structure is locally conformally balanced if and only if $DV = \mathrm{div}(V)$.

If we focus on quasi abelian Lie algebras $(\mathbb{R}^8, \mathcal{E})$ with $h = 0$, the problem of determining whether or not the Lee form of a structure is homothetic to a unit-length 1-form θ becomes an eigenvalue problem.

As Proposition 1.48 states, the Lee form of the Spin(7) structure defined by η is homothetic to a 1-form $E^* \in \mathfrak{h}^*$ determined by the equation $\gamma \cdot_{\mathfrak{h}} \eta = 3E \cdot_{\mathfrak{h}} \eta$, where γ is the 2-form associated to the skew-symmetric part of $\mathcal{E}$. For a unit-length 1-form θ, the condition $\gamma \cdot_{\mathfrak{h}} \eta = -\lambda\theta \cdot_{\mathfrak{h}} \eta$ for some $\lambda \neq 0$ is equivalent to $(\theta\gamma - \lambda) \cdot_{\mathfrak{h}} \eta = 0$, that is, the endomorphism of Δ^+ given by $\phi \longmapsto \theta \cdot_{\mathfrak{h}} \gamma \cdot_{\mathfrak{h}} \phi$ has λ as an eigenvalue.

This argument enables us to prove that if a nilpotent quasi abelian $\mathfrak{g}$ Lie algebra is decomposable, that is $\mathfrak{g} = \mathfrak{g}' \oplus \langle W \rangle$ as Lie algebras, then W is homothetic to the Lee form of a Spin(7) structure.

Lemma 1.55. *Let $(\mathbb{R}^8, \mathcal{E})$ be a unimodular quasi abelian Lie algebra. If $\mathcal{E}_{24} \neq 0$ and $\mathcal{E}_{24}(W) = 0$ for some unit-length vector $W \in \mathbb{R}^7$, then $(\mathbb{R}^8, \mathcal{E})$ admits a spinor η whose associated* Spin(7) *structure has Lee form homothetic to W^*.*

In particular, if a decomposable quasi abelian Lie algebra $\mathfrak{g} = \mathfrak{g}' \oplus \langle W \rangle$ is non-abelian and nilpotent, it admits a Spin(7) *structure whose Lee form is homothetic to W^*.*

Proof. First note that $\gamma \in \Lambda^2 \langle W^* \rangle^\perp$ so that $(W^*\gamma) \cdot_{\mathfrak{h}} \phi = (W^* \wedge \gamma) \cdot_{\mathfrak{h}} \phi$ for all $\phi \in \Delta^+$. But the product by an element of $\Lambda^3(\mathbb{R}^7)^*$ is a symmetric endomorphism of Δ^+. Therefore, the condition $\mathcal{E}_{24} \neq 0$ guarantees the existence of a non-zero eigenvalue of the product by $W^* \wedge \gamma$ and therefore, a spinor η whose associated Spin(7) structure has Lee form homothetic to W^*.

Suppose that a decomposable quasi abelian Lie algebra $\mathfrak{g} = \mathfrak{g}' \oplus \langle W \rangle$ is non-abelian and nilpotent. It is straightforward to check that W lies in the abelian ideal $\mathfrak{h}$. Thus, if we take a metric g with e_0 perpendicular to $\mathfrak{h}$ and W perpendicular to $\mathfrak{h} \cap \mathfrak{g}'$ then $(\mathfrak{g}, g)$ is identified in terms of Lemma 1.45 with a pair $(\mathbb{R}^8, \mathcal{E})$ such that $\mathcal{E}_{24}(W) = 0$. In addition, $\mathcal{E}_{24} \neq 0$ because the algebra is non-abelian and nilpotent. □

A more detailed analysis of the eigenvalue problem provides the following result:

Lemma 1.56. *Let $(\mathbb{R}^8, \mathcal{E})$ be a unimodular quasi abelian Lie algebra and suppose that $\mathcal{E}_{24}$ is associated to $(\lambda_1, \lambda_2, \lambda_3)$ as in Lemma 1.47 with $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$ with $\lambda_3 \leq \lambda_1 + \lambda_2$. Then, each $\theta \in (\mathbb{R}^7)^*$ is homothetic to the Lee form of a* Spin(7) *structure.*

Proof. Let θ in $(\mathbb{R}^7)^*$ and take $(X_1, \ldots, X_7)$ an orthonormal oriented basis of $\mathbb{R}^7$ such that:

$$\begin{aligned} \gamma &= \lambda_1 X_1^* \wedge X_2^* + \lambda_2 X_3^* \wedge X_4^* + \lambda_3 X_5^* \wedge X_6^*, \\ \theta^\sharp &= \mu_1 X_1 + \mu_3 X_3 + \mu_5 X_5 + \mu_7 X_7. \end{aligned}$$

To obtain such formulas we may diagonalize γ as in Lemma 1.47. A rotation on each of the eigenspaces allows to obtain a basis $(X_1, X_2, \ldots, X_7)$ such that the projection of $\theta^\sharp$ to the plane $\langle X_{2i-1}, X_{2i}\rangle$ is parallel to X_{2i-1} for $1 \leq i \leq 3$.

Let ρ be the representation of Cl_7 constructed in the proof of Theorem 1.49. The characteristic polynomial of the matrix $\rho(\theta^\sharp)\rho(\lambda_1 X_1 X_2 + \lambda_2 X_3 X_4 + \lambda_3 X_5 X_6)$ is $p(t) = (t^4 + a_2 t^2 + a_1 t + a_0)^2$, where:

$$\begin{aligned}
a_0 =& -(\lambda_1+\lambda_2+\lambda_3)(-\lambda_1+\lambda_2+\lambda_3)(\lambda_1-\lambda_2+\lambda_3)(\lambda_1+\lambda_2-\lambda_3),\\
a_1 =& 8\lambda_1\lambda_2\lambda_3\mu_7,\\
a_2 =& -2(\mu_1^2(-\lambda_1^2+\lambda_2^2+\lambda_3^2)+\mu_3^2(\lambda_1^2-\lambda_2^2+\lambda_3^2)+\mu_5^2(\lambda_1^2+\lambda_2^2-\lambda_3^2)+\mu_7^2(\lambda_1^2+\lambda_2^2+\lambda_3^2)).
\end{aligned}$$

Therefore,

1. If $\lambda_3 < \lambda_1 + \lambda_2$ then $a_0 < 0$ so that $p(t)$ has a non-zero real root.
2. If $\lambda_3 = \lambda_1 + \lambda_2$ then $p(t) = t^2(t^3 + a_2 t + a_1)^2$ with $a_2 < 0$. Therefore, p has a non-zero real root.

□

1.9.1 Quasi abelian nilpotent Lie algebras and Spin(7) structures

Quasi abelian nilpotent Lie algebras are classified by the adjoint action a vector which is transverse to the abelian ideal. Therefore, each isomorphism type is associated to a unique element of $\mathcal{N}_7/\mathrm{Gl}(7)$, where $\mathcal{N}_7$ is the set of nilpotent matrices of $\mathbb{R}^7$ and $\mathrm{Gl}(7)$ acts via conjugation. The orbits are matrices with the same Jordan normal form, and therefore, classified by the dimensions of those blocks. There are 15 types that we denote by $(n_1, \ldots, n_k)$ with $n_i \leq n_{i+1}$ and $\sum_{i=1}^k n_i = 7$.

We determine those which admit a balanced $\mathrm{Spin}(7)$ structure or a $\mathrm{Spin}(7)$ structure with closed Lee form in the cohomology of the algebra. Note that the last type induces strict locally conformally balanced structures on each nilmanifold associated to the algebra because the cohomology of the algebra is isomorphic to the cohomology of any associated nilmanifold. In this context we say that the $\mathrm{Spin}(7)$ structure of a nilpotent Lie algebra is strict locally conformally balanced.

First of all observe that the abelian Lie algebra only admits parallel structures. Next, we analyze the algebras $L_3 \oplus A_5$ and $L_4 \oplus A_4$, where L_3 denotes the Lie algebra of the 3-dimensional Heisenberg group, L_4 the unique irreducible 4-dimensional nilpotent Lie algebra and A_j the j-dimensional abelian Lie algebra. In our previous notation, they are associated to $(2, 1, 1, 1, 1, 1)$ and $(3, 1, 1, 1, 1)$.

Proposition 1.57. *The Lie algebras $A_4 \oplus L_3$ and $A_3 \oplus L_4$ do not admit any balanced structure. However, both of them admit strict locally conformal balanced structures.*

Proof. Let $\mathfrak{h}$ be an abelian ideal of $\mathfrak{g}$ and let g be a metric. Take a vector e_0 orthogonal to $\mathfrak{h}$ and denote $\mathcal{E} = \mathrm{ad}(e_0)|_{\mathfrak{h}}$. We write in both cases the endomorphism $\mathcal{E}$ with respect to a suitable orthonormal basis $(e_1, \ldots, e_7)$ of $\mathfrak{h}$:

1. If $\mathfrak{g} = A_4 \oplus L_3$ we can suppose that $\ker(\mathcal{E}) = \langle e_1, \ldots, e_6\rangle$ and $\mathcal{E}(e_7) = -\lambda e_6$ for some $\lambda \neq 0$. Thus, $\gamma = \lambda e^{67}$ so that $\gamma\eta \neq 0$ for all η.
2. If $\mathfrak{g} = A_3 \oplus L_4$ we can suppose that $\ker(\mathcal{E}) = \langle e_1, \ldots, e_5\rangle$, $\mathcal{E}(e_6) = -\lambda_1 e_5$ and $\mathcal{E}(e_7) = -\lambda_2 e_4 - \lambda_3 e_5 - \lambda_4 e_6$, where $\lambda_1\lambda_4 \neq 0$. Thefore, $\gamma = \lambda_1 e^{56} + (\lambda_2 e^4 + \lambda_3 e^5) \wedge e^7 + \lambda_4 e^{67}$. The spinor $\lambda_4 e^{67}\eta$ is non-zero and orthogonal to $(\lambda_1 e^{56} + (\lambda_2 e^4 + \lambda_3 e^5) \wedge e^7)\eta$. Therefore, $\gamma\eta \neq 0$ for all η.

The existence of strict locally conformally balanced structures is a consequence of Lemma 1.55. □

Now, we focus on types associated to matrices with two different Jordan blocks of dimension greater than 1, which are $(5,2)$, $(4,3)$, $(4,2,1)$, $(3,3,1)$, $(3,2,2)$, $(3,2,1,1)$, $(2,2,2,1)$ and $(2,2,1,1,1)$.

Proposition 1.58. *Nilpotent quasi abelian Lie algebras with two different Jordan blocks of dimension greater than* 1 *admit a metric with both a balanced and a strict locally conformally balanced* $\mathrm{Spin}(7)$ *structure.*

Proof. Let e_0 be transversal to the abelian ideal $\mathfrak{h}$ and observe that there is a splitting $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$ with $\dim \mathfrak{h}_2 \in \{2,3\}$, $\mathfrak{h}_3$ abelian and $\mathrm{ad}(e_0)(\mathfrak{h}_i) \subset \mathfrak{h}_i$. Observe that $\mathfrak{h}_3$ may be $\{0\}$. We consider a metric g which makes e_0 perpendicular to $\mathfrak{h}$ and $g|_{\mathfrak{h}} = g_1 + g_2 + g_3$ where g_i are metrics on $\mathfrak{h}_i$.

Therefore $\mathcal{E}$ is a block matrix $\begin{pmatrix} \mathcal{E}_1 & 0 & 0 \\ 0 & \mathcal{E}_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with respect to an orthonormal basis adapted to the splitting $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$.

Obviously, for each $\lambda > 0$ there exists an upper triangular matrix of dimension 2 or 3, conjugated to a Jordan block of dimension 2 or 3, such that its skew-symmetric part has eigenvalues $\pm\lambda i$ or $0, \pm\lambda i$. Therefore, once obtained the eigenvalues of the skew-symmetric part of $\mathcal{E}_1$ with respect to any metric g_1 we can change g_2 so that g satisfies the balanced condition.

Except for $(2,2,1,1,1)$, $(3,2,1,1)$ and $(3,3,1)$, we can change g_1 so that the skew-symmetric part of $\mathcal{E}_1$ has two different eigenvalues. Lemma 1.56 ensures the existence of strict locally conformally balanced structures. Finally, the algebras considered except $(5,2)$ and $(4,3)$ satisfy that $\mathcal{E}_{24}(W) = 0$ for some non-zero vector W so that Lemma 1.55 ensures the existence of a strict locally conformally balanced structure associated to the metric that we defined previously. □

Remark 1.59. A similar construction ensures the existence of metrics without associated balanced structures that admit strict locally conformally balanced structures.

Finally we analyze the case of the algebras associated to $(4,1,1,1)$, $(5,1,1)$, $(6,1)$, (7).

Proposition 1.60. *The quasi abelian nilpotent Lie algebras associated to* $(4,1,1,1)$*,* $(5,1,1)$*,* $(6,1)$*,* (7) *have both a balanced and a strict locally conformally balanced* $\mathrm{Spin}(7)$ *structure.*

Proof. Lemma 1.55 guarantees the existence of strict locally conformally balanced structures in the algebras associated to $(4,1,1,1)$, $(5,1,1)$, $(6,1)$. We prove that all of them admit a balanced structure giving an explicit example of an structure of the type $(\mathbb{R}^8, \mathcal{E})$. In the case of (7), we have $\dim(\mathcal{E}_{24}(\mathbb{R}^7) = 6$ so that the same metric also admits a strict locally conformally balanced $\mathrm{Spin}(7)$ structure as Lemma 1.56 states. Define:

$$\mathcal{E} = -\begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & c & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1+a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1+b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $a = b = c = 0$, the Lie algebra is associated to $(4, 1, 1, 1)$, if $a = b = 0$ and $c \neq 0$ to $(5, 1, 1)$, if $a = 0$, $b \neq 0$ and $c \neq 0$ to $(6, 1)$ and if $a \neq 0$, $b \neq 0$ and $c \neq 0$, to (7). The skew-symmetric part of $\mathcal{E}$ is associated to the 2-form:

$$\gamma = ae^{12} + be^{23} + ce^{25} + ce^{34} - e^{45} - e^{47} + (1 + a)e^{56} + (1 + b)e^{67}.$$

Let η be the spinor whose associated 4-form is the standard $\mathrm{Spin}(7)$ form Ω_0. Then, the equality $\gamma\eta = 0$ follows from the equalities:

$$e^{67}\eta = e^{45}\eta, \quad e^{56}\eta = e^{47}\eta, \quad e^{34}\eta = -e^{25}\eta, \quad e^{23}\eta = -e^{67}\eta, \quad e^{12}\eta = -e^{56}\eta.$$

□

Our discussion proves:

Theorem 1.61. 1. *Every* $\mathrm{Spin}(7)$ *structure on the abelian Lie algebra* A_8 *is parallel.*

2. *The Lie algebras* $\mathfrak{g} = A_5 \oplus L_3$ *or* $\mathfrak{g} = A_3 \oplus L_4$ *admit strict locally conformally balanced* $\mathrm{Spin}(7)$ *structures. They do not admit balanced* $\mathrm{Spin}(7)$ *structures.*

3. *The rest of quasi abelian nilpotent Lie algebras admit a balanced structure and a strict locally conformally balanced structure.*

CHAPTER 2

Spin-harmonic structures and nilmanifolds

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Abstract

We introduce spin-harmonic structures, a class of geometric structures on Riemannian manifolds of low dimension which are defined by a harmonic unit-length spinor. Such structures are related to SU(2) ($\dim = 4, 5$), SU(3) ($\dim = 6$) and G_2 ($\dim = 7$) structures; in dimension 8, a spin-harmonic structure is equivalent to a balanced Spin(7) structure. As an application, we obtain examples of compact 8-manifolds endowed with non-integrable Spin(7) structures of balanced type.

MSC classification [2010]: Primary 57N16 ; Secondary 15A66, 53C27, 22E25.

Key words: Spinors, geometric structures, Dirac operator, nilmanifolds

2.1 Introduction

In 1980 Thomas Friedrich proved a remarkable inequality involving the scalar curvature of a compact, spin Riemannian manifold and the first eigenvalue of the Dirac operator, see [53]. This triggered a deep analysis of spin Riemannian manifolds; particular emphasis was put on which compact manifolds admitted parallel, twistor or Killing spinors, see for instance [7, 9, 92]. In particular, it was soon clarified that Riemannian manifolds endowed with a parallel spinor are related to Riemannian manifolds with *special* holonomy, i.e. Riemannian manifolds whose Riemannian holonomy is contained in $SU(m)$, $Sp(k)$, G_2 or Spin(7); notice that the Ricci curvature of a compact Riemannian manifold endowed with a parallel spinor vanishes.

Relaxing the requirement to have a parallel spinor, it was later shown that many non-integrable G structures, $G \subset SO(n)$ being a closed subgroup, can be understood in terms of nowhere vanishing spinors, generalizing the case of parallel spinors. For instance, in [1] the authors described SU(3) and G_2 structures in dimensions 6 and 7 respectively using a unit-length spinor. Not only does the spinorial approach offer an alternative framework for telling apart different classes of such structures, but also provides a unifying language showing how the same spinor is responsible for the emerging of both structures.

SU(2) structures in dimension 5 have been introduced by Conti and Salamon in [35] and classified by Bedulli and Vezzoni in [16] in terms of the exterior derivatives of the corresponding defining forms – see Section 2.4. In [35], the study of SU(2) structures in dimension 5 was certainly motivated by spinors, concretely, generalized Killing spinors. However, no spinorial description of such structures is available; the first goal of this paper is to tackle this question. We do this in Section 2.4.

As for Spin(7) structures on 8-dimensional manifolds, they can be described in terms of a triple cross product on each tangent space; an equivalent description can be given in terms of the so-called fundamental 4-form Ω. The different types of Spin(7) structures were classified by Fernández in [43] using the triple cross product: there exist two pure classes, called *balanced* and *locally conformally parallel*. An equivalent classification is obtained by considering the fundamental form: balanced Spin(7) structures are characterized by the equation $\star(d\Omega)\wedge\Omega = 0$, while the 4-form of a locally conformally parallel Spin(7) structure satisfies $d\Omega = \theta\wedge\Omega$ for a closed 1-form θ, called the *Lee form*. In [69] Ivanov discovered that the unit-length spinor which characterizes balanced Spin(7) structures is *harmonic*, that is, it lies in the kernel of the Dirac operator $\not{D}$, but gave no further application of this fact. Notice that Hitchin proved in [67] than every compact spin 8-manifold carries a harmonic spinor; not much is known, however, about zeroes of harmonic spinors (see [8]).

A systematic spinorial approach to Spin(7), along the lines of [1], was taken by the second author in [86]. In particular, the observation that balanced Spin(7) structures are equivalent to unit-length harmonic spinors was exploited in [86] to construct examples of balanced Spin(7) structures on 8-dimensional nilmanifolds and solvmanifolds. There it became clear that the spinorial approach has some practical advantages over the "classical" one, which uses the 4-form. The principle we follow in this paper is that albeit both the equation $\not{D}\eta = 0$ for a unit-length spinor and the equation $\star(d\Omega)\wedge\Omega = 0$ for a 4-form are non-linear, the first one seems to be more tractable, at least if one is interested in constructing examples of balanced Spin(7) structures on compact quotients of simply connected nilpotent and solvable Lie groups, that is, on nilmanifolds and solvmanifolds.

Indeed, the second goal of this paper is to construct examples of balanced Spin(7) structures on 8-dimensional nilmanifolds. The first known example of such a structure is a nilmanifold described by Fernández in [46]. Further examples are discussed in [69, 82]. Notice that the classification of 8-dimensional nilpotent Lie algebras is not known. Even if it were, however, it is not immediately clear how to sift through them in order to find those admitting balanced Spin(7) structures (for instance, the balanced condition is not of cohomological type).

We describe briefly the idea behind the construction. As we pointed out, it is very natural to consider Spin(7) structures in dimension 8 defined by a chiral unit-length harmonic spinor. Nothing hinders, however, to consider G_2, SU(3) and SU(2) structures in dimensions 7, 6 and 5 respectively, such that the defining spinor is harmonic. Using the spinorial approach of [1], one can precisely track which classes of G_2 and SU(3) are defined by harmonic spinors; moreover, our spinorial description also allows to pinpoint which classes of SU(2) structures arise from a harmonic spinor. While Spin(7) structures defined by a harmonic spinor form a pure class, the same is not true in lower dimensions; for instance, in dimension 5, the requirement to be harmonic for the corresponding spinor turns out to be quite loose.

Viceversa, beginning with an SU(2) structure on a 5-manifold (resp. an SU(3) structure on a 6-manifold, or a G_2 structure on a 7-manifold), defined by a harmonic spinor, one can multiply by a flat torus T^k, $k = 3, 2, 1$, to obtain a Spin(7) structure in dimension 8 defined by a harmonic spinor, that is, a balanced structure.

In order to construct such examples, we need a formula for the Dirac operator acting on a particular class of spinors on a nilmanifold $\Gamma\backslash G$; namely, we restrict to *left-invariant* spinors, those which come from left-invariant spinors on the Lie group G; for more details, we refer the reader to Section 2.5. The following formula is obtained in Proposition 2.41 and expresses the Dirac operator on invariant spinors in a purely algebraic way:

$$4\not{D}\phi = -\sum_{i=1}^{n}(e^i \wedge de^i + i(e_i)de^i)\phi\,.$$

In Section 2.6 we rely on the existing classification of nilpotent Lie algebras up to dimension 6 (see for instance [13]) for solving the equation $\not{D}\eta = 0$ in the space of left-invariant spinors on low dimensional nilmanifolds. In particular, we show which metric nilpotent Lie algebras in dimensions 4, 5, and 6 admit a harmonic spinor – see Theorems 2.49, 2.53 and 2.58, and Subsection 2.6.3. We point out here that, although the proof is achieved by a case-by-case analysis, ours is the first systematic spinorial approach to the study of geometric structures on nilmanifolds.

This paper is organized as follows: in Section 2.2 we review the necessary preliminaries on Clifford algebras and spinor bundles. Section 2.3 reviews the spinorial description of Spin(7), G_2 and SU(3) structures; we introduce the notion of a *spin-harmonic* geometric structure, that is, a geometric structure defined by a harmonic unit-length spinor. In Section 2.4 we carry out the spinorial classification of SU(2) structures on 5-manifolds. In Section 2.5 we consider left-invariant spinors on simply connected Lie groups, finding a general formula for the Dirac operator – see Proposition 2.41 – which we specialize to the case of nilpotent and (a certain kind of) solvable Lie groups. Using this formula, in Section 2.6 we tackle nilpotent Lie algebras (and nilmanifolds) in dimensions 4, 5, and 6. In dimension 4, a non-abelian nilpotent Lie algebra admits no metric with harmonic spinors. In dimension 5 we classify metric nilpotent Lie algebras and determine those which admit harmonic spinors. Finally, in dimension 6, either we provide a metric on the Lie algebra which admits harmonic spinors, or we show that no such metric exists.

Acknowledgements. We are grateful to Anna Fino for useful conversations. The authors were partially supported by Project MINECO (Spain) MTM2015-63612-P. The first author was supported by a *Juan de la Cierva - Incorporación* Fellowship of Spanish Ministerio de Ciencia, Innovación y Universidades. The second author acknowledges financial support by an FPU Grant (FPU16/03475).

2.2 Preliminaries

In this section we recall some basic aspects about the representation theory of Clifford algebras, in the real and the complex case, as well as generalities on spinor bundles; further details can be found in [54] and [79].

2.2.1 Representations of the real Clifford algebra

If $n \not\equiv 3 \pmod 4$, the real Clifford algebra Cl_n of $\left(\mathbb{R}^n, \sum_{j=1}^n x_j^2\right)$ is isomorphic to the algebra of l-dimensional matrices with coefficients in the (skew) field $\mathbf{k}$, $\mathbf{k} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$; we denote this algebra by $\mathbf{k}(l)$. If $n \equiv 3 \pmod 4$, Cl_n is isomorphic to $\mathbf{k}(l) \oplus \mathbf{k}(l)$. In low dimensions, the following isomorphisms hold (see [79, Chapter 1, Theorem 4.3]):

- $\mathrm{Cl}_1 = \mathbb{C}$;
- $\mathrm{Cl}_2 = \mathbb{H}$;
- $\mathrm{Cl}_3 = \mathbb{H} \oplus \mathbb{H}$;
- $\mathrm{Cl}_4 = \mathbb{H}(2)$;
- $\mathrm{Cl}_5 = \mathbb{C}(4)$;
- $\mathrm{Cl}_6 = \mathbb{R}(8)$;
- $\mathrm{Cl}_7 = \mathbb{R}(8) \oplus \mathbb{R}(8)$;
- $\mathrm{Cl}_8 = \mathbb{R}(16)$.

Isomorphisms in higher dimensions are determined by the property $\mathrm{Cl}_{n+8} = \mathrm{Cl}_n \otimes \mathrm{Cl}_8$. As a consequence, there is a unique equivalence class of irreducible representations of Cl_n if $n \not\equiv 3 \pmod 4$ and two different ones if $n \equiv 3 \pmod 4$; these are determined by the image of the volume form, which can be I or $-$I [79, Chapter 1, Proposition 5.9].

By construction, the even part of the Clifford algebra Cl_n, denoted Cl_n^0, is isomorphic to the Clifford algebra Cl_{n-1}; using this, one can construct irreducible representations of Cl_{n-1} from irreducible representations of Cl_n by using the following result, which is essentially a reformulation of [79, Chapter 1, Proposition 5.12].

Proposition 2.1. *Let W be a $\mathbf{k}$-vector space and let $\rho_n\colon \mathrm{Cl}_n \to \mathrm{End}_{\mathbf{k}}(W)$ be an irreducible representation. Write $\mathbb{R}^n = \mathbb{R}^{n-1} \oplus \mathbb{R}$, where the second factor is generated by a unit-length vector e_n, and denote by $i_{n-1}\colon \mathrm{Cl}_{n-1} \to \mathrm{Cl}_n^0$ the extension to Cl_{n-1} of the map $\mathbb{R}^{n-1} \to \mathrm{Cl}_n^0$, $v \mapsto ve_n$; define $\rho_{n-1} = \rho_n \circ i_{n-1}\colon \mathrm{Cl}_{n-1} \to \mathrm{End}_{\mathbf{k}}(W)$. Then,*

1. *If $n \equiv 0 \pmod 4$ the representation ρ_{n-1} splits into two irreducible representations $\rho_{n-1}^{\pm}$, that are inequivalent. These are the eigenspaces $W^{\pm}$ of the endomorphism $\rho_n(\nu_n)\colon W \to W$, where ν_n is the volume form in $\mathbb{R}^n$.*

2. *If $n \equiv 1, 2 \pmod 8$, the representation ρ_{n-1} splits into two irreducible equivalent representations.*

3. *If $n \equiv 3, 5, 6, 7 \pmod 8$, the representation ρ_{n-1} is irreducible.*

In this paper, we work with the following 6-dimensional real representation of Cl_6:

$$
\begin{aligned}
e_1 &= +E_{18} + E_{27} - E_{36} - E_{45}, & e_4 &= -E_{15} - E_{26} - E_{37} - E_{48},\\
e_2 &= -E_{17} + E_{28} + E_{35} - E_{46}, & e_5 &= -E_{13} - E_{24} + E_{57} + E_{68},\\
e_3 &= -E_{16} + E_{25} - E_{38} + E_{47}, & e_6 &= +E_{14} - E_{23} - E_{58} + E_{67},
\end{aligned}
$$

where the matrix E_{ij} denotes the skew-symmetric endomorphism of $\mathbb{R}^8$ that maps the i^{th} vector of the canonical basis to the j^{th} one and is zero on the orthogonal complement.

2.2.2 Representations of the complex Clifford algebra

Let $\mathbb{C}\mathrm{l}_n$ be the complex Clifford algebra of $\left(\mathbb{C}^n, \sum_{j=1}^n z_j^2\right)$. A construction of an irreducible representation of $\mathbb{C}\mathrm{l}_n$ can be found in [54]. There exist a 2^k-dimensional complex vector space Δ_{2k} and isomorphisms

$$
\begin{aligned}
\kappa_{2k}&\colon \mathbb{C}\mathrm{l}_{2k} \to \mathrm{End}_{\mathbb{C}}(\Delta_{2k}),\\
\tilde{\kappa}_{2k+1}&\colon \mathbb{C}\mathrm{l}_{2k+1} \to \mathrm{End}_{\mathbb{C}}(\Delta_{2k}) \oplus \mathrm{End}_{\mathbb{C}}(\Delta_{2k})\,.
\end{aligned}
$$

Let $\mathrm{pr}_1\colon\ \mathrm{End}_{\mathbb{C}}(\Delta_{2k}) \oplus \mathrm{End}_{\mathbb{C}}(\Delta_{2k}) \to \mathrm{End}_{\mathbb{C}}(\Delta_{2k})$ be the projection onto the first summand. The complex representation of $\mathbb{C}\mathrm{l}_n$ is defined as κ_n if $n = 2k$ or $\mathrm{pr}_1 \circ \tilde{\kappa}_n$ if $n = 2k+1$.

Then Δ_{2k} is irreducible as a representation of $\mathbb{C}\mathrm{l}_n$ and is used to define the complex spin representation: this is the restriction of κ_n to $\mathrm{Spin}(n) \subset \mathbb{C}\mathrm{l}_n^0$. This representation is faithful and irreducible if $n = 2k+1$; however, if $n = 2k$, it splits into two irreducible summands $\Delta_{2k}^{\pm}$, which are the eigenspaces of eigenvalue ± 1 of the $\mathrm{Spin}(n)$-equivariant endomorphism $\kappa_n(\nu_n^{\mathbb{C}})$, where $\nu_n^{\mathbb{C}} = \mathbf{i}^k \nu_n$.

Depending on the dimension, the complex vector space Δ_{2k} is endowed with a real structure φ or a quaternionic structure j_2. These are antilinear endomorphisms of Δ_{2k} such that $\varphi^2 = \mathrm{I}$ and $\mathrm{j}_2^2 = -\mathrm{I}$; they commute or anticommute with the Clifford product, determining a real or quaternionic representation of $\mathrm{Spin}(n)$. The precise result is contained in the following proposition (see [54, Chapter 1]):

Proposition 2.2. *Suppose $n = 2k + r$, with $r \in \{0, 1\}$.*

1. *If $k \equiv 0, 3 \pmod 4$, then Δ_{2k} has a real structure φ with $\varphi \circ \kappa_n(v) = (-1)^{k+1}\kappa_n(v) \circ \varphi$ for any $v \in \mathbb{R}^n$.*

2. *If $k \equiv 1, 2 \pmod 4$, then Δ_{2k} has a quaternionic structure j_2 that satisfies $\mathrm{j}_2 \circ \kappa_n(v) = (-1)^{k+1}\kappa_n(v) \circ \mathrm{j}_2$ for any $v \in \mathbb{R}^n$.*

If Δ_{2k} is decomposable as a $\mathrm{Spin}(n)$ representation, then

- $\varphi(\Delta^{\pm}_{8p}) = \Delta^{\pm}_{8p}$;
- $\mathrm{j}_2(\Delta^{\pm}_{8p+2}) = \Delta^{\mp}_{8p+2}$;
- $\varphi(\Delta^{\pm}_{8p+6}) = \Delta^{\mp}_{8p+6}$;
- $\mathrm{j}_2(\Delta^{\pm}_{8p+4}) = \Delta^{\pm}_{8p+4}$.

We denote also by $(\Delta^{+}_{8p})_{\pm}$, $(\Delta^{-}_{8p})_{\pm}$ and $(\Delta_{8p+6})_{\pm}$ the eigenspaces of eigenvalue ± 1 of φ on Δ^{+}_{8p}, Δ^{-}_{8p} and $(\Delta_{8p+6})_{\pm}$ respectively. If $n = 8p + q$ with $0 \le q \le 7$ then Cl_n is isomorphic via $\tilde{\kappa}_n$ if $k \equiv 1 \pmod 2$, or via κ_n otherwise, to:

$q = 0$: $\mathrm{End}_{\mathbb{R}}((\Delta^{+}_{8p})_{+} \oplus (\Delta^{-}_{8p})_{-})$,

$q = 1$: $\mathrm{End}_{\mathbb{C}}(\Delta_{8p})$,

$q = 2$: $\mathrm{End}_{\mathbb{H}}(\Delta_{8p+2})$,

$q = 3$: $\mathrm{End}_{\mathbb{H}}(\Delta_{8p+2}) \oplus \mathrm{End}_{\mathbb{H}}(\Delta_{8p+2})$,

$q = 4$: $\mathrm{End}_{\mathbb{H}}(\Delta_{8p+4})$,

$q = 5$: $\mathrm{End}_{\mathbb{C}}(\Delta_{8p+4})$,

$q = 6$: $\mathrm{End}_{\mathbb{R}}((\Delta_{8p+6})_{+})$,

$q = 7$: $\mathrm{End}_{\mathbb{R}}((\Delta_{8p+6})_{+}) \oplus \mathrm{End}_{\mathbb{R}}((\Delta_{8p+6})_{+})$.

Remark 2.3. If $n \equiv 2, 3 \pmod 8$ then j_2 is a quaternionic structure that commutes with the Clifford product and if $n \equiv 4 \pmod 8$ then $\nu_4 \mathrm{j}_2$ has the same property. That explains the notations $\mathrm{End}_{\mathbb{H}}(\Delta_{8p+2})$ and $\mathrm{End}_{\mathbb{H}}(\Delta_{8p+4})$.

In addition, the representation Δ_{2k} is equipped with a hermitian product h that makes the Clifford product by vectors on $\mathbb{R}^{2k}$ and $\mathbb{R}^{2k+1}$ a skew-symmetric endomorphism. We construct from it a scalar product on the irreducible representation of the Clifford algebra using standard results of real and quaternionic structures on irreducible representations applied to the $\mathrm{Spin}(2k+1)$ module Δ_{2k}.

1. If $k \equiv 0, 3 \pmod 4$ the restriction of h to $(\Delta_{2k})_{\pm}$ is real valued. Moreover, the spaces $\Delta^{\pm}_{2k}$ are orthogonal if $k \equiv 0 \pmod 4$ because the multiplication by $\nu^{\mathbb{C}}_n$ is a unitary transformation.

2. If $k \equiv 1, 2 \pmod 4$ then $h(\mathrm{j}_2\phi, \mathrm{j}_2\eta) = \overline{h(\phi, \eta)}$, hence j_2 is an isometry for the real part of h.

In both cases, we denote by $\langle \cdot, \cdot \rangle$ the real part of h.

2.2.3 Spinor bundles

Let (M, g) be an oriented n-dimensional spin manifold and let $\mathrm{Ad}\colon \mathrm{P}_{\mathrm{Spin}}(M) \to \mathrm{P}_{\mathrm{SO}}(M)$ be a spin structure. Let W be a $\mathbf{k}$ vector space and $\rho_n\colon \mathrm{Cl}_n \to \mathrm{End}_{\mathbf{k}}(W)$ an irreducible representation. Recall that for $n \equiv 0 \pmod 4$ there is a splitting $W = W^{+} \oplus W^{-}$ into $\mathrm{Spin}(n)$ irreducible representations (see Proposition 2.1).

Definition 2.4. A *real spinor bundle* over M is $\Sigma(M) = \mathrm{P}_{\mathrm{Spin}}(M) \times_{\rho_n} W$, for an irreducible representation $\rho_n\colon \mathrm{Cl}_n \to \mathrm{End}_{\mathbf{k}}(W)$. If $n \equiv 0 \pmod 4$, the *positive* and *negative subbundles* are $\Sigma^{\pm}(M) = \mathrm{P}_{\mathrm{Spin}}(M) \times_{\rho_n} W^{\pm}$.

Let $\mathrm{Cl}(M)$ denote the bundle whose fiber over $p \in M$ is the Clifford algebra of (T_pM, g_p); the spinor bundle is a $\mathrm{Cl}(M)$-module with the Clifford product by a vector field $X \in \mathfrak{X}(M)$ given by

$$X[\tilde{F}, v] = \left[\tilde{F}, \sum_i X^i \rho_n(e_i)v\right];$$

here X^i are the coordinates of X with respect to the orthonormal frame $F = \mathrm{Ad}(\tilde{F})$. The Clifford multiplication extends to $\Lambda^k T^*M$ in the following way:

- the product with a covector is defined by $X^*\phi = X\phi$, with canonical identification between the tangent and the cotangent bundle given by the metric: $X^* = g(X, \cdot)$.
- If the product is defined on $\Lambda^l T^*M$ when $l \leq k$, we define

  $$(X^* \wedge \beta)\phi = X(\beta\phi) + (i(X)\beta)\phi,$$

  where $i(X)\beta$ denotes the contraction, $\beta \in \Lambda^k T^*M$ and $X \in \mathfrak{X}(M)$. This product is extended linearly to $\Lambda^{k+1}T^*M$.

The relation among representations of Cl_n determine relations among spinor bundles. For instance, we have the following result:

Lemma 2.5. *Let (M, g) be an n-dimensional spin manifold with $n = 8p + 8 - m$ and $4 \leq m < 8$. Consider the Riemannian manifold $(M \times \mathbb{R}^m, g + g_m)$, where g_m is the canonical metric on $\mathbb{R}^m$ with orthonormal basis $(e_{n+1}, \dots, e_{8p+8})$. Denote by $\mathrm{pr}_1 \colon M \times \mathbb{R}^m \to M$ the canonical projection.*

1. *There is a bijection between spin structures on M and spin structures on $M \times \mathbb{R}^m$.*
2. *The spinor bundles are related by $\Sigma^+(M \times \mathbb{R}^m) = \mathrm{pr}_1^* \Sigma(M)$ with Clifford product $X(\phi, t) = (Xe_{n+1}\phi, t)$ for $X \in \mathfrak{X}(M)$.*

Proof. Denote by $i \colon M \hookrightarrow M \times \mathbb{R}^m$ the canonical inclusion. First of all, $\mathrm{P}_{\mathrm{SO}}(M \times \mathbb{R}^m) = \mathrm{pr}_1^* \mathrm{P}_{\mathrm{SO}}(M)$. Therefore, each spin structure on M determines a spin structure on $M \times \mathbb{R}^m$ by $\mathrm{P}_{\mathrm{Spin}}(M \times \mathbb{R}^m) = \mathrm{pr}_1^* \mathrm{P}_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}(n)} \mathrm{Spin}(8p + 8)$. Conversely, given a spin structure $\mathrm{P}_{\mathrm{Spin}}(M \times \mathbb{R}^m)$ on $M \times \mathbb{R}^m$, we have that $i^*(\mathrm{P}_{\mathrm{Spin}}(M \times \mathbb{R}^m))$ is a $\mathrm{Spin}(8p + 8)$ structure. Taking the preimage of $\mathrm{P}_{\mathrm{SO}}(M) \subset \mathrm{P}_{\mathrm{SO}(8p+8)}(M)$, we get a spin structure on M.

There is an isomorphism between the bundles $\mathrm{P}_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}(n)} W^+$ and $\mathrm{P}_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}(n)} \mathrm{Spin}(8p + 8) \times_{\mathrm{Spin}(8p+8)} W^+$, given by $[\tilde{F}, v] \mapsto [[\tilde{F}, 1], v]$. Thus, taking into account Proposition 2.1, we get $\Sigma^+(M \times \mathbb{R}^m) = \mathrm{pr}_1^* \Sigma(M)$.

The relation between Clifford products follows from the equality $\rho_n(v) = \rho_{8p+8}(ve_{n+1})$, for $v \in \mathbb{R}^n$; this is obtained using the definition of ρ_n in Proposition 2.1 as follows:

$$\rho_n(v) = \rho_{n+1}(ve_{n+1}) = \rho_{n+2}(ve_{n+2}e_{n+1}e_{n+2}) = \rho_{n+2}(ve_{n+1}) = \cdots = \rho_{8p+8}(ve_{n+1}).$$

□

The scalar product $\langle \cdot, \cdot \rangle$ on W defines a scalar product on the spinor bundle that we also denote by $\langle \cdot, \cdot \rangle$; the Clifford product with a vector field is a skew-symmetric endomorphism. The Levi-Civita connection $\overline{\nabla}$ of g induces a connection ∇ on the spinor bundle which is $\langle \cdot, \cdot \rangle$-metric and acts as a derivation with respect to the Clifford product with a vector field. Moreover, the complex and quaternionic structures on W determine complex and quaternionic structures on the spinor bundle, which are isometries of $\langle \cdot, \cdot \rangle$ and parallel with respect to ∇.

Definition 2.6. The *Dirac operator* is the differential operator $\not{D}\colon \Gamma(\Sigma(M)) \to \Gamma(\Sigma(M))$ given locally by the expression

$$\not{D}\phi = \sum_{i=1}^{n} X_i \nabla_{X_i}\phi\,,$$

where $(X_1, \ldots, X_n)$ is a local orthonormal frame of M.

Definition 2.7. A spinor $\eta \in \Gamma(\Sigma(M))$ is called *harmonic* if $\not{D}\eta = 0$.

There is a relation between positive harmonic spinors in different dimensions; we follow the notation of Lemma 2.5:

Lemma 2.8. *For $m \in \{1, 2, 3, 4\}$, let (M, g) be an $(8p - m)$-dimensional spin Riemannian manifold. Let ϕ be a unit-length harmonic spinor of M. Then, $\eta = \mathrm{pr}_1^*\,\phi$ is a unit-length harmonic spinor on $M \times \mathbb{R}^m$.*

Proof. Let $(X_1, \ldots, X_n)$ be a local orthonormal frame of TM and let $(e_{n+1}, \ldots, e_{8p+8})$ be an orthonormal basis of $\mathbb{R}^m$; observe that $\overline{\nabla}^{M\times\mathbb{R}^m}_{X_i} X_j = \overline{\nabla}^{M}_{X_i} X_j$, $\overline{\nabla}^{M\times\mathbb{R}^m}_{e_i} X_j = \overline{\nabla}^{M\times\mathbb{R}^m}_{X_j} e_i = 0$ and $\overline{\nabla}^{\mathbb{R}^m}_{e_i} e_j = 0$. Therefore, $\nabla^{M\times\mathbb{R}^m}_{X_i}\eta = \mathrm{pr}_1^*(\nabla^{M}_{X_i}\phi)$ and $\nabla^{M\times\mathbb{R}^m}_{e_i}\eta = 0$. From the relation between $\Sigma(M)$ and $\Sigma^+(M \times \mathbb{R}^m)$ proved in Lemma 2.5 we deduce:

$$e_{n+1}\not{D}\eta = \sum_{i=1}^{n} e_{n+1} X_i \nabla^{M\times\mathbb{R}^m}_{X_i}\eta = -\,\mathrm{pr}_1^*\,\not{D}\phi\,.$$

The spinor η is harmonic because the multiplication by e_{n+1} is an isometry. □

2.3 Spinors and geometric structures

The purpose of this paper is to study geometric structures defined by unit-length harmonic spinors on Riemannian manifolds. This is interesting because a unit-length harmonic spinor defines different geometric structures according to the dimensions. We shall focus on dimensions 4, 5, 6, 7 and 8. In these dimensions, the relation between unit-length spinors and geometric structures on manifolds is summarized in the following result:

Proposition 2.9. *Let $\rho_n\colon \mathrm{Cl}_n \to \mathrm{End}_{\mathbf{k}}(W)$ an irreducible representation and let $\eta \in W$ be a unit-length spinor.*

1. *If $n = 8$ and $\eta \in W^{\pm}$ then* $\mathrm{Stab}_{\mathrm{Spin}(8)}(\eta) = \mathrm{Spin}(7)$.
2. *If $n = 7$ then* $\mathrm{Stab}_{\mathrm{Spin}(7)}(\eta) = \mathrm{G}_2$.
3. *If $n = 6$ then* $\mathrm{Stab}_{\mathrm{Spin}(6)}(\eta) = \mathrm{SU}(3)$.
4. *If $n = 5$ then* $\mathrm{Stab}_{\mathrm{Spin}(5)}(\eta) = \mathrm{SU}(2)$.
5. *If $n = 4$ then* $\mathrm{Stab}_{\mathrm{Spin}(4)}(\eta) = \mathrm{SU}(2)$.

This proposition means that a unit-length spinor in dimension 8 determines a Spin(7) structure on the underlying manifold, and similarly for the other dimensions.

Motivated by Definition 2.7, we give the following definition:

Definition 2.10. Let (M, g) be a Riemannian spin manifold of dimension $n \in \{4, \ldots, 8\}$, and let $\eta \in \Gamma(\Sigma(M))$ be a unit-length section. We say that η determines a *spin-harmonic structure* on M if $\not{D}\eta = 0$. Moreover, if $n \equiv 0 \pmod 4$, we say that the spin-harmonic structure is *positive* or *negative* if $\eta \in \Gamma(\Sigma^{\pm}(M))$.

Remark 2.11. For dimensions $n > 8$, the action of $\mathrm{Spin}(n)$ on the sphere of unit-length spinors is not transitive. Therefore the stabilizers of the spinors may be different groups, so it makes no sense to define a geometric structure via a unit-length spinor unless we require the constancy of the stabilizer (this happens for instance when one has a parallel spinor).

From now on, we denote a generic spinor by ϕ and a fixed unit-length spinor by η.

More precisely, our motivation is constructing 8-dimensional nilmanifolds with invariant balanced Spin(7) structures. As we shall see later, these structures are characterized by the presence of a positive spin-harmonic structure. Lemma 2.8 guarantees that if $n \in \{4, 5, 6, 7\}$, M is an n-dimensional spin manifold with a spin-harmonic structure and T^{8-n} is an $(8-n)$-dimensional flat torus, then $M \times T^{8-n}$ has a Spin(7) balanced structure. In section 2.6 we construct such spin-harmonic structures on low dimensional nilmanifolds.

Spin-harmonic structures have already appeared, under disguise, in the papers [1] and [86]; we proceed to review the relevant results and to relate spin-harmonic structures with the different kinds of Spin(7), G_2 and SU(3) structures. There is no spinorial description of SU(2) structures in dimension 5; we carry out this classification in Section 2.4. We do not study the condition in dimension 4; in fact, as we shall see in Theorem 2.49, there are no invariant harmonic spinors on 4-dimensional nilmanifolds.

2.3.1 Positive spin-harmonic Spin(7) structures in dimension 8

Let (M, g) be an 8-dimensional Riemannian manifold; a Spin(7) structure is characterized by the presence of a triple cross product on each tangent space; in turn, this is determined by a 4-form Ω (see [104, Definition 6.13]).

As usual, a way to measure the lack of integrability of a geometric structure is provided by its intrinsic torsion (see [105]). In this case, the intrinsic torsion of a Spin(7) structure is a section of the bundle $T^*M \otimes \mathfrak{spin}(7)^\perp$, which is isomorphic to $\Lambda^3 T^*M$ via the alternating map. The Hodge star defines an isomorphism $\star\colon \Lambda^3 T^*M \to \Lambda^5 T^*M$. Therefore, the different classes of Spin(7) structures are determined by the exterior derivative of Ω.

For a fixed Spin(7) form Ω on $\mathbb{R}^8$, the decomposition of the space of 3-forms of $\mathbb{R}^8$ into irreducible Spin(7) invariant subspaces is given by (see [104, Theorem 9.8]):

$$\Lambda^3(\mathbb{R}^8)^* = \Lambda^3_8(\mathbb{R}^8)^* \oplus \Lambda^3_{48}(\mathbb{R}^8)^*.$$

where $\Lambda^3_8(\mathbb{R}^8)^* = i(\mathbb{R}^8)\Omega$ and $\Lambda^3_{48}(\mathbb{R}^8)^* = \{\tau \in \Lambda^3(\mathbb{R}^8)^* \mid \tau \wedge \Omega = 0\}$. We denoted by $\Lambda^k_l(\mathbb{R}^8)^*$ an l-dimensional invariant subspace of $\Lambda^k(\mathbb{R}^8)^*$; moreover, the induced bundle on M is denoted by $\Lambda^k_l T^*M$. According to this discussion, there exist $\tau_1 \in \Lambda^1 T^*M$ and $\tau_3 \in \Lambda^3_{48} T^*M$ such that:

$$d\Omega = \tau_1 \wedge \Omega + \star\tau_3\,.$$

In [43], Fernández distinguished Spin(7) structures in the following pure classes:

Definition 2.12. A Spin(7)-structure given by Ω is said to be:

1. *parallel*, if $d\Omega = 0$;
2. *locally conformally parallel*, if $\tau_3 = 0$;
3. *balanced*, if $\tau_1 = 0$.

A Riemannian manifold (M, g) admitting a Spin(7) structure is spin and the positive part of its spinor bundle has a unit-length section. Conversely, a spin 8-dimensional manifold whose spinor bundle admits a positive unit-length section η can be endowed with a Spin(7) structure by the formula

$$\Omega(W,X,Y,Z) = \frac{1}{2}\langle(-WXYZ + WZYX)\eta, \eta\rangle \,.$$

As for spin-harmonic structures, the following result was proved by the second author in [86]:

Theorem 2.13. *The spinor η determines a positive spin-harmonic structure if and only if the induced* Spin(7) *structure is balanced.*

Remark 2.14. Spin-harmonic structures are thus especially relevant in dimension 8, since they represent a pure class of Spin(7) structures.

2.3.2 Spin-harmonic G_2 structures in dimension 7

A G_2 structure on a Riemannian 7-dimensional manifold (M, g) is characterized by the presence of a cross product on (TM, g), which is determined by a 3-form Ψ (see [104, Lemma 2.6])

The torsion of a G_2 structure is a section of the bundle $T^*M \otimes \mathfrak{g}_2^\perp$. The splitting of $\mathbb{R}^7 \otimes \mathfrak{g}_2^\perp$ into four G_2 invariant irreducible subspaces determines four subbundles, $\chi_1, \chi_2, \chi_3, \chi_4$ which, in turn, determine pure types of G_2 structures.

Such classes are completely determined by differential equations for Ψ and $\star\Psi$. In order to state the precise result, we recall the decomposition of $\Lambda^2(\mathbb{R}^7)^*$ and $\Lambda^3(\mathbb{R}^7)^*$ into G_2 irreducible parts for a fixed G_2 form Ψ of $\mathbb{R}^7$ (see [104, Theorem 8.5]):

$$\begin{aligned}\Lambda^2(\mathbb{R}^7)^* =& \Lambda^2_7(\mathbb{R}^7)^* \oplus \Lambda^2_{14}(\mathbb{R}^7)^*,\\ \Lambda^3(\mathbb{R}^7)^* =& \Lambda^3_1(\mathbb{R}^7)^* \oplus \Lambda^3_7(\mathbb{R}^7)^* \oplus \Lambda^3_{27}(\mathbb{R}^7)^*\,,\end{aligned}$$

where $\Lambda^2_7(\mathbb{R}^7)^* = i(\mathbb{R}^7)\Psi$, $\Lambda^2_{14}(\mathbb{R}^7)^* = \mathfrak{g}_2$, $\Lambda^3_1(\mathbb{R}^7)^* = \langle\Psi\rangle$, $\Lambda^3_7(\mathbb{R}^7)^* = i(\mathbb{R}^7)(\star\Psi)$ and $\Lambda^3_{27}(\mathbb{R}^7)^* = \{\omega \mid \Psi \wedge \omega = 0, \star\Psi \wedge \omega = 0\}$. Then we have (see [23, Proposition 1]):

Proposition 2.15. *There exist $\tau^1 \in C^\infty(M)$, $\tau^4 \in \Lambda^1 T^*M$, $\tau^2 \in \Lambda^2_{14} T^*M$ and $\tau^3 \in \Lambda^3_{27} T^*M$ such that:*

$$\begin{aligned}d\Psi =& \tau^1(\star\Psi) + 3\tau^4 \wedge \Psi + \star\tau^3,\\ d(\star\Psi) =& 4\tau^4 \wedge (\star\Psi) + \tau^2 \wedge \Psi.\end{aligned}$$

Moreover, the torsion is a section of χ_j if and only if $\tau^k = 0$ for $k \neq j$.

A Riemannian manifold (M, g) admitting a G_2 structure is spin and its spinor bundle has a unit-length section. Conversely, the spinor bundle $\Sigma(M)$ of a spin 7-manifold M has a unit-length section η and the 3-form of the G_2 structure is given by [1]:

$$\Psi(X, Y, Z) = \langle XYZ\eta, \eta\rangle \,.$$

The relationship between G_2-structures and harmonic spinors is characterized by the following result:

Theorem 2.16. *[1, Theorem 4.8] The spinor η determines a spin-harmonic structure if an only if the induced* G_2 *structure is of type $\chi_2 \oplus \chi_3$.*

2.3.3 Spin-harmonic SU(3) structures in dimension 6

Let (M, g) be a 6-dimensional Riemannian manifold. An SU(3) structure on M consists of a compatible almost complex structure J and a complex volume form Θ (see [68, 105]). We denote by Θ_+ and Θ_- the real and imaginary part of Θ and we define the fundamental 2-form ω by $\omega(X, Y) = g(JX, Y)$ for $X, Y \in \mathfrak{X}(M)$.

The space $\mathbb{R}^6 \otimes \mathfrak{su}(3)^\perp$ decomposes into seven SU(3)-invariant irreducible subspaces; accordingly the intrinsic torsion of an SU(3) structure, which is a section of $T^*M \otimes \mathfrak{su}(3)^\perp$, decomposes into the subbundles χ_1, $\chi_{\bar{1}}$, χ_2, $\chi_{\bar{2}}$, χ_3, χ_4, χ_5 (see [32]).

These are related to differential equations for ω, Θ_+ and Θ_-. Before formulating the result, we recall the decomposition of $\Lambda^2(\mathbb{R}^6)^*$ and $\Lambda^3(\mathbb{R}^6)^*$ into SU(3) irreducible representations. For this, we consider the U(3) decomposition $\Lambda^n(\mathbb{C}^6)^* = \oplus_{p+q=n}\Lambda^{p,q}(\mathbb{C}^6)^*$ and we denote the real part of a complex vector space V by $[\![V]\!]$. For a fixed SU(3) structure $(\omega, \Theta_+, \Theta_-)$ on $\mathbb{R}^6$, the splitting is:

$$\begin{aligned}\Lambda^2(\mathbb{R}^6)^* =&\langle\omega\rangle \oplus [\![\Lambda_0^{1,1}(\mathbb{C}^6)^*]\!] \oplus i(\mathbb{R}^6)\Theta_+,\\ \Lambda^3(\mathbb{R}^6)^* =&\langle\Theta_+\rangle \oplus \langle\Theta_-\rangle \oplus [\![\Lambda_0^{2,1}(\mathbb{C}^6)^*]\!] \oplus \mathbb{R}^6 \wedge \omega_0.\end{aligned}$$

where $\Lambda_0^{1,1}(\mathbb{C}^6)^*$ and $\Lambda_0^{2,1}(\mathbb{C}^6)^*$ are the spaces of primitive forms, that is, forms of $\Lambda^{1,1}(\mathbb{C}^6)^*$ and $\Lambda^{2,1}(\mathbb{C}^6)^*$ which are orthogonal to ω and $\omega \wedge (\mathbb{C}^6)^*$, respectively. The associated bundles of M are denoted respectively by $[\![\Lambda_0^{1,1}(T^*M \otimes \mathbb{C})]\!]$ and $[\![\Lambda_0^{2,1}(T^*M \otimes \mathbb{C})]\!]$.

Proposition 2.17. *[15, Section 2.5] There exist $\tau^1, \tau^{\bar{1}} \in C^\infty(M)$, $\tau^4, \tau^5 \in \Lambda^1 T^*M$, $\tau^2, \tau^{\bar{2}} \in [\![\Lambda_0^{1,1}(T^*M \otimes \mathbb{C})]\!]$ and $\tau^3 \in [\![\Lambda_0^{2,1}(T^*M \otimes \mathbb{C})]\!]$ such that:*

$$\begin{aligned} d\omega &= -\frac{3}{2}\tau^{\bar{1}}\Theta_+ + \frac{3}{2}\tau^1\Theta_- + \tau^3 + \tau^4 \wedge \omega,\\ d\Theta_+ &= \tau^1\omega^2 - \tau^2 \wedge \omega + \tau^5 \wedge \Theta_+,\\ d\Theta_- &= \tau^{\bar{1}}\omega^2 - \tau^{\bar{2}} \wedge \omega + J\tau^5 \wedge \Theta_+ . \end{aligned}$$

Moreover, the intrinsic torsion is a section of χ_j if and only if $\tau^k = 0$ for $k \neq j$.

A Riemannian manifold (M, g) with an SU(3) structure is spin and its spinor bundle has a unit-length section. Conversely, a spin 6-dimensional manifold has a unit-length spinor; the following proposition explains how the spinor induces the SU(3) structure.

Proposition 2.18. *[1, Section 2] The spinor bundle of M splits as*

$$\Sigma(M) = \langle\eta\rangle \oplus \langle \mathrm{j}\eta\rangle \oplus TM\eta .$$

The fundamental form ω and the real part of the complex 3-form Θ_+ of the SU(3) *structure determined by η are given by*

$$\omega(X, Y) = \langle \mathrm{j}X\eta, Y\eta\rangle \quad \textit{and} \quad \Theta_+ = -\langle XYZ\eta, \eta\rangle .$$

Proposition 2.18 guarantees the existence and uniqueness of $S \in \mathrm{End}(TM)$ and $\gamma \in T^*M$ such that:

$$\nabla_X\eta = S(X)\eta + \gamma(X)\mathrm{j}\eta .$$

The relation between harmonic spinors and SU(3) structures is given by the following result:

Theorem 2.19. *[1, Theorem 3.7] The spinor η determines a spin-harmonic structure if an only if its induced* SU(3) *structure lies in the class $\chi_{2\bar{2}345}$ and satisfies $\delta\omega = -2\gamma$.*

We finally relate Theorem 2.19 and Proposition 2.17.

Corollary 2.20. *The* SU(3) *structure is spin-harmonic if and only if it lies on* $\chi_{2\bar{2}345}$ *and satisfies* $\tau^4 = \tau^5$.

Proof. First, $\delta\omega = -\star(\tau^4 \wedge \omega^2) = J\tau^4$. To find an expression for γ in terms of the torsion forms we first observe that, according to [1, Theorem 3.13], it only depends on the projection of the intrinsic torsion Γ to χ_5. Therefore, we assume that $\gamma \in \chi_5$ for this computation; observe that in this case $d\Theta_+ = \tau^5 \wedge \Theta_+$, due to Proposition 2.17.

If $\Gamma \in \chi_5$ then $\nabla_X \eta = \gamma(X)\mathrm{j}\eta$ and therefore, for orthonormal vectors: $\nabla_W \Theta_+(X,Y,Z) = -2\gamma(W)\langle XYZ\eta, \mathrm{j}\eta\rangle = 2\gamma(W)\langle J(X)YZ\eta, \eta\rangle = -2\gamma(W)\Theta_-(X,Y,Z)$. For the penultimate equality we took into account that $X\mathrm{j}\eta = -\mathrm{j}X\eta = -J(X)\eta$. For the last, we used that $\Theta_-(X,Y,Z) = \Theta_+(J(X),Y,Z)$. Therefore,

$$\begin{aligned} d\Theta_+(W,X,Y,Z) &= \\ &= \nabla_W\Theta_+(X,Y,Z) - \nabla_X\Theta_+(W,Y,Z) + \nabla_Y\Theta_+(X,W,Z) - \nabla_Z\Theta_+(X,Y,W) \\ &= -2\gamma \wedge \Theta_-(W,X,Y,Z). \end{aligned}$$

In addition, one can observe that $\alpha\wedge\Theta_- = -J\alpha\wedge\Theta_+$ for $\alpha \in \xi^*$; this implies that, $\tau^5 = 2J\gamma$. Therefore, the equality $\delta\omega = -2\gamma$ is equivalent to $\tau^4 = \tau^5$. □

2.4 Spin-harmonic SU(2) structures on 5-dimensional manifolds

2.4.1 SU(2) structures

An SU(2) structure on a Riemannian manifold (M, g) is determined by an orthogonal splitting $TM = \langle \alpha^\sharp \rangle \oplus \xi$, where α is a unit-length 1-form and the distribution $\xi = \ker \alpha$ is endowed with three almost complex structures $J_k\colon \xi \to \xi$, $k = 1, 2, 3$ which are isometries with respect to the induced metric, and satisfy $J_1 \circ J_2 = J_3$ and $J_k \circ J_l = -J_l \circ J_k$ for $k \neq l$. The vector field $\alpha^\sharp$ is denoted by R. The three fundamental 2-forms are given by $\omega_k(X,Y) = g(J_kX, Y)$, $k = 1, 2, 3$, $X, Y \in \mathfrak{X}(M)$.

In fact, SU(2) structures are characterized by the forms $(\alpha, \omega_1, \omega_2, \omega_3)$, as the following result states:

Proposition 2.21. *[35, Proposition 1]* SU(2) *structures on a* 5*-manifold are in one-to-one correspondence with* $(\alpha, \omega_1, \omega_2, \omega_3) \in \Lambda^1T^*M \times (\Lambda^2T^*M)^3$, *such that:*

1. $\omega_i \wedge \omega_j = 0$ *for* $i \neq j$, $\omega_1^2 = \omega_2^2 = \omega_3^2$ *and* $\alpha \wedge \omega_1^2 \neq 0$,
2. *If* $i(X)\omega_1 = i(Y)\omega_2$, *then* $\omega_3(X,Y) \geq 0$.

Proposition 2.22. *[35, Corollary 3] Let* $(\alpha, \omega_1, \omega_2, \omega_3)$ *be an* $SU(2)$ *structure on a 5-manifold. There is a local frame of the cotangent bundle,* $(e^1, \dots, e^5)$, *such that* $\alpha = e^5$, $\omega_1 = e^{12} + e^{34}$, $\omega_2 = e^{13} - e^{24}$, $\omega_3 = e^{14} + e^{23}$.

An almost complex structure $J_k\colon \xi \to \xi$ defines an almost complex structure on ξ^* by $(J_k\beta)(X) = \beta(J_kX)$ for $\beta \in \xi^*$ and $X \in \xi$; one has $(J_k \circ J_l)\beta = -(J_l \circ J_k)\beta$, but $(J_1 \circ J_2)\beta = -J_3\beta$. The next lemma will be used in the next section:

Lemma 2.23. *For* $\beta \in \xi^*$, $\star_\xi(\beta \wedge \omega_k) = -J_k\beta$.

Proof. We compute the equality for $\beta = e^1$. Using that $J_ke^1 = -(J_ke_1)^*$ and that $\omega_k = -(\mathrm{I} + \star_\xi)(e^1 \wedge J_ke^1)$, we get: $\star_\xi(e^1 \wedge \omega_k) = -\star_\xi(e^1 \wedge \star_\xi(e^1 \wedge J_ke^1)) = -(i(e_1)(e^1 \wedge J_ke^1)) = -J_ke^1$. □

As usual, SU(2) structures are classified by the intrinsic torsion, which is a section of $T^*M \otimes \mathfrak{su}(2)^\perp$. In the following, we denote the intrinsic torsion by an SU(2) equivariant map,

$$\Xi\colon \mathrm{P}_{\mathrm{SO}}(M) \to T^*M \otimes \mathfrak{su}(2)^\perp,$$

where $\mathrm{P}_{\mathrm{SO}}(M)$ is the frame bundle of M. Proposition 2.25 below shows that Ξ is determined by $(d\alpha, d\omega_1, d\omega_2, d\omega_3)$. In order to state it, we recall the irreducible decomposition of some SU(2) modules (see [16]).

Proposition 2.24. *Let $\mathbb{R}^5$ be endowed with the* SU(2) *structure $(\alpha, \omega_1, \omega_2, \omega_3)$. Then*

1. $\Lambda^1(\mathbb{R}^5)^* = \langle\alpha\rangle \oplus \xi^*$,
2. $\Lambda^2(\mathbb{R}^5)^* = \alpha \wedge \xi^* \oplus (\oplus_{k=1}^3 \langle\omega_k\rangle) \oplus \mathfrak{su}(2)$,
3. $\Lambda^3(\mathbb{R}^5)^* = \Lambda^3\xi^* \oplus (\oplus_{k=1}^3 \langle\alpha \wedge \omega_k\rangle) \oplus \alpha \wedge \mathfrak{su}(2)$,
4. $\mathrm{End}(\xi) = \langle \mathrm{I}\rangle \oplus (\oplus_{k=1}^3 \sigma_k(\xi)) \oplus (\oplus_{k=1}^3 \langle J_k\rangle) \oplus \mathfrak{su}(2)$, *where*

$$\sigma_k(\xi) = \left\{S \in \mathrm{Sym}_0(\xi) \mid SJ_l = (-1)^{\delta_k^l+1} J_l S,\ l = 1,2,3\right\},\ k = 1,2,3\,.$$

Moreover, the map $E_k\colon \sigma_k(\xi) \to \mathfrak{su}(2)$, $E_k(S) = i(S)\omega_k$ is an isomorphism.

Proposition 2.25. *[35, Proposition 9] As an* SU(2)*-module, $\mathbb{R}^5 \otimes \mathfrak{su}(2)^\perp$ decomposes as:*

$$\mathbb{R}^5 \otimes \mathfrak{su}(2)^\perp = 7\mathbb{R} \oplus 4(\mathbb{R}^4)^* \oplus 4\mathfrak{su}(2),$$

where $7\mathbb{R}$ means 7 copies of the trivial representation $\mathbb{R}$, and so on. Let $\tau_0^l, \tau_0^{kl} \in C^\infty(M)$, $k,l = 1,2,3$, $\tau_1^k \in \xi^*$ and $\tau_2^k \in \mathfrak{su}(2)$, $k = 1,2,3,4$, be such that

$$d\alpha = \sum_{l=1}^3 \tau_0^l \omega_l + \alpha \wedge \tau_1^4 + \tau_2^4\,,$$
$$d\omega_k = \sum_{l=1}^3 \tau_0^{kl} \alpha \wedge \omega_l + \tau_1^k \wedge \omega_k + \alpha \wedge \tau_2^k\,,$$

Then $\tau_0^{kk} = \tau_0^{ll}$ and $\tau_0^{kl} = -\tau_0^{lk}$ for $l \neq k$. Moreover,

$$\Xi(u) = ((\tau_0^{11}, \tau_0^{jk}, \tau_0^l), (u^*\tau_0^j, u^*\tau_1^4), (u^*\tau_0^j, u^*\tau_2^4))\,.$$

2.4.2 Spinorial point of view

Let $\rho_5\colon \mathrm{Cl}_5 \to \mathrm{End}_{\mathbb{C}}(W)$ be an irreducible representation with complex structure $\mathrm{j}_1 = \rho_5(\nu_5)$. Take also a quaternionic stucture j_2 that anticommutes with the Clifford product, and define $\mathrm{j}_3 = \mathrm{j}_1 \circ \mathrm{j}_2$.

Let (M, g) be a spin Riemannian manifold and let $\mathrm{Ad}\colon \mathrm{P}_{\mathrm{Spin}(5)}(M) \to \mathrm{P}_{\mathrm{SO}(5)}(M)$ be a spin structure. The spinor bundle $\Sigma(M) = \mathrm{P}_{\mathrm{Spin}(5)}(M) \times_{\rho_5} W$ has a unit-length section η. Define $\mathrm{Stab}(\eta)$ as the subbundle whose fiber at $p \in M$ is the stabilizer of the spinor $\eta(p)$ under the action of Spin(5). It is an SU(2) reduction of $\mathrm{P}_{\mathrm{Spin}(5)}(M)$, and the projection $\mathrm{Ad}(\mathrm{Stab}(\eta))$ is an SU(2) structure because the kernel of Ad is ± 1 and $-1 \notin \mathrm{Stab}(\eta_p)$.

We first explain the decomposition of the spinor bundle of M and write the forms that determine the structure by means of spinors. For that purpose consider the map $\rho_\eta\colon \mathrm{Spin}(5) \to W$, $\rho_\eta(g) = g\eta$, whose differential is $d\rho_\eta\colon \Lambda^2\mathbb{R}^5 \to W$, $d\rho_\eta(\gamma) = \gamma\eta$.

Lemma 2.26. *The restriction* $d\rho_\eta\colon \mathfrak{su}(2)^\perp \to \langle\eta\rangle^\perp$ *is an isomorphism, hence there is a decomposition of* $\langle\eta\rangle^\perp$ *with respect to the* SU(2) *structure determined by* η, $(\alpha, \omega_1, \omega_2, \omega_3)$*:*

$$\Sigma(M) = \langle\eta\rangle \oplus (\oplus_{k=1}^3 \langle\omega_k\eta\rangle) \oplus \xi^*\eta.$$

Proof. The kernel of $d\rho_\eta$ is $\mathfrak{su}(2)$ because $\mathrm{Stab}(\eta) = \mathrm{SU}(2)$ and $\mathrm{Im}(d\rho_\eta) \subset \langle\eta\rangle^\perp$. By Proposition 2.24(2), we have $\Sigma(M) = \langle\eta\rangle \oplus (\oplus_{k=1}^3\langle\omega_k\eta\rangle) \oplus (\alpha\wedge\xi^*)\eta$. Now $(\alpha\wedge\xi^*)\eta = \xi^*\eta$ because these are irreducible representations of the same dimension. □

We can write the forms that determine the SU(2) structure in terms of spinors. For that purpose recall that $\mathrm{Cl}_5 \cong \mathbb{C}(4)$. According to Propositon 2.2, the spin representation has a quaternionic structure j_2 that anticommutes with the Clifford product. Therefore, the space of spinor is $\mathbb{R}^8$ endowed with the complex structures, $(\mathrm{j}_1, \mathrm{j}_2, \mathrm{j}_3)$, where j_1 is the complex structure determined by the isomorphism $\mathrm{Cl}_5 \cong \mathbb{C}(4)$, and $\mathrm{j}_3 = \mathrm{j}_1\circ\mathrm{j}_2$. Of course, $\mathrm{j}_k\mathrm{j}_l = -j_lj_k$ if $k \neq l$. The complex structure j_1 commutes with the Clifford product by a vector and both j_2 and j_3 anticommute. According to this, we define $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_3 = -1$; we have that $\mathrm{j}_kX\phi = \varepsilon_kX\mathrm{j}_k\phi$, for every spinor ϕ.

Lemma 2.27. *The spinors* η, $\mathrm{j}_1\eta$, $\mathrm{j}_2\eta$, $\mathrm{j}_3\eta$ *are orthogonal and the spaces* $\mathbb{H}_\eta = \langle\eta, \mathrm{j}_1\eta, \mathrm{j}_2\eta, \mathrm{j}_3\eta\rangle$ *and* $\mathbb{H}_\eta^\perp$ *are* j_k*-invariant,* $k = 1, 2, 3$.

Moreover, there exists a subspace $\xi \subset \mathbb{R}^5$ *such that* $\xi\eta = \mathbb{H}_\eta^\perp$; ξ *inherits a quaternionic structure determined by* $\mathrm{j}_k(X\eta) = J_k(X)\eta$.

Proof. The orthogonality of the mentioned spinors follows from the fact that the endomorphisms j_k are isometries. It also follows from this property that the subspace $\mathbb{H}_\eta^\perp$ is j_k-invariant.

In addition, $\mathbb{H}_\eta^\perp$ is SU(2)-irreducible as a consequence of Lemma 2.26, and the map $X \mapsto X\eta$ is injective and SU(2)-equivariant. Being $\mathbb{R}^5 = \mathbb{R}\oplus\mathbb{C}^2$ as SU(2) modules, necessarily $\mathbb{H}_\eta^\perp = \xi\eta$ for some 4-dimensional subspace $\xi \subset \mathbb{R}^5$. Finally, the endomorphisms J_k define a quaternionic structure on ξ, because j_2 is a quaternionic structure on $\mathbb{H}_\eta^\perp$. □

Definition 2.28. Let (M, g) be a Riemannian manifold with a spin structure and let $\eta \in \Sigma(M)$ be a unit spinor. The SU(2) structure $(\alpha, \omega_1, \omega_2, \omega_3)$ defined by η is given by:

1. $\omega_k(X, Y) = g(J_kX_\xi, Y_\xi)$, where Z_ξ is the orthogonal projection of a vector field Z to ξ.
2. $\mathbb{R}^5 \cong \langle R\rangle \oplus \xi$ as oriented vector spaces, where ξ is oriented by $\omega_1^2|_\xi$, and $R = \alpha^\sharp$.

Lemma 2.29. *Let* ν *be the unit-lenght volume form. The following equalities hold:*

1. $\omega_k\eta = -2\varepsilon_k\mathrm{j}_k\eta$ *for* $\varepsilon_1 = 1$ *and* $\varepsilon_2 = \varepsilon_3 = -1$.
2. $\alpha\eta = -\mathrm{j}_1\eta$,
3. $\alpha\mathrm{j}_2\eta = -\mathrm{j}_3\eta$ *and* $\alpha\mathrm{j}_3\eta = \mathrm{j}_2\eta$,
4. $\nu\eta = -j_1\eta$.

Proof. Let $(e_1, e_2, e_3, e_4, e_5)$ be an orthonormal oriented frame such that $\omega_1 = e^{12} + e^{34}$, $\omega_2 = e^{13} - e^{24}$, $\omega_3 = e^{14} + e^{23}$ and $\alpha = e^5$. Taking into account that $J_1(e_1) = e_2$ and $J_1(e_3) = e_4$, we obtain:

$$\omega_1\eta = (e_1e_2 + e_3e_4)\eta = e_1J_1(e_1)\eta + e_3J_1(e_3)\eta = \mathrm{j}_1(e_1^2 + e_3^2)\eta = -2\mathrm{j}_1\eta\,.$$

For $k \in \{2, 3\}$ the computation is similar, but one has to take into account that j_2 and j_3 anticommute with the Clifford product with a vector.

Finally, $e_{12}\eta = -\mathrm{j}_1\eta = e_{34}\eta$ implies $\nu_5\eta = -e_5\eta$. The second and third equalities are a consequence of the previous one, together with the fact that $\mathrm{j}_1\mathrm{j}_2 = \mathrm{j}_3$. For instance, $\alpha\mathrm{j}_2\eta = -\mathrm{j}_2\alpha\eta = \mathrm{j}_2\mathrm{j}_1\eta = -\mathrm{j}_3\eta$.

For the last equality, observe that in terms of the previous frame we have: $\nu = e^{12345} = e^1 \wedge (J_1(e_1))^* \wedge e^3 \wedge (J_1(e_3))^* \wedge e^5$. Taking into account the previous equalities and that $(e^k \wedge (J_1(e^k))^*)\eta = -\mathrm{j}_1\eta$ as before, we obtain:

$$\nu\eta = -\mathrm{j}_1 e^1 J_1(e^1) e^3 J_1(e^3)\eta = -\mathrm{j}_1\eta.$$

□

Remark 2.30. The subspaces $\Lambda^2\xi^*\eta$ and $\xi^*\eta$ are orthogonal.

Lemma 2.31. *For $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_3 = -1$, $\omega_k(X,Y) = \varepsilon_k\langle X\mathrm{j}_k\eta, Y\eta\rangle$. Moreover, $\alpha(X) = -\langle X\eta, \mathrm{j}_1\eta\rangle$.*

Proof. The tensor $(X,Y) \mapsto \langle X\mathrm{j}_k\eta, Y\eta\rangle$ is skew-symmetric because j_k is an isometry, $\mathrm{j}_k^2 = -\mathrm{Id}$ and $\langle \mathrm{j}_k\eta, \eta\rangle = 0$. If $X, Y \in \xi$,

$$\omega_k(X,Y) = g(J_kX, Y) = \langle J_kX\eta, Y\eta\rangle = \varepsilon_k\langle X\mathrm{j}_k\eta, Y\eta\rangle.$$

Moreover, $\omega_k(R,Y) = 0 = \varepsilon_k\langle R\mathrm{j}_k\eta, Y\eta\rangle$, because $R\mathrm{j}_k\eta \in \mathbb{H}_\eta$ and $Y\eta \in \mathbb{H}_\eta^\perp$. Finally, $\alpha(X) = \langle X\eta, R\eta\rangle = -\langle X\eta, \mathrm{j}_1\eta\rangle$. □

We now compute the Dirac operator of η in order to relate it with the torsion of the SU(2) structure. We first introduce some notation.

Definition 2.32. Lemmas 2.26 and 2.29 guarantee the existence and uniqueness of $S \in \mathrm{End}(\xi)$, $V_\xi \in \xi$, $\Theta_l \in \xi^*$ and $\phi_l \in C^\infty(M)$, $l = 1, 2, 3$, such that:

$$\nabla_X\eta = S(X_\xi)\eta + \alpha(X)V_\xi\eta + \sum_{l=1}^{3}(\Theta_l(X_\xi) + \alpha(X)\phi_l)\mathrm{j}_l\eta\,, \tag{2.1}$$

where $X = X_\xi + \alpha(X)R$.

Definition 2.33. According to Proposition 2.24, there is a decomposition of $S \in \mathrm{End}(\xi)$:

$$S(X) = \mu\,\mathrm{I} + \sum_{l=1}^{3} S_l + \sum_{l=1}^{3}\lambda_l J_l + S_0\,,$$

where $S_k \in \sigma_k(\xi)$ and $S_0 \in \mathfrak{su}(2)$.

We now compute the Dirac operator of η in terms of the tensors we introduced; we use the notation of Definition 2.32.

Proposition 2.34. *Let $\eta \in \Sigma(M)$ be a unit-length spinor. The Dirac operator is*

$$\begin{aligned}\not{D}\eta =& (-4\mu + \phi_1)\eta - 4\lambda_1\mathrm{j}_1\eta + (4\lambda_2 + \phi_3)\mathrm{j}_2\eta + (4\lambda_3 - \phi_2)\mathrm{j}_3\eta \\ &+ (J_1(V_\xi + \Theta_1^\sharp) - J_2(\Theta_2^\sharp) - J_3(\Theta_3^\sharp))\eta.\end{aligned}$$

Proof. Let (e_1, e_2, e_3, e_4, R) be an oriented orthonormal local frame. From (2.1), we have

$$\not{D}\eta = m(S) + RV_\xi\eta + \sum_{k=1}^{3}\left(\left(\sum_{i=1}^{4}\Theta_k(e_i)e_i\right) + \phi_k R\right)\mathrm{j}_k\eta\,,$$

where $m\colon \mathrm{End}(\xi) \to \Sigma(M)$, $e_i \otimes e_j^* \mapsto e_i e_j \eta$. Observe that m is SU(2) equivariant and $\mathrm{Im}(m) = \mathbb{H}_\eta$. Taking into account Proposition 2.24, we obtain $\ker(m) = \mathfrak{su}(2) \oplus (\oplus_{k=1}^3 \sigma_k(\xi))$. Moreover, $m(\mathrm{I}) = -4\eta$ and $m(J_k) = -4\varepsilon_k \mathrm{j}_k \eta$.

In addition, $RV_\xi \eta = J_1(V_\xi)\eta$. Finally,

$$\sum_{i=1}^4 \Theta_k(e_i) e_i \mathrm{j}_k \eta = \varepsilon_k J_k \Theta_k^\sharp \eta \quad \text{and} \quad \sum_{k=1}^3 \phi_k R \mathrm{j}_k \eta = \phi_1 \eta - \phi_2 \mathrm{j}_3 \eta + \phi_3 \mathrm{j}_2 \eta \,.$$

□

We now write the torsion in terms of the forms $(\alpha, \omega_1, \omega_2, \omega_3)$ defined by a unit-length spinor $\eta \in \Sigma(M)$ as in Lemma 2.31.

Proposition 2.35. *The covariant derivatives of the forms $(\alpha, \omega_1, \omega_2, \omega_3)$ are governed by the formulas*

$$\begin{aligned} (\overline{\nabla}_Z \omega_k)(X, Y) &= \varepsilon_k \langle \nabla_Z \eta, (XY - YX)\mathrm{j}_k \eta \rangle, \quad k = 1, 2, 3\,, \\ (\overline{\nabla}_Z \alpha)(X) &= 2\langle \nabla_Z \eta, X \mathrm{j}_1 \eta \rangle\,, \end{aligned}$$

where $\overline{\nabla}$ is the Levi-Civita connection and ∇ is the spinorial connection.

Proof. Take $X, Y, Z \in T_pM$ and extend them to vector fields with $\overline{\nabla} X|_p = \overline{\nabla} Y|_p = \overline{\nabla} Z|_p = 0$. Then, according to Lemma 2.31 we have:

$$\begin{aligned} (\overline{\nabla}_Z \omega_k)(X, Y) &= Z(\omega_k(X, Y)) = \varepsilon_k \langle \mathrm{j}_k X \nabla_Z \eta, Y\eta \rangle + \varepsilon_k \langle \mathrm{j}_k X \eta, Y \nabla_Z \eta \rangle \\ &= \varepsilon_k \langle \nabla_Z \eta, (XY - YX)\mathrm{j}_k \eta \rangle, \\ (\overline{\nabla}_Z \alpha)(X) &= Z(\alpha(X)) = -\langle X \nabla_Z \eta, \mathrm{j}_1 \eta \rangle - \langle X\eta, \mathrm{j}_1 \nabla_Z \eta \rangle \\ &= 2\langle \nabla_Z \eta, X \mathrm{j}_1 \eta \rangle. \end{aligned}$$

□

Before computing the differentials, we prove a technical result:

Lemma 2.36. *For $X, Y \in \xi$, one has:*

$$\begin{aligned} \omega_1(S(X), Y) - \omega_1(S(Y), X) &= 2(\mu\omega_1 - \lambda_3\omega_2 + \lambda_2\omega_3 + i(S_1)\omega_1)(X, Y)\,, \\ \omega_2(S(X), Y) - \omega_2(S(Y), X) &= 2(\lambda_3\omega_1 + \mu\omega_2 - \lambda_1\omega_3 + i(S_2)\omega_2)(X, Y)\,, \\ \omega_3(S(X), Y) - \omega_3(S(Y), X) &= 2(-\lambda_2\omega_1 + \lambda_1\omega_2 + \mu\omega_3 + i(S_3)\omega_3)(X, Y)\,. \end{aligned}$$

Proof. We prove the first equality, the others being similar. We analyze each irreducible part separately. It is clear that $\omega_1(\mu X, Y) - \omega_1(Y, \mu X) = 2\mu\omega_1(X, Y)$. Taking into account that $S_k J_1 = \varepsilon_k J_1 S_k$, we obtain that $S_k J_1$ is skew-symmetric for $k = 1$ and symmetric for $k \in \{2, 3\}$. Therefore,

$$\sum_{k=1}^3 g(J_1 S_k(X), Y) - g(J_1 S_k(Y), X) = 2\omega_1(S_1(X), Y)\,.$$

Finally we conclude:

$$\begin{aligned} \sum_{k=1}^3 \lambda_k g(J_1 J_k(X), Y) - \lambda_k g(J_1 J_k(Y), X) &= -2\lambda_3 g(J_2(X), Y) + 2\lambda_2 g(J_3(X), Y) \\ &= 2(-\lambda_3\omega_2 + \lambda_2\omega_3)(X, Y). \end{aligned}$$

The equality $\omega_1(S_0(X), Y) + \omega_1(X, S_0(Y)) = 0$ follows from the fact that $S_0 \in \mathfrak{su}(2)$. □

Proposition 2.37. *Let $\eta \in \Sigma(M)$ be a unit-length spinor and let α be the 1-form of the* SU(2) *structure determined by η. Then (with the notations of Proposition 2.25),*

$$d\alpha = \alpha \wedge \tau_1^4 + \sum_{k=1}^{3} \tau_0^k \omega_k + \tau_2^4 \,,$$

where:

- $\tau_0^1 = -4\mu$, $\tau_0^2 = 4\lambda_3$, $\tau_0^3 = -4\lambda_2$,
- $\tau_1^4 = 2J_1 V_\xi^*$,
- $\tau_2^4 = -4i(S_1)\omega_1$.

Proof. Proposition 2.35 implies that $\frac{1}{2}d\alpha(X,Y) = \langle \nabla_X \eta, Y \mathrm{j}_1 \eta \rangle - \langle \nabla_Y \eta, X \mathrm{j}_1 \eta \rangle$. In order to compute $d\alpha|_\xi$, we first consider $X, Y \in \xi$; according to equation (2.1), the orthogonal projection of $\nabla_X \eta$ to $\xi\eta$ is $S(X)\eta$. So that $\langle \nabla_X \eta, Y \mathrm{j}_1 \eta \rangle = \langle S(X)\eta, J_1(Y)\eta \rangle$. Taking into account the previous observation, and Lemma 2.36 we obtain:

$$\begin{aligned} \frac{1}{2} d\alpha(X,Y) &= \langle X\eta, J_1 S(Y)\eta \rangle - \langle Y\eta, J_1 S(X)\eta \rangle \\ &= -2(\mu\omega_1 - \lambda_3\omega_2 + \lambda_2\omega_3 + i(S_1)\omega_1)(X,Y). \end{aligned}$$

Finally, we compute $d\alpha(R,Y)$; arguing as before, equation (2.1) implies that $\langle \nabla_R \eta, \mathrm{j}_1 Y \eta \rangle = \langle V_\xi \eta, \mathrm{j}_1 Y \eta \rangle$. In addition, $\langle \nabla_Y \eta, j_1 R \eta \rangle = \langle \nabla_Y \eta, \eta \rangle = 0$, according to Lemma 2.29. Thus,

$$\frac{1}{2} d\alpha(R,Y) = \langle V_\xi \eta, \mathrm{j}_1 Y \eta \rangle - \langle \mathrm{j}_1 R \eta, \nabla_Y \eta \rangle = \langle V_\xi \eta, J_1(Y)\eta \rangle.$$

□

Proposition 2.38. *Let $\eta \in \Sigma(M)$ be a unit-length spinor and let $(\omega_1, \omega_2, \omega_3)$ be the 2-forms of the* SU(2) *structure determined by η. Then*

$$d\omega_k = \alpha \wedge \tau_2^k + \sum_{l=1}^{3} \tau_0^{kl} \alpha \wedge \omega_l + \tau_1^k \wedge \omega_k \,,$$

where:

- $\tau_0^{kk} = 4\lambda_1$, $\tau_0^{12} = 4\lambda_2 + 2\phi_3$, $\tau_0^{13} = 4\lambda_3 - 2\phi_2$, $\tau_0^{23} = 4\mu - 2\phi_1$,
- $\tau_1^k = -2 \sum_{l \neq k} \varepsilon_k J_l \Theta_l$,
- $\tau_2^1 = 4i(S_0)g$, $\tau_2^2 = 4i(S_3)\omega_3$, $\tau_2^3 = -4i(S_2)\omega_2$.

Proof. Suppose that X, Y, Z are orthonormal; then according to Proposition 2.35 we have $\overline{\nabla}_Z \omega(X,Y) = 2\varepsilon_k \langle \nabla_Z \eta, XY \mathrm{j}_k \eta \rangle$, thus:

$$\varepsilon_k \frac{1}{2} d\omega_k(X,Y,Z) = \langle \nabla_X \eta, YZ \mathrm{j}_k \eta \rangle - \langle \nabla_Y \eta, XZ \mathrm{j}_k \eta \rangle + \langle \nabla_Z \eta, XY \mathrm{j}_k \eta \rangle. \tag{2.2}$$

We first assume that $X, Y, Z \in \xi$. Then,

$$\varepsilon_k \frac{1}{2} d\omega_k(X,Y,Z) = \langle X\nabla_X \eta + Y\nabla_Y \eta + Z\nabla_Z \eta, XYZ \mathrm{j}_k \eta \rangle.$$

We now let $W \in \xi$ be the unit-length vector, orthogonal to $\langle X, Y, Z \rangle$ such that (X, Y, Z, W, R) is positively oriented. We observe the following:

1. $X\nabla_X\eta + Y\nabla_Y\eta + Z\nabla_Z\eta = \not{D}\eta - W\nabla_W\eta - R\nabla_R\eta$,
2. The unit-lenght volume form is $\nu = X^* \wedge Y^* \wedge Z^* \wedge W^* \wedge R^*$. From the equality $\nu\eta = -j_1\eta = R\eta$ (see Lemma 2.29 (2) and (4)) we obtain $XYZW\eta = \eta$ and thus, $XYZ\eta = -W\eta$. Therefore,
$$XYZ\mathrm{j}_k\eta = \varepsilon_k\mathrm{j}_kXYZ\eta = -\varepsilon_k\mathrm{j}_kW\eta = -\varepsilon_kJ_k(W)\eta.$$

These observations imply:
$$\frac{1}{2}d\omega_k(X,Y,Z) = -\langle\not{D}\eta, J_kW\eta\rangle + \langle W\nabla_W\eta, J_k(W)\eta\rangle + \langle R\nabla_R\eta, J_k(W)\eta\rangle\,.$$

From Proposition 2.34 we obtain that the orthogonal projection of $-\not{D}\eta$ to $\xi\eta$ is $(-J_1(V_\xi + \Theta_1^\sharp) + J_2(\Theta_2^\sharp) + J_3(\Theta_3^\sharp))\eta$. Since $J_l(\alpha^\sharp)^* = -J_l(\alpha)$ if $\alpha\in\xi^*$ we have:
$$-\langle\not{D}\eta, J_k(W)\eta\rangle = (-J_1(V_\xi)^* + \sum_{l=1}^{3}\varepsilon_lJ_l(\Theta_l))(J_k(W)).$$
Morever, $\langle W\nabla_W\eta, J_k(W)\eta\rangle = \varepsilon_k\langle\nabla_W\eta, \mathrm{j}_k\eta\rangle = \varepsilon_k\Theta_k(W)$ according to equation (2.1). In addition, taking into account equation (2.1), and that the spinor $Rj_k\eta = -\varepsilon_k\mathrm{j}_k\mathrm{j}_1\eta$ is perpendicular to $\xi\eta$, we obtain $\langle R\nabla_{R\eta}\eta, J_kW\eta\rangle = \langle J_1V_\xi\eta, J_kW\eta\rangle = (J_1V_\xi)^*(J_kW)\,.$

From the previous discussion, we deduce:
$$\frac{1}{2}d\omega_k(X,Y,Z) = \sum_{l=1}^{3}\varepsilon_l(J_l\Theta_l)(J_kW) + \varepsilon_k\Theta_k(W) = \sum_{l\neq k}\varepsilon_lJ_l\Theta_l(J_kW)\,.$$

The previous equality implies that $\star_\xi(\tau_1^k\wedge\omega_k) = 2\sum_{l\neq k}\varepsilon_lJ_k(J_l\Theta_l)$, because the frame (X,Y,Z,W) of ξ is positively oriented. Taking into account Lemma 2.23, we obtain $\tau_1^k = -2\sum_{l\neq k}\varepsilon_lJ_l\Theta_l$.

Consider orhonormal vectors $X, Y\in\xi$; we now compute $i(R)d\omega$ by using equation (2.2). To arrange the second and the third summands of equation (2.2), we observe that if $Z\in\xi$, then:
$$\alpha Z\mathrm{j}_k\eta = \alpha\varepsilon_kJ_k(Z)\eta = \varepsilon_kJ_k(Z)\mathrm{j}_1\eta = \varepsilon_k(J_1(J_k(Z)))\eta.$$

Thus,
$$\frac{1}{2}d\omega_1(R,X,Y) = \varepsilon_k\langle\nabla_R\eta, XY\mathrm{j}_k\eta\rangle - \langle S(X)\eta, J_1(J_k(Y))\eta\rangle + \langle S(Y)\eta, J_1(J_k(X))\eta\rangle.$$

We first deal with the summand $\varepsilon_k\langle\nabla_R\eta, XY\mathrm{j}_k\eta\rangle$. According to equation (2.1) we have: $\langle\nabla_R\eta, XY\mathrm{j}_k\eta\rangle = \langle V_\xi\eta, XY\mathrm{j}_k\eta\rangle + \sum_{l=1}^{3}\phi_l\langle\mathrm{j}_l\eta, XY\mathrm{j}_k\eta\rangle$. Due to Remark 2.30, $\langle V_\xi\eta, XY\mathrm{j}_k\eta\rangle = \langle -J_k(V_\xi)\eta, XY\eta\rangle = 0$. We now observe that $\langle\mathrm{j}_l\eta, XY\mathrm{j}_k\eta\rangle = \varepsilon_k\varepsilon_l\langle J_k(J_l(X))\eta, Y\eta\rangle$ and we compute:
$$\begin{aligned}\varepsilon_1\langle\nabla_R\eta, XY\mathrm{j}_1\eta\rangle &= \phi_3\omega_2 - \phi_2\omega_3\,,\\ \varepsilon_2\langle\nabla_R\eta, XY\mathrm{j}_2\eta\rangle &= -\phi_3\omega_1 - \phi_1\omega_3\,,\\ \varepsilon_3\langle\nabla_R\eta, XY\mathrm{j}_3\eta\rangle &= +\phi_2\omega_1 + \phi_1\omega_2\,.\end{aligned}$$

We now deal the summand $T^k(X,Y) = -\langle S(X)\eta, J_1(J_k(Y))\eta\rangle + \langle S(Y)\eta, J_1(J_k(X))\eta\rangle$. From Definition 2.33, one can check:
$$\begin{aligned}T^1(X,Y) =& 2\langle S_0(X)\eta, Y\eta\rangle + 2\sum_{k=1}^{3}\lambda_k\langle J_k(X)\eta, Y\eta\rangle\\ =& 2(i(S_0)g + \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3)(X,Y).\end{aligned}$$

In addition, $T^2(X,Y) = \omega_3(S(X),Y) - \omega_3(S(Y),X)$ and $T^3(X,Y) = -(\omega_2(S(X),Y) - \omega_3(S(Y),X))$. Taking into account Lemma 2.36 we obtain:

$$T^2(X,Y) = 2(-\lambda_2\omega_1 + \lambda_1\omega_2 + \mu\omega_3 + i(S_3)\omega_3)(X,Y)\,,$$
$$T^3(X,Y) = 2(-\lambda_3\omega_1 - \mu\omega_2 + \lambda_1\omega_3 - i(S_2)\omega_2)(X,Y)\,.$$

In sum, $i(R)d\omega_1 = 4i(S_0)g + 4\lambda_1\omega_1 + (4\lambda_2 + 2\phi_3)\omega_2 + (4\lambda_3 - 2\phi_2\omega_3)$. Thus, $\tau_0^{kk} = 4\lambda_1$, $\tau_0^{12} = 4\lambda_2 + 2\phi_3$, $\tau_0^{13} = 4\lambda_3 - 2\phi_2$ and $\tau_2^0 = 4i(S_0)g$. The remaining equalities are obtained similarly. □

Our previous results allow us to write the equations for SU(2) structures induced by a harmonic spinor. We equate $\not{D}\eta = 0$ in Proposition 2.34, and use the values of $d\alpha$ and $d\omega_k$ computed in Propositions 2.37 and 2.38. Rewriting with the notations of Proposition 2.25, we obtain:

Corollary 2.39. *The spinor η is harmonic if and only if* SU(2) *structure determined by η, $(\alpha,\omega_1,\omega_2,\omega_3)$, satisfies:*

$$d\alpha = \tau_0^{23}\omega_1 + \tau_0^{13}\omega_2 - \tau_0^{12}\omega_3 + \frac{1}{2}\sum_{k=1}^{3}(\alpha\wedge\tau_1^k) + \tau_2^4\,,$$
$$d\omega_1 = +\tau_0^{12}\alpha\wedge\omega_2 + \tau_0^{13}\alpha\wedge\omega_3 + \tau_1^1\wedge\omega_1 + \alpha\wedge\tau_2^1\,,$$
$$d\omega_2 = -\tau_0^{12}\alpha\wedge\omega_1 + \tau_0^{23}\alpha\wedge\omega_3 + \tau_1^2\wedge\omega_2 + \alpha\wedge\tau_2^2\,,$$
$$d\omega_3 = -\tau_0^{13}\alpha\wedge\omega_1 - \tau_0^{23}\alpha\wedge\omega_2 + \tau_1^3\wedge\omega_3 + \alpha\wedge\tau_2^3\,.$$

Proof. We equate $\not{D}\eta = 0$ in Proposition 2.34, and we obtain $4\mu = \phi_1$, $\lambda_1 = 0$, $4\lambda_2 = -\phi_3$, $4\lambda_3 = \phi_2$, and $-J_1(V_1^*) = \sum_{k=1}^3 \varepsilon_k J_k(\Theta_k)$. According to Propositions 2.37 and 2.38, the 0-forms are related as follows:

$$\tau_0^{kk} = 4\lambda_1 = 0,$$
$$\tau_0^{12} = 4\lambda_2 + 2\phi_3 = -4\lambda_2 = -\tau_0^3,$$
$$\tau_0^{13} = 4\lambda_3 - 2\phi_2 = -4\lambda_3 = \tau_0^2,$$
$$\tau_0^{23} = 4\mu - 2\phi_1 = -4\mu = \tau_0^1.$$

In addition, $\tau_1^4 = 2J_1(V_\xi^*) = -2\sum_{k=1}^3 \varepsilon_k J_k(\Theta_k) = \frac{1}{2}\sum_{k=1}^3 \tau_1^k$. □

In [35, Definition 1.5], the authors defined *hypo* SU(2) structures as those satisfying

$$d\omega_1 = 0 \quad \text{and} \quad d(\alpha\wedge\omega_k) = 0, \quad k = 2,3\,.$$

The intersection between hypo and spin-harmonic stuctures is characterized by the equations:

- $d\alpha = -\tau_0^{23}\omega_1 + \tau_2^4$;
- $d\omega_1 = 0$;
- $d\omega_2 = +\tau_0^{23}\alpha\wedge\omega_3 + \alpha\wedge\tau_2^2$;
- $d\omega_3 = -\tau_0^{23}\alpha\wedge\omega_2 + \alpha\wedge\tau_2^3$.

In section 2.6 we present three nilmanifolds that admit SU(2) invariant structures in this intersection.

2.5 Dirac operator of invariant spinors on Lie groups

2.5.1 Spin structures on Lie groups

Let (G, g) be an n-dimensional connected, simply connected Lie group endowed with a left-invariant metric. Fix an orthonormal left-invariant frame $(e_1, \dots, e_n)$; the frame bundle of G is $\mathrm{P}_{\mathrm{SO}}(\mathrm{G}) = \mathrm{G} \times \mathrm{SO}(n)$ and its unique spin structure is $\mathrm{P}_{\mathrm{Spin}}(\mathrm{G}) = \mathrm{G} \times \mathrm{Spin}(n)$. Fix also an irreducible representation $\rho\colon \mathrm{Cl}_n \to \mathrm{End}_{\mathbf{k}}(W)$. The spinor bundle of G is $\Sigma(\mathrm{G}) = \mathrm{G} \times W$ and the Clifford multiplication by a vector field $X(x) = \sum_{i=1}^n X^i(x) e_i(x)$ is given by $X(x)\phi(x) = \sum_{i=1}^n X^i(x)\rho(e_i)\phi(x)$ where $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$. Each spinor is identified with a map $\phi\colon \mathrm{G} \to W$ and we call the spinor ϕ *left-invariant* if it is constant.

Let Γ be a discrete subgroup of G and $\pi\colon \mathrm{G} \to \Gamma\backslash\mathrm{G}$ be the canonical projection. We endow $\Gamma\backslash\mathrm{G}$ with the metric, that we also denote by g, which pulls back to g under π.

Lemma 2.40. *There is a bijective correspondence between homomorphisms $\varepsilon\colon \Gamma \to \{\pm 1\}$ and spin structures on $\Gamma\backslash\mathrm{G}$:*

$$\varepsilon \longmapsto \mathrm{P}_{\mathrm{Spin}}(\Gamma\backslash\mathrm{G})^{\varepsilon} = \Gamma\backslash(\mathrm{G} \times \mathrm{Spin}(n)) \,,$$

where the action is $y \cdot (x, \tilde{h}) = (yx, \varepsilon(y)\tilde{h})$, for $y \in \Gamma$.

Proof. Spin structures on $\Gamma\backslash\mathrm{G}$ are in a bijective correspondence with liftings of the action $\Gamma \times \mathrm{P}_{\mathrm{SO}}(\mathrm{G}) \to \mathrm{P}_{\mathrm{SO}}(\mathrm{G}), y \cdot F_x = d(\mathrm{L}_y)_x(F_x)$ where L_y denotes the left multiplication by y (see [54, page 43]). This action commutes with the action of $\mathrm{SO}(n)$ on $\mathrm{P}_{\mathrm{SO}}(\mathrm{G})$ and therefore a lifting of this action commutes with the action of $\mathrm{Spin}(n)$ on $\mathrm{P}_{\mathrm{Spin}}(\mathrm{G})$.

According to the identification $\mathrm{P}_{\mathrm{SO}}(\mathrm{G}) = \mathrm{G} \times \mathrm{SO}(n)$ given by $(e_1, \dots, e_n)$, the action is $y \cdot (x, h) = (yx, h)$. A lifting of the action to $\mathrm{P}_{\mathrm{Spin}}(\mathrm{G}) = \mathrm{G} \times \mathrm{Spin}(n)$ should satisfy $y \cdot (x, 1) = (yx, \varepsilon(y)1)$ for some map $\varepsilon\colon \Gamma \to \{\pm 1\}$, which is necessarily a homomorphism. The previous discussion shows that this property determines the action. □

The spinor bundle associated to $\mathrm{P}_{\mathrm{Spin}}(\Gamma\backslash\mathrm{G})^{\varepsilon}$ is $\Sigma(\Gamma\backslash\mathrm{G})^{\varepsilon} = \mathrm{P}_{\mathrm{Spin}}(\Gamma\backslash\mathrm{G})^{\varepsilon} \times_{\rho} W$, which is isomorphic to $\Gamma\backslash(\mathrm{G} \times W)$ via the induced action $y \cdot (x, v) = (yx, \varepsilon(y)v)$. Spinors are then identified with maps $\phi\colon \mathrm{G} \to W$ such that $\phi(yx) = \varepsilon(y)\phi(x)$ for $x \in \mathrm{G}$, $y \in \Gamma$, and Clifford multiplication of a spinor $\phi\colon \mathrm{G} \to W$ with a vector field $X \in \mathfrak{X}(\Gamma\backslash\mathrm{G})$ such that $X(\pi(x)) = \sum_{i=1}^n X^i(x) d\pi_x(e_i(x))$ is determined by $X\phi(x) = \sum_{i=1}^n X^i(x)\rho(e_i)\phi(x)$. Moreover, a spinor $\phi \in \Sigma(\Gamma\backslash\mathrm{G})^{\varepsilon}$ lifts to a unique spinor $\bar{\phi} \in \Sigma(\mathrm{G})$ and both are identified with the same map $\mathrm{G} \to W$. Using this identification, for a left-invariant vector field $X \in \mathfrak{X}(\mathrm{G})$ we have $\nabla_{d\pi_x(X)}\phi(x) = \nabla_X \bar{\phi}(x)$ and, according to [54, page 60],

$$\nabla_X \bar{\phi} = d_X \bar{\phi} + \frac{1}{2} \sum_{j<k} g(\nabla_X e_j, e_k) e_j e_k \bar{\phi}. \tag{2.3}$$

In the sequel we focus on quotients $\Gamma\backslash\mathrm{G}$ and on spinors that lift to left-invariant spinors on G; we call those *left-invariant spinors*. Of course, they are associated to the trivial spin structure and they are constant. Special examples are given by *nilmanifolds*, where G is nilpotent, and *solvmanifolds*, where G is solvable.

In particular, we restrict our attention to left-invariant *harmonic* spinors. Mind that the non existence of left-invariant harmonic spinors does not imply the non existence of harmonic spinors associated to the trivial spin structure. For instance, from Proposition 2.41 one can deduce that a 3-dimensional nilmanifold, quotient of the Heisenberg group, does not admit left-invariant harmonic spinors; however, Corollary 3.2 in [3] implies that every spin structure on such a nilmanifold admits a left-invariant metric with non-zero harmonic spinors.

2.5.2 Dirac operator

Let (G, g) be a Lie group endowed with a left-invariant metric, let $(e_1, \dots, e_n)$ be a left-invariant orthonormal frame with dual coframe $(e^1, \dots, e^n)$. Let Γ be a discrete subgroup of G and consider the spin structure associated to the trivial action on $\Gamma\backslash\mathrm{G}$. We follow the notation of the previous subsection.

Proposition 2.41. *Let ϕ be a left-invariant spinor. Then*

$$4\not{D}\phi = -\sum_{i=1}^{n} (e^i \wedge de^i + i(e_i)de^i)\phi\,. \tag{2.4}$$

Proof. First we compute the covariant derivative of ϕ according to formula (2.3). Observe that $d_{e_i}\phi = 0$ because ϕ is left-invariant. The Koszul formula allows us to obtain

$$2\overline{\nabla}_{e_i} e_j = (i(e_i)de^j + i(e_j)de^i)^\sharp - \sum_k de^k(e_i, e_j)e_k\,,$$

where $\overline{\nabla}$ is the Levi-Civita connection and ∇ is the spinor connection. Therefore,

$$\begin{aligned}\nabla_{e_i}\phi &= \frac{1}{4}\left(\sum_{j<k}\Big(de^j(e_i, e_k) + de^i(e_j, e_k) - de^k(e_i, e_j)\Big)e_je_k\right)\phi\\ &= \frac{1}{4}\left(de^i\phi - 2\sum_{j,k} de^k(e_i, e_j)e_je_k\phi + 2\sum_k de^k(e_k, e_i)\right)\phi\,.\end{aligned}$$

From this we deduce:

$$\begin{aligned}4\not{D}\phi &= \sum_{i=1}^{n} e^i de^i\phi - 2\sum_{i<j,k} de^k(e_i, e_j)e_ie_je_k\phi + 2\sum_{i,k} de^k(e_k, e_i)e_i\phi\\ &= \sum_{i=1}^{n}(e^i de^i - 2de^ie_i + 2i(e_i)de^i)\phi = -\sum_{i=1}^{n}(e^i \wedge de^i + i(e_i)de^i)\phi\,,\end{aligned}$$

where we have used that $e^ide^i\phi = (e^i\wedge de^i - i(e_i)de^i)\phi$ and $(de^i)e^i\phi = (e^i\wedge de^i + i(e_i)de^i)\phi$. ☐

Since our focus is on nilmanifolds and solvmanifolds, we specialize Proposition 2.41 to this setting. Recall that a frame $(e_1, \dots, e_n)$ of a nilpotent Lie group is called *nilpotent* if

$$[e_i, e_j] = \sum_{k>i,j} c_{ij}^k e_k\,.$$

Corollary 2.42. *Let G be a nilpotent Lie group and let $(e_1, \dots, e_n)$ be an orthonormal nilpotent left-invariant frame. Let $\phi\colon \mathrm{G} \to W$ be a left-invariant spinor; then*

$$4\not{D}\phi = -\sum_{i=1}^{n}(e^i \wedge de^i)\phi\,. \tag{2.5}$$

In particular, the operator $\not{D}$ is $\langle\cdot,\cdot\rangle$-symmetric on the space of invariant spinors.

Next, suppose that $\mathfrak{g}$ is a rank-1 extension of a nilpotent Lie algebra $\mathfrak{n}$, and let G and N be the associated simply connected Lie groups. As vector spaces $\mathfrak{g} = \langle e_0\rangle \oplus \mathfrak{n}$; the Lie bracket in $\mathfrak{g}$ is given by

$$[e_0, X]_{\mathfrak{g}} = \mathcal{D}(X), \quad [X, Y]_{\mathfrak{g}} = [X, Y]_{\mathfrak{n}} \quad \text{for } X, Y \in \mathfrak{n}\,,$$

where $\mathcal{D}\colon \mathfrak{n} \to \mathfrak{n}$ is a derivation. In terms of covectors, $\mathcal{D}$ can be seen as a linear map $\mathfrak{n}^* \to \mathfrak{n}^*$ such that $d_\mathfrak{n} \circ \mathcal{D} = \mathcal{D} \circ d_\mathfrak{n}$, where $d_\mathfrak{n}\colon \Lambda^k\mathfrak{n}^* \to \Lambda^{k+1}\mathfrak{n}^*$ is the Chevalley-Eilenberg differential. Extending $\alpha \in \Lambda^k\mathfrak{n}^*$ by zero to $\langle e_0\rangle$, one has

$$d_\mathfrak{g}\alpha = d_\mathfrak{n}\alpha + (-1)^{k+1}\mathcal{D}(\alpha) \wedge e^0\,, \tag{2.6}$$

where $d_\mathfrak{g}\colon \Lambda^k\mathfrak{g}^* \to \Lambda^{k+1}\mathfrak{g}^*$ is the Chevalley-Eilenberg differential. We also suppose that G is endowed with an invariant metric which makes e^0 orthogonal to $\mathfrak{n}^*$.

Corollary 2.43. *Suppose that $(e_1,\ldots,e_n)$ is an orthonormal left-invariant frame of* N *and let $\phi\colon \mathrm{G} \to W$ be a left-invariant spinor. Then*

$$4\not{D}\phi = -\sum_{i=1}^{n}(e^i \wedge d_\mathfrak{n}e^i + i(e_i)d_\mathfrak{n}e^i + e^0 \wedge e^i \wedge \mathcal{D}(e^i))\phi - \mathrm{tr}(\mathcal{D})e^0\phi\,. \tag{2.7}$$

In particular if $\mathcal{D}$ is symmetric and $(e_1,\ldots,e_n)$ is a basis of eigenvectors then $4\not{D}\phi = -\sum_{i=1}^n (e^i \wedge d_\mathfrak{n}e^i) + i(e_i)d_\mathfrak{n}e^i\phi - \mathrm{tr}(\mathcal{D})e^0\phi$.

Proof. The formula is deduced from Proposition 2.41 and (2.6). In addition, if $\mathcal{D}$ is symmetric and $(e^1,\ldots,e^n)$ is a basis of eigenvectors of $\mathcal{D}$, then $e^i \wedge \mathcal{D}(e^i) = 0$. □

2.5.3 The operator $\not{D}^2$ on nilmanifolds

The square of the Dirac operator is elliptic and has positive eigenvalues. In this subsection we fix the trivial spin structure on a nilmanifold $\Gamma\backslash\mathrm{G}$ associated to the trivial action and obtain a formula for the square of the Dirac operator over the space of left-invariant spinors. This allows us to understand the eigenvalues of the 5-dimensional Dirac operator in Section 2.6. A straightforward computation gives the following result:

Lemma 2.44. *Suppose that $(e_1,\ldots,e_n)$ is an orthonormal nilpotent left-invariant frame of* G *and $\phi\colon \mathrm{G} \to W$ a left-invariant spinor, then:*

$$16\not{D}^2\phi = \left(\sum_i -(de^i)^2 + \sum_{i<j}(e^{ij}de^ide^j - de^jde^ie^{ij})\right)\phi\,. \tag{2.8}$$

We discuss each summand of (2.8). We use the juxtaposition of indices to denote Clifford products, for instance $e_{ij} = e_ie_j$. Moreover, each $\beta = \sum_{i_1<\cdots<i_k}\beta_{i_1,\ldots,i_k}e^{i_1\ldots i_k} \in \Lambda^k\mathfrak{g}^*$ is identified with the element $\sum_{i_1<\cdots<i_k}\beta_{i_1,\ldots,i_k}e_{i_1\ldots i_k}$ of the Clifford algebra. This identification does not depend on the orthonormal basis chosen. We also set

$$\gamma_{ij} = e^{ij}de^ide^j - de^jde^ie^{ij}\,.$$

Lemma 2.45. *Consider $\omega \in \Lambda^2\mathfrak{g}^*$; in terms of the previous identifications,*

$$\omega\cdot\omega = -\|\omega\|^2 + \omega\wedge\omega\,.$$

Proof. Let $(e_1,\ldots,e_n)$ be an orthonormal basis and write $\omega = \sum_{i<j}\omega_{ij}e_{ij}$. If i,j,k,l are distinct indices, then it is easy to obtain that $e_{ij}e_{ik} + e_{ik}e_{ij} = 0$ and that $e_{ijkl} + e_{klij} = 2e_{ijkl}$. A combination of these properties leads to the equality:

$$\left(\sum_{i<j}\omega_{ij}e_{ij}\right)^2 = -\sum_{i<j}\omega_{ij}^2 + 2\sum_{i<j<k<l}(\omega_{ij}\omega_{kl} + \omega_{il}\omega_{jk} - \omega_{ik}\omega_{jl})e_{ijkl}\,,$$

which proves the lemma. □

Remark 2.46. The operator $e_{ijkl}\cdot$ satisfies $(e_{ijkl}\cdot)^2 = \mathrm{I}$ and it is not an homotethy. Let $\Delta_\pm$ be the eigenspace associated to ± 1 and consider $\phi_\pm \in \Delta_\pm$. Then,

$$(\omega_{ij}e^{ij} + \omega_{kl}e^{kl})^2\phi_\pm = -(\omega_{ij} \mp \omega_{kl})^2\phi_\pm\,.$$

The endomorphism $(\omega_{ij}e^{ij} + \omega_{kl}e^{kl})$ is invertible except when $\omega_{ij} = \pm\omega_{kl}$; in this case the kernel is $\Delta_\pm$.

Lemma 2.47. *Let* $(e_1, \dots, e_n)$ *be an orthonormal nilpotent left-invariant frame of* G *and* $i < j$. *Then*

$$\gamma_{ij} = -2\, de^i \wedge i(e_i)de^j \wedge e^j + 2\sum_{k<i} i(e_k)de^i \wedge i(e_k)(de^j|_{\langle e_i\rangle^\perp}) \wedge e^{ij}\,.$$

Proof. We denote $\alpha = i(e_i)de^j \in \mathfrak{g}^*$ and $\beta = de^j|^{\perp}_{\langle e_i\rangle} \in \Lambda^2\langle e^i\rangle^\perp$, that is, $de^j = e^i \wedge \alpha + \beta$. Observe that $e_{ij}de^i de^j = e_{ij}de^i(e^i\wedge\alpha+\beta) = de^i(-e^i\wedge\alpha+\beta)e_{ij}$ and that $e_i\beta = \beta e_i$. Therefore,

$$\gamma_{ij} = \left(de^i(-e^i \wedge \alpha + \beta) - (e^i \wedge \alpha + \beta)de^i\right) e_{ij} = -(de^i\alpha + \alpha de^i)e_j + (de^i\beta - \beta de^i)e_{ij}.$$

We now identify the terms in the summand. On the one hand, if we write $de^i = \alpha \wedge \alpha' + \beta'$ where $\alpha' = i(\alpha^\sharp)de^i$ and $\beta' = de^i|_{\langle\alpha^\sharp\rangle^\perp}$, we obtain:

$$(de^i\alpha + \alpha de^i)e_j = 2(\beta'\alpha)e^j = 2de^i \wedge \alpha \wedge e^j\,.$$

On the other hand, it is sufficient to prove $(de^i\beta - \beta de^i) = 2\sum_{k<i} i(e_k)de^i \wedge i(e_k)\beta$ in the case that $de^i = e^{pq}$ and $\beta = e^{lm}$ with $l < m$ and $p < q$. We distinguish two cases:

1. If $(p, q) = (l, m)$ or $p, q \notin \{l, m\}$, then $e^{pq}e^{lm} - e^{lm}e^{pq} = 0$. In addition, we have $\sum_{k=1}^{j-1} i(e_k)e^{pq} \wedge i(e_k)e^{lm} = 0$.
2. In other case; for instance if $p = l$ and $q \neq m$, then $e^{pq}e^{pm} - e^{pm}e^{pq} = 2e^{qm}$ and $2\sum_{k=1}^{j-1} i(e_k)e^{pq} \wedge i(e_k)e^{pm} = 2e^{qm}$. The other instances are similar.

□

From this we obtain:

Corollary 2.48. *Let* $(e_1, \dots, e_n)$ *be a nilpotent orthonormal left-invariant frame of* G *and let* ϕ *be a left-invariant spinor; then,*

$$\begin{aligned}16\not{D}^2\phi = &\sum_{i=1}^{n} (\|de^i\|^2 - de^i \wedge de^i)\phi - 2\sum_{i<j} (de^i \wedge i(e_i)de^j \wedge e^j)\phi \\ &+ 2\sum_{k<i<j} i(e_k)de^i \wedge i(e_k)(de^j|_{\langle e_i\rangle^\perp}) \wedge e^{ij}\phi.\end{aligned}$$

2.6 Spin-harmonic structures on nilmanifolds

In order to determine left-invariant harmonic structures on nilmanifolds one has to compute the Dirac operator associated to each left-invariant metric and study its kernel. In dimension 4 and 5 we give a list of all left-invariant metrics and compute the eigenvalues of the Dirac operator by means of the metric using Corollary 2.48. We also give a list of 6-dimensional nilmanifolds that admit left-invariant harmonic structures and list one such metric on each algebra.

Note that the existence of left-invariant harmonic spinors on a nilmanifold $\Gamma\backslash\mathrm{G}$ depends on the Lie algebra $\mathfrak{g}$. For this reason, we sometimes write that the Lie algebra $\mathfrak{g}$ admits harmonic spinors.

For Lie algebras we use Salamon's notation: $(0, 0, 12, 13)$ denotes the 4-dimensional Lie algebra with basis (e_1, e_2, e_3, e_4) and dual basis (e^1, e^2, e^3, e^4), with differentials $de^1 = de^2 = 0$, $de^3 = e^{12}$ and $de^4 = e^{13}$. The list of nilmanifolds up to dimension 6 can be found in [13].

2.6.1 4-dimensional nilmanifolds

In terms of an orthonormal nilpotent basis, the list of non-abelian 4-dimensional metric nilpotent Lie algebras is:

		de^3	de^4
$L_3 \oplus A_1$	$(0,0,0,12)$	0	$\mu_{12}e^{12}$
L_4	$(0,0,12,13)$	$\mu_{12}e^{12}$	$e^1(\lambda_{12}e^2 + \mu_{13}e^3)$

Here μ_{ij} denote structure constants which are necessarily non-zero, while λ_{ij} may vanish.

Theorem 2.49. *4-dimensional non-abelian nilmanifolds have no left-invariant harmonic spinors.*

Proof. The Dirac operator on $L_3 \oplus A_1$ is $\not{D}\phi = \mu_{12}e^{124}\phi$, and the square of the Dirac operator on L_4 is $16\not{D}^2\phi = (\mu_{12}^2 + \mu_{13}^2 + \lambda_{12}^2)\phi$. Both are invertible. □

2.6.2 5-dimensional nilmanifolds

As in Section 2.4.2, we fix an irreducible representation of Cl_5, $\rho_5 \colon Cl_5 \to \mathrm{End}_{\mathbb{C}}(W)$, with complex structure $j_1 = \rho_5(\nu_5)$ and a quaternionic stucture j_2 that anticommutes with the Clifford product; define $j_3 = j_1 \circ j_2$. For instance, let ρ_6 be the representation of the real 6-dimensional Clifford algebra described on subsection 2.4.2 and define $\rho_5 = \rho_6 \circ i_5$, as in Proposition 2.1. Then, $j_1 = \rho_5(\nu_5)$ and $j_2 = \rho_6(e_6)$.

We first use Corollary 2.48 to obtain the eigenvalues of the Dirac operator. In the presence of a harmonic spinor η, we can relate the operator $16\not{D}^2$ with the 1-form α of the SU(2) structure defined by η.

Proposition 2.50. *Let $(e_1, \ldots, e_5)$ be an orthonormal nilpotent left-invariant frame of $\mathfrak{g}$ and let ϕ be a left-invariant spinor. Then $16\not{D}^2\phi = \mu\phi + v j_1\phi$ where $\mu = \sum \|de^i\|^2$ and*

$$v^\sharp = \star\,(de^5 \wedge de^5) + 2\star\left(\sum_{i=3}^{4} de^i \wedge i(e_i)de^5 \wedge e^5\right) - 2\star\left(\sum_{i=3}^{4}\sum_{k=1}^{3} i(e_k)de^i \wedge i(e_k)(de^5|_{\langle e_i\rangle^\perp}) \wedge e^{i5}\right).$$

In addition, $\mu \geq \|v\|$ and the restriction of the operator $4\not{D}$ to the space of invariant spinors has four complex eigenspaces, associated to $\pm(\mu \pm \|v\|)^{\frac{1}{2}}$. The endomorphism j_2 maps the eigenspace associated to $(\mu \pm \|v\|)^{\frac{1}{2}}$ to the eigenspace associated to $-(\mu \pm \|v\|)^{\frac{1}{2}}$. In particular, there exist left-invariant harmonic spinors if and only if $\mu = \|v\|$.

Proof. First observe that if $\gamma \in \Lambda^4\mathfrak{g}^*$, then $\gamma\phi = -(\star\gamma)j_1\phi$. This computation is straightforward for simple forms and is extended to $\Lambda^4\mathfrak{g}^*$ by linearity. Note also that the nilpotency property guarantees that $de^j \wedge de^j = 0$ for $j \leq 4$ and that $\gamma_{34} = 0$. Those remarks and Corollary 2.48 allow us to conclude the first statement. From this we get that the eigenvalues of $16\not{D}^2$ are $\mu \pm \|v\| \geq 0$ and the eigenvalues of $4\not{D}$ are therefore, $\pm(\mu \pm \|v\|)^{\frac{1}{2}}$. Finally, the equality $\nabla_X j_k\phi = j_k\nabla_X\phi$, implies $\not{D}j_k = \varepsilon_k j_k\not{D}$ which is sufficient to conclude the rest. □

Proposition 2.51. *Let $(\alpha, \omega_1, \omega_2, \omega_3)$ be the SU(2) structure determined by a left-invariant unit-length spinor η. Let $(e_1, \ldots, e_5)$ be an orthonormal nilpotent frame and consider μ and v defined as in Proposition 2.50. The spinor η is harmonic if and only if $\|v\| = \mu$ and $v = -\mu\alpha^\sharp$.*

Proof. Decompose $v = \lambda\alpha^\sharp + w$ according to the orthogonal decomposition $\langle\alpha^\sharp\rangle \oplus \xi$. By Corollary 2.48, $\not{D}^2\eta = \mu\eta + (\lambda\alpha^\sharp + w)\mathrm{j}_1\eta = (\mu+\lambda)\eta + w\mathrm{j}_1\eta$, using that $\alpha^\sharp\mathrm{j}_1\eta = \mathrm{j}_1\alpha^\sharp\eta = \mathrm{j}_1(-\mathrm{j}_1\eta) = \eta$, from Lemma 2.29(2). This implies, according to Lemma 2.27, that $w = 0$ and $\mu = -\lambda$. Thus, $v = -\mu\alpha^\sharp$. □

From these results we observe that on a nilpotent Lie algebra, the component of v on the subspace $\langle e^5\rangle$ depends on the non-degeneracy of de^5. Moreover, taking into account the structure equations of 5-dimensional nilpotent Lie algebras given in Lemma 2.52, one deduces that the component of v on $\langle e^4\rangle$ is always 0. In any case, the vector v is determined in Theorem 2.53.

The non-abelian nilpotent 5-dimensional Lie algebras are the following:

- $\mathrm{L}_3 \oplus \mathrm{A}_2$, $(0,0,0,0,12)$
- $\mathrm{L}_4 \oplus \mathrm{A}_1$, $(0,0,0,12,14)$
- $\mathrm{L}_{5,1}$, $(0,0,0,0,12+34)$
- $\mathrm{L}_{5,2}$, $(0,0,0,12,13)$
- $\mathrm{L}_{5,3}$, $(0,0,0,12,14+23)$
- $\mathrm{L}_{5,5}$, $(0,0,12,13,23)$
- $\mathrm{L}_{5,4}$, $(0,0,12,13,14)$
- $\mathrm{L}_{5,6}$, $(0,0,12,13,14+23)$

Lemma 2.52. *The following table contains a list of non-abelian* 5*-dimensional metric nilpotent Lie algebras in terms of an orthonormal nilpotent basis* $(e_1,\ldots,e_5)$ *with dual basis* $(e^1,\ldots,e^5)$*. Here* μ_{ij} *denote structure constants which are non-zero, while* λ_{ij} *or* $\lambda_{ij;k}$ *denote those which may be zero.*

	de^3	de^4	de^5
$\mathrm{L}_3 \oplus \mathrm{A}_2$	0	0	$\mu_{12}e^{12}$
$\mathrm{L}_4 \oplus \mathrm{A}_1$	0	$\mu_{12}e^{12}$	$e^1(\lambda_{12}e^2 + \lambda_{13}e^3 + \mu_{14}e^4)$
$\mathrm{L}_{5,1}$	0	0	$\mu_{12}e^{12} + \mu_{34}e^{34}$
$\mathrm{L}_{5,2}$	0	$\mu_{12}e^{12}$	$\mu_{13}e^{13}$
$\mathrm{L}_{5,3}$	0	$\mu_{12}e^{12}$	$e^1(\lambda_{12}e^2 + \lambda_{13}e^3 + \mu_{14}e^4) + \mu_{23}e^{23}$
$\mathrm{L}_{5,5}$	$\mu_{12}e^{12}$	$e^1(\lambda_{12;4}e^2 + \mu_{13}e^3)$	$\lambda_{12;5}e^{12} + \mu_{23}e^{23}$
$\mathrm{L}_{5,4}$	$\mu_{12}e^{12}$	$e^1(\lambda_{12;4}e^2 + \mu_{13}e^3)$	$e^1(\lambda_{12;5}e^2 + \lambda_{13}e^3 + \mu_{14}e^4)$
$\mathrm{L}_{5,6}$	$\mu_{12}e^{12}$	$e^1(\lambda_{12;4}e^2 + \mu_{13}e^3)$	$e^1(\lambda_{12;5}e^2 + \lambda_{13}e^3 + \mu_{14}e^4) + \mu_{23}e^{23}$

Theorem 2.53. *If a 5-dimensional nilmanifold* $\Gamma\backslash\mathrm{G}$ *admits left-invariant harmonic spinors, then* $\mathfrak{g} = \mathrm{L}_{5,j}$*,* $j = 1,2,3,4,6$.

Proof. Following the notation of Lemma 2.52, we compute μ and v defined as in Proposition 2.50. Obviously, μ is the sum of the squares of the parameters involved. In order to compute the vector v, we suppose that the nilpotent basis is positively oriented. This assumption does not depend on the existence of harmonic spinors. We summarize the result in the following table:

	v
$L_3 \oplus A_2$	0
$L_4 \oplus A_1$	$-2\mu_{12}\lambda_{13}e_1$
$L_{5,1}$	$2\mu_{12}\mu_{34}e_5$
$L_{5,2}$	$-2\mu_{12}\mu_{13}e_1$
$L_{5,3}$	$2(-\mu_{12}\lambda_{13}e_1 - \mu_{12}\mu_{23}e_2 + \mu_{14}\mu_{23}e_5)$
$L_{5,5}$	$2(\mu_{13}\lambda_{12;5}e_1 + \lambda_{12;4}\mu_{23}e_2 - \mu_{12}\mu_{13}e_3)$
$L_{5,4}$	$2(\mu_{12}\mu_{14} - \lambda_{12;4}\lambda_{13} + \mu_{13}\lambda_{12;5})e_1$
$L_{5,6}$	$2((\mu_{12}\mu_{14} - \lambda_{12;4}\lambda_{13} + \mu_{13}\lambda_{12;5})e_1 - \mu_{23}(\lambda_{12;4}e_2 + \mu_{13}e_3) + \mu_{14}\mu_{23}e_5)$

We now study, on each Lie algebra, the equation that determines the presence of left-invariant harmonic spinors: $\mu = \|v\|$.

$L_3 \oplus A_2$ and $L_4 \oplus A_1$ do not admit any left-invariant harmonic spinor because $\mu > \|v\|$. Left-invariant metrics admitting left-invariant harmonic spinors on $L_{5,1}$ are characterized by the equation $\mu_{12} = \pm\mu_{34}$. On the algebra $L_{5,2}$ are characterized by $\mu_{12} = \pm\mu_{13}$.

On the algebra $L_{5,3}$, the smallest eigenvalue of $16\not{D}^2$ is

$$\lambda_{12}^2 + \mu_{12}^2 + \lambda_{13}^2 + \mu_{14}^2 + \mu_{23}^2 - 2(\mu_{12}^2(\lambda_{13}^2 + \mu_{23}^2) + \mu_{14}^2\mu_{23}^2)^{\frac{1}{2}} \geq 0.$$

If the metric has harmonic spinors, necessarily $\lambda_{12} = 0$. In addition, the previous condition leads us to $\lambda_{13}^2 = \mu_{12}^2 - \mu_{13}^2 - \mu_{14}^2 \pm 2(\mu_{14}^2\mu_{23}^2 - \mu_{14}^2\mu_{12}^2)^{\frac{1}{2}}$, whose solutions are $\lambda_{13} = 0$, $\mu_{23}^2 > \mu_{12}^2$ and $\mu_{14}^2 = \mu_{23}^2 - \mu_{12}^2$.

On $L_{5,5}$ the smallest eigenvalue of $16\not{D}^2$ is,

$$\mu_{12}^2 + \lambda_{12;4}^2 + \mu_{13}^2 + \lambda_{12;5}^2 + \mu_{23}^2 - 2(\mu_{13}^2\mu_{23}^2 + \lambda_{12;4}^2\mu_{23}^2 + \lambda_{12;5}^2\mu_{13}^2)^{\frac{1}{2}} \geq 0.$$

Since this value is non-negative for every choice of the parameters, necessarily $\lambda_{12;4}^2 + \mu_{13}^2 + \lambda_{12;5}^2 + \mu_{23}^2 - 2(\mu_{13}^2\mu_{23}^2 + \lambda_{12;4}^2\mu_{23}^2 + \lambda_{12;5}^2\mu_{13}^2)^{\frac{1}{2}} \geq 0$. The smallest eigenvalue is therefore greater or equal to $\mu_{12}^2 > 0$. Consequently, the metric has no left-invariant harmonic spinors.

On $L_{5,4}$ the eigenvalues of $16\not{D}^2$ are:

$$(\mu_{12} \mp \mu_{14})^2 + (\lambda_{12;4} \pm \lambda_{13})^2 + (\mu_{13} \mp \lambda_{12;5})^2.$$

Metrics which admit left-invariant harmonic spinors are such that: $\mu_{12} = \pm\mu_{14}$, $\lambda_{12;4} = \mp\lambda_{13}$ and $\mu_{13} = \pm\lambda_{12;5}$.

Finally, a metric on $L_{5,6}$ has left-invariant harmonic spinors if and only if:

$$\begin{aligned}(\mu_{12}^2 + \lambda_{12;4}^2 + \mu_{13}^2 + \lambda_{12;5}^2 + \lambda_{13}^2 + \mu_{14}^2 + \mu_{23}^2)^2 = \\ = 4\left(\mu_{14}^2\mu_{23}^2 + (-\mu_{13}\lambda_{12;5} + \lambda_{13}\lambda_{12;4} - \mu_{12}\mu_{14})^2 + \lambda_{12;4}^2\mu_{23}^2 + \mu_{13}^2\mu_{23}^2\right).\end{aligned}$$

We now show that this equation has solutions. If we suppose that $\lambda_{12;4} = 0$ then the condition $\lambda_{13} = 0$ is necessary for the presence of harmonic spinors. Moreover, the previous equation leads us to: $\mu_{23}^2 = \mu_{13}^2 + \mu_{14}^2 - \lambda_{12;5}^2 + 2\mathbf{i}(\lambda_{12;5}\mu_{14} - \mu_{12}\mu_{13})$. Therefore, $\mu_{14}^2 > \mu_{12}^2$, $\lambda_{12;5}^2 = \frac{\mu_{13}^2\mu_{12}^2}{\mu_{14}^2}$ and $\mu_{23}^2 = \frac{1}{\mu_{14}^2}(\mu_{14}^2 - \mu_{12}^2)(\mu_{13}^2 + \mu_{14}^2)$. □

Lemma 2.52 is a list in which one fixes an orthonormal basis of $\mathbb{R}^5$ and varies the Lie bracket within an isomorphism class of Lie brackets. According to Lemma 2.51 and the proof of Theorem 2.53, the 1-forms α of two different spin-harmonic structures on a nilmanifold with universal covering $L_{5,1}$, $L_{5,2}$ or $L_{5,4}$ are proportional. We give an example of the forms that determine the structure on each case; we compute them using the representation we fixed at the beginning of the section. We also suppose that the basis $(e_1, e_2, e_3, e_4, e_5)$ is positively oriented.

On the algebra $L_{5,1}$, α is parallel to e^5, in particular, if $\mu_{12} = \pm\mu_{34}$ then $\alpha = \mp e^5$. Then α is contact because $d\alpha = \mu_{34}(\pm e^{12} + e^{34})$. Moreover, $\xi = \langle e_1, \dots, e_4\rangle$ and therefore, $d\omega_k = 0$ for $k = 1, 2, 3$.

If $\mu_{12} = -\mu_{34}$, then $\ker(\mathrm{j} + \alpha\cdot) = \ker(\mathrm{j} + e_5\cdot) = \langle\phi_1, \phi_2, \phi_3, \phi_4\rangle$. Consider $\eta = \phi_1$; then $\omega_1 = e^{12} + e^{34}$, $\omega_2 = e^{14} + e^{23}$ and $\omega_3 = e^{13} - e^{24}$. Thus, $d\alpha = \tau_2^4 \in \mathfrak{su}(2)$ with $\tau_2^4 = \mu_{12}(e^{12} - e^{34})$. The structure is hypo because $d\omega_1 = 0$ and $d(\alpha \wedge \omega_2) = d(\alpha \wedge \omega_3) = 0$. In the same manner, when $\mu_{12} = \mu_{34}$ we consider $\eta = \phi_5$ and obtain $\omega_1 = -e^{12} + e^{34}$, $\omega_2 = e^{14} - e^{23}$ and $\omega_3 = -e^{13} + e^{24}$. Again, $d\alpha = \tau_2^4 \in \mathfrak{su}(2)$ with $\tau_2^4 = \mu_{12}(e^{12} + e^{34})$.

On the algebras $L_{5,2}$ and $L_{5,4}$, α is parallel to e^1 and, consequently, $d\alpha = 0$. These algebras are quasi-abelian, that is, they have a codimension-1 abelian ideal, which is $\xi = \langle e_2, e_3, e_4, e_5\rangle$. In particular, taking into account the equations in terms of forms of harmonic structures, $d\omega_k = \alpha \wedge \tau_2^k$. Thus, $d(\omega_k \wedge \alpha) = 0$. If $\alpha = -e^1$ we choose $\eta = 2^{-\frac{1}{2}}(\phi_1 + \phi_5) \in \ker(\mathrm{j} - e_1)$. Therefore, $\omega_1 = -e^{25} + e^{34}$, $\omega_2 = e^{23} - e^{45}$ and $\omega_3 = e^{24} + e^{35}$. On the one hand, the nilpotency of the basis implies that $i(e_5)d\omega_1 = 0$. On the other, $i(-e_1)d\omega_1 = \tau_2^1$ which is 0 or non-degenerate on ξ. Thus, $d\omega_1 = 0$. The same argument holds for $d\omega_3$ on $\mathcal{N}_{5,5}$ because e^3 is closed. The structure is hypo and the torsions which may be non-zero are τ_2^2 and τ_2^3; we compute them:

1. On $L_{5,2}$ the condition $\alpha = -e^1$ implies $\mu_{12} = -\mu_{13}$. Then, $d\omega_2 = \mu_{13}(e^{125} + e^{134})$ so that the unique non-zero torsion is $\tau_2^2 = \mu_{13}(e^{25} + e^{34})$.

2. On $L_{5,4}$ the condition $\alpha = e^1$ implies $\mu_{12} = \mu_{14}$, $\lambda_{13} = -\lambda_{12;4}$ and $\mu_{13} = \lambda_{12;5}$. Then, $d\omega_1 = 0$, $d\omega_2 = e^1(\lambda_{13}(e^{25} + e^{34}) + \mu_{13}(e^{24} - e^{35}))$ and $d\omega_3 = \mu_{12}(e^{25} + e^{34})$.

On $L_{5,3}$ metrics with harmonic spinors satisfy $\lambda_{12} = \lambda_{13} = 0$ and $\mu_{14}^2 = \mu_{23}^2 - \mu_{12}^2 > 0$. Therefore, $v = 2(-\mu_{12}\mu_{23}e_2 \pm \mu_{23}(\mu_{23}^2 - \mu_{12}^2)^{\frac{1}{2}} e_5)$. Thus, $d\alpha$ is proportional to $\mu_{14}e^{14} + \mu_{23}e^{23}$ and harmonic invariant structures are contact.

Remark 2.54. Fernández' first example of a balanced Spin(7)-manifold was a nilmanifold $\Gamma\backslash\mathrm{G}$ with $\mathfrak{g} = L_{5,2} \oplus A_3$.

2.6.3 6-dimensional nilmanifolds

We fix the irreducible representation of Cl_6 described in Section 2.2.1 and denote by j the Clifford multiplication by the volume form, which anticommutes with the Clifford product with a vector. As in the 5-dimensional case we have the following:

Proposition 2.55. *Let $(e_1, \dots, e_6)$ be an orthonormal nilpotent left-invariant frame of* G *and let ϕ be a left-invariant spinor. Then $16\not{D}^2\phi = \mu\phi + \gamma j\phi$, where $\mu = \sum \|de^i\|^2$ and*

$$\gamma = \sum_{l=5}^{6} \star \left((de^l \wedge de^l) + \sum_{i=3}^{4} de^i \wedge i(e_i)de^l \wedge e^l \right)$$
$$- \sum_{l=5}^{6} \star \left(\sum_{i=3}^{4} \sum_{k=1}^{3} i(e_k)de^i \wedge i(e_k)(de^l|_{\langle e_i\rangle^\perp}) \wedge e^{il} \right)$$
$$+ \star(de^5 \wedge i(e_5)de^6 \wedge e^6) - \star\left(\sum_{k=1}^{4} i(e_k)de^5 \wedge i(e_k)(de^6|_{\langle e_5\rangle^\perp}) \right).$$

In addition, the restriction of the operator $\not{D}^2$ *over the space of left-invariant spinors has eight eigenspaces,* Δ_j*, associated to* $\pm\lambda_1, \pm\lambda_2, \pm\lambda_3, \pm\lambda_4$ *for some* $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ *and* j *restricts to a map,* $\mathrm{j}\colon \Delta_{\lambda_j} \to \Delta_{-\lambda_j}$.

Decomposable algebras

Except for $\mathrm{L}_3 \oplus \mathrm{L}_3 = (0,0,0,0,12,34)$, the structure constants of decomposable Lie algebras can easily be obtained by those in dimension 5, listed above. We proceed to obtain a metric classification of such Lie algebras, characterizing the structure equations in terms of an orthonormal basis.

Lemma 2.56. *The list of* 6*-dimensional decomposable metric nilpotent algebras is:*

	de^4	de^5	de^6
$\mathrm{L}_3 \oplus \mathrm{A}_3$	0	0	$\mu_{12}e^{12}$
$\mathrm{L}_3 \oplus \mathrm{L}_3$	0	$\mu_{12}e^{12} + \lambda_{13;5}e^{13}$	$e^3(\mu_{14}e^4 + \lambda_{13;6}e^1 + \lambda_{23}e^2)$
$\mathrm{L}_4 \oplus \mathrm{A}_2$	0	$\mu_{12}e^{12}$	$e^1(\lambda_{12}e^2 + \lambda_{13}e^3 + \mu_{15}e^5)$
$\mathrm{L}_{5,1} \oplus \mathrm{A}_1$	0	0	$\mu_{12}e^{12} + \mu_{34}e^{34}$
$\mathrm{L}_{5,2} \oplus \mathrm{A}_1$	0	$\mu_{12}e^{12}$	$\mu_{13}e^{13}$
$\mathrm{L}_{5,3} \oplus \mathrm{A}_1$	0	$\mu_{12}e^{12}$	$e^1(\lambda_{12}e^2 + \lambda_{13}e^3 + \lambda_{14}e^4 + \mu_{15}e^5) + \mu_{23}e^{23}$
$\mathrm{L}_{5,5} \oplus \mathrm{A}_1$	$\mu_{12}e^{12}$	$\lambda_{12;5}e^{12} + \lambda e^{23} + \mu_{14}e^{14}$	$\lambda_{12;6}e^{12} + \lambda e^{13} + \mu_{24}e^{24}$
$\mathrm{L}_{5,4} \oplus \mathrm{A}_1$	$\mu_{12}e^{12}$	$e^1(\lambda_{12;5}e^2 + \mu_{14}e^4 + \lambda_{13}e^3)$	$e^1(\lambda_{12;6}e^2 + \lambda_{13}e^3 + \lambda_{14}e^4 + \mu_{15}e^5)$
$\mathrm{L}_{5,6} \oplus \mathrm{A}_1$	$\mu_{12}e^{12}$	$e^1(\lambda_{12;5}e^2 + \mu_{14}(\lambda_{13;4}e^3 + e^4))$	$e^1(\lambda_{12;6}e^2 + \lambda_{13;6}e^3 + \lambda_{14}e^{14} + \mu_{15}e^5) + \mu_{24}e^2(\lambda_{13}e^3 + e^4)$

Proof. The equations for $\mathrm{L}_3 \oplus \mathrm{L}_3$ are obtained from a basis $(x^1, \dots, x^6)$ associated to the stucture equations $(0,0,0,0,12,34)$. First observe that we can suppose that x^i is orthogonal to x^{i+1} for $i \in \{1,3\}$ and that x^1 is orthogonal to x^3. The Gram-Schmidt process allows us to obtain an orthonormal basis $e^1 = \frac{x^1}{\|x^1\|}$, $e^3 = \frac{x^3}{\|x^3\|}$, $e^2 = \mu_{22}x^2 + \mu_{23}e^3$ and $e^4 = \mu_{44}x^4 + \lambda_{14}e^1 + \lambda_{24}e^2 + \lambda_{34}e^3$.

Finally take two orthogonal and unit-length forms $e^5, e^6 \in \ker(d)^{\perp}$ with $de^5 = x^{12}$.

The remaining algebras can be decomposed as $\mathrm{L}_5 \oplus \mathrm{A}_1$, where L_5 is a 5-dimensional nilpotent Lie algebra. Let d_5 be the corresponding differential. Let dt be a generator of A_1^* and observe that $\ker(d) = \ker(d_5) \oplus \langle dt \rangle$ and $d\colon d^{-1}(\Lambda^2 \ker(d)) \to \Lambda^2 \ker(d_5)$. Therefore, a unit-length 1-form $\alpha \in \ker(d)$ orthogonal to $\ker(d_5)$ satisfies $i(\alpha^\sharp)d\beta = 0$ for all $\beta \in d^{-1}(\Lambda^2 \ker d)$.

If the Lie algebra is 2-step, the decomposition $\mathrm{L}_5 \oplus \mathrm{A}_1$ is orthogonal and the equations follow from Lemma 2.52.

The equations for $\mathrm{L}_{5,6} \oplus \mathrm{A}_1$, $\mathrm{L}_{5,4} \oplus \mathrm{A}_1$ and $\mathrm{L}_{5,3} \oplus \mathrm{A}_1$ can be arranged using the Gram-Schmidt process, starting with an orthonormal basis $(e^1, \dots, e^k, \alpha)$ with $e^i \in \ker(d_5)$.

To obtain the equations for $\mathrm{L}_{5,5} \oplus A_1$ consider $F_1 = d^{-1}(\Lambda^2 \ker d) \cap \ker(d)^{\perp}$ and $F_2 = d^{-1}(\Lambda^2 F_1) \cap F_1^{\perp}$. Let π the plane generated in $(\mathrm{L}_{5,5} \oplus A_1)^*$ by dF_1 and observe that there is an isomorphism $\tilde{d}\colon F_2 \longmapsto \pi \otimes F_1$ obtainted from d and the projection of the space of closed forms to $\pi \otimes F_1$. Take $e^4 \in F_1$ unit-length and let $e^5, e^6 \in F_2$ and $e^1, e^2 \in \pi$ orthonormal such that $\tilde{d}e^5 = \mu_{14}e^{14}$ and $\tilde{d}e^6 = \mu_{24}e^{24}$. Define the map $\pi \to \pi$, $\beta \longmapsto \star p(d\tilde{d}^{-1}(\beta \otimes e^4))$, where $\star$ is the Hodge star and $p\colon \Lambda^2 \ker(d) \oplus (\pi \otimes F_1) \to \Lambda^2 \ker(d) \cap dF_1^{\perp}$ is the orthogonal projection. This map is diagonal with eigenvalue λ (see [13, pp. 1017-1018]), so that $de^5 = \lambda_{12;5}e^{12} + \lambda e^{23} + \mu_{14}e^{14}$ and $de^6 = \lambda_{12;6}e^{12} + \lambda e^{13} + \mu_{24}e^{24}$. □

We describe the set of metrics on $\mathrm{L}_3 \oplus \mathrm{L}_3$ with harmonic spinors.

Lemma 2.57. *Following the notation of Lemma 2.56, metrics with harmonic spinors on* $\mathrm{L}_3 \oplus \mathrm{L}_3$ *are those which satisfy one of the following conditions:*

1. $\lambda_{23} = 0$, $\lambda_{13;6} = \sigma_1\mu_{12}$ *and* $\lambda_{13;5} = \sigma_2\mu_{34}$*, for some* $\sigma_1, \sigma_2 \in \{\pm 1\}$.

2. $4\lambda_{23}^2(\lambda_{13}^2+\mu_{12}^2) = \mu_{12}^2+\lambda_{13;5}^2+\lambda_{13;6}^2+\lambda_{23}^2+\mu_{34}^2-4(\sigma\mu_{12}\lambda_{13;6}+\lambda_{13;5}\mu_{34})^2$ *for some* $\sigma \in \{\pm 1\}$.

Proof. Consider an orthonormal basis $(e^1,\dots,e^6)$ associated to the structure equations given in Lemma 2.56. Then, μ is the sum of the squares of the parameters involved. If we assume that the basis is positively oriented, then:

$$\gamma = -2(\mu_{12}\lambda_{13;6}e^{14}+\lambda_{13;5}\lambda_{23}e^{34}+\mu_{12}\lambda_{23}e^{24}-\lambda_{13;5}\mu_{34}e^{23}).$$

Observe that the operators $e^{14}\mathrm{j}$ and $e^{23}\mathrm{j}$ commute. Define the operator

$$A = -2(\lambda_{13;5}\lambda_{23}e^{34}+\mu_{12}\lambda_{23}e^{24})\mathrm{j}\cdot$$

and observe that it anticommutes with the previous operators and that $A^2 = 4\lambda_{23}^2(\lambda_{13;5}^2+\mu_{12}^2)\mathrm{I}$. We distinguish two cases:

- If $\lambda_{23}=0$ then $A=0$ and the eigenvalues of $\not{D}^2$ are $(\mu_{12}^2 \pm \lambda_{13;6})^2+(\lambda_{13;5}\pm\mu_{24})^2$. Therefore, the metric has harmonic spinors if $\lambda_{13;6}=\pm\mu_{12}\neq 0$ and $\lambda_{13;5}=\pm\mu_{34}\neq 0$.

- If $\lambda_{23}\neq 0$ then A is invertible. Denote $\mu = \mu_{12}^2+\lambda_{13;5}^2+\lambda_{13;6}^2+\lambda_{23}^2+\mu_{34}^2$. Let $\Delta_\pm$ be the eigenspaces associated to the eigenvalue ± 1 of $e^{14}\mathrm{j}$ and decompose $\Delta_\pm = \Delta_\pm^+ \oplus \Delta_\pm^-$ according to the eigenspaces of $e^{23}\mathrm{j}$. Note that $A(\Delta_\pm^+)=\Delta_\mp^-$ and that $A^2 = 4\lambda_{23}^2(\lambda_{13}^2+\mu_{12}^2)\mathrm{I}$. Thus, the eigenvalues are of the form $\phi_\pm^+ + \phi_\mp^-$ with $\phi_\pm^+\in\Delta_\pm^+$ and $\phi_\mp^-\in\Delta_\mp^-$. The eigenvalue 0 occurs on $\Delta_+^+\oplus\Delta_-^-$ if and only if:

  $$\begin{aligned} A\phi_+^+ &= (\mu-2\mu_{12}\lambda_{13;6}+2\lambda_{13;5}\mu_{34})\phi_-^-\,, \\ A\phi_-^- &= (\mu+2\mu_{12}\lambda_{13;6}-2\lambda_{13;5}\mu_{34})\phi_+^+\,. \end{aligned}$$

  This implies that $4\lambda_{23}^2(\lambda_{13}^2+\mu_{12}^2) = (\mu-2\mu_{12}\lambda_{13;6}+2\lambda_{13;5}\mu_{34})(\mu+2\mu_{12}\lambda_{13;6}-2\lambda_{13;5}\mu_{34})$. Moreover, if this equation holds we can take $\phi_+^+\in\Delta_\pm^+$, define $\phi_-^- = (\mu+2\mu_{12}\lambda_{13;6}-2\lambda_{13;5}\mu_{34})A^{-1}\phi_+^+$. Then,

  $$\begin{aligned} A\phi_+^+ &= (\mu+2\mu_{12}\lambda_{13;6}-2\lambda_{13;5}\mu_{34})^{-1}A^2\phi_-^- \\ &= (\mu-2\mu_{12}\lambda_{13;6}+2\lambda_{13;5}\mu_{34})\phi_-^-\,. \end{aligned}$$

  We can do a similar analysis on $\Delta_+^-\oplus\Delta_-^+$ to conclude that the metric has harmonic spinors if and only if

  $$\begin{aligned} &4\lambda_{23}^2(\lambda_{13}^2+\mu_{12}^2) \\ &\quad = \mu_{12}^2+\lambda_{13;5}^2+\lambda_{13;6}^2+\lambda_{23}^2+\mu_{34}^2-4(\sigma\mu_{12}\lambda_{13;6}+\lambda_{13;5}\mu_{34})^2\,, \end{aligned}$$

  for some $\sigma\in\{\pm1\}$. If $\mu_{12}=1$, this equation has solutions if and only if, $1+\lambda_{13;5}^2+\lambda_{13;6}^2+\mu_{34}^2-4(\lambda_{13;6}+\lambda_{13;5}\mu_{34})^2>0$. This inequality holds taking the parameters small enough.

□

The other decomposable cases can be obtained by taking into account the results of the previous sections. It is clear from Theorem 2.53 and Lemma 2.56 that the algebras $\mathrm{L}_3\oplus\mathrm{A}_3$ and $\mathrm{L}_4\oplus\mathrm{A}_2$ do not admit left-invariant harmonic spinors and that $\mathrm{L}_{5,j}\oplus\mathrm{A}_1$ has harmonic spinors for $j\neq 5$. Finally take an orthonormal basis $(e^1,\dots,e^6)$ associated to the structure equations of $\mathrm{L}_{5,5}\oplus\mathrm{A}_1$ given in Lemma 2.56 and suppose $\mu_{12}=1$. Now we write the Dirac operator using the formula obtained in Corollary 2.42 and then we use the fixed representation to obtain an endomorphism of the spinor bundle. The metric has left-invariant harmonic spinors if and only if the determinant of the endomorphism is 0. Solving the equation we get:

$$\lambda = \frac{1}{2}(1+(\mu_{14}+\mu_{24})^2)^{-\frac{1}{2}}((1+\lambda_{12;5}^2+\lambda_{12;6}^2+\mu_{14}^2-\mu_{24}^2)^2-4\lambda_{12;6}^2+4\mu_{24}^2)^{\frac{1}{2}}\,.$$

But the number on the square root is obviously positive if $\lambda_{12;6}=0$. Therefore, there are metrics with harmonic spinors.

We have proved:

Theorem 2.58. *Let* $\Gamma\backslash\mathrm{G}$ *be a non-abelian* 6*-dimensional nilmanifold with* $\mathfrak{g}$ *decomposable. Then, unless* $\mathfrak{g}$ *equals* $\mathrm{L}_3\oplus\mathrm{A}_3$ *or* $\mathrm{L}_4\oplus\mathrm{A}_2$*,* $\Gamma\backslash\mathrm{G}$ *admits an invariant metric with left-invariant harmonic spinors.*

Non-decomposable algebras

Using the fixed representation of Cl_6 we are able to find a metric with harmonic spinors on each nilmanifold associated to a non-decomposable Lie algebra. We follow the same procedure that we used to determine metrics with left-invariant harmonic spinors on $\mathrm{L}_{5,5}\oplus\mathrm{A}_1$. In many cases we are not able to determine the roots of the polynomial in terms of the parameters. Therefore, we make some choices as the following example explains:

We consider the algebra $\mathrm{L}_{6,7}$, which has structure equations $(0,0,0,12,13,15+24)$. We first declare the canonical basis orthonormal and compute the Dirac operator. One can show that this metric does not have left-invariant harmonic spinors. Neither does any metric constructed by declaring orthonormal a basis which is obtained by rescaling the canonical basis.

Now we proceed to write the structure equations by means of an orthonormal basis with respect to a metric. First, write $F_1=\ker(d)$, $F_2=d^{-1}(\Lambda^2F_1)$ and $F_3=d^{-1}(\Lambda^2F_2)=\mathrm{L}_{6,7}$. One can take an orthonormal basis of F_2 such that $de^4=\mu_{13}e^{12}$ and $de^5=\mu_{13}e^{13}$. Now take e^6 orthogonal to F_2, then according to [13], de^6 is a closed form of Λ^2F_2 such that $e^1\wedge(de^6)^2=0$, $e^1\wedge de^6\notin\Lambda^3F_1$ and $de^6\notin\ker(d)\otimes F_2$. Those equations imply:

$$\begin{aligned} de^6 &= \lambda_{12}e^{12}+\lambda_{13}e^{13}+\lambda_{14}e^{14}+\lambda_{15}e^{15}\\ &\quad+\lambda_{23}e^{23}+\lambda_{24}e^{24}+\lambda_{35}e^{35}+\left(\frac{\lambda_{24}\lambda_{35}}{\mu_{12}\mu_{13}}\right)^{\frac{1}{2}}(\mu_{13}e^{34}+\mu_{12}e^{25}),\end{aligned}$$

with $\lambda_{24}\lambda_{35}\geq 0$ and $-\lambda_{14}\left(\frac{\lambda_{24}\lambda_{35}}{\mu_{12}\mu_{13}}\right)^{\frac{1}{2}}\mu_{12}+\lambda_{15}\lambda_{24}\neq 0$. We choose $\lambda_{35}=0$ and therefore, $de^6=\lambda_{12}e^{12}+\lambda_{13}e^{13}+\lambda_{14}e^{14}+\lambda_{15}e^{15}+\lambda_{24}e^{24}$ with $\lambda_{15}\lambda_{24}\neq 0$. We fix $1=\mu_{13}=\mu_{12}=\lambda_{15}=\lambda_{24}$ and vary the rest of the parameters.

The choice $\lambda_{12}=1=\lambda_{23}$ leads to the condition that λ_{13} is a root of the polynomial $Z^8+8(\lambda_{14}^2+8)Z^6+(16\lambda_{14}^2+24\lambda_{14}^2+32)Z^4+32\lambda_{14}^3Z^3+4(\lambda_{14}^6+24\lambda_{14}^4+128\lambda_{14}^2)Z^2+(16\lambda_{14}^5+128\lambda_{14}^3)Z+\lambda_{14}^8+8\lambda_{14}^6+32\lambda_{14}^4$. Hence, $(\lambda_{13},\lambda_{14})=(0,0)$ is a solution.

We finish with a list of the non-decomposable metric nilpotent Lie algebras in dimension 6 which admit a harmonic spinor.

		de^3	de^4	de^5	de^6
$L_{6,1}$	$(0,0,0,0,12,13+24)$	0	0	e^{12}	$2e^{13}+e^{24}$
$L_{6,2}$	$(0,0,0,0,13-24,14+23)$	0	0	$e^{13}-e^{24}$	$e^{14}+e^{23}$
$L_{6,3}$	$(0,0,0,0,12,15+34)$	0	0	e^{12}	$e^{14}+e^{15}+e^{34}$
$L_{6,4}$	$(0,0,0,12,13,23)$	0	e^{12}	e^{13}	$2e^{23}$
$L_{6,5}$	$(0,0,0,12,13,14)$	0	e^{12}	$2^{\frac{1}{2}}e^{13}$	e^{14}
$L_{6,6}$	$(0,0,0,12,13,24)$	0	e^{12}	e^{13}	$2e^{13}+3^{\frac{1}{2}}e^{24}+e^{23}$
$L_{6,7}$	$(0,0,0,12,13,15+24)$	0	e^{12}	e^{13}	$e^{12}+e^{15}+e^{23}+e^{24}+e^{23}$
$L^{+}_{6,8}$	$(0,0,0,12,13,24+35)$	0	e^{12}	e^{13}	$e^{24}+e^{35}$
$L^{-}_{6,8}$	$(0,0,0,12,13,24-35)$	0	e^{12}	e^{13}	$-2e^{23}+e^{24}-e^{35}$
$L_{6,9}$	$(0,0,0,12,13,14+23)$	0	e^{12}	e^{13}	$e^{14}+e^{23}+(2(2^{\frac{1}{2}}-1))^{\frac{1}{2}}e^{12}$
$L_{6,10}$	$(0,0,0,12,14,23+24)$	0	e^{12}	e^{14}	$e^{23}+e^{24}$
$L_{6,11}$	$(0,0,0,12,14,13+24)$	0	e^{12}	e^{14}	$e^{13}+e^{24}$
$L_{6,12}$	$(0,0,0,12,14+23,13-24)$	0	$2^{-\frac{1}{2}}e^{12}$	$2^{\frac{1}{2}}e^{14}+e^{23}$	$e^{13}-2^{\frac{1}{2}}e^{24}$
$L_{6,13}$	$(0,0,0,12,14,15+23)$	0	e^{12}	e^{14}	$e^{15}+2^{\frac{1}{2}}e^{13}+e^{23}$
$L_{6,14}$	$(0,0,0,12,14,15+23+24)$	0	e^{12}	$e^{14}-\frac{7}{4}e^{13}$	$e^{15}+e^{24}-\frac{3}{4}e^{23}+2e^{12}$
$L_{6,15}$	$(0,0,0,12,14+23,15-34)$	0	e^{12}	$e^{14}+e^{23}$	$\frac{1}{4}(e^{15}+e^{34})$
$L_{6,16}$	$(0,0,12,13,23,14)$	e^{12}	e^{13}	e^{23}	e^{14}
$L^{+}_{6,17}$	$(0,0,12,13,23,14+25)$	e^{12}	e^{13}	e^{23}	$e^{14}+e^{24}+e^{12}+2^{\frac{1}{2}}e^{23}$
$L^{-}_{6,17}$	$(0,0,12,13,23,14-25)$	e^{12}	e^{13}	e^{23}	$e^{14}-e^{25}-e^{12}+2^{\frac{1}{2}}e^{23}$
$L_{6,18}$	$(0,0,12,13,14,15)$	e^{12}	e^{13}	$\frac{1}{5}(e^{14}+e^{12})$	$\frac{1}{5}(e^{12}+e^{14}+46^{\frac{1}{2}}e^{15})$
$L_{6,19}$	$(0,0,12,13,14,15+23)$	e^{12}	e^{13}	e^{14}	$e^{15}+e^{23}+e^{12}$
$L_{6,20}$	$(0,0,12,13,14,15-34)$	e^{12}	e^{13}	e^{14}	$e^{25}-e^{34}+5^{\frac{1}{2}}e^{12}$
$L_{6,21}$	$(0,0,12,13,14+23,15+24)$	e^{12}	e^{13}	$\frac{1}{m}(e^{14}+e^{23})$	$me^{15}+e^{24}$
$L_{6,22}$	$(0,0,12,13,14+23,15-34)$	e^{12}	e^{13}	$e^{14}+e^{23}$	$e^{25}-e^{34}+(1+5^{\frac{1}{2}})e^{12}$

where $m = \dfrac{\sqrt{3}\left((459+12\sqrt{177})^{\frac{1}{3}}((459+12*\sqrt{177})^{\frac{2}{3}}+6(459+12\sqrt{177})^{\frac{1}{3}}+57)\right)^{\frac{1}{2}}}{3(459+12\sqrt{177})^{\frac{1}{3}}}$.

2.6.4 8-dimensional nilmanifolds with balanced Spin(7) structures

The results collected so far allow us to obtain examples of invariant balanced Spin(7)-structures on nilmanifolds $N_k \times T^{8-k}$ with N_k a k-dimensional nilmanifold, $k = 5, 6$. By considering $N_k \times T^{7-k}$, one obtains a 7-dimensional nilmanifold with a spin-harmonic G_2-structure. If M is any 7-dimensional manifold endowed with a spin-harmonic G_2-structure, then $M \times S^1$ admits a balanced Spin(7)-structure. According to Theorem 2.16, every closed G_2-structure is spin-harmonic and a coclosed G_2-structure is spin-harmonic if and only if it is of pure type χ_3. Now 7-dimensional nilpotent Lie algebras with closed and coclosed G_2-structures are classified by Conti-Fernández [34] and Bagaglini [6] respectively. We show that not all our examples of balanced Spin(7) nilmanifolds can be obtained by Conti-Fernández and Bagaglini. To do this we compare decomposable 7-dimensional Lie algebras admitting closed, coclosed and spin-harmonic G_2-structures in the table below.

We have seen in Theorem 2.58 that $L_3 \oplus A_3$ and $L_4 \oplus A_2$ do not admit any metric with harmonic spinors; we show that the same happens when we add abelian factors of dimension 1 and 2 to these Lie algebras.

Proposition 2.59. *The Lie algebras $L_3 \oplus A_4$, $L_3 \oplus A_5$, $L_4 \oplus A_3$ and $L_4 \oplus A_4$ do not admit any metric with harmonic spinors.*

Proof. We prove the result for $L_3 \oplus A_4$ and $L_4 \oplus A_3$, the other cases being similar. Let us write the structure equations in term of a suitable orthonormal basis $(e^1, \ldots, e^7)$ of each Lie algebra.

1. For $L_3 \oplus A_4$ the structure equations are $de^i = 0$ for $i = 1, \ldots, 6$, and $de^7 = \mu e^{12}$, for some $\mu \neq 0$. One computes that $\not{D}\phi = \mu e^{127}\phi$, which has no kernel.

2. The structure equations of $L_4 \oplus A_3$ are $de^i = 0$ for $i = 1, \ldots, 5$

$$de^6 = \mu_{12}e^{12} \quad \text{and} \quad de^7 = \mu_{16}e^{16} + \lambda_{12}e^{12} + \lambda_{13}e^{13}\,.$$

   With this one computes $\not{D}\phi = e^1(\mu_{12}e^{26}+\mu_{16}e^{67}+\lambda_{12}e^{27}+\lambda_{13}e^{37})\phi$. Note that $\mu_{16}e^{167}\phi$ is orthogonal to $e^1(\mu_{12}e^{26} + \lambda_{12}e^{27} + \lambda_{13}e^{37})\phi$ hence, since $\mu_{16} \neq 0$, the kernel of the Dirac operator is trivial.

□

	closed	coclosed	spin-harmonic
$\mathrm{L}_3 \oplus \mathrm{A}_4$	×	✓	×
$\mathrm{L}_3 \oplus \mathrm{L}_3 \oplus A_1$	✓	✓	✓
$\mathrm{L}_4 \oplus \mathrm{A}_3$	×	×	×
$\mathrm{L}_{5,1} \oplus \mathrm{A}_2$	×	✓	✓
$\mathrm{L}_{5,2} \oplus \mathrm{A}_2$	✓	✓	✓
$\mathrm{L}_{5,3} \oplus \mathrm{A}_2$	×	✓	✓
$\mathrm{L}_{5,5} \oplus \mathrm{A}_2$	×	×	✓
$\mathrm{L}_{5,4} \oplus \mathrm{A}_2$	×	✓	✓
$\mathrm{L}_{5,6} \oplus \mathrm{A}_2$	×	✓	✓
$\mathrm{L}_{6,1} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}_{6,2} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}_{6,3} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}_{6,4} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}_{6,5} \oplus \mathrm{A}_1$	×	×	✓
$\mathrm{L}_{6,6} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}_{6,7} \oplus \mathrm{A}_1$	×	×	✓
$\mathrm{L}^+_{6,8} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}^-_{6,8} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}_{6,9} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}_{6,10} \oplus \mathrm{A}_1$	×	×	✓
$\mathrm{L}_{6,11} \oplus \mathrm{A}_1$	×	×	✓
$\mathrm{L}_{6,12} \oplus \mathrm{A}_1$	×	×	✓
$\mathrm{L}_{6,13} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}_{6,14} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}_{6,15} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}_{6,16} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}^+_{6,17} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}^-_{6,17} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}_{6,18} \oplus \mathrm{A}_1$	×	×	✓
$\mathrm{L}_{6,19} \oplus \mathrm{A}_1$	×	×	✓
$\mathrm{L}_{6,20} \oplus \mathrm{A}_1$	×	×	✓
$\mathrm{L}_{6,21} \oplus \mathrm{A}_1$	×	✓	✓
$\mathrm{L}_{6,22} \oplus \mathrm{A}_1$	×	×	✓

CHAPTER 3

Resolution of 4-dimensional symplectic orbifolds

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Abstract

We give a method to resolve 4-dimensional symplectic orbifolds making use of techniques from complex geometry and gluing of symplectic forms. We provide some examples to which the resolution method applies.

MSC classification [2010]: Primary 57R18, 57K43; Secondary 53C25, 53D35.

Key words: Symplectic orbifolds, Symplectic resolution

3.1 Introduction

An orbifold is a space which is locally modelled on balls of $\mathbb{R}^n$ quotiented by a finite group. These have been very useful in many geometrical contexts [109]. In the setting of symplectic geometry, symplectic orbifolds have been introduced mainly as a way to construct symplectic manifolds by resolving their singularities via symplectic blow-up. This method has served to construct many symplectic manifolds with interesting properties such as being simply-connected and non-Kahler and/or simply-connected and non-formal, e.g. see [11], [50], [103]. On the other hand, symplectic and Kähler orbifolds also have interest in their own right, for instance they play an important role in Sasakian and K-contact geometry. We refer to the book [20] for an extensive account on these subjects.

The problem of resolution of singularities and blow-up in the symplectic setting was posed by Gromov in [60]. A few years later, the symplectic blow-up was rigorously defined by McDuff [89] and it was used to construct a simply-connected symplectic manifold with no Kähler structure. The concept of symplectic blow-up was later generalized to the orbifold setting in [57].

McCarthy and Wolfson developed in [88] a symplectic resolution for isolated singularities of orbifolds in dimension 4. Later on, Cavalcanti, Fernández and Muñoz gave a method of performing symplectic resolution of *isolated* orbifold singularities in all dimensions [28]. This was used in [50] to give the first example of a simply-connected symplectic 8-manifold which is non-formal, as the resolution of a suitable symplectic 8-orbifold. This manifold was proved to have also a complex structure in [14].

Bazzoni, Fernández and Muñoz [11] gave the first construction of a symplectic resolution of an orbifold of dimension 6 with isotropy sets of dimension 0 and 2, although the construction is ad hoc for the particular example at hand as it satisfies that the normal bundle to the 2-dimensional isotropy set is trivial. This was used to give the first example of a simply-connected non-Kähler manifold which is simultaneously complex and symplectic.

Niederkrüger and Pasquotto provided methods for resolving different types of symplectic orbifold singularities in [96, 97]. The second deals with orbifolds arising as symplectic reductions of Hamiltonian circle actions; these singularities are cyclic and might not be isolated. In dimension 4, the previous work in [93] serves to resolve symplectic 4-orbifolds whose isotropy set consists of codimension 2 *disjoint* submanifolds. In such case the orbifold is topologically a manifold (the isotropy points are non-singular), so the question only amounts to change the orbifold symplectic form into a smooth symplectic form.

In a more general setting, one can try to develop a resolution method for orbifolds with any given geometric structure. This has yielded [71],[72],[75] remarkable results for constructing Riemannian manifolds with holonomy G_2, which have been extended for closed G_2-structures in [47] and [85].

In this paper we give an elementary and self-contained method to resolve arbitrary symplectic 4-orbifolds. For the symplectic part, we make use of techniques for gluing symplectic forms. These include the so called *inflation procedure* introduced by Thurston in [108], and the notion of *positivity* (or tameness) with respect to an almost complex structure, studied in detail in the book [90]. For the topological part (the resolution of quotient singularities), we mainly make use of complex local models from [28], and tools coming from invariant theory. There is however an essential difficulty when dealing with non-isolated isotropy points; this comes from the fact that the (local) resolution of the topologically-singular points must be made compatible with the resolution of the isotropy divisors (real codimension 2) of the orbifold. To overcome this difficulty, the desingularization of the isotropy divisors has to be made with care. The method in [94] starts with a manifold and constructs on it an orbifold atlas with isotropy along a configuration of divisors. This construction has to be reversed, but with an essential change: mainly, that the orbifold and the manifold structures along the divisors must be related through a *holomorphic* map.

The main result is:

Theorem 3.1. *Let (X, ω) be a compact symplectic 4-orbifold. There exists a symplectic manifold $(\tilde{X}, \tilde{\omega})$ and a smooth map $\pi : (\tilde{X}, \tilde{\omega}) \to (X, \omega)$ which is a symplectomorphism outside an arbitrarily small neighborhood of the isotropy set of X.*

Actually, the compactness hypothesis in the above theorem can be relaxed: it suffices that every connected component $S \subset X$ of the set of isotropy surfaces has compact closure $\bar{S}$ in X.

In addition, Theorem 3.1 can be used to construct a 4-dimensional simply connected symplectic manifold as the symplectic resolution of a suitable 4-orbifold. This symplectic orbifold is a quotient of a Kähler manifold $M_\gamma(\Sigma_2) \times S^1$ by an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$, where $M_\gamma(\Sigma_2)$ is a non-trivial mapping torus of the genus 2 surface. The isotropy set of the action consists of 8 isolated points and 3 tori that have 4 intersection points, so this symplectic orbifold cannot be resolved with the methods of [28, 93].

In the recent paper [30] by Chen, it is given an alternative method for resolving arbitrary symplectic 4-orbifolds. The techniques used in [30] for constructing the resolution (e.g. symplectic reduction and symplectic fillings) are rather involved technically, and differ completely from the ones used here.

This paper is organized as follows. In section 3.2 we review the necessary preliminaries on symplectic orbifolds. Section 3.3 studies the isotropy set of 4-dimensional symplectic

orbifolds, giving special local models for the isotropy surfaces. With these tools at hand we prove Theorem 3.1 in section 3.4. Finally, in section 3.5 we provide some examples to which the symplectic resolution of Theorem 3.1 applies.
Acknowledgements. We are grateful to Vicente Muñoz and Giovanni Bazzoni for useful conversations.
The first author acknowledges financial support by a FPU Grant (FPU16/03475).

3.2 Symplectic orbifolds

In this section we introduce some aspects about orbifolds and symplectic orbifolds, which can be found in [93],[106], [109].

3.2.1 Orbifolds

Definition 3.2. An n-dimensional orbifold is a Hausdorff and second countable space X endowed with an atlas $\{(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)\}$, where $\{V_\alpha\}$ is an open cover of X, $U_\alpha \subset \mathbb{R}^n$, $\Gamma_\alpha < \mathrm{Diff}(U_\alpha)$ is a finite group acting by diffeomorphisms, and $\phi_\alpha : U_\alpha \to V_\alpha \subset X$ is a Γ_α-invariant map which induces a homeomorphism $U_\alpha/\Gamma_\alpha \cong V_\alpha$.

There is a condition of compatibility of charts for intersections. For each point $x \in V_\alpha \cap V_\beta$ there is some $V_\delta \subset V_\alpha \cap V_\beta$ with $x \in V_\delta$ so that there are group monomorphisms $\rho_{\delta\alpha} : \Gamma_\delta \hookrightarrow \Gamma_\alpha$, $\rho_{\delta\beta} : \Gamma_\delta \hookrightarrow \Gamma_\beta$, and open differentiable embeddings $\imath_{\delta\alpha} : U_\delta \to U_\alpha$, $\imath_{\delta\beta} : U_\delta \to U_\beta$, which satisfy $\imath_{\delta\alpha}(\gamma(x)) = \rho_{\delta\alpha}(\gamma)(\imath_{\delta\alpha}(x))$ and $\imath_{\delta\beta}(\gamma(x)) = \rho_{\delta\beta}(\gamma)(\imath_{\delta\beta}(x))$, for all $\gamma \in \Gamma_\delta$.

The concept of change of charts in orbifolds is borrowed from its analogue in manifolds.

Definition 3.3. For an orbifold X, a *change of charts* is the map

$$\psi^\delta_{\alpha\beta} = \imath_{\delta\beta} \circ \imath^{-1}_{\delta\alpha} : \imath_{\delta\alpha}(U_\delta) \to \imath_{\delta\beta}(U_\delta).$$

Note that $\imath_{\delta\alpha}(U_\delta) \subset U_\alpha$ and $\imath_{\delta\beta}(U_\delta) \subset U_\beta$, so $\psi^\delta_{\alpha\beta}$ is a change of charts from U_α to U_β. A change of charts between U_α and U_β depends on the inclusion of a third chart U_δ. This dependence is up to the action of an element in Γ_δ. In general this dependence is irrelevant, so we may abuse notation and write $\psi_{\alpha\beta}$ for any change of charts between U_α and U_β. We may further abuse notation and write

$$\psi_{\alpha\beta} : U_\alpha \to U_\beta$$

for a change of charts as above, even though its domain and range do not equal in general all U_α and U_β but an open subset of them.

We can refine the atlas of an orbifold X in order to obtain better properties; given a point $x \in X$, there is a chart (U, V, ϕ, Γ) with $U \subset \mathbb{R}^n$, $U/\Gamma \cong V$, so that the preimage $\phi^{-1}(\{x\}) = \{u\}$ is only a point, and the group Γ acting on U leaves the point u fixed, i.e. $\gamma(u) = u$ for all $\gamma \in \Gamma$. We call Γ the *isotropy group* at x, and we denote it by Γ_x. This group is well defined up to conjugation by a diffeomorphism of a small open set of $\mathbb{R}^n$. In addition, using a Γ_x-invariant metric and the exponential chart one can prove:

Proposition 3.4. *Around any point $x \in X$ there exists an orbifold chart (U, V, ϕ, Γ) with $\Gamma_x = \Gamma < \mathrm{O}(n)$.*

Definition 3.5. The *isotropy subset* of X is $\Sigma = \{x \in X \text{ s.t. } \Gamma_x \neq \{1\}\}$.

As we shall see, the isotropy set is stratified into suborbifolds; this notion is also similar to the concept of a submanifold:

Definition 3.6. Let X be an orbifold of dimension n. A *suborbifold of dimension* d or d*-suborbifold* of X is defined to be a subspace $Y \subset X$ such that for each $p \in Y$ there exists an orbifold chart (U, V, ϕ, Γ) of X around p with $\Gamma < \mathrm{O}(n)$, $\phi(p) = 0$, and such that $U' = U \cap (\mathbb{R}^d \times \{0\})$ satisfies $\phi(U') = Y \cap V$.

Let $Y \subset X$ be a suborbifold. Then Y has an orbifold structure inherited from X, as follows. Consider the chart (U, V, ϕ, Γ) of the above definition and let us identify $\mathbb{R}^d \cong \mathbb{R}^d \times \{0\} \subset \mathbb{R}^n$. Consider $\tilde{\Gamma} = \{\gamma \in \Gamma \text{ s.t. } \gamma(\mathbb{R}^d) \subset \mathbb{R}^d\} < \Gamma$ the subgroup of elements leaving invariant $\mathbb{R}^d$. Consider the representation given by $\varrho : \widetilde{\Gamma} \to \mathrm{End}(\mathbb{R}^d)$; its image is a subgroup $\Gamma' = \mathrm{Im}(\varrho) \cong \widetilde{\Gamma}/\ker(\varrho)$. Let us denote $V' = Y \cap V = \phi(U')$, and $\phi' = \phi|_{U'} : U' \to V'$. The orbifold chart of Y around p is defined to be (U', V', ϕ', Γ'). Clearly, U' is a Γ'-invariant set and satisfies $U'/\Gamma' \cong Y \cap V$.

Let us state a notion of equivalence between groups of diffeomorphisms that is useful for orbifolds.

Definition 3.7. Let $H < \mathrm{Diff}(U)$, $H' < \mathrm{Diff}(U')$ be two groups of diffeomorphisms of open sets U, U' of $\mathbb{R}^{2n}$. We say that the germs (U, H) and (U', H') are *equivalent* if there exists a diffeomorphism $f : U \to U'$ such that $f \circ H \circ f^{-1} = H'$. In this case we write $(U, H) \cong (U', H')$.

Note that the above gives an equivalence relation on the set of germs of diffeomorphisms of $\mathbb{R}^{2n}$. If (U, V, Γ, ϕ) is an orbifold chart, a diffeomorphism $f : U \to U'$ gives an induced orbifold chart (U', V, Γ', ϕ'), where $\Gamma' = f \circ \Gamma \circ f^{-1}$ and $\phi' = \phi \circ f^{-1}$. Hence, all the germs (U', Γ') equivalent to (U, Γ) induce the same orbifold chart. We shall also specify this notion for finite subgroups of $\mathrm{O}(n)$.

Definition 3.8. Two finite subgroups Γ, Γ' of $\mathrm{O}(n)$ are *equivalent* if there exists open sets $U, U' \subset \mathbb{R}^n$ containing 0 such that the germs (U, Γ) and (U', Γ') are equivalent. We denote $\Gamma \cong \Gamma'$ in this case.

Proposition 3.9. *[93, Proposition 4] Let X be an orbifold, and let Σ be its isotropy subset. For every equivalence class H of finite subgroup $H < \mathrm{O}(n)$, we can define the set*

$$\Sigma_H = \{x \in X \text{ s.t. } \Gamma_x \cong H\}.$$

Then the closure $\overline{\Sigma}_H$ is a suborbifold of X, and $\Sigma_H = \overline{\Sigma}_H - \bigcup_{H<H'} \Sigma_{H'}$ is a submanifold of X.

Definition 3.10. An orbifold function $f : X \to \mathbb{R}$ is a continuous function such that $f \circ \phi_\alpha : U_\alpha \to \mathbb{R}$ is smooth for every α.

Note that this is equivalent to giving smooth functions f_α on U_α which are Γ_α-equivariant and which agree under the changes of charts. An *orbifold partition of unity subordinated to the open cover* $\{V_\alpha\}$ *of* X consists of orbifold functions $\rho_\alpha : X \to [0, 1]$ such that the support of ρ_α lies inside V_α and the sum $\sum_\alpha \rho_\alpha \equiv 1$ on X.

Proposition 3.11. *[93, Proposition 5] Let X be an n-orbifold. For any sufficiently refined open cover $\{V_\alpha\}$ of X there exists an orbifold partition of unity subordinated to $\{V_\alpha\}$.*

Orbifold tensors are defined in the same way as functions are. That is, an orbifold tensor on X is a collection of Γ_α-invariant tensors on each U_α which agree under the changes of charts. In particular, there is a notion of orbifold differential forms $\Omega_{orb}(X)$ and the exterior derivative is also well-defined.

3.2.2 Symplectic orbifolds

Definition 3.12. A symplectic orbifold is a $2n$-dimensional orbifold X equipped with an orbifold 2-form $\omega \in \Omega^2_{orb}(X)$ such that $d\omega = 0$ and ω^n is nowhere-vanishing.

The proof of the existence of an almost Kähler structure on a manifold (see [25]) easily carries over to the orbifold case:

Proposition 3.13. *[93, Proposition 8] Let (X, ω) be a symplectic orbifold. Then (X, ω) admits an almost Kähler orbifold structure (X, ω, J, g).*

We denote $(\omega_0, \mathfrak{j}, g_0)$ the standard Kahler structure on $\mathbb{C}^n$.

Corollary 3.14. *Let (X, ω) be a symplectic $2n$-orbifold. Every point in X admits a chart $(U, V, \phi, \Gamma, \omega)$ with $\Gamma < \mathrm{U}(n)$.*

Proof. Put any almost Kähler structure (ω, J, g) on X as provided by Proposition 3.13. Fix a chart (U, V, ϕ, Γ) around p such that $\phi(0) = p$, Γ acts linearly, and the almost Kähler structure $(\omega_p, J_p, g_p) = (\omega_0, \mathfrak{j}, g_0)$ at $p = 0$ is standard. As J is an orbifold almost complex structure, Γ preserves J; in particular at the point $0 \in U$ we have $d_0\gamma \circ \mathfrak{j} = \mathfrak{j} \circ d_0\gamma$ for all $\gamma \in \Gamma$. As γ is linear, we have that $d_0\gamma = \gamma$, hence γ preserves the complex structure of $\mathbb{C}^n = (\mathbb{R}^{2n}, \mathfrak{j})$. This means that $\Gamma < \mathrm{Gl}(n, \mathbb{C})$. Similarly, since γ preserves the standard metric g_0, one sees that $\Gamma < \mathrm{O}(2n)$. The conclusion is that $\Gamma < \mathrm{Gl}(n, \mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{U}(n)$. ☐

For symplectic (almost Kähler) orbifolds, the isotropy set inherits a symplectic (almost Kähler) structure.

Corollary 3.15. *[93, Corollary 9] The isotropy set Σ of (X, ω) consists of immersed symplectic suborbifolds $\overline{\Sigma}_H$. Moreover, if we endow X with an almost Kähler orbifold structure (ω, J, g), then the sets $\overline{\Sigma}_H$ are almost Kähler suborbifolds.*

The following result is a Darboux theorem for symplectic orbifolds.

Proposition 3.16. *[93, Proposition 10] Let (X, ω) be a symplectic orbifold and $x_0 \in X$. There exists an orbifold chart (U, V, ϕ, Γ) around x_0 with local coordinates $(x_1, y_1, \ldots, x_n, y_n)$ such that the symplectic form has the expression $\omega = \sum dx_i \wedge dy_i$ and $\Gamma < \mathrm{U}(n)$ is a subgroup of the unitary group.*

Any orbifold almost Kähler structure can be perturbed to make it standard around any chosen point. We include a proof below.

Corollary 3.17. *Let (X, ω) be a symplectic orbifold, and let (J, g) be a compatible almost Kähler structure. Let $p \in X$ a point and $(U, V, \phi, \Gamma, \omega_0)$ a Darboux chart around p. Choose V_1 a neighborhood of p such that $\overline{V_1} \subset V$, and let $U_1 = \phi^{-1}(V_1) \subset U$. There exists another compatible almost Kähler structure (J', g') such that $J' = J$ and $g' = g$ outside V, and (J', g') is the standard $(\mathfrak{j}, g_0)$ when lifted to the chart $U_1 \subset U$.*

Proof. Take a bump function ρ which equals 1 on V_1 and 0 outside V. Consider the metric $g_1 = \rho g_0 + (1 - \rho) g$, where ρg_0 coincides with the standard metric when lifted to U_1, and extends as 0 to all of X. If we use the metric g_1 as auxiliary metric in the proof of Proposition 3.13 and construct a compatible almost Kähler structure (J', g'), we find that $J' = \mathfrak{j}$, $g' = g_0$ when lifted to U_1 because both ω and the auxiliary metric g_1 are standard in U_1. ☐

Let us recall a result from symplectic linear algebra that will be useful later. Consider the retraction

$$r\colon \mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{U}(n), \qquad r(A) = A(A^tA)^{-1/2} \tag{3.1}$$

The fact that $A(A^tA)^{-1/2} \in \mathrm{U}(n) = \mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{O}(2n)$ for any $A \in \mathrm{Sp}(2n,\mathbb{R})$ can be seen as follows. First, since $A^t\Omega_0 A = \Omega_0$, with Ω_0 the matrix of the standard symplectic form on $\mathbb{R}^{2n}$, it is easy to check that $(A^tA)^t\Omega_0 A^tA = \Omega_0$ so $A^tA \in \mathrm{Sp}(2n,\mathbb{R})$. Then, by expressing the square root $S^{1/2}$ as a power series in S, for S a positive definite symmetric matrix, one sees that $(A^tA)^{1/2} \in \mathrm{Sp}(2n,\mathbb{R})$, hence so does its inverse and it follows that $r(A) = A(A^tA)^{-1/2} \in \mathrm{Sp}(2n,\mathbb{R})$. Finally, using that S and $S^{1/2}$ commute, it follows that $r(A)^t r(A) = \mathrm{Id}$, so $r(A) \in \mathrm{O}(2n)$.

This retraction satisfies the following. If there is a group $\Gamma < \mathrm{U}(k)$ and an isomorphism $\rho\colon \Gamma \to \Gamma' < \mathrm{U}(k)$, such that $A \in \mathrm{Sp}(2n,\mathbb{R})$ is Γ-equivariant in the sense that $A \circ \gamma = \rho(\gamma) \circ A$ for all $\gamma \in \Gamma$, then $r(A)$ is also Γ-equivariant, i.e. $r(A) \circ \gamma = \rho(\gamma) \circ r(A)$ for all $\gamma \in G$. This property is a consequence of the following result:

Lemma 3.18. *[93, Lemma 21] Let $A, C \in \mathrm{U}(k)$ and $B \in \mathrm{Sp}(2n,\mathbb{R})$ such that $A = B^{-1}CB$. Then $A = r(B)^{-1}C\,r(B)$.*

3.3 Symplectic orbifolds in dimension 4.

3.3.1 The isotropy set in dimension 4.

Let (X,ω) be a symplectic orbifold of dimension 4, and let $x \in X$. Put a compatible orbifold almost complex structure on (X,ω), obtaining an almost Kähler orbifold (X,ω,J). By the equivariant Darboux Theorem, around any point we have an orbifold chart $(U,V,\phi,\Gamma,\omega_0)$ such that $U = B_\varepsilon(0) \subset \mathbb{C}^2$ is a ball and $\phi^{-1}(\{x\}) = \{0\}$, and $\Gamma = \Gamma_x < \mathrm{U}(2)$ acts on U by unitary matrices. Unless otherwise stated, from now on we assume that every orbifold chart of (X,ω) has the form above. We will write $(U,V,\phi,\Gamma,\omega_0)$ if the symplectic form is standard in the chart, and analogously for another tensors like g_0 and $\mathfrak{j}$.

In dimension 4 the isotropy set can be expressed as a union $\Sigma = \Sigma^0 \cup \Sigma^* \cup \Sigma^1$ of three subsets with distinct properties. These are determined by a geometric condition that depends on the action of the isotropy groups $\Gamma_x < \mathrm{U}(2)$ in $\mathbb{C}^2$, as follows.

Case 1: $x \in \Sigma^0$ if the action of Γ_x on $\mathbb{C}^2 - \{0\}$ is free.

Case 2: $x \in \Sigma^*$ if there exists a complex line $L \subset \mathbb{C}^2$ such that for every non-identity element $\gamma \in \Gamma_x$ we have $\mathrm{Fix}(\gamma) = L$.

Case 3: $x \in \Sigma^1$ if there exist at least two complex lines $L_1, L_2 \subset \mathbb{C}^2$ and non-identity elements $\gamma_1, \gamma_2 \in \Gamma_x$ so that $L_1 = \mathrm{Fix}(\gamma_1)$ and $L_2 = \mathrm{Fix}(\gamma_2)$.

Note the following:

- If $x \in \Sigma^0$, then x is an isolated point of Σ. That is why the points of Σ^0 are called *isolated singular points*.

- If $x \in \Sigma^*$ then $D = \phi(L)$ is contained on Σ^* and every point on this line has constant isotropy Γ_x. The connected components of Σ^* are therefore surfaces S_i such that all its points have the same isotropy group Γ_i.

- The points of Σ^1 are also isolated; in addition these lie in the closure of some surfaces $S_i \subset \Sigma^*$. Given $x \in \Sigma^1$, let us call I_x the set of indices i such that the surface S_i accumulates to x and write Γ_i the isotropy set of S_i.

We have the following result:

Lemma 3.19. *Let $p \in \Sigma^1$ and let (U, V, Γ) be an orbifold chart around p, with $\Gamma = \Gamma_p < \mathrm{U}(2)$. Let $\Gamma^* = \langle \Gamma_i \text{ s.t. } i \in I_p \rangle \lhd \Gamma$ be the normal subgroup generated by the isotropy groups of all the surfaces S_i accumulating at p.*

1. *The space $U' = U/\Gamma^*$ is a topological manifold and inherits naturally a complex orbifold structure with isotropy set the surfaces S_i.*
2. *The quotient group $\Gamma' = \Gamma/\Gamma^*$ has an induced action on $U' = U/\Gamma^*$. Moreover, U'/Γ' and U/Γ are canonically isomorphic (as orbifolds).*

Proof. We check first that Γ^* is a normal subgroup of Γ. Take $g_i \in \Gamma_i$, and $\gamma \in \Gamma$. Then $\gamma g_i \gamma^{-1}$ leaves fixed all the points in the surface $\gamma(S_i) \subset U$. Hence $\gamma g_i \gamma^{-1}$ belongs to the isotropy group of some of the surfaces $S_j = \gamma(S_i)$. This means that $\Gamma \circ \Gamma_i \circ \Gamma^{-1} \subset \bigcup_j \Gamma_j \subset \Gamma^*$. If we take now a generic element of Γ^*, i.e. finite product $\prod_k g_{i_k}$ with $g_{i_k} \in \Gamma_{i_k}$, then for any $\gamma \in \Gamma$ we have $\gamma(\prod_k g_{i_k})\gamma^{-1} = \prod_k (\gamma g_{i_k} \gamma^{-1}) \in \Gamma^*$, and this proves that $\Gamma \circ \Gamma^* \circ \Gamma^{-1} \subset \Gamma^*$, so Γ^* is a normal subgroup.

The complex orbifold structure on U/Γ^* exists because Γ acts on $U \subset \mathbb{C}^2$ by biholormorphisms, so it acts j-equivariantly. To see that $U' = U/\Gamma^*$ is a topological manifold, observe that the group Γ^* acts on $\mathbb{C}^2$ and it is generated by complex reflections. Hence the algebra $\mathbb{C}[z_1, z_2]^{\Gamma^*}$ of Γ^*-invariant polynomials is a polynomial algebra generated by 2 elements, say f, g. This is proved for real reflections in [31], but the proof carries over to complex reflections also, see [101]. Consider

$$H \colon \mathbb{C}^2 \to \mathbb{C}^2, \qquad H(z) = (f(z_1, z_2), g(z_1, z_2)).$$

This map induces a homeomorphism $\bar{H} \colon \mathbb{C}^2/\Gamma^* \to \mathbb{C}^2$. This ensures that U/Γ^* is a topological manifold.

Now consider $\Gamma' = \Gamma/\Gamma^* = \{\gamma\Gamma^* \text{ s.t. } \gamma \in \Gamma\}$, and define its action on $U' = U/\Gamma^* = \{\Gamma^* u \text{ s.t. } u \in U\}$ by $(\gamma\Gamma^*) \cdot (\Gamma^* u) = \Gamma^*(\gamma u)$ for $u \in U$ and $\gamma \in \Gamma$. This is well defined since for $\gamma' = \gamma\gamma_1^*$ and $u' = \gamma_2^* u$, other representatives of $\gamma\Gamma^*$ and $\Gamma^* u$, we have $\gamma' u' = (\gamma\gamma_1^*)(\gamma_2^* u) = \gamma(\gamma_1^*\gamma_2^*)u = \gamma\gamma^* u = c_\gamma(\gamma^*)\gamma u$, where $c_\gamma \colon \Gamma \to \Gamma$ is conjugation by γ and $\gamma^* = \gamma_1^*\gamma_2^* \in \Gamma^*$. Taking into account that $c_\gamma(\Gamma^*) = \Gamma^*$ we obtain $\Gamma^*(\gamma' u') = \Gamma^*(\gamma u)$. It is immediate to check that this gives an action. Moreover, the orbit of $\Gamma^* u$ in U'/Γ' is given by $\Gamma' \cdot (\Gamma^* u) = \{\Gamma^*(\gamma u) | \gamma \in \Gamma\}$ so it equals Γu, the orbit of u in U/Γ. □

The following lemma proves the existence of a suitable orbifold almost Kähler structure in dimension 4. It gives a local Kähler model around any point in $\Sigma^1 \cup \Sigma^0$.

Lemma 3.20. *Let (X, ω) be a 4-dimensional symplectic orbifold. There exists an almost Kähler structure (X, ω, J, g) such that:*

1. *For each point $p \in \Sigma^0$, there is an orbifold chart $(U, V, \Gamma, \omega_0, \mathrm{j}, g_0)$ around p.*
2. *For each point $p \in \Sigma^1$ there is an orbifold chart $(U, V, \phi, \Gamma, \omega_0, \mathrm{j}, g_0)$, and each surface S_i that accumulates to p lifts to $\phi^{-1}(S_i)$ which is a union of disjoint complex curves in the chart (U, j).*

Proof. We use Corollary 3.17 to put an almost Kähler structure (J, g) so that there are flat Kähler charts around any point in $\Sigma^1 \cup \Sigma^0$. Now, using Corollary 3.15, both statements are clear. □

3.3.2 Tubular neighbourhood of singular surfaces

With respect to the orbifold almost Kähler structure of above, given a surface $S \subset \Sigma^*$, note that $TS^{\perp\omega} = TS^{\perp g}$, i.e. for every $z \in S$, the symplectic and metric orthogonal spaces to T_zS are the same. The following lemma gives an orbifold atlas of X such that a tubular neighborhood of any surface $S \subset \Sigma^*$ inherits an atlas of an orbifold disc-bundle with structure group in $\mathrm{U}(1)$.

Lemma 3.21. *The symplectic orbifold (X, ω) admits an atlas $\mathcal{A}$ such that for any $S \subset \Sigma^*$, some neighborhood $D_{\varepsilon_0}(\bar{S})$ of $\bar{S}$ in X admits an open cover $D_{\varepsilon_0}(\bar{S}) = \cup_\alpha V_\alpha$ such that for each α there is an orbifold chart $(U_\alpha, V_\alpha, \Gamma_\alpha, \phi_\alpha, \omega_\alpha) \in \mathcal{A}$, satisfying:*

1. *If $V_\alpha \cap \Sigma^1 = \emptyset$, then $U_\alpha = S_\alpha \times D_{\varepsilon_0}$ is a product, with $S_\alpha \subset S$ open, $D_{\varepsilon_0} \subset \mathbb{C}$ a disc, and the group $\Gamma_\alpha = \Gamma$ is the isotropy group of the surface S. For any other V_β with $V_\beta \cap \Sigma^1 = \emptyset$, the orbifold change of charts are given by*
$$\psi_{\alpha\beta} = (\psi^1_{\alpha\beta}, \psi^2_{\alpha\beta})\colon U_\alpha \to U_\beta, \qquad (z,w) \mapsto (\psi^1_{\alpha\beta}(z), A_{\alpha\beta}(z)w)$$
with
$$A_{\alpha\beta}\colon S_\alpha \to \mathrm{U}(1), \qquad z \mapsto A_{\alpha\beta}(z)$$
a smooth function taking values in the unit circle $\mathrm{U}(1)$. The group $\Gamma < \mathrm{U}(1)$ acts in U_α and U_β by a rotation in D_{ε_0}, in particular it is isomorphic to $\mathbb{Z}_m$.

2. *For each $p \in \Sigma^1 \cap \bar{S}$ denote V_p an open set of the cover that contains p. Then the corresponding chart $(U_p, V_p, \Gamma_p, \phi_p, \omega_0)$ satisfies that $H_p \times D_{\varepsilon_0} \subset U_p$ with $H_p \subset \mathbb{R}^2$ open and $\phi_p(H_p \times \{0\}) = \bar{S} \cap V^p$. Moreover if V_α does not contain p, the change of charts is given by*
$$\psi_{\alpha p}\colon U_\alpha \to U_p, \qquad (z,w) \mapsto (\psi^1_{\alpha p}(z), A_{\alpha p}(z)w)$$
with $A_{\alpha p}(z) \in \mathrm{U}(1)$, and its image is $\psi_{\alpha p}(U_\alpha) = H_\alpha \times D_{\varepsilon_0} \subset H_p \times D_{\varepsilon_0}$, with $\phi_p(H_\alpha \times \{0\}) = S \cap V_\alpha \cap V_p$. If we denote $\rho_{\alpha p}\colon \Gamma_\alpha = \Gamma \hookrightarrow \Gamma_p$ the associated monomorphism of isotropy groups, then the subgroup $\rho_{\alpha p}(\Gamma) < \Gamma_p$ acts on $H_\alpha \times D_{\varepsilon_0}$ as a rotation in D_{ε_0}.

Proof. Consider an orbifold almost Kähler structure (ω, J, g) on X as in Lemma 3.20. To see (1), take an initial cover $\cup_\alpha V_\alpha$ of S with orbifold charts $(U'_\alpha, V_\alpha, \Gamma, \omega_0, J_\alpha, g_\alpha)$ such that $V_\alpha \cap \Sigma^1 = \emptyset$. Let (z,w) be coordinates in U'_α, such that $S_\alpha = S \cap U'_\alpha = \{w = 0\}$. Recall that we have for $z = (z,0) \in S_\alpha$ an identification $(T_zS_\alpha)^\perp = \{z\} \times \mathbb{C}$. The change of charts are given by

$$\begin{aligned} \epsilon_{\alpha\beta}\colon\ & U'_\alpha \to U'_\beta \\ & (z,w) \mapsto (\epsilon^1_{\alpha\beta}(z,w), \epsilon^2_{\alpha\beta}(z,w) = (z', w')) \end{aligned}$$

with $\epsilon^2_{\alpha\beta}(z,0) = 0$ for all $(z,0) \in S_\alpha$. Consider now $U_\alpha = S_\alpha \times \mathbb{C}$, and the maps

$$\begin{aligned} \phi_{\alpha\beta}\colon\ & U_\alpha = S_\alpha \times \mathbb{C} \to U_\beta = S_\beta \times \mathbb{C} \\ & (z,u) \mapsto (\phi^1_{\alpha\beta}(z), A'_{\alpha\beta}(z)u) = (z', u') \end{aligned}$$

with $\phi^1_{\alpha\beta}(z) = \epsilon^1_{\alpha\beta}(z,0)$, and $A'_{\alpha\beta}(z) = \partial_w \epsilon^2_{\alpha\beta}|_{(z,0)}$. Here $\partial_w \epsilon^2_{\alpha\beta}$ stands for the Jacobian matrix of $\epsilon^2_{\alpha\beta}$ in the variable w. Now we use the exponential map to identify U'_α and $U_\alpha = S_\alpha \times D_\varepsilon$, where $D_\varepsilon \subset \mathbb{C}$ is a small disc. To this end let us consider the maps

$$e_\alpha\colon U_\alpha = S_\alpha \times D_\varepsilon \to U'_\alpha, \qquad (z,u) \mapsto \exp_z(u) = (z,w)$$

which are diffeomorphisms for all $\varepsilon \leq \varepsilon_0$, maybe reducing U'_α. The induced action of the group Γ in $U_\alpha = S_\alpha \times D_\varepsilon$ is given by complex multiplication in D_ε. Now, it is easy to check that the maps $\phi_{\alpha\beta}$ are the induced change of charts with respect to the new coordinates (z, u) and (z', u') in U_α and U_β. In other words, $\phi_{\alpha\beta} = e_\beta^{-1} \circ \epsilon_{\alpha\beta} \circ e_\alpha$. Hence we can take the maps $\phi_{\alpha\beta}$ as new orbifold change of charts. The matrices

$$A'_{\alpha\beta}(z)\colon ((T_z S_\alpha)^\perp, h_\alpha|) \to ((T_{z'} S_\beta)^\perp, h_\beta|)$$

are isometries with respect to the orbifold hermitian metrics $h_\alpha = g_\alpha + i\omega_0(\cdot, J_\alpha \cdot)$ and $h_\beta = g_\beta + i\omega_0(\cdot, J_\beta \cdot)$ restricted to the orthogonal spaces to S (we use the notation $h_\alpha|$ to express this restriction). In particular $A'_{\alpha\beta}(z) \in \mathrm{Sp}(2)$ are symplectic matrices. Take orthonormal bases of $((T_z S_\alpha)^\perp, h_\alpha|)$ and $((T_{z'} S_\beta)^\perp, h_\beta|)$ so that $h_\alpha|_z$ and $h_\beta|_z$ become the standard hermitian metric h_0, and denote $P_\alpha(z), P_\beta(z') \in \mathrm{Sp}(2, \mathbb{R})$ the matrices of change of basis. Call the new coordinates $(z, v) = (z, P_\alpha(z)u)$ and $(z', v') = (z', P_\beta(z')u')$. The change of trivializations in the new coordinates are given by the matrices

$$A''_{\alpha\beta}(z) = P_\beta(z') \cdot A'_{\alpha\beta}(z) \cdot (P_\alpha(z))^{-1} \in \mathrm{U}(1).$$

These matrices are unitary as we want, but the isotropy groups act via

$$\Gamma_z = P_\alpha(z) \cdot \Gamma \cdot P_\alpha(z)^{-1}, \qquad \Gamma_{z'} = P_\beta(z') \cdot \Gamma \cdot P_\beta(z')^{-1}$$

so they are groups acting non-linearly. To fix this, consider $r\colon \mathrm{Sp}(2, \mathbb{R}) \to \mathrm{U}(1)$ the retraction given in (3.1). By Lemma 3.18 we have

$$\Gamma_z = r(P_\alpha(z)) \cdot \Gamma \cdot r(P_\alpha(z))^{-1}, \qquad \Gamma_{z'} = r(P_\beta(z')) \cdot \Gamma \cdot r(P_\beta(z'))^{-1}.$$

So if we introduce further coordinates w in U'_α by $(z, w) = (z, r(P_\alpha(z))^{-1} v)$ and w' in U'_β by $(z', w') = (z', r(P_\beta(z'))^{-1} v')$ then the corresponding transition matrices are given by

$$A_{\alpha\beta}(z) = r(P_\beta(z'))^{-1} \cdot A''_{\alpha\beta}(z) \cdot r(P_\alpha(z)) \in \mathrm{U}(1)$$

and moreover the varying groups Γ_z and $\Gamma_{z'}$ become Γ again. This shows what we wanted. The sought transition maps $\psi_{\alpha\beta}$ are given by $\psi^1_{\alpha\beta}(z) = \phi^1_{\alpha\beta}(z)$ and $\psi^2_{\alpha\beta}(z, w) = A_{\alpha\beta}(z) w$.

Now let us see (2). Suppose that S accumulates at $p \in \Sigma^1$, and let (U, V, ϕ, Γ) a chart around $p = \phi(0)$ with coordinates (z, w) such that (ω, g, J) is the standard Kähler structure in this chart. After a complex rotation on U (which preserves the whole structure) we can suppose that $\bar{S} \cap V = \phi(\{w = 0\})$. In this case, $e_U(z, w) = (z, w)$ so that (U, V, ϕ, Γ) remains invariant after the process described before. □

Remark 3.22. The proof of this lemma shows that, given the Kähler chart $\phi\colon U_p \to V_p$ of a point $p \in \Sigma^1$, the atlas for the tubular neighborhood $D_{\varepsilon_0}(S)$ of a singular surface S with $p \in \overline{S}$ can be constructed making a complex rotation of the preimage $\phi^{-1}(V_p \cap D_{\varepsilon_0}(S))$ so that $S = \{w = 0\}$.

Remark 3.23. Near $p \in \Sigma^1 \cap \bar{S}$ we can define a compatible orbifold chart from $(U_p, V_p, \Gamma_p, \phi_p)$: we let $\varepsilon_p > 0$ be such that $B_{3\varepsilon_p}(p) \subset V_p$ and let $\varepsilon_0 > 0$ such that

$$\phi_p((B_{3\varepsilon_p}(0) - B_{\varepsilon_p}(0)) \cap (\mathbb{C} \times D_{\varepsilon_0})) \subset X - \cup_{S' \neq S} D_{\varepsilon_0}(\bar{S}').$$

There is a compatible orbifold chart $(A^3_p, V_{A^3_p}, \tilde{\Gamma}, \phi_p)$ with $A^3_p = (B_{3\varepsilon_p}(0) - B_{\varepsilon_p}(0)) \cap (\mathbb{C} \times D_{\varepsilon_0})$, $V_{A^3_p} = \phi_p(A^3_p)$ and $\tilde{\Gamma} = \{\gamma \in \Gamma_p \text{ s.t. } \gamma(z, 0) = (z', 0)\} < \mathrm{U}(1) \times \mathrm{U}(1)$.

Moreover, if Γ_S is the isotropy of S then $A_p/\Gamma_S \to A_p/\tilde{\Gamma}$ is a covering with deck group $\tilde{\Gamma}/\Gamma_S$. In addition, given $\phi_p(z, 0) \in U_{A^3_p}$ one can restrict sufficiently the previous chart to obtain an orbifold chart of X with isotropy Γ_S.

Remarks 3.24.

1. The symplectic forms $\omega_\alpha = e_\alpha^*\omega_0$ of the atlas above may not be standard in the charts $U'_\alpha = S_\alpha \times D_\varepsilon$, but they are standard at the points of S, so we have in coordinates $(z,w) \in U'_\alpha$ the expression
$$\omega_\alpha = -\tfrac{\mathrm{i}}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w}) + O(|w|).$$

2. The atlas $\mathcal{A}$ constructed above can be refined so that for any $p \in \Sigma^1$ and any neighborhood W^p of p in X, there is an orbifold chart $(U^p, V^p, \phi_p, \Gamma_p)$ in $\mathcal{A}$ with $p \in V^p \subset W^p$. Also, we can assume that only one of the open sets of the atlas contains the point $p \in \Sigma^1$.

Consider an orbifold almost Kähler structure (X, ω, J, g) of lemma 3.20. Let $S \subset \Sigma^*$ be an isotropy surface, and $D_{\varepsilon_0}(\bar{S})$ a neighborhood of $\bar{S}$ in X as in Lemma 3.21, with an open cover $D_{\varepsilon_0}(\bar{S}) = \cup_\alpha V_\alpha$ and orbifold charts $(U_\alpha, V_\alpha, \Gamma_\alpha, \phi_\alpha, \omega_\alpha)$. For $p \in \bar{S} \cap \Sigma^1$ let $(U_p, V_p, \Gamma_p, \phi_p, \omega_0)$ be the unique orbifold chart covering p. Denote $\pi\colon D_{\varepsilon_0}(\bar{S}) \to \bar{S}$ the projection. The following lemma shows the existence of an orbifold connection 1-form on $D_{\varepsilon_0}(\bar{S}) - (\cup_{p\in\Sigma^1\cap\bar{S}} B_{\varepsilon_p}(p) \cup \bar{S})$, where ε_p satisfies that $B_{3\varepsilon_p}(0) \subset U_p$ and $3\varepsilon_p < \varepsilon_0$.

Lemma 3.25. *Notations as above. There exists an orbifold* 1*-form* $\eta = \eta_S \in \Omega^1_{orb}(D_{\varepsilon_0}(\bar{S}) - (\cup_{p\in\Sigma^1\cap\bar{S}} B_{\varepsilon_p}(p) \cup \bar{S}))$ *such that:*

1. *If* $V_\alpha \cap \Sigma^1 = \emptyset$*, the liftings* η_α *in the orbifold charts* $U_\alpha = S_\alpha \times D_{\varepsilon_0}$ *have the form* $\eta_\alpha = d\theta + \pi^*\nu_\alpha$ *for* $\nu_\alpha \in \Omega^1(S_\alpha)$*, with* θ *the angular coordinate in* D_{ε_0}*.*

2. *For* $p \in \Sigma^1 \cap \bar{S}$*, let* $H_p \times D_{\varepsilon_0} \subset U_p$ *with* $\phi(H_p \times \{0\}) = \bar{S} \cap V_p$*. Then, the lifting of* η *in* $U_p - B_{\varepsilon_p}(p)$ *equals* $d\theta$ *in* $V_{A_p^2}$*, with* $V_{A_p^2} = \phi_p(A_p^2)$ *and* $A_p^2 = (B_{2\varepsilon_p}(0) - B_{\varepsilon_p}(0)) \cap (\mathbb{C} \times D_{\varepsilon_0})$*.*

Proof. Consider $\pi_\alpha\colon S_\alpha \times D_{\varepsilon_0} \to D_{\varepsilon_0}$, and the angular function $\pi_\alpha^*\theta$ which measures the angle in each fiber D_{ε_0}. We have that $\pi_\alpha^*\theta - \pi_\beta^*\theta = \pi^*\xi_{\alpha\beta}$ in the intersections, where $\xi_{\alpha\beta} = \xi_{\alpha\beta}(z)$ is a function on S.

The 1-forms $d\pi_\alpha^*\theta - d\pi_\beta^*\theta = \pi^* d\xi_{\alpha\beta} = \pi^*\nu_{\alpha\beta}$ are Γ-invariant since Γ acts on the angle θ as a translation in the charts. The argument carries also on the chart $(A_p^3, V_{A_p^3}, \tilde{\Gamma}, \phi_p)$ defined on Remark 3.23; the angular form is $\tilde{\Gamma}$-invariant because each element of $\tilde{\Gamma}$ can be expressed as the composition of a map $(z,w) \to (e^{\frac{2\pi i}{k}} z, w)$ which preserves the angle, and a map that acts on the angle θ as a translation.

We take a cover of $\bar{S} - \cup_{p\in\Sigma^1\cap\bar{S}} B_{\varepsilon_p}(p)$ formed by coordinate open sets and such that all points of $V_{A_p^2}$ are covered only by $V_{A_p^3}$. We denote it by $\{V_\alpha\}_{\alpha\in\Delta}$. Now, taking a partition of unity ρ_α subordinated to the cover $\{V_\alpha\}$ we can define $\eta_\alpha = \sum_\alpha \pi^*\rho_\alpha \cdot \pi_\alpha^*(d\theta)$. If we fix a chart $U_\beta = S_\beta \times D_{\varepsilon_0}$, then the lifting of η to U_β is given by
$$\eta|_{U_\beta} = \sum_\alpha \pi^*\rho_\alpha \cdot (\pi_\beta^*(d\theta) + \pi^*\nu_{\alpha\beta}) = \pi_\beta^*(d\theta) + \sum_\alpha \pi^*(\rho_\alpha \cdot \nu_{\alpha\beta}).$$
This proves that η restricts to $d\theta$ on each fiber and (1).

Take $p \in \Sigma^1$. Since points on $\phi_p((B_{2\varepsilon_p}(0) - B_{\varepsilon_p}(0)) \cap \mathbb{C} \times D_{\varepsilon_0})$ are covered by a unique open set of the covering, the connection is trivial over it. This proves (2). □

3.4 Resolution.

In this section we shall be explicit about the atlas that we consider in the space X. Let $\mathcal{A}$ be the orbifold atlas of X that satisfies Lemma 3.21 and denote the symplectic orbifold structure by $(X, \omega, \mathcal{A})$.

First observe that we can suppose that $\Sigma^0 = \emptyset$ by employing the method described in [28] to resolve isolated singularities; we briefly describe it in subsection 3.4.1. In order to perform the resolution we first endow X with the structure of a symplectic orbifold $(X, \widehat{\mathcal{A}}, \widehat{\omega})$ without changing the underlying topological manifold. The isotropy points of the new structure are isolated and consist of $\Sigma^0 \cup \Sigma^1$. For that purpose, we first construct a manifold atlas of $X - \Sigma^1$ and replace ω with a closed 2-form ω_a^*, which is zero on a closed neighbourhood of Σ^1 and symplectic away from it. After this we extend the orbifold structure to Σ^1 to obtain the desired orbifold structure $(X, \widehat{\mathcal{A}})$. The orbifold form ω' naturally extends to $(X, \widehat{\mathcal{A}})$; we finally use a gluing lemma (see Lemma 3.37) to contruct $\widehat{\omega}$.

The extension process is inspired in Lemma 3.19. Following its notation, if $p \in \Sigma^1$ and (U, V, Γ, ϕ) is an orbifold chart then $V = U/\Gamma = (U/\Gamma^*)/\Gamma'$. A holomorphic homeomorphism $H \colon U/\Gamma^* \to \widehat{U} \subset \mathbb{C}^2$ allows us to resolve the singularities of $\Sigma^* \cap V$; and $\widehat{U}/\Gamma'$ has an isolated singularity at 0. This structure must be compatible with the atlas defined on $X - \Sigma^1$; for that reason we resolve the singularities on Σ^* using complex transformations. Riemann extension theorem will ensure the compatibility of both structures away from Σ^1.

We finally resolve the isolated isotropy locus of $(X, \widehat{\mathcal{A}}, \widehat{\omega})$ using again the method of [28]. This process yields a resolution of $(X, \mathcal{A}, \omega)$ as follows:

Theorem 3.26. *Let (X, ω) be a symplectic 4-orbifold such that the closure of each conected component $S \subset \Sigma^*$ is compact. There exists a symplectic manifold $(\tilde{X}, \tilde{\omega})$ and a smooth map $\pi \colon (\tilde{X}, \tilde{\omega}) \to (X, \omega)$ which is a symplectomorphism outside an arbitrarily small neighborhood of the isotropy set of X.*

3.4.1 Resolution of isolated singularities

We briefly outline the process of resolving an isolated singularity, which can be found in [28]. As one should observe, this method is valid for symplectic orbifolds of arbitrary dimension; but we restrict to the case that the dimension is 4.

Let $p \in \Sigma^0$ be an isolated singular point and let $(U, V, \Gamma, \phi, \omega_0, \mathrm{j}, g_0)$ be a Kähler Darboux chart around p with $V \cong U/\Gamma$, $\Gamma < \mathrm{U}(2)$. The space U/Γ is an affine variety because one can consider $\langle P_1, \ldots, P_N \rangle$ a basis of the finitely generated $\mathbb{C}$-algebra of polynomials that are invariant by the action of Γ, and define the holomorphic embedding:

$$\iota \colon \mathbb{C}^2/\Gamma \to \mathbb{C}^N, \qquad \iota(x) = (P_1, \ldots, P_N)(x).$$

The model $\iota(\mathbb{C}^2/\Gamma)$ is then used to perform the resolution of singularities. This consists of a finite number of blow-ups [65], [66]. The resolution $b \colon F \to \iota(\mathbb{C}^2/\Gamma)$ is quasi-projective and consequently Kähler. We shall denote by ω_F and j_F the symplectic form and the complex structure on the resolution.

Then we replace $B_\varepsilon(p) = \phi(B_\varepsilon(0)) \subset V$ by a small ball around the exceptional set $E = b^{-1}(0)$ in F; that is, define:

$$X' = (X - B_\varepsilon(p)) \cup_{\bar{\phi} \circ b} b^{-1}(B_\varepsilon(0)/\Gamma),$$

To endow X' with a symplectic form we interpolate $b^*\omega_0$ and $\lambda\omega_F$ on $A = b^{-1}(B_{3\delta}(0) - B_\delta(0)/\Gamma)$, where λ is small enough. The interpolation is allowed due to the fact that A is a lens space and thus $H^2(A, \mathbb{R}) = 0$; in order to do so one has to replace the Kähler potential r^2 of ω_0 with a radial Kähler potential on $\mathbb{C}^2 - B_\delta(0)$ that vanishes on $\overline{B_\delta(0)}$ and coincides with r^2 on $\mathbb{C}^2 - B_{2\delta}(0)$, obtaining a form ω_1. If $d\eta = \omega_F - b^*\omega_1$ on A, λ is small enough, and ρ is a radial bump function which is 1 on $B_{2\delta}(0)$ and 0 on $\mathbb{C}^2 - B_{3\delta}(0)$, then the 2-form:

$$\omega_\lambda = b^*\omega_1 + \lambda d((\rho \circ b)\eta)$$

extends to a symplectic form on $b^{-1}(B_{2\varepsilon}(0)/\Gamma)$ and interpolates the desired forms; this is similar to the gluing process described in Lemma 3.37. To ensure that ω_λ is symplectic on $\delta < r \leq 2\delta$ we use the fact that both $b^*\omega_1$ and ω_F are positive with respect to the complex structure j_F on F.

3.4.2 Construction of $(X - \Sigma^1, \widehat{\mathcal{A}}, \omega')$.

In this first step we resolve each surface $S \subset \Sigma^*$ separately; working away from Σ^1. We split the construction in two parts: we first do a preparation on the orbifold $(X, \mathcal{A}, \omega)$ and then change the symplectic orbifold structure.

Preparation

In order to construct a smooth atlas $\widehat{\mathcal{A}}$ of $X - \Sigma^1$ we shall modify $\mathcal{A}$ around singular surfaces. For this, we use the basic fact that the map $q \colon \mathbb{C} \to \mathbb{C}$, $q(z) = z^m$ gives a homeomorphism between $\mathbb{C}/\mathbb{Z}_m$ and $\mathbb{C}$. This map applied to the fibers $\{z\} \times D_{\varepsilon_0} \subset D_{\varepsilon_0}(\bar{S})$ yields a manifold atlas of $D_{\varepsilon_0}(\bar{S}) - \Sigma^1 \cap \bar{S}$, hence providing the sought manifold atlas $\widehat{\mathcal{A}}$ on $X - \Sigma^1$.

But the symplectic form ω is singular on Σ^* with respect to the atlas $\widehat{\mathcal{A}}$ of $X - \Sigma^1$. For this reason we replace ω on the orbifold $(X, \mathcal{A})$ with a form ω_a^* that is degenerate on each $S \subset \Sigma^*$, but it is symplectic on the manifold $(X - \overline{B_\varepsilon(\Sigma^1)}, \widehat{\mathcal{A}})$. Here $B(\Sigma^1)$ stands for a neighborhood of Σ^1 which is a union of balls around each $p \in \Sigma^1$ that are contained in V_p, where $(U_p, V_p, \Gamma_p, \omega_0) \in \mathcal{A}$ is the Darboux chart as usual. More precisely, given $p \in \Sigma^1$ the ball is $\phi(B_{\varepsilon_p}(0))$ where $\varepsilon_p > 0$ satisfies $B_{3\varepsilon_p}(0) \subset U_p$. In addition, $\omega_a^* = 0$ on $\overline{B(\Sigma^1)}$.

As a first step, we need an orbifold symplectic form ω^0 on X which is constant on the fibers of $D_{\varepsilon_0}(\bar{S})$ for each $S \subset \Sigma^*$. For that purpose we first introduce some notations; let S be an isotropy surface, we denote $\pi \colon D_{\varepsilon_0}(\bar{S}) \to \bar{S}$ the projection. By Lemma 3.25 we have an orbifold connection 1-form η on $D_{\varepsilon_0}(\bar{S}) - \bar{S} \times \{0\}$ which equals $d\theta$ in each punctured fiber $\{z\} \times (D_\varepsilon - \{0\})$, $z \in \bar{S}$. Denote $\omega_S = \iota^*\omega \in \Omega^2(S)$ the symplectic form on S, with $\iota \colon \bar{S} \hookrightarrow D_{\varepsilon_0}(\bar{S}) \subset X$ the inclusion.

Lemma 3.27. *For any choice of $\delta > 0$ small enough, there exists an orbifold symplectic form $\omega^0 = \omega^0(\delta)$ on X such that $\omega^0 = \omega$ in*

$$(X - \cup_{S \subset \Sigma^*} D_{2\delta}(\bar{S})) \cup_{p \in \Sigma^1} V_p,$$

and for every singular surface $S \subset \Sigma^*$, $\omega^0 = \pi^*\omega_S + r dr \wedge \eta + \frac{1}{2} r^2 d\eta$ in $D_\delta(\bar{S})$; where $\pi \colon D_\delta(\bar{S}) \to \bar{S}$ denotes the projection.

Proof. Let $S \subset \Sigma^*$ be a singular surface; we define an orbifold 2-form by

$$\omega' = \pi^*\omega_S + r dr \wedge \eta + \tfrac{1}{2} r^2 d\eta \in \Omega^2_{orb}(D_{\varepsilon_0}(\bar{S}))$$

where r is the function in $D_{\varepsilon_0}(\bar{S})$ measuring the radius of the fiber D_{ε_0}. A simple computation shows that ω' is smooth for $r = 0$, and that $d\omega' = 0$. In addition, given $p \in \Sigma^1 \cap \overline{S}$, it holds that $\eta = d\theta$ and ω is the standard Kähler form on the set $H_p \times D_\varepsilon$; therefore ω' coincides with ω.

It is clear that ω' is non-degenerate at every point of the zero section $\bar{S} \times \{0\}$, so it is non-degenerate in a maybe smaller neighborhood which we call again $D_{\varepsilon_0}(\bar{S})$. Now we interpolate ω' and ω to obtain the sought orbifold symplectic form ω^0 on X. Since $\iota^*(\omega' - \omega) = 0$ and $D_{\varepsilon_0}(\bar{S})$ retracts onto $\bar{S}$, we have that $\omega' - \omega = d\beta$ for some orbifold 1-form β defined in $D_{\varepsilon_0}(\bar{S})$ which is 0 on $H_p \times D_\varepsilon$. By Remark 3.24 we have $|\omega' - \omega| = O(r)$ in $D_{\varepsilon_0}(\bar{S})$. We can take a primitive β of $\omega' - \omega = d\beta$ such that $|\beta| = O(r^2)$. Indeed, we can write $\omega' - \omega = \alpha_0 \wedge dr + \alpha_1$

for α_0 a 1-form and α_1 a 2-form with $\alpha_1(\partial_r, \cdot) = 0$. Then we set $\beta = \int_0^r \alpha_0 dr$, which is smooth and a primitive for $\omega' - \omega$ such that

$$|\beta| \leq Cr|\alpha_0| = Cr|(\omega' - \omega)(\partial_r, \cdot)| \leq Cr|\omega - \omega'||\partial_r| = O(r^2)$$

since $|\partial_r|$ is bounded. Now consider a bump function $\rho_\delta(r)$ which equals 1 on $D_\delta(\bar{S})$ and 0 outside $D_{2\delta}(\bar{S})$ and such that $|\rho'_\delta| \leq \frac{3}{\delta}$. Here $\delta < \frac{\varepsilon_0}{2}$ is small, to be fixed later. Define $\omega^0 = \omega + d(\rho_\delta \beta)$. We have that

$$|\omega^0 - \omega| = |\rho'_\delta(r) dr \wedge \beta + \rho_\delta d\beta| = O(\tfrac{r^2}{\delta}) + O(r)$$

so $\omega^0 - \omega = O(\delta)$ in $D_{2\delta}(S)$. Hence ω^0 is symplectic on $D_{2\delta}(\bar{S})$ for δ small. Outside $D_{2\delta}(S)$ we have $\omega^0 = \omega$ so it is also symplectic, and then ω^0 is a global orbifold symplectic form on X. Note that ω^0 equals ω' on $D_\delta(S)$, as desired. Finally, it is clear from the construction of ω^0 that $\omega = \omega^0$ on a neighborhood $\cup_p V_p$ of Σ^1. □

We now modify ω' in order to obtain an intermediate form, $\omega_a \in \Omega^2(X - B(\Sigma^1))$. This is a closed form which is symplectic on $X - (\Sigma^* \cup B(\Sigma^1))$ but $\omega_a \neq 0$ on $\partial B(\Sigma^*)$; we contruct later the desired form ω_a^* from ω_a. The construction of ω_a follows the ideas of the proof of Lemma 3.27 and consists of defining a symplectic form which is adapted to a splitting of the tangent bundle of each singular surface $S \subset \Sigma^*$ into two distributions that we now introduce.

Recall that our atlas provides a well-defined radial function around S. The connection 1-form η defined in Lemma 3.25 allows us to define the horizontal subbundle $\mathcal{H} = \ker(rdr \wedge \eta)$ and the vertical subbundle $\mathcal{V} = \ker(d\pi)$; these can be endowed with an almost Kähler structure:

1. On the horizontal space we consider the symplectic form $\pi^*\omega_S$; if J_S tames ω_S on S, we extend it to $\mathcal{H}$ via the isomorphism $d\pi$ and continue denoting it with the same name. This extension tames $\pi^*\omega_S$ because $(d\pi)^t(\omega_S) = \pi^*\omega_S$.
2. On the vertical bundle $\mathcal{V}$ we consider the standard metric $g_{\mathcal{V}} = dr^2 + r^2 d\theta^2$ and the complex structure $J_{\mathcal{V}}$ induced by the complex multiplication by i in the atlas $(U_\alpha, V_\alpha, \Gamma_\alpha, \phi_\alpha)$. The induced form is $\omega_{\mathcal{V}} = rdr \wedge d\theta = rdr \wedge \eta|_{\mathcal{V}}$.

Note that $\mathcal{H}^* \cong \mathrm{Ann}(\mathcal{V}) = \mathcal{C}^\infty \otimes \pi^*(\Omega^1(S))$ and that $\mathcal{V}^* \cong \mathrm{Ann}(\mathcal{H}) = \mathcal{C}^\infty \otimes \langle dr, \eta \rangle$, so we can extend any tensor initially constructed in the horizontal (vertical) distribution as being zero in the vertical (horizontal) distribution respectively. This applies especially to $J_{\mathcal{V}}$.

Before stating the result in which we construct the form ω_a, we introduce some notations. Consider the neighborhoods $D_\delta(\bar{S})$ for $0 < \delta \leq \varepsilon_0$; there exists $\delta_p^{\Sigma^*} > 0$ such that for any $0 < \delta < \delta_p^{\Sigma^*}$ it holds $D_\delta(\bar{S}) \cap D_\delta(\bar{S}') \subset B(\Sigma^1)$ for any pair of singular surfaces S, S'. Fix a singular surface S, define for $0 < \delta < \delta_p^{\Sigma^*}$ the δ-normal neighborhood of $S - B(\Sigma^1)$

$$N_\delta(S) = \bigcup_{\alpha \in \Lambda_S} \phi_\alpha(S_\alpha \times B_\delta(0)),$$

where Λ_S denotes the set of indexes α such that $V_\alpha \cap S \neq \emptyset$ and $V_\alpha \subset X - B(\Sigma^1)$. To ease notation, we assume that ε_0 is chosen so that $\varepsilon_0 < \delta_p^{\Sigma^*}$. Hence the neighborhoods $N_{\varepsilon_0}(S)$ are disjoint for the different surfaces S.

Proposition 3.28. *For every isotropy surface S there exist $0 < \delta_0^S < \frac{1}{2}\delta_p^{\Sigma^*}$, $\delta_2^S < \frac{1}{3}\delta_0^S$, and $a_2^S > 0$ such that for every $a < \min_S\{a_2^S\}$ there is a closed form $\omega_a \in \Omega^2(X - B(\Sigma^1))$ which is* *non-degenerate on* $X-(B(\Sigma^1)\cup\Sigma^*)$, *such that* $\omega_a=\omega$ *on* $X-\Big(\cup_{p\in\Sigma^1}B_{\varepsilon_p}(p)\cup_{S\in\Sigma^*}N_{2\delta_0^S}(S)\Big)$, *and on* $N_{\delta_2^S}(S)$ *we have:*

$$\omega_a=\pi^*\omega_S-\frac{1}{4}dJ_{\mathcal{V}}d(r^{2m_S}+a^2)^{\frac{1}{m_S}}$$

where $\mathbb{Z}_{m_S}$ *is the isotropy group of every* $x\in S$. *On* $\cup_{p\in\Sigma^1}(B_{2\varepsilon_p}(p)-B_{\varepsilon_p}(p))-\Sigma^*$ *the form* ω_a *is* $\mathfrak{j}$*-tamed and Kähler.*

Proof. We describe the process in a neighborhood of a fixed singular surface S. To ease notations, we denote the order of its cyclic isotropy group by m instead of m_S. Note that $J_{\mathcal{V}}(dr)=-r\eta$ for $r\neq 0$, so in particular

$$\frac{1}{2}d(r^2\eta)=-\frac{1}{2}d(rJ_{\mathcal{V}}dr)=-\frac{1}{4}dJ_{\mathcal{V}}dr^2.$$

Let ω^0 be the symplectic form of Lemma 3.27 such that

$$\omega^0=\pi^*(\omega_S)+\frac{1}{2}d(r^2\eta)=\pi^*(\omega_S)-\frac{1}{4}dJ_{\mathcal{V}}dr^2$$

on $N_{\delta_0}(S)$ and $\omega^0=\omega$ on $X-N_{2\delta_0}(S)$, for some δ_0 with $0<\delta_0<\frac{1}{2}\delta_p^{\Sigma^*}$.

Define the 2-form:

$$\omega_a^0=\pi^*(\omega_S)-\frac{1}{4}dJ_{\mathcal{V}}d(f(r^2,a)),$$

where $f(r,a)=(r^m+a^2)^{\frac{1}{m}}$.

Given a function $\bar{f}\colon\mathbb{R}\to\mathbb{R}$, the 2-form $-\frac{1}{4}dJ_{\mathcal{V}}d(\bar{f}(r^2))$ is expressed as follows:

$$\begin{aligned}-\tfrac{1}{4}dJ_{\mathcal{V}}d\bar{f}(r^2)=&-\tfrac{1}{2}dJ_{\mathcal{V}}(r\bar{f}'(r^2)dr)=\tfrac{1}{2}d(\bar{f}'(r^2)r^2\eta)\\ =&\tfrac{1}{2}r^2\bar{f}'(r^2)\pi^*\kappa+(r^2\bar{f}''(r^2)+\bar{f}'(r^2))rdr\wedge\eta,\end{aligned}$$

where $\pi^*(\kappa)=d\eta$ is the curvature of the connection. In addition, we observe:

1. The projection of $-\frac{1}{4}dJ_{\mathcal{V}}d\bar{f}$ to the space $\Lambda^2\mathcal{V}^*$ is

$$-\tfrac{1}{4}dJ_{\mathcal{V}}d\bar{f}|_{\mathcal{V}}=(r^2\bar{f}''(r^2)+\bar{f}'(r^2))rdr\wedge\eta.$$

   It is $J_{\mathcal{V}}$-tamed on an annulus $R_0\leq r\leq R_1$ as long as $x\bar{f}''(x)+\bar{f}'(x)>0$ for $x\in[R_0^2,R_1^2]$.

2. Denote $\|\cdot\|$ the norm with respect to the metric $g_S+g_{\mathcal{V}}$. If $r\leq 1$ then,

$$\|\tfrac{1}{4}dJ_vd\bar{f}\|\leq\tfrac{1}{2}|\bar{f}'(r^2)|\|\pi^*\kappa\|+|\bar{f}''(r^2)|+|\bar{f}'(r^2)|.$$

   In particular, if $\Delta=[\bar{\delta}_1,\bar{\delta}_2]\subset[0,1]$ and $\bar{f}_a$ is a family of functions such that $\bar{f}_a|_\Delta$ tends uniformly to 0 as $a\to 0$ in the C^2 norm, then given $\varepsilon>0$ one can choose $a_0>0$ small enough such that for $a<a_0$, $\|\frac{1}{4}dJ_vd\bar{f}_a\|<\varepsilon$ on $r\in\Delta$.

We now check that we can choose $\delta_1<\frac{1}{2}\delta_0$ and $a_1>0$ such that for every $a<a_1$ the form ω_a^0 is non-degenerate on $0<r<\delta_1$. The vertical part $\omega_a^0|_{\mathcal{V}}=-\frac{1}{4}dJ_{\mathcal{V}}df(r^2,a)|_{\mathcal{V}}$ is non-degenerate and $J_{\mathcal{V}}$-tamed on $r\neq 0$ because:

$$\begin{aligned}\left(\frac{d}{dr}f\right)(r,a)=&r^{m-1}(r^m+a^2)^{\frac{1}{m}-1}>0\\ \left(\frac{d}{dr^2}f\right)(r,a)=&a^2(m-1)r^{m-2}(r^m+a^2)^{\frac{1}{m}-2}>0\,.\end{aligned}$$

The horizontal part is $\omega_a^0|_{\mathcal{H}} = \pi^*(\omega_S) + \frac{1}{2}r^2\left(\frac{d}{dr}f\right)(r^2,a)\pi^*\kappa$, whose first summand $\pi^*(\omega_S)$ is non-degenerate and J_S-tamed on $\mathcal{H}$; since $r^2\left(\frac{d}{dr}f\right)(r^2,0) = r^2$ we conclude the existence of $\delta_1 < \frac{1}{2}\delta_0$ and $a_1 > 0$ such that $\omega_a^0|_{\mathcal{H}}$ is non-degenerate and J_S-tamed on $\mathcal{H}$ for $r < \delta_1$ and $a < a_1$.

Choose $\delta_2 < \frac{1}{3}\delta_1$; we now show that there exists $a_2 < a_1$ such that for every $a < a_2$ there is a form ω_a on X with $\omega_a = \omega_a^0$ if $0 \leq r \leq \delta_2$, $\omega_a = \omega^0$ if $r > 2\delta_2$ and such that ω_a is $J_{\mathcal{V}} + J_{\mathcal{H}}$ tamed on $\delta_2 < r < 2\delta_2$. Let $\rho = \rho(x)$ be a smooth function such that $\rho = 1$ if $x \leq 1$ and $\rho = 0$ if $x \geq 4$ and define $\rho_\delta(x) = \rho(\frac{x}{\delta^2})$. We also define $h(x,a) = f(x,a) - x$, $H(x,a) = \rho_{\delta_2}(x)h(x,a)$ and the closed form

$$\omega_a = \omega^0 - \frac{1}{4}dJ_{\mathcal{V}}d(H(r^2,a)).$$

We now show that this is $J_{\mathcal{V}} + J_{\mathcal{H}}$ tamed on $\delta_2 < r < 2\delta_2$. Note that the function H is smooth on $(x,a) \in (0,\infty) \times \mathbb{R}$ and satisfies that $H(x,0) = \rho_{\delta_2}(x)h(x,0) = 0$. Thus, the family $\bar{f}_a(x) = H(x,a)$ converges uniformly to 0 in the C^2 norm on the domain $x \in [\delta_2^2, 4\delta_2^2]$.

Let $\varepsilon > 0$ be such that an ε-ball with respect to $g_S + g_{\mathcal{V}}$ around ω^0 is $(J_S + J_{\mathcal{V}})$-tamed on $\delta_2 \leq r \leq 2\delta_2$. Our previous observation ensures the existence of $a_2 > 0$ such that for every $a < a_2$:

$$\|\omega_a - \omega^0\| = \|\tfrac{1}{4}dJ_{\mathcal{V}}d(H_\delta(r^2,a))\| < \varepsilon$$

on $\delta_2 \leq r \leq 2\delta_2$, and thus ω_a is $J_{\mathcal{V}} + J_{\mathcal{H}}$-tamed on $\delta_2 \leq r \leq 2\delta_2$, so it is a symplectic form there.

Note also that on the chart $B_{2\varepsilon_p}(p) \subset V_p$ the connection is flat, i.e. $\eta = d\theta$, and moreover (ω_S, J_S) becomes the standard Kähler structure on $S \cap V_p$, so that $J_{\mathcal{V}} + J_{\mathcal{H}} = \mathfrak{j}$ becomes standard on $U_p \subset \mathbb{C}^2$. Thus, the computation above proves that ω_a is $\mathfrak{j}$-tamed and Kähler on $\cup_{p\in\Sigma^1}(B_{2\varepsilon_p}(p) - B_{\varepsilon_p}(p)) - \Sigma^*$. □

Remark 3.29. For a fixed surface S, the formula defining ω_a near S clearly extends to a non-degenerate closed 2-form on $D_{\varepsilon_0}(\bar{S}) - \Sigma^1$. However, for different surfaces S, S' these extended 2-forms may differ in $D_{\varepsilon_0}(\bar{S}_i) \cap D_{\varepsilon_0}(\bar{S}_j) \subset B(\Sigma^1)$. That is why we restrict the definition of ω_a to $X - B(\Sigma^1)$.

To construct ω_a^* we interpolate ω_a with 0 near Σ^1; for that purpose we first prove that ω_a admits a Kähler potential on a neighbourhood of Σ^1. This neighbourhood consists of the union of the annuli $A_p = B_{2\varepsilon_p}(p) - B_{\varepsilon_p}(p) \subset X$ for each $p \in \Sigma^1$; these are covered by the orbifold charts $U_{A_p} = B_{2\varepsilon_p}(0) - B_{\varepsilon_p}(0)$.

Proposition 3.30. *Let $p \in \Sigma^1$; there is a Kähler potential $F_a\colon A_p \to [0,\infty)$ for the lifting of ω_a to the chart U_{A_p}. That is, in U_{A_p} we have*

$$\omega_a = \frac{\mathrm{i}}{2}\partial\bar{\partial}F_a.$$

In addition, a can be chosen so that there exists $0 < t_0 < t_1$ such that:

$$B_{\varepsilon_p}(0) \subset F_a^{-1}([0,t_0)) \subset B_{3\varepsilon_p/2}(0) \subset F_a^{-1}([0,t_1)) \subset B_{2\varepsilon_p}(0)\,.$$

Proof. First of all recall that the preparation of Lemma 3.27 does not alter $\omega|_{V_p}$, being $V = V_p$ a neighborhood of p containing $B_{2\varepsilon_p}(p)$. To ease notation let us suppose from now on that $V = B_{2\varepsilon_p}(p)$, so V is covered by an orbifold chart $U = B_{2\varepsilon_p}(0) \to V$ with coordinates (z,w). Fix a surface $S \subset \Sigma^*$. We cover $V - B_{\varepsilon_p}(p)$ with the charts $W = U_p - (B_{\varepsilon_p}(0) \cup_S N_\delta(S))$ and $W' = \cup_S W'_S$, with $W'_S = N_{2\delta}(S) \cap V$, so $V = W \cup W'$. The Kähler potential over W is of course:

$$F_a|_W = |z|^2 + |w|^2.$$

We now look for the Kähler potential near a singular surface S. Consider a rotation of V in which S corresponds to $w = 0$. By Remark 3.22, on the set $W'_S = N_{2\delta}(S) \cap V$ the expression of ω_a is:

$$\omega_a = \frac{\mathrm{i}}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w}) - \frac{1}{4} dJ_V dH(|w|^2, a)$$

where $H(x) = \rho_{\delta_2}(x)h(x,a)$. In addition, $dJ_{\mathcal{V}} dH(|w|^2, a) = d\mathrm{j}dH(|w|^2, a)$ because $J_{\mathcal{V}} + J_S = \mathrm{j}$, and $dH(|w|^2, a) \in \mathcal{V}^*$. Moreover, taking into account that $\mathrm{j}(d\zeta) = \mathrm{i}d\zeta$ and $\mathrm{j}(d\bar{\zeta}) = -\mathrm{i}d\bar{\zeta}$ for a complex variable ζ, we get $\mathrm{j}\partial = \mathrm{i}\partial$ and $\mathrm{j}\bar{\partial} = -\mathrm{i}\bar{\partial}$. Hence we obtain:

$$d\mathrm{j}d = (\partial + \bar{\partial})\mathrm{j}(\partial + \bar{\partial}) = -2\mathrm{i}\partial\bar{\partial}\,.$$

Thus, $-\frac{1}{4} dJ_V dH(|w|^2, a) = \frac{\mathrm{i}}{2}\partial\bar{\partial}H(|w|^2, a)$ and the Kähler potential is:

$$F_a|_{W'_S} = |z|^2 + |w|^2 + H(|w|^2, a).$$

Note that if $|w| > 2\delta_2$ then $H(|w|^2, a) = 0$; thus $F_a|_{|w|>2\delta_2} = |z|^2 + |w|^2$.

If we consider another singular surface S' with $p \in \bar{S}'$, we make another rotation in V and repeat the process to construct F_a near S'. Since transition functions are rotations, these functions glue together and give a function F_a well-defined on A. Note that, as discussed above, the global expression of F_a in A depends on both the radius $r^2 = |z|^2 + |w|^2$ and the distance d_S from a surface S. That is, we have in global coordinates $(z, w) \in A$ the expression:

$$F_a(z, w) = |z|^2 + |w|^2 + \sum_S H(d_S(z, w)^2, a)$$

where each $H(d_S(z, w)^2)$ extends as 0 outside $N_{2\delta_2}(S)$. Finally, the choice of $0 < t_0 < t_1$ with

$$B_{\varepsilon_p}(0) \subset F^{-1}([0, t_0)) \subset B_{3\varepsilon_p/2}(0) \subset F^{-1}([0, t_1)) \subset B_{2\varepsilon_p}(0)$$

can be made for a small enough. Indeed, the function $|z|^2 + |w|^2$ satisfies the above property for $t_0 = \frac{5}{4}\varepsilon_p$ and $t_1 = \frac{7}{4}\varepsilon_p$, and $H_a(x) = H(x, a)$ are positive functions that converge uniformly to 0 as $a \to 0$. □

We now prove a technical result that enables us to perform the desired interpolation.

Lemma 3.31. *Let $V \subset \mathbb{C}^n$ open, and $F\colon V \to \mathbb{R}$ a smooth function such that $\frac{\mathrm{i}}{2}\partial\bar{\partial}F$ is j-semipositive. Let $h\colon \mathbb{R} \to \mathbb{R}$ smooth with $h' \geq 0$, $h'' \geq 0$. Denote $\omega = \frac{\mathrm{i}}{2}\partial\bar{\partial}F$, $\omega_h = \frac{\mathrm{i}}{2}\partial\bar{\partial}(h \circ F)$.*

Then the form ω_h is j-semipositive. Moreover, ω_h is j-positive on the subset of V where $\omega = \frac{\mathrm{i}}{2}\partial\bar{\partial}F$ is j-positive and $h'(F) > 0$.

Proof. A computation in the complexified tangent bundle $TV \otimes \mathbb{C}$ gives that

$$\tfrac{\mathrm{i}}{2}\partial\bar{\partial}(h \circ F) = \tfrac{i}{2}h''(F)\partial F \wedge \bar{\partial}F + \tfrac{i}{2}h'(F)\partial\bar{\partial}F.$$

On the other hand denote

$$\beta = \partial F \wedge \bar{\partial}F = \sum_{i,j} (\partial_{z_i}F)(\partial_{\bar{z}_j}F) dz_i \wedge d\bar{z}_j\,.$$

Recall that $\beta(v, \mathrm{j}v) = -\frac{\mathrm{i}}{2}\beta(v - \mathrm{i}\mathrm{j}v, v + \mathrm{i}\mathrm{j}v)$ for every $v \in TV$, with $v - \mathrm{i}\mathrm{j}v \in T^{1,0}V$. Take a vector $u = v - \mathrm{i}\mathrm{j}v = \sum_i a_i\partial_{z_i} \in T^{1,0}V$ and compute:

$$\begin{aligned}\beta(u, \bar{u}) &= \sum_{i,j} (\partial F \wedge \bar{\partial}F)(a_i\partial_{z_i}, \bar{a}_j\partial_{\bar{z}_j}) = \sum_{i,j} a_i\bar{a}_j(\partial_{z_i}F)(\partial_{\bar{z}_j}F) \\ &= \sum_{i,j} (a_i\partial_{z_i}F)(\bar{a}_j\partial_{\bar{z}_j}F) = |\sum_i a_i\partial_{z_i}F|^2 = |\partial F(u)|^2\,.\end{aligned}$$

Here we have taken into account that $\partial_{\bar{z}_j} F = \overline{\partial_{z_j} F}$ because F is real. This shows that $\frac{\mathrm{i}}{2}\beta(v, \mathrm{j}v) = \frac{1}{4}\beta(u, \bar{u}) = \frac{1}{4}|\partial F(u)|^2$ for $v \in TV$. Finally, since $\omega_h = \frac{\mathrm{i}}{2}h''(F)\beta + h'(F)\omega$, the result is clear. □

Consider the Kähler potential for ω_a in the chart U_A, given by $F_a \colon U_A \to [0, \infty)$. As shown in Proposition 3.30, we can take numbers $t_1 > t_0 > 0$ so that

$$B_{\varepsilon_p}(0) \subset F_a^{-1}([0, t_0)) \subset B_{3\varepsilon_p/2}(0) \subset F_a^{-1}([0, t_1)) \subset B_{2\varepsilon_p}(0)\,.$$

Let $h \colon \mathbb{R} \to \mathbb{R}$ be a function which vanishes for $t \leq t_0$, such that $h(t) = t + c$ for $t \geq t_1$, and with $h', h'' \geq 0$. For instance one can take a bump function ϱ with $\varrho' \geq 0$ so that ϱ vanishes in $(-\infty, t_0)$ and equals 1 in $(t_1, +\infty)$, and then define $h(t) = \int_{-\infty}^{t} \varrho$.

Let us define $\omega_a^* = \frac{\mathrm{i}}{2}\partial\bar{\partial}(h \circ F_a)$. This gives a closed 2-form in $U_A = B_{2\varepsilon_p}(0) - B_{\varepsilon_p}(0)$ which is j-semipositive by Lemma 3.31 above; moreover it extends to $B_{\varepsilon_p}(0)$ as zero. The global formula on U_A for the Kahler potential F_a shows that F_a is invariant by the isotropy group Γ_p, therefore $h \circ F_a$ is also Γ_p-invariant. On the other hand, as Γ_p acts by holomorphic maps, we have that $\bar{\partial}\gamma^* = \gamma^*\bar{\partial}$ and $\partial\gamma^* = \gamma^*\partial$ as operators acting on forms, for any $\gamma \in \Gamma_p$.

It follows that $\omega_a^* = \partial\bar{\partial}(h \circ F_a)$ is Γ_p-invariant in U_p. Since ω_a^* equals ω_a outside $B_{2\varepsilon_p}(0)$, we see that ω_a^* is a global orbifold 2-form defined on X. We summarize the above discussion in the following:

Corollary 3.32. *There exists a closed orbifold 2-form ω_a^* in X satisfying:*

- *It vanishes on $B_{\varepsilon_p}(p)$.*
- *It is j-positive on $B_{2\varepsilon_p}(p) - (B_{\varepsilon_p}(p) \cup \Sigma^*)$. In fact, $\omega_a^* = \partial\bar{\partial}(h \circ F_a)$ there.*
- *It coincides with ω_a outside $B_{2\varepsilon_p}(p)$.*

Desingularisation

As explained before, we now define a smooth atlas $\widehat{\mathcal{A}}$ on $X - \Sigma^1$ that makes the map $\mathrm{Id} \colon (X - \Sigma^1, \mathcal{A}) \to (X - \Sigma^1, \widehat{\mathcal{A}})$ differentiable; we also prove that $\widehat{\omega}_a^* = \mathrm{Id}_*(\omega_a^*)$ is the desired 2-form. In order to make the presentation clearer, we first check in Proposition 3.33 that $\widehat{\omega}_a = \mathrm{Id}_*(\omega_a)$ endows $(X - \cup_{p \in \Sigma^1} B_{\varepsilon_p}(p), \widehat{\mathcal{A}})$ with the structure of a symplectic manifold. For simplicity let us denote $B(\Sigma^1) = \cup_{p \in \Sigma^1} B_{\varepsilon_p}(p)$.

Proposition 3.33. *Notations and hipotheses as above. The following holds:*

1. *There is a manifold atlas $\widehat{\mathcal{A}} = \{(\widehat{U}_\alpha, \widehat{V}_\alpha, \widehat{\phi}_\alpha, \widehat{\Gamma}_\alpha)\}$ on $X - B(\Sigma^1)$ (i.e. an orbifold atlas with isotropy $\widehat{\Gamma}_\alpha = \{1\}$) such that the identity*
$$\mathrm{Id} \colon (X - \Sigma^1, \mathcal{A}) \to (X - \Sigma^1, \widehat{\mathcal{A}}),$$
is a smooth orbifold map, and it is a diffeomorphism away from Σ^*.
2. *The push-forward $\widehat{\omega}_a = (\mathrm{Id})_*(\omega_a)$ is smooth on $(X - B(\Sigma^1), \widehat{\mathcal{A}})$, and is a symplectic form for $a < a_2$. In addition, on $(\cup_{p \in \Sigma^1}(B_{2\varepsilon_p}(p) - B_{\varepsilon_p}(p)), \widehat{\mathcal{A}})$ we have that $\widehat{\omega}_a$ is tamed by j.*

Proof. We shall modify some orbifold charts of $\mathcal{A}$ to obtain $\widehat{\mathcal{A}}$. First, if $x \notin \Sigma^*$ we consider an orbifold chart $(U_x, V_x, \phi_x, \{1\}) \in \mathcal{A}$ around x with $V_x \cap \Sigma^* = \emptyset$ and we take this as a chart of x in $\widehat{\mathcal{A}}$. Now, given a singular surface S with isotropy isomorphic to $\mathbb{Z}_m$, we consider the cover of $D_{\varepsilon_0}(\bar{S})$ as in Lemma 3.21. Take $(U_\alpha, V_\alpha, \Gamma_\alpha, \phi_\alpha)$ in this cover with $U_\alpha = S_\alpha \times D_{\varepsilon_0}$ and $p \notin V_\alpha$. We define $\widehat{U}_\alpha = S_\alpha \times D_{(\varepsilon_0)^m}$, $\widehat{V}_\alpha = V_\alpha$, and $\widehat{\phi}_\alpha(z', w') = \phi_\alpha(z', w'^{\frac{1}{m}})$. Despite the fact that $w'^{\frac{1}{m}}$ is not well-defined on $\mathbb{C}$, the composition $\phi_\alpha \circ (z', w'^{\frac{1}{m}})$ is because ϕ_α is a Γ_α-invariant map. The manifold coordinates (z', w') of $\widehat{\mathcal{A}}$ and the orbifold coordinates (z, w) of $\mathcal{A}$ are related by $w' = w^m, z' = z$. We now check that the change of charts of $\widehat{\mathcal{A}}$ are smooth. Denote

$$\psi_{\alpha\beta} = (\psi^1_{\alpha\beta}, \psi^2_{\alpha\beta}) \colon U_\alpha \to U_\beta$$

the change of charts of the atlas $\mathcal{A}$. Let $V_\alpha \subset D_{\varepsilon_0}(\bar{S})$ be a chart from $\mathcal{A}$ not containing any $p \in \Sigma^1$, and take $(\widehat{U}_\alpha, \widehat{V}_\alpha, \widehat{\phi}_\alpha, \widehat{\Gamma}_\alpha)$ another chart in $\widehat{\mathcal{A}}$. Two cases arise:

1. If $\widehat{V}_\beta \subset D_{\varepsilon_0}(\bar{S}) - \Sigma^1$ we have induced transition functions given by

$$\widehat{\psi}_{\alpha\beta} \colon \widehat{U}_\alpha \to \widehat{U}_\beta\,, \quad (z', w') \mapsto (\psi_{\alpha\beta}(z'), A_{\alpha\beta}(z')^m w') = \psi_{\alpha\beta}(z', w'^{\frac{1}{m}})$$

   because

$$\widehat{\phi}_\beta(\widehat{\psi}_{\alpha\beta}(z', w')) = \widehat{\phi}_\beta(\psi_{\alpha\beta}(z', w'^{\frac{1}{m}})) = \phi_\alpha(z', w'^{\frac{1}{m}}) = \widehat{\phi}_\alpha(z', w').$$

   The map $\widehat{\psi}_{\alpha\beta}$ is a diffeomorphism because $A_{\alpha\beta}(z) \in \mathrm{U}(1)$.

2. If $\widehat{V}_\beta \not\subset D_{\varepsilon_0}(\bar{S})$ and $\widehat{V}_\beta \cap D_{\varepsilon_0}(\bar{S}) \neq \emptyset$, then by construction $\widehat{V}_\beta \cap \Sigma^* = \emptyset$. The induced change of chart in the atlas $\widehat{\mathcal{A}}$ is

$$\widehat{\psi}_{\alpha\beta}(z', w') = (\psi^1_{\alpha\beta}(z', w'), \psi^2_{\alpha\beta}(z', w')^m)\,;$$

   this is a local diffeomorphism since $V_\beta \cap \Sigma^* = \emptyset$ and therefore $\psi^2_{\alpha\beta}(z, w) \neq 0$.

The identity map Id restricted to $D_{\varepsilon_0}(\bar{S})$ is covered by the local maps:

$$\mathrm{Id}_\alpha \colon U_\alpha \to \widehat{U}_\alpha, \quad (z, w) \to (z, w^m) = (z', w'),$$

which are diffeomorphisms outside $w = 0$. Note that the radial function $r' = |w'|$ is again well-defined on $(D_{\varepsilon_0^m}(\bar{S}), \widehat{\mathcal{A}})$ and $\mathrm{Id}^*(r') = r^m = |w|^m$.

We now consider the symplectic form around a singular surface S; we follow the notation of Proposition 3.28. First observe that if $\eta_\alpha = \pi^*(\nu_\alpha) + d\theta$, then $(\mathrm{Id})_*(\eta_\alpha) = \widehat{\pi}^*(\nu_\alpha) + m\, d\theta$, where $\widehat{\pi} \colon D_{\varepsilon_0^m}(\bar{S}) \to \bar{S}$ is the projection. If we define $\eta' = \frac{1}{m}\mathrm{Id}_*(\eta)$, then η' is a connection form on $D_{\varepsilon_0^m}(\bar{S})$. Again, one can define smooth distributions $H' = \ker(r'dr' \wedge \eta')$ and $V' = \ker d\widehat{\pi} = \mathrm{Id}_*(\ker d\widehat{\pi})$, and almost Kähler structures as before: $(\widehat{\pi}^*\omega_S, J'_S)$, $(r'dr' \wedge \eta', J'_V = \mathrm{i})$. Taking into account that $\mathrm{Id}^*(r') = r^m$ and the fact that $(\mathrm{Id}_*)J_V = J'_V$ (since Id is holomorphic), we obtain:

$$(\mathrm{Id})^*(\pi^*\omega_S - \frac{1}{4} dJ'_V d(r'^{\,2} + a^2)^{\frac{1}{m}}) = \omega_a, \quad r' \leq \delta_2^m.$$

Therefore $\widehat{\omega}_a = \mathrm{Id}_*(\omega_a)$ extends smoothly to $(X - B_{\varepsilon_p}(p), \widehat{\mathcal{A}})$, it is closed, and it is non-degenerate outside of S. Moreover, near S it has the form:

$$\widehat{\omega}_a = \pi^*\omega_S - \frac{1}{4} dJ'_V d(r'^{\,2} + a^2)^{\frac{1}{m}}\,.$$

At every point of $S = \{r' = 0\}$, taking into account the formula obtained for $dJ_V d\bar{f}(r^2)$ in Proposition 3.28, the form $\widehat{\omega}_a$ coincides with:

$$\widehat{\pi}^*\omega_S + \frac{1}{m a^{1-\frac{1}{m}}} r' dr' \wedge \eta',$$

which is $J'_V + J'_H$-tamed.

Take the set $V_S^* = (B_{2\varepsilon_p}(p) - B_{\varepsilon_p}(p)) \cap N_{\delta_2}(S)$, covered by the chart $U_S^* = (B_{2\varepsilon_p}(0) - B_{\varepsilon_p}(0)) \cap (\mathbb{C} \times D_{\delta_2}) \in \mathcal{A}$; this has isotropy $\tilde{\Gamma} = \{\gamma \in \Gamma_p \text{ s.t. } \gamma(z,0) = (z',0)\}$. Consider the induced chart $\widehat{U}_S^* \in \widehat{\mathcal{A}}$ which has coordinates (z', w') given by

$$\mathrm{Id}_{U_S^*} \colon U_S^* \to \widehat{U}_S^*, \qquad (z,w) \mapsto (z, w^m) = (z', w') .$$

Its isotropy is $\tilde{\Gamma}/\mathbb{Z}_m$, that acts without fixed points on U_S^*. We claim that $\widehat{\omega}_a$ is tamed by the standard complex structure j on $\widehat{U}_S^*$. Moreover, we can push-forward an almost complex structure on $(X - \Sigma^1, \mathcal{A})$ to $(X - \Sigma^1, \widehat{\mathcal{A}})$, and near Σ^1 this push-forward gives the standard almost complex structure. Note that ω_a is j-tamed in U_S^* and, outside S, $\widehat{\omega}_a$ coincides with ω_a via the local biholomorphism $\mathrm{Id}_{U_S^*}$. Indeed, we saw that the form ω_a was tamed on U_S^* by $J_H + J_V = \mathrm{j}$; outside S we have

$$J'_V + J'_H = (\mathrm{Id}_{U_S^*})_*(J_V + J_H) = (\mathrm{Id}_{U_S^*})_*(\mathrm{j}) = \mathrm{j},$$

the last equality since $\mathrm{Id}_{U_S^*}$ is holomorphic. Hence $\widehat{\omega}_a$ is j-tamed in $\widehat{U}_S^* - S$, and also in $S \cap \widehat{U}_S^*$ because it is $J'_V + J'_H$-tamed on S and $J'_V + J'_H = \mathrm{j}$ near p. □

To finish this section we extend the form $\widehat{\omega}_a$ by zero as we did with ω_a in Corollary 3.32:

Corollary 3.34. *There exists a closed orbifold 2-form $\widehat{\omega}_a^*$ in $(X - \Sigma^1, \widehat{\mathcal{A}})$ satisfying:*

- *It vanishes on $B(\Sigma^1)$.*
- *It is j-positive on $\cup_{p \in \Sigma^1} B_{2\varepsilon_p}(p) - B_{\varepsilon_p}(p)$.*
- *It coincides with $\widehat{\omega}_a$ outside $\cup_{p \in \Sigma^1} \bar{B}_{2\varepsilon_p}(p)$. In particular it is symplectic there.*

Proof. Consider the orbifold symplectic form ω_a^* on $(X, \mathcal{A})$ of Corollary 3.32. We need to check that the form $\mathrm{Id}_*(\omega_a^*|_{X-\Sigma^*})$ extends to a closed 2-form $\widehat{\omega}_a^*$ on $(X - B(\Sigma^1), \widehat{\mathcal{A}})$; the extension has the required properties. As $\omega_a^* = \omega_a$ outside $B_{2\varepsilon_p}(p)$ and $\mathrm{Id}_*(\omega_a) = \widehat{\omega}_a$, we only need to check that the push-forward of ω_a^* extends on $B_{2\varepsilon_p}(p)$.

Let us consider an isotropy surface $S \subset \Sigma^*$ and denote by $\widehat{\phi} \colon \widehat{U}_S^* \to V_S^*$ the manifold chart in $\widehat{\mathcal{A}}$ that we constructed in the proof of Proposition 3.33 in order to desingularize $V_S^* = (V - B_{\varepsilon_p}(p)) \cap N_\delta(S)$. The restriction of the identity map

$$\mathrm{Id} \colon (V_S^*, \mathcal{A}) \to (V_S^*, \widehat{\mathcal{A}})$$

is holomorphic, and its inverse is holomorphic on $V_S^* - S$. This leads to the following equality on $V_S^* - S$:

$$\mathrm{Id}_*(\omega_a^*|_{X-\Sigma^*}) = \frac{\mathrm{i}}{2} \partial \bar{\partial} (h \circ \widehat{F}_a),$$

where $h \colon \mathbb{R} \to \mathbb{R}$ is the smooth function constructed in Corollary 3.32 and

$$\widehat{F}_a(z,w) = |z|^2 + |w|^{\frac{2}{m}} + \rho_{\delta_2}(|w|^{\frac{2}{m}}) \left((|w|^2 + a)^{\frac{1}{m}} - |w|^{\frac{2}{m}} \right).$$

The function $\widehat{F}_a$ has a smooth extension defined on V_S^* because near $w = 0$ the expression of $\widehat{F}_a$ is $\widehat{F}_a(z,w) = |z|^2 + (|w|^{\frac{2}{m}} + a)^{\frac{1}{m}}$. Thus, we can extend $\mathrm{Id}_*(\omega_a^*)$ over S. □

3.4.3 Symplectic orbifold structure with only isolated singularities.

Now first extend our manifold atlas $\widehat{\mathcal{A}}$ of $X - \Sigma^1$ to an orbifold atlas of X with only isolated singularities, and then we extend the symplectic form; ending up with $(X, \widehat{\mathcal{A}}, \widehat{\omega})$ a symplectic orbifold with only isolated singularities.

Extension of the orbifold structure.

Let $p \in \Sigma^1$ and let $(U, V, \Gamma, \mathrm{j}, \omega_0)$ be a Kähler orbifold chart of $(X, \mathcal{A})$ around p. We have $\Gamma^* \lhd \Gamma < \mathrm{U}(2)$, with Γ^* the isotropy group of the surfaces $S \subset \Sigma^*$ accumulating at p and $\Gamma' = \Gamma/\Gamma^*$ the quotient, which acts in U/Γ^*. The manifold $(V - \{p\}, \widehat{\mathcal{A}})$ has a complex structure induced from the orbifold chart $(U - \{0\}, \mathrm{j}) \in \mathcal{A}$, as was shown in Proposition 3.33. On the other hand, $V \cong U/\Gamma$ has the structure of a complex orbifold induced by $\mathcal{A}$. The identity map $\mathrm{Id}\colon (V - \{p\}, \mathcal{A}) \to (V - \{p\}, \widehat{\mathcal{A}})$ is holomorphic and a biholomorphism outside of Σ^*. In both cases, the complex structure is the restriction to $U - \{0\}$ of the standard complex structure j on $\mathbb{C}^2$.

We also have a covering map $(U - \{0\})/\Gamma^* \to (U - \{0\})/\Gamma$ because Γ' acts freely on $(U - \{0\})/\Gamma^*$. This allows us to consider the complex manifold $(U - \{0\}/\Gamma^*, \widehat{\mathcal{A}})$ and the complex orbifold $(U - \{0\}/\Gamma^*, \mathcal{A})$; the complex structure is again in both cases induced from $\mathbb{C}^2$, and the identity map $(U - \{0\}/\Gamma^*, \mathcal{A}) \to (U - \{0\}/\Gamma^*, \widehat{\mathcal{A}})$ is holomorphic and biholomorphic outside Σ^*. The next proposition shows that the orbifold $(U - \{0\}/\Gamma^*, \mathcal{A})$ can be naturally seen as an open set of $\mathbb{C}^2$, allowing us to extend the complex structure $(U - \{0\}/\Gamma^*, \widehat{\mathcal{A}})$ at the point 0.

Proposition 3.35. *The complex manifold structure on $((U - \{0\})/\Gamma^*, \widehat{\mathcal{A}})$ can be naturally extended to a complex manifold structure on U/Γ^* so that the group $\Gamma' = \Gamma/\Gamma^*$ acts by biholomorphisms in the complex manifold $(U/\Gamma^*, \widehat{\mathcal{A}})$.*

In addition, there is an open set $\widehat{U} \subset \mathbb{C}^2$ containing 0, a group Γ'' acting on $\widehat{U}$ by biholomorphisms, and a biholomorphic map $G\colon (\widehat{U}, \mathrm{j}) \to (U/\Gamma^*, \widehat{\mathcal{A}})$ such that G is (Γ'', Γ')-equivariant.

Proof. As explained in the proof of Lemma 3.19 there is a homeomorphism,

$$H\colon \mathbb{C}^2 \to \mathbb{C}^2, \qquad H(z) = (f(z_1, z_2), g(z_1, z_2)). \tag{3.2}$$

where $\{f, g\}$ is a basis of the algebra $\mathbb{C}[z_1, z_2]^{\Gamma^*}$ of Γ^*-invariant polynomials. This map induces a homeomorphism $\bar{H}\colon \mathbb{C}^2/\Gamma^* \to \mathbb{C}^2$ which is holomorphic as an orbifold map and a biholomorphism outside of the singular locus Σ^*; here we have considered $\mathbb{C}^2/\Gamma^*$ as a complex orbifold, covered by a unique chart $(\mathbb{C}^2, \Gamma^*)$. The structure that $(U - \{0\})/\Gamma^*$ inherits when viewed as an open subset of $\mathbb{C}^2/\Gamma^*$ is precisely the orbifold structure determined by $\mathcal{A}$. Let us call $G' = \bar{H}^{-1}$, define $\widehat{U} = H(U) \subset \mathbb{C}^2$, so $U/\Gamma^* \cong \widehat{U}$ via $\bar{H}$. Let $G = \mathrm{Id} \circ G'\colon \widehat{U} \to U/\Gamma^*$ and consider the restriction

$$G|\colon (\widehat{U} - \{0\}, \mathrm{j}) \xrightarrow{G'} (U - \{0\}/\Gamma^*, \mathcal{A}) \xrightarrow{\mathrm{Id}} ((U - \{0\})/\Gamma^*, \widehat{\mathcal{A}}),$$

which is bijective and biholomorphic outside of $G|^{-1}(\Sigma^*)$, and can be extended as a homeomorphism from $\widehat{U}$ to U/Γ^*. The inverse $G|^{-1}$ is holomorphic outside of Σ^*, being $\Sigma^* \cap U$ a union of complex hyperplanes. Also, $G|^{-1}$ is a homeomorphism onto the set $\widehat{U} - \{0\} \subset \mathbb{C}^2$ which is bounded. By the Riemann extension theorem, $G|^{-1}$ is holomorphic. The inverse function theorem ensures that $G|$ is a biholomorphism. This shows that the complex manifold structure on $((U - \{0\})/\Gamma^*, \widehat{\mathcal{A}})$ can be extended naturally to all U/Γ^*, in such a way that $G\colon (\widehat{U}, \mathrm{j}) \to (U/\Gamma^*, \widehat{\mathcal{A}})$ is a global complex chart, hence a biholomorphism.

We consider $\Gamma'' = \{\gamma'' = G^{-1} \circ [\gamma] \circ G \colon [\gamma] \in \Gamma' = \Gamma/\Gamma^*\}$. Observe that, since $\gamma \in U(2)$, the action of Γ' in $(U/\Gamma^*, \widehat{\mathcal{A}})$ is holomorphic outside the isotropy, i.e. on $((U - \Sigma)/\Gamma^*, \widehat{\mathcal{A}})$ with $\Sigma = \Sigma^* \cup \{0\}$ a complex subvariety. Again by the Riemann extension theorem, the action of Γ' must be holomorphic on all $(U/\Gamma^*, \widehat{\mathcal{A}})$. Since G is a biholomorphism, every $\gamma'' \in \Gamma''$ is a biholomorphism of $(\widehat{U}, \mathrm{j})$. Hence Γ'' acts on $\widehat{U}$ by biholomorphisms.

We call $q\colon U/\Gamma^* \to (U/\Gamma^*)/\Gamma' \cong U/\Gamma$ the quotient map. Consider $Y = (U/\Gamma^*, \widehat{\mathcal{A}})$ a complex manifold and $\Gamma' \cong \Gamma''$ equivalent groups acting on Y by biholomorphisms, so the space Y/Γ' is a complex orbifold. In addition, $(\widehat{U}, Y/\Gamma', \phi_0, \Gamma'')$ gives a global orbifold chart of Y/Γ', with $\phi_0 = q \circ G\colon \widehat{U} \to Y/\Gamma'$ the orbifold chart that induces $\bar{\phi}_0\colon \widehat{U}/\Gamma'' \to Y/\Gamma'$ a homeomorphism.

Finally, using the homeomorphism $h\colon Y/\Gamma' \to V$ given by $Y/\Gamma' = (U/\Gamma^*)/\Gamma' \cong U/\Gamma \cong V$ we have $(\widehat{U}, V, \widehat{\phi}, \Gamma'')$ with $\widehat{\phi} = h \circ \phi_0$; this gives an orbifold chart around the point $p \in X$ which is compatible with the manifold structure $(X - \Sigma^1, \widehat{\mathcal{A}})$. □

Corollary 3.36. *The map G induces an orbifold chart $(\widehat{U}, V, \widehat{\phi}, \Gamma'')$ of $V = V^p$ which is compatible with the manifold structure $(X - \Sigma^1, \widehat{\mathcal{A}})$.*

Symplectic form on $(X, \widehat{\mathcal{A}})$

Adding to $\widehat{\mathcal{A}}$ the charts defined in Corollary 3.36 we obtain an orbifold atlas $\mathcal{A}'$ on X with isolated singularities. We also have a symplectic form $\widehat{\omega}_a^*$ on $(X - B(\Sigma^1), \widehat{\mathcal{A}})$ given by Proposition 3.33. The last step now is extending the symplectic form $\widehat{\omega}_a^*$ to all the orbifold $(X, \mathcal{A}')$. The following lemma is useful for our purpose:

Lemma 3.37. *Denote $B_r = B_r(0)$ a ball of radius r, and let $U \subset \mathbb{C}^n$ be an open set containing B_{r_0}. Let $\omega_1, \omega_2 \in \Omega^2(U)$ be closed 2-forms so that:*

- *The form ω_1 vanishes on $\bar{B}_{\varepsilon_1}$, it is $\jmath$-semipositive in $B_{\varepsilon_2} - \bar{B}_{\varepsilon_1}$, and it is $\jmath$-positive in $U - B_{\varepsilon_2}$, for some $\varepsilon_1 < \varepsilon_2 < r_0$.*
- *The form ω_2 is non-degenerate in U and $\jmath$-tamed.*

Then, for any choice of ε_3 with $r_0 > \varepsilon_3 > \varepsilon_2$ there is a $\jmath$-tamed symplectic form ω in U so that $\omega|_{B_{\varepsilon_1}} = \delta\omega_2$ for some $\delta > 0$ small, $\omega = \omega_1$ outside B_{ε_3}.

Proof. Let $\rho = \rho_\varepsilon(r)$ be a radial bump function which equals 1 in $0 \le r \le \varepsilon_2$ and equals 0 in $r \ge \varepsilon_3$. Let $\beta \in \Omega^1(B_{r_0})$ such that $d\beta = \omega_2$. Let us define $\omega = \omega_\delta = \omega_1 + \delta d(\rho\beta)$. We have that $\omega = \delta\omega_2$ on $r \le \varepsilon_1$, so it is symplectic and j-tamed there. On $\varepsilon_1 \le r \le \varepsilon_2$ we have $\omega = \omega_1 + \delta\omega_2$; as ω_1 is j-semipositive and ω_2 is j-positive in $B_{\varepsilon_2} - B_{\varepsilon_1}$, we see that ω is j-positive in $B_{\varepsilon_2} - B_{\varepsilon_1}$. Also, $\omega = \omega_1$ on $r \ge \varepsilon_3$.

Finally, on $\varepsilon_2 \le r \le \varepsilon_3$ we have $\omega = \omega_1 + \delta d\rho \wedge \beta + \delta\rho\,\omega_2$. Since ω_1 is j-positive on the compact annulus $\varepsilon_2 \le r \le \varepsilon_3$, there exists a constant $C > 0$ with $\omega_1(u, \jmath u) \ge C|u|^2$ for all $u \in \mathbb{R}^{2n}$ and all points in the annulus. Hence

$$\begin{aligned}
|\omega(u, \jmath u)| &= |\omega_1(u, \jmath u) + \delta d\rho \wedge \beta(u, \jmath u) + \delta\rho\,\omega_2(u, \jmath u)| \\
&\ge \omega_1(u, \jmath u) + \delta\rho\,\omega_2(u, \jmath u) - |\delta d\rho \wedge \beta(u, \jmath u)| \\
&\ge C|u|^2 - \delta\|d\rho \wedge \beta\||u|^2 \\
&= (C - \delta\|d\rho \wedge \beta\|)|u|^2
\end{aligned}$$

so if $\delta < \frac{C}{\|d\rho\wedge\beta\|+1}$ then ω is j-tamed and symplectic. □

Now recall the closed 2-form $\widehat{\omega}_a^*$ of Corollary 3.34. The form $\widehat{\omega}_a^*$ is defined on $(X, \mathcal{A}')$, vanishes on $B(\Sigma^1)$, coincides with $\widehat{\omega}_a$ outside $B_{2\varepsilon_p}(p)$, and it is j-positive on $\cup_{p\in\Sigma^1}(B_{2\varepsilon_p}(p) - B_{\varepsilon_p}(p))$. By Lemma 3.37 we can glue $\widehat{\omega}_a^*$ with the standard symplectic form ω_0 near p to construct an orbifold symplectic form on $(X, \mathcal{A}')$ extending $\widehat{\omega}_a$.

Corollary 3.38. *There exists an orbifold symplectic form $\bar{\omega}_a$ on $(X, \mathcal{A}')$ which coincides with $\widehat{\omega}_a$ outside of some neighborhood of Σ^1.*

Proof. Consider the orbifold chart $(\widehat{U}, V, \widehat{\phi}, \Gamma'')$ around a point $p \in \Sigma^1$ of Corollary 3.36. Consider a local representative of $\widehat{\omega}_a^*$ in the chart $\widehat{U}$, denote it $\omega_1 = \widehat{\omega}_a^*$. Consider also $\omega_2 = -\frac{\mathrm{i}}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w})$ the standard symplectic form on $\widehat{U} \subset \mathbb{C}^2$. Take balls $B_{\varepsilon_i} \subset \widehat{U}$ so that

$$B_{\varepsilon_1} \subset \widehat{\phi}^{-1}(B_{\varepsilon_p}(p)) \subset \widehat{\phi}^{-1}(B_{2\varepsilon_p}(p)) \subset B_{\varepsilon_2} \subset B_{\varepsilon_3}\,.$$

We have that ω_1 vanishes on B_{ε_1}, it is j-semipositive on $B_{\varepsilon_2} - B_{\varepsilon_1}$ and coincides with $\widehat{\omega}_a$ outside B_{ε_2}, so it is j-positive there. We are in the hypothesis of Lemma 3.37, and this gives our desired symplectic form $\bar{\omega}_a$ in $\widehat{U}$ with $\bar{\omega}_a = \widehat{\omega}_a$ outside B_{ε_3}. The only point is that $\bar{\omega}_a$ may not be Γ''-invariant; in case it is not, replace it by its average over Γ'', which is also j-tamed because diffeomorphisms on Γ'' are holomorphic. Being ω_1 invariant under Γ'', the average coincides with $\widehat{\omega}_a$ outside B_{ε_3}. ☐

Corollary 3.39. *The symplectic orbifold $(X, \mathcal{A}', \bar{\omega}_a)$ has only isolated singularities.*

3.4.4 Cohomology groups of the resolution

The computation of the cohomology groups of the resolution can be obtained from the results in [28].

Proposition 3.40. *Let $\pi\colon (\widetilde{X}, \widehat{\omega}) \to (X, \omega)$ be a symplectic resolution of a symplectic orbifold. Define the subset of Σ^1*

$$\Delta = \{x \in \Sigma^1 \ s.t.\ \Gamma_x/\Gamma_x^* \neq \{1\}\},$$

where Γ_x^* is the subgroup of Γ_x generated by the isotropy surfaces accumulating at x. For each $p \in \Delta \cup \Sigma^0$, let $E_p = \pi^{-1}(p)$ be the exceptional set. For $k > 0$ there is a short exact sequence:

$$0 \to H^k(X) \xrightarrow{\pi^*} H^k(\widetilde{X}) \xrightarrow{i^*} \bigoplus_{p \in \Sigma^0 \cup \Delta} H^k(E_p) \to 0.$$

Proof. The symplectic resolution of $(X, \mathcal{A}, \omega)$ is divided into two steps; we first perform a partial resolution $(X, \widehat{\mathcal{A}}, \widehat{\omega}) \to (X, \mathcal{A}, \omega)$. The underlying topological space of the partial resolution does not change but its singularities are isolated and consists precisely of the points in $\Delta \cup \Sigma^0$. After this, we construct a resolution $(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\omega}) \to (X, \widehat{\mathcal{A}}, \widehat{\omega})$ employing the method described in [28, Theorem 3.3]. The cohomology ring of $\widetilde{X}$ was computed in [28, Proposition 3.4] and implies the statement. ☐

3.5 Examples

In this section we give some examples of 4-orbifolds to which the resolution described above can be applied.

Products of orbifolds

Let (S, ω) be a compact symplectic 2-dimensional orbifold. Its isotropy set consists of an isolated set of points $\{p_0, \ldots, p_n\}$; we denote the isotropy group of p_j by G_j.

Consider the product orbifold $(S \times S, \omega + \omega)$ and the symplectic involution $R(x, y) = (y, x)$. Let us define the symplectic orbifold $X = (S \times S)/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{R, \mathrm{Id}\}$, and denote $q\colon S \times S \to X$ the projection to the orbit space. The isotropy set of X is $\Sigma^* \cup \Sigma^1$, where:

1. $\Sigma^* = q(\cup_{j=1}^n (S - \{p_1, \ldots, p_n\}) \times \{p_j\}) \cup q(\{(x, x), x \in S - \{p_1, \ldots, p_n\}\})$,
2. $\Sigma^1 = q(\{(p_j, p_k), \quad 1 \leq j \leq k \leq n\})$.

The isotropy group of points on $q((S - \{p_1, \dots, p_n\}) \times \{p_j\})$ is G_j, and for points on $q(\{(x,x), x \in S - \{p_1, \dots, p_n\}\})$ it is $\mathbb{Z}_2$. If $j < k$, the isotropy group of (p_j, p_k) is $\mathrm{G}_{jk} = \mathrm{G}_j \times \mathrm{G}_k$. If $j = k$ a presentation of the isotropy group G_{jj} is $\langle \xi, \mathrm{G}_j \times \mathrm{G}_j | \quad \xi^2 = 1, \quad \xi(\gamma, \gamma') = (\gamma', \gamma)\xi \rangle$. Indeed if (U, V, ϕ, Γ_j) is an orbifold chart around p_j on S, then an orbifold chart around $q(p_j, p_j)$ on X is:

$$(U \times U, q(V \times V), q \circ (\phi \times \phi), G_{jj})$$

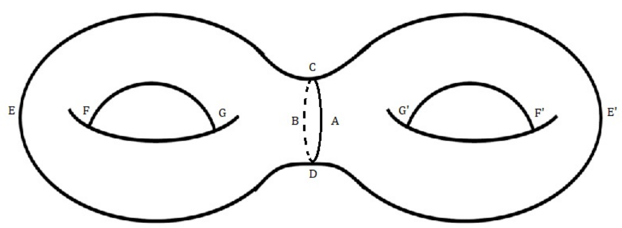
where the action of ξ is given by $\xi(z, w) = (w, z)$, and the action of $\mathrm{G}_j \times \mathrm{G}_j$ is $(\gamma, \gamma')(z, w) = (\gamma z, \gamma' w)$.

Theorem 3.1 allows us to obtain a symplectic resolution of the orbifold (X, ω); this resolution is homeomorphic to X because one can check that $\mathrm{G}'_{jk} = \{1\}$, following the notation of Lemma 3.19.

Now let (S', ω') be a compact 2-dimensional symplectic orbifold (possibly different from (S, ω)), with singularities $\{p'_0, \dots, p'_m\}$ and isotropy groups $\Gamma'_1, \dots, \Gamma'_m$. We can also consider the product orbifold $(S \times S', \omega + \omega')$. The isotropy set is $\Sigma^* \cup \Sigma^1$, with $\Sigma^1 = \{(p_j, p'_k), \quad 1 \leq j \leq n, 1 \leq k \leq m\}$. The isotropy group of (p_j, p_k) is $G_{jk} = \Gamma_j \times \Gamma'_k$ and satisfy that $G'_{jk} = \{1\}$. The orbifold $(S \times S', \omega + \omega')$ satisfies the hypothesis of Theorem 3.1, and its resolution is homeomorphic to $S \times S'$. Note that one could also have constructed the resolution as $(\widetilde{S} \times \widetilde{S}', \widetilde{\omega} + \widetilde{\omega}')$, where $q\colon (\widetilde{S}, \widetilde{\omega}) \to (S, \omega)$ and $q\colon (\widetilde{S}, \widetilde{\omega}) \to (S, \omega)$ are the symplectic resolutions provided in [28].

Mapping torus over a surface of genus 2

Consider Σ_2 a genus 2 surface smoothly embedded in $\mathbb{R}^3$ with coordinates $(x, y, z) \in \mathbb{R}^3$. We require that Σ_2 is symmetric with respect to the planes $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$. Consider the symplectic form in Σ_2 given by $\omega_{\Sigma_2} = \iota_N(\mathrm{vol}_3)|_{\Sigma_2}$, being N the outer unit lenght normal to Σ_2, and $\mathrm{vol}_3 = dx \wedge dy \wedge dz$ the volume form of $\mathbb{R}^3$. Consider the maps $\phi(x, y, z) = (-x, y, -z)$, $\gamma(x, y, z) = (-x, -y, z)$; these restrict to symplectomorphisms of $(\Sigma_2, \omega_{\Sigma_2})$ since they preserve N and vol_3.



Consider $M_\gamma(\Sigma_2)$ the mapping torus of Σ_2 by γ; that is, $M_\gamma(\Sigma_2) = (\Sigma_2 \times I)/\sim$ where $(p, 1) \sim (\gamma(p), -1)$ and $I = [-1, 1]$. In the space $M_\gamma(\Sigma_2) \times S^1$ we lift the action of ϕ as

$$\phi([p, t], s) = ([\phi(p), t], s)$$

for $[p, t] \in M_\gamma(\Sigma_2)$, $s \in S^1 = [-1, 1]/\sim$.

Note that this action is well defined because if we take $(p, 1)$ and $(\gamma(p), -1)$ two representatives of the same class, they get mapped to $(\phi(p), 1)$ and $(\phi(\gamma(p)), -1) = (\gamma(\phi(p)), -1)$, so their images represent the same class also. Take also the map ξ acting on $M_\gamma(\Sigma_2) \times S^1$ as

$$\xi([p, t], s) = ([p, -t], -s) .$$

The above action is well-defined because $(p,1,s)$ and $(\gamma(p),-1,s)$ are mapped to $(p,-1,-s)$ and $(\gamma(p),1,-s)$, and $(\gamma(p),1)\sim(p,-1)$ since $\gamma^2=\mathrm{Id}$. On the other hand let us consider the symplectic form on $M_\gamma(\Sigma_2)\times S^1$ given in coordinates as

$$\omega=\omega_{\Sigma_2}+dt\wedge ds.$$

Near a point $([p,1],s)=([\gamma(p),-1],s)\in M_\gamma(\Sigma_2)\times S^1$ we consider a chart of the form

$$(U^p\times(1-\varepsilon,1]\times(s-\varepsilon,s+\varepsilon))\cup(\gamma(U^p)\times[-1,-1+\varepsilon)\times(s-\varepsilon,s+\varepsilon)),$$

where the above expression for ω is well-defined, since γ is a symplectomorphism of Σ_2.

We can describe $M_\gamma(\Sigma_2)\times S^1$ in an alternative manner. Consider $Y=\Sigma_2\times\mathbb{C}^2$ and the isometries of Y, $\tau_1(p,w)=(\gamma(p),w+1)$, $\tau_2(p,w)=(p,w+i)$. These determine a $\mathbb{Z}^2$-Kähler action on Y and $M_\gamma(\Sigma_2)\times S^1=Y/\mathbb{Z}^2$, hence $M_\gamma(\Sigma_2)\times S^1$ is Kähler.

Note that in the symplectic manifold $(M_\gamma(\Sigma_2)\times S^1,\omega)$ the group $\Gamma=\langle\phi,\xi\rangle\cong\mathbb{Z}_2\times\mathbb{Z}_2$ acts by symplectomorphisms. We define a 4-orbifold X as

$$X=\frac{M_\gamma(\Sigma_2)\times S^1}{\langle\phi,\xi\rangle}$$

so (X,ω) is a symplectic orbifold.

Let us study the isotropy subset of X. We may abuse notation and identify the isotropy points of X with the isotropy points of the action of $\langle\phi,\xi\rangle$ in $M_\gamma(\Sigma_2)\times\Sigma^1$; the context should clarify each case. The maps $\phi,\gamma,\gamma\circ\phi\colon\Sigma_2\to\Sigma_2$ have the following fixed points

$$\mathrm{Fix}(\phi)=\{A,B\}\,,\ \mathrm{Fix}(\gamma)=\{C,D\}\,,\ \mathrm{Fix}(\gamma\circ\phi)=\{E,F,G,E',F',G'\}\subset\Sigma_2$$

with $A=(0,1,0),B=(0,-1,0),C=(0,0,1),D=(0,0,-1)$, and $\mathrm{Fix}(\gamma\circ\phi)$ corresponds to the six points of intersection of Σ_2 with the x-axis. Note also that $\gamma(A)=B,\gamma(B)=A$, $\phi(C)=D$, $\phi(D)=C$, and $\phi(E')=E$, $\phi(F')=F$, $\phi(G')=G$.

The isotropy points for the group $\langle\phi,\xi\rangle$ acting on $M_\gamma(\Sigma_2)\times S^1$ are as follows:

- Isotropy surfaces given by

$$\begin{aligned} S_\phi&=\{([A,t],s)\text{ s.t. }(t,s)\in I^2\}\cup\{([B,t],s)\text{ s.t. }(t,s)\in I^2\}\ ,\\ S_\xi^1&=\{([p,0],0)\text{ s.t. }p\in\Sigma_2\}\,,\\ S_\xi^2&=\{([p,0],1)\text{ s.t. }p\in\Sigma_2\}\,.\end{aligned}$$

  Note that S_ϕ is a torus, since $(A,1,s)\sim(B,-1,s)$, and S_ξ^i are surfaces with genus 2, identified with Σ_2. The generic points of S_ϕ have isotropy $\langle\phi\rangle\cong\mathbb{Z}_2$, and those of S_ξ have isotropy $\langle\xi\rangle\cong\mathbb{Z}_2$.

- The intersection of S_ϕ and the S_ξ^i are the points $A_0=([A,0],0)$, $B_0=([B,0],0)$, $A_1=([A,0],1)$, and $B_1=([B,0],1)$; these are points of isotropy $\langle\phi,\xi\rangle\cong\mathbb{Z}_2\times\mathbb{Z}_2$.

- Eight isolated isotropy points. Two of them, $C_1=([C,1],1)$ and $D_1=([D,1],1)$, have isotropy $\langle\xi\rangle\cong\mathbb{Z}_2$; the rest of them are the points $E_1=([E,1],1)$, $F_1=([F,1],1)$, $G_1=([G,1],1)$, $E_1'=([E',1],1)$, $F_1'=([F',1],1)$, $G_1'=([G',1],1)$, all with isotropy $\langle\phi\circ\xi\rangle\cong\mathbb{Z}_2$.

Of the above fixed points in $M_\gamma(\Sigma_2)\times S^1$ not all of them are different in the quotient X: we have $E_1\sim E_1'$, $F_1\sim F_1'$, $G_1\sim G_1'$. Moreover S_ξ^i becomes a torus $\Sigma_2/\langle\phi\rangle$ in X, and S_ϕ becomes a sphere.

Following the previous notation for the isotropy points of an orbifold X, the isotropy subset Σ of X decomposes as $\Sigma = \Sigma^* \cup \Sigma^1 \cup \Sigma^0$, with $\Sigma^1 = \{A_0, B_0, A_1, B_1\}$, $\Sigma^* = (S_\phi \cup S^1_\xi \cup S^2_\xi) - \Sigma^1$, and $\Sigma^0 = \{C_1, D_1, E_1, F_1, G_1\}$.

Now we compute the Betti numbers of X. For this it is useful to express X in an alternative way. Recall that the quotient $T = \Sigma_2/\langle\phi\rangle$ is a torus; its fundamental domain being $D_T = \Sigma_2 \cap \{x \geq 0\} \subset \mathbb{R}^3$ with identifications $(0, y, z) \sim (0, y, -z)$. The map $\gamma\colon \Sigma_2 \to \Sigma_2$ commutes with ϕ, so it descends to a homeomorphism of T. Consider the mapping torus

$$M_\gamma(T) = (T \times [-1, 1])/\sim$$

where $([p], 1) \sim ([\gamma(p)], -1)$. It is immediate to check that $X = (M_\gamma(T) \times S^1)/\langle\xi\rangle$.

The following lemma is necessary for the computation of the fundamental group of X.

Lemma 3.41. *Let T be a CW-complex, and $\gamma\colon T \to T$ a homeomorphism which fixes a point $x_0 \in T$. Let $M_\gamma(T) = T \times [0, 1]/\sim$ with $(x, 0) \sim (\gamma(x), 1)$. Then $\pi_1(M_\gamma(T)) \cong \pi_1(S^1) \ltimes_{\gamma_*} \pi_1(T)$.*

Proof. Recall first that the operation in $\pi_1(S^1) \ltimes_{\gamma^*} \pi_1(T)$ is

$$(n, g) \cdot (n', g') = (n + n', g \cdot \gamma_*^n(g')) ,$$

where $\gamma_*\colon \pi_1(T, x_0) \to \pi_1(T, x_0)$ is the induced map.

We have a bundle structure on $M_\gamma(T)$ given by $T \xrightarrow{i} M_\gamma(T) \xrightarrow{\pi_*} S^1$, where $i(x) = [x, 0]$ and $\pi([x, t]) = t$. This gives a short exact sequence

$$1 \to \pi_1(T) \xrightarrow{i_*} \pi_1(M_\gamma(T)) \xrightarrow{\pi_*} \pi_1(S^1) \to 1 .$$

There is a section $s\colon S^1 \to M_\gamma(T), t \mapsto (x_0, t)$; it is well-defined because $\gamma(x_0) = x_0$. This gives $s_*\colon \pi_1(S^1) \to \pi_1(M_\gamma(T))$ a right inverse for π, which gives a splitting of the above short exact sequence; then $\pi_1(M_\gamma(T))$ is the semi-direct product of $\pi_1(T)$ and $\pi_1(S^1)$, where the action of $\pi_1(S^1)$ in $\pi_1(T)$ is by conjugation.

Let us call $\alpha = s_*(1)$, where $1 \in \pi_1(S^1)$ is the generator. Note that $\alpha(t) = [(x_0, t)]$, $t \in [0, 1]$. It only remains to see that every $g \in \pi_1(T)$ satisfies that $\alpha g \alpha^{-1} = \gamma_*(g)$ in $\pi_1(M_\gamma(T))$.

Consider the homotopy $H\colon S^1 \times [0, 1] \to M_\gamma(T)$ given as

$$H_s(t) = \begin{cases} (x_0, 3ts), & t \in [0, \frac{1}{3}], \\ (\gamma(g(3t-1)), s), & t \in [\frac{1}{3}, \frac{2}{3}], \\ (x_0, 3(1-t)s), & t \in [\frac{2}{3}, 1]. \end{cases}$$

It is immmediate to check that $[H_0] = \gamma_*(g)$ and $[H_1] = \alpha g \alpha^{-1}$, proving the lemma. □

Now we compute the fundamental group of $M_\gamma(T)$, with $T = \Sigma_2/\langle\phi\rangle$ as above. Take as base point $[C] = [D] \in T$, which is a fixed point by γ, and choose generators a, b for $\pi_1(T)$ so that a representative for $a = [\alpha]$ in the fundamental domain D_T is the circle

$$\alpha = D_T \cap \{z = 1\} = \{(2 + \cos t, \sin t, 1) : 0 \leq t \leq 2\pi\} .$$

Similarly, a representative for $b = [\beta]$ is a semicircle

$$\beta = \{(\cos t, 0, \sin t) : \pi/2 \leq t \leq 3\pi/2\}$$

going from C to D in $D_T \cap \{y=0\}$; β descends to a loop in the quotient $T = D_T/\sim$. By Lemma 3.41, the fundamental group of $M_\gamma(T)$ is

$$\pi_1(M_\gamma(T)) \cong \pi_1(S^1) \ltimes_{\gamma_*} \pi_1(T) \cong \mathbb{Z} \ltimes_{\gamma_*} \mathbb{Z}^2$$

with operation $(n,x)\cdot(n',x') = (n+n', x+(\gamma_*)^n(x'))$, being $\gamma_*\colon \pi_1(T)\to\pi_1(T)$ the automorphism induced by $\gamma\colon T\to T$ in $\pi_1(T)=\pi_1(T,[C])$.

In order to compute γ_*, we take the representatives in D_T of a and b described above and compute their image by γ_*; note that γ seen as a map in Σ_2 does not map D_T to itself, but $\phi\circ\gamma(x,y,z)=(x,-y,-z)$ does, and both maps induce the same map on the quotient $T=\Sigma_2/\langle\phi\rangle$. The loop $a=[(2+\cos t,\sin t,1)]$, $0\le t\le 2\pi$, is mapped to $\phi\circ\gamma(a) = [(2+\cos t,-\sin t,-1)]$, and this is a circle in $D_T\cap\{z=-1\}$ homotopic to a but with the opposite orientation as a, so $\gamma_*(a)=-a$. Similarly, $b=[(\cos t,0,\sin t)]$, $\pi/2\le t\le 3\pi/2$, is mapped to $\phi\circ\gamma(b)=[(\cos t,0,-\sin t)]$, again the same circle but with opposite orientation, so $\gamma_*(b)=-b$. We conclude that

$$\gamma_* = -\mathrm{Id}\colon \pi_1(T)\to\pi_1(T),\ x\mapsto -x\,.$$

It follows that $\pi_1(M_\gamma(T))\cong\mathbb{Z}\ltimes\mathbb{Z}^2$ with operation given by

$$(n,x)\cdot(n',x') = (n+n', x+(-1)^n x')\,.$$

We claim that the abelianization of this group is $H_1(M_\gamma(T),\mathbb{Z})\cong\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_2$. Indeed, if we impose the condition that $(1,x)\cdot(0,x)$ and $(0,x)\cdot(1,x)$ coincide we get that $(1,0)$ equals $(1,2x)$, hence $2x=0$ for all x in the abelianization. This applies to the generators a,b. Once we impose that in the abelianization every x equals $-x$, the operation $\cdot$ becomes commutative, hence the claim.

From this it follows that

$$\pi_1(M_\gamma(T)\times S^1)\cong(\mathbb{Z}\ltimes\mathbb{Z}^2)\times\mathbb{Z}$$
$$H_1(M_\gamma(T)\times S^1,\mathbb{Z})\cong\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}\cong\mathbb{Z}_2^2\times\mathbb{Z}^2\,.$$

Note that torsion part of the homology comes from the torus T and the free part comes from the two circles associated to the coordinates (s,t). When passing to real coefficients we can consider de Rham cohomology and we get $H^1(M_\gamma(T)\times S^1,\mathbb{R})=\langle dt,ds\rangle$. As the action of ξ in $M_\gamma(T)\times S^1$ sends dt,ds to $-dt,-ds$, it follows that the cohomology of the orbifold $X=(M_\gamma(T)\times S^1)/\langle\xi\rangle$ is the ξ-invariant part of $\langle dt,ds\rangle$, i.e. $H^1(X,\mathbb{R})=0$.

Now let us compute the fundamental group of X. Recall that $M_\gamma(T)\times S^1$ is a torus bundle over a torus, i.e. $T\to M_\gamma(T)\times S^1\to S^1\times S^1$ where fibers are given by $T=\{([p,t_0],s_0)$ s.t. $p\in T\}$ and the bundle map sends $([p,t],s)$ to (t,s). We have a short exact sequence

$$1\to\pi_1(T)\xrightarrow{i_*}\pi_1(M_\gamma(T)\times S^1)\xrightarrow{\pi_*}\pi_1(S^1\times S^1)\to 1$$

where $i\colon T\to M_\gamma(T)\times S^1$, $p\mapsto([p,t],s)$ is the inclusion of the fiber $F_{(t,s)}\cong T$, and the bundle map is $\pi\colon M_\gamma(T)\times S^1\to S^1\times S^1$, $([p,t],s)\mapsto(t,s)$. Consider

$$q\colon M_\gamma(T)\times S^1\to X=(M_\gamma(T)\times S^1)/\langle\xi\rangle$$

the quotient map. Take as base points A_0 and $q(A_0)$ respectively. Since A_0 is fixed by ξ, we have $q^{-1}(q(A_0))=\{A_0\}$. This gives that $q_*\colon\pi_1(M_\gamma(T)\times S^1)\to\pi_1(X)$ is an epimorphism by [21, Corollary 6.3].

In $M_\gamma(T)\times S^1$ there are two fibers invariant by the action of ξ and not formed by fixed points, namely $F_{(1,0)}$ and $F_{(1,1)}$. Let us take as base points A, $([A,1],0)$ and $(1,0)$ respectively.

Call $F \cong T$ any of these fibers. Under the quotient map q, F is mapped to $q(F) \cong T/\langle\gamma\rangle$. This is so because

$$\xi([p,1],0) = ([p,-1],0) = ([\gamma(p),1],0),$$
$$\xi([p,1],1) = ([p,-1],-1) = ([\gamma(p),1],1);$$

hence $q \circ i(p) = q \circ i(\gamma(p))$ for $p \in T$, being $i\colon T \to F \subset M_\gamma(T) \times S^1$ the inclusion. Recall that $q(F) = F/\langle\gamma\rangle \cong \Sigma_2/\langle\phi,\gamma\rangle \cong S^2$ is topologically a sphere, so we call $S^2 = T/\langle\gamma\rangle$. The map $q_* \circ i_*\colon \pi_1(F) \to \pi_1(M_\gamma(T) \times S^1)$ factors through $\pi_1(S^2) = \{1\}$, so it is constant. Hence $\mathrm{Im}(i_*) = \ker(\pi_*) \subset \ker(q_*)$, so the map q_* induces a map $\bar{q}_*\colon \pi_1(S^1 \times S^1) \to \pi_1(X)$ in the quotient $\pi_1(M_\gamma(T) \times S^1)/\pi_1(F) \cong \pi_1(S^1 \times S^1)$.

Note that $\pi_1(S^1 \times S^1)$ can be seen as a subgroup of $\pi_1(M_\gamma(T) \times S^1)$ via the section

$$f\colon S^1 \times S^1 \to M_\gamma(T) \times S^1,\ (t,s) \mapsto \begin{cases} ([A, 1+2t], s)\,, & t \in [-1,0], \\ ([B, -1+2t], s)\,, & t \in [0,1]. \end{cases}$$

The image of f is precisely the isotropy surface S_ϕ, whose image by q is $q(S_\phi) = S_\phi/\langle\xi\rangle \cong S^2$, homeomorphic to a sphere. As $\bar{q}_* = q_* \circ f_*$ factors through $\pi_1(q(S_\phi)) = 1$, we see that $\bar{q}_* = 1$, so $q = 1$ and X is simply connected.

Now let us compute the second homology of X over $\mathbb{R}$.

Proposition 3.42.

$$H^2(X,\mathbb{R}) = \langle \omega_{\Sigma_2}, dt \wedge ds \rangle$$

Proof. First of all one can prove that $H^2(X,\mathbb{R}) \cong H^2(M_\gamma(\Sigma_2) \times S^1, \mathbb{R})^{\langle\phi,\xi\rangle}$ by averaging closed forms. The Künneth formula ensures that

$$H^2(M_\gamma(\Sigma_2) \times S^1, \mathbb{R}) = H^1(M_\gamma(\Sigma_2),\mathbb{R}) \wedge \langle ds \rangle \oplus H^2(M_\gamma(\Sigma_2),\mathbb{R}).$$

The first summand is of course equal to $\langle dt \wedge ds\rangle$; to compute the second we take into account [10, Lemma 12]:

$$\begin{aligned} H^2(M_\gamma(\Sigma_2)) = {} & \ker(\mathrm{Id} - \gamma^*\colon H^2(\Sigma_2,\mathbb{R}) \to H^2(\Sigma_2,\mathbb{R})) \\ & \oplus \mathrm{Coker}(\mathrm{Id} - \gamma^*\colon H^1(\Sigma_2,\mathbb{R}) \to H^1(\Sigma_2,\mathbb{R})) \wedge \langle dt \rangle. \end{aligned}$$

On the one hand, $\gamma^* = \mathrm{Id}\colon H^2(\Sigma_2) \to H^2(\Sigma_2)$ because $\gamma_*(\omega_{\Sigma_2}) = \omega_{\Sigma_2}$, as was previously argued. On the other, $\gamma^* = -\mathrm{Id}\colon H^1(\Sigma_2,\mathbb{R}) \to H^1(\Sigma_2,\mathbb{R})$; this can be deduced from the fact that $\gamma_* = -\mathrm{Id}$. Thus,

$$H^2(M_\gamma(\Sigma_2)) = \langle \omega_{\Sigma_2} \rangle.$$

The proof concludes by observing that both ω_{Σ_2} and $dt \wedge ds$ are invariant under the action of $\langle \phi, \xi \rangle$. □

Proposition 3.43. *Let $\pi\colon \widetilde{X} \to X$ the symplectic resolution of X. Denote $\Sigma^0 = \{p_1, \ldots, p_5\}$; then $E_j = \pi^{-1}(p_j)$ is diffeomorphic to $\mathbb{CP}^1$. In addition,*

1. $\pi_1(\widetilde{X}) = \{1\}$.
2. $H^2(\widetilde{X},\mathbb{R}) = \langle \pi^*(\omega_{\Sigma_2}), \pi^*(dt \wedge ds), \omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \rangle$, *where ω_j is the Thom class of E_j.*

Proof. First observe that $\Delta = \emptyset$, where Δ is defined as in Proposition 3.40. In addition, if $p \in \Sigma^0$ is an isolated singularity then $\Gamma_p = \mathbb{Z}_2$; the Kähler local model around p is necessarily of the form $\mathbb{C}^2/\mathbb{Z}_2$, with $\mathbb{Z}_2 = \langle \mathrm{Id}, -\mathrm{Id} \rangle$. The algebraic resolution of this space is $\widetilde{\mathbb{C}}^2/\mathbb{Z}_2$, where $\widetilde{\mathbb{C}}^2$ stands for the blow-up of 0 in $\mathbb{C}^2$; that is:

$$\widetilde{\mathbb{C}}^2/\mathbb{Z}_2 = \{(v,l) \in \mathbb{C}^2 \times \mathbb{CP}^1 \text{ s.t. } v \in l\}/(v,l) \sim (-v,l).$$

We compute $\pi_1(\widetilde{X})$ using the Seifert-Van Kampen theorem. Let B_j^ε be an ε-ball centered at p_j with ε small enough to ensure that B_j^ε are pairwise disjoint. Let N_j be a neighbourhood of a path between p_j and p_{j+1} that does not intersect B_j^ε for $k \neq j, j+1$. Define:

$$U = \left(\cup_{j=1}^5 B_j{}^\varepsilon\right) \cup \left(\cup_{j=1}^4 N_j\right), \qquad V = X - \cup_{j=1}^5 \overline{B_j}^{\frac{\varepsilon}{2}}.$$

The space $U \cap V$ is pathwise connected and has the homotopy type of $\bigvee_{j=1}^5 S_j^3/\mathbb{Z}_2$, where we denoted a copy of S^3 as S_j^3. Its fundamental group is the free product of 5 copies of $\mathbb{Z}_2$. Being U contractible, it holds that $1 = \pi_1(X) = \pi_1(V)/i_*(\pi_1(U \cap V))$, with $i\colon U \cap V \to V$.

In addition define $\widetilde{U} = \pi^{-1}(U)$, $\widetilde{V} = \pi^{-1}(V)$. The space $\widetilde{U}$ has the homotopy type of $\bigvee_{j=1}^5 \mathbb{CP}_j^1$; which is simply connected. Thus, $\pi_1(\widetilde{X}) = \pi_1(\widetilde{V})/j_*(\pi_1(\widetilde{U} \cap \widetilde{V}))$, with $j\colon \widetilde{U} \cap \widetilde{V} \to \widetilde{V}$. Taking into account that $\pi\colon (\widetilde{V}, \widetilde{U} \cap \widetilde{V}) \to (V, U \cap V)$ is a homeomorphism of pairs; this ensures that $\pi_1(\widetilde{X}) = \pi_1(X) = \{1\}$.

We finally compute $H^2(\widetilde{X}, \mathbb{R})$. By Propositions 3.40 and 3.42 there is a short exact sequence:

$$0 \to \langle \omega_{\Sigma_2}, dt \wedge ds \rangle \xrightarrow{\pi^*} H^2(\widetilde{X}, \mathbb{R}) \xrightarrow{i^*} \sum_{j=1}^5 H^2(E_j, \mathbb{R}) \to 0.$$

The restriction of ω_j to E_j is a volume form of E_j because the bundle $\widetilde{\mathbb{C}}^2 \to \mathbb{CP}^1$ is non-trivial. This yields a splitting: $i^*(\omega_j) \longmapsto \omega_j$. This finishes the proof.

□

CHAPTER 4

A compact non-formal closed G_2 manifold with $b_1 = 1$

Lucía Martín-Merchán

Abstract

We construct a compact manifold with a closed G_2 structure not admitting any torsion-free G_2 structure, which is non-formal and has first Betti number $b_1 = 1$. We develop a method of resolution for orbifolds that arise as a quotient $M/\mathbb{Z}_2$ with M a closed G_2 manifold under the assumption that the singular locus carries a nowhere-vanishing closed 1-form.

MSC classification [2010]: Primary 53C38, 53C15; Secondary 17B30, 22E25.

Key words: G_2 orbifold resolution, formality.

4.1 Introduction

A G_2 structure on a 7-dimensional manifold M is a reduction of the structure group of its frame bundle to the exceptional Lie group G_2. Such a structure determines an orientation, a metric g and a non-degenerate 3-form φ; these define a cross product $\times$ on TM by means of the expression

$$\varphi(X, Y, Z) = g(X \times Y, Z).$$

The group G_2 appears on Berger's list [17] of possible holonomy groups of simply connected, irreducible and non-symmetric Riemannian manifolds. Non-complete metrics with holonomy G_2 were given by Bryant in [22] and complete metrics were obtained by Bryant and Salamon in [24]. First compact examples were constructed in 1996 by Joyce in [71] and [72]. More compact manifolds with holonomy G_2 were constructed later by Kovalev [77], Kovalev and Lee [78], Corti, Haskins, Nordström and Pacini [36] and recently by Joyce and Karigiannis [75].

The torsion of a G_2 structure (M, φ, g) is defined as $\nabla\varphi$, the covariant derivative of φ. Fernández and Gray [48] classified G_2 structures into 16 different types according to equations involving the torsion of the structure. In this paper we focus on two of them, namely *torsion-free* and *closed* G_2 structures. A G_2 structure is called torsion-free if the holonomy of g is contained in G_2, that is $\nabla\varphi = 0$ or equivalently $d\varphi = 0$ and $d \star \varphi = 0$, where $\star$ denotes the Hodge star. A G_2 structure is said to be closed if it satisfies $d\varphi = 0$; these are also named *calibrated*. Metrics defined by such types of G_2 structures have interesting properties; while torsion-free G_2 manifolds are Ricci-flat, closed G_2 manifolds have non-positive scalar curvature and both the scalar-flatness and the Einstein condition are equivalent to the fact that the structure is torsion-free (see [23] and [33]).

This paper contributes to understanding topological properties of compact manifolds with a closed G_2 structure that cannot be endowed with a torsion-free G_2 structure. First examples of these were provided by Fernández in [44] and [45]; the example in [44] is a nilmanifold and the examples in [45] are solvmanifolds. Nilmanifolds and solvmanifolds arise as compact quotients of Lie groups by lattices; these Lie groups are nilpotent in the first case and solvable in the second. In both examples the G_2 structure is induced by a closed left-invariant G_2 form on the Lie group. The solvmanifolds in [45] have $b_1 = 3$. In [34] the authors classify nilpotent Lie algebras that admit a closed G_2 structure; this list provides more examples of compact manifolds with $b_1 \geq 2$ endowed with a closed G_2 structure but not admitting torsion-free G_2 structures. In [81] the author develops a method that allows to construct 7-dimensional solvable Lie groups endowed with a closed G_2 structure and as an application provided an example with $b_1 = 1$. Recently in [47] the authors construct another example that has $b_1 = 1$. Their starting point is a nilmanifold M with $b_1 = 3$ that admits a closed G_2 structure and an involution that preserves it. The quotient $X = M/\mathbb{Z}_2$ is an orbifold with $b_1 = 1$ and its isotropy locus consists of 16 disjoint tori. Then they resolve the singularities to obtain a smooth manifold.

Being this the geography of such manifolds, this paper provides an example of a compact manifold carrying a closed G_2 structure. Its topological properties are different from those that the already mentioned ones have, as we shall discuss later. Our construction consists of resolving an orbifold; for that purpose we first develop a resolution method that is summarized in the following result:

Theorem 4.1. *Let (M, φ, g) be a closed G_2 structure on a compact manifold. Suppose that $\mathrm{j}\colon M \to M$ is an involution such that $\mathrm{j}^*\varphi = \varphi$ and consider the orbifold $X = M/\mathrm{j}$. Let $L = \mathrm{Fix}(\mathrm{j})$ be the singular locus of X and suppose that there is a nowhere-vanishing closed 1-form $\theta \in \Omega^1(L)$. Then, there exists a compact G_2 manifold endowed with a closed G_2 structure $(\widetilde{X}, \widetilde{\varphi}, \widetilde{g})$ and a map $\rho\colon \widetilde{X} \to X$ such that:*

1. *The map $\rho\colon \widetilde{X} - \rho^{-1}(L) \to X - L$ is a diffeomorphism.*
2. *There exists a small neighbourhood U of L such that $\rho^*(\varphi) = \widetilde{\varphi}$ on $\widetilde{X} - \rho^{-1}(U)$.*

The fixed point locus L is an oriented 3-dimensional manifold (see Lemma 4.10); the existence of a nowhere-vanishing closed $\theta \in \Omega^1(L)$ is equivalent to the fact that each connected component of L is a mapping torus of an orientation-preserving diffeomorphism of an oriented surface. In our example, the singular locus is formed by 16 disjoint nilmanifolds whose universal covering is the Heisenberg group.

The resolution method follows the ideas of Joyce and Karigiannis in [75], where they develop a method to resolve $\mathbb{Z}_2$ singularities induced by the action of an involution on manifolds endowed with a torsion-free G_2 structure in the case that the singular locus L has a nowhere-vanishing harmonic 1-form. The local model of the singularity being $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\})$, the resolution is constructed by replacing a tubular neighbourhood of the singular locus with a bundle over L with fibre the Eguchi-Hanson space. Then they construct a 1-parameter family of closed G_2 structures on the resolution; these have small torsion when the value of the parameter is small. Then they apply a theorem of Joyce [74, Th. 11.6.1] which states that if one can find a closed G_2 structure φ on a compact 7-manifold M whose torsion is sufficiently small in a certain sense, then there exists a torsion-free G_2 structure which is close to φ and it determines the same de Rham cohomology class. This method provides a torsion-free G_2 structure on the resolution; if its fundamental group is finite then its holonomy is G_2.

The main difficulty of their construction relies on the fact that two of the three pieces that they glue, namely an annulus around the singular set of the orbifold and a germ of resolution, do not come naturally equipped with torsion-free G_2 structures. However, there is a canonical way to define a G_2 structure on them and to obtain a closed G_2 structure by making a small perturbation. The torsion of the structure is too large so that they need to make additional corrections. We shall follow the same ideas to perform the resolution; the method is simplified because we avoid these technical difficulties.

In this paper we are interested in the interplay between closed G_2 manifolds with small first Betti number and the condition of being formal. Formal manifolds are those whose rational cohomology algebra is described by its rational model. This is a notion of rational homotopy theory and has been successfully applied in some geometric situations. The Thurston-Weinstein problem is a remarkable example in the context of symplectic geometry; this consists in constructing symplectic manifolds with no Kähler structure. Deligne, Griffiths, Morgan and Sullivan proved in [40] that compact Kähler manifolds are formal; thus, non-formal symplectic manifolds are solutions of this problem. Formality is less understood in the case of exceptional holonomy; in particular, the problem of deciding whether or not manifolds with holonomy G_2 and Spin(7) are formal is still open. There are some partial results for holonomy G_2 manifolds; in [38] authors proved that compact non-formal manifolds with holonomy G_2 must have second Betti number $b_2 \geq 4$. In addition, in [29] authors proved that compact manifolds with holonomy G_2 are *almost formal*; this condition implies that triple Massey products $\langle \xi_1, \xi_2, \xi_3 \rangle$ are trivial except perhaps for the case that the degree of ξ_1, ξ_2 and ξ_3 is 2. Non-trivial Massey products are obstructions to formality but there are examples of non-formal compact 7-manifolds that only have trivial triple Massey products (see [38]). However, the presence of a geometric structure makes the situation different; for instance in [95] the authors prove that simply-connected 7-dimensional Sasakian manifolds are formal if and only if its triple Massey products are trivial.

Formal examples of closed G_2 manifolds that do not admit any torsion-free G_2 structure are the solvmanifolds provided in [45] and [81], and the compact manifold with $b_1 = 1$ provided in [47]. Non-formal examples are the nilmanifolds obtained in [34]; these have $b_1 \geq 2$. In this paper we prove:

Theorem 4.2. *There exists a compact non-formal closed* G_2 *manifold with* $b_1 = 1$ *that cannot be endowed with a torsion-free* G_2 *structure.*

The manifold $\widetilde{X}$ that we construct is the resolution of a closed G_2 orbifold X, obtained as the quotient of a nilmanifold M by the action of the group $\mathbb{Z}_2$. The orbifold has $b_1 = 1$ and a non-trivial Massey product coming from M. The resolution process does not change the first Betti number; in addition the non-trivial Massey product on X lifts to a non-trivial Massey product on $\widetilde{X}$.

This paper is organized as follows. In section 4.2 we review some necessary preliminaries on orbifolds, G_2 structures and formality. Section 4.3 is devoted to prove Theorem 4.1, and in section 4.4 we characterise the cohomology ring of the resolution. With these tools at hand we finally construct in section 4.5 the non-formal compact closed G_2 manifold with $b_1 = 1$.

Acknowledgements. I am grateful to my thesis advisors Giovanni Bazzoni and Vicente Muñoz for suggesting this problem to me and for useful conversations. I acknowledge financial support by a FPU Grant (FPU16/03475).

4.2 Preliminaries

4.2.1 Orbifolds

We first introduce some aspects about orbifolds, which can be found in [28] and [93].

Definition 4.3. An n-dimensional orbifold is a Hausdorff and second countable space X endowed with an atlas $\{(U_\alpha, V_\alpha, \psi_\alpha, \Gamma_\alpha)\}$, where $\{V_\alpha\}$ is an open cover of X, $U_\alpha \subset \mathbb{R}^n$, $\Gamma_\alpha < \mathrm{Diff}(U_\alpha)$ is a finite group acting by diffeomorphisms, and $\psi_\alpha \colon U_\alpha \to V_\alpha \subset X$ is a Γ_α-invariant map which induces a homeomorphism $U_\alpha/\Gamma_\alpha \cong V_\alpha$.

There is a condition of compatibility of charts for intersections. For each point $x \in V_\alpha \cap V_\beta$ there is some $V_\delta \subset V_\alpha \cap V_\beta$ with $x \in V_\delta$ so that there are group monomorphisms $\rho_{\delta\alpha} \colon \Gamma_\delta \hookrightarrow \Gamma_\alpha$, $\rho_{\delta\beta} \colon \Gamma_\delta \hookrightarrow \Gamma_\beta$, and open differentiable embeddings $\imath_{\delta\alpha} \colon U_\delta \to U_\alpha$, $\imath_{\delta\beta} \colon U_\delta \to U_\beta$, which satisfy $\imath_{\delta\alpha}(\gamma(x)) = \rho_{\delta\alpha}(\gamma)(\imath_{\delta\alpha}(x))$ and $\imath_{\delta\beta}(\gamma(x)) = \rho_{\delta\beta}(\gamma)(\imath_{\delta\beta}(x))$, for all $\gamma \in \Gamma_\delta$.

We can refine the atlas of an orbifold X in order to obtain better properties; given a point $x \in X$, there is a chart (U, V, ψ, Γ) with $U \subset \mathbb{R}^n$, $U/\Gamma \cong V$, so that the preimage ${}^{-1}(\{x\}) = \{u\}$, and satisfies $\gamma(u) = u$ for all $\gamma \in \Gamma$. We call Γ the *isotropy group* at x, and we denote it by Γ_x. This group is well defined up to conjugation by a diffeomorphism of a small open set of $\mathbb{R}^n$. The singular locus of X is the set $S = \{x \in X \text{ s.t. } \Gamma_x \neq \{1\}\}$, and of course, $X - S$ is a smooth manifold.

We now describe the de Rham complex of an n-dimensional orbifold X. First of all, a k-form η on X consists of a collection of differential k-forms $\{\eta_\alpha\}$ such that:

1. $\eta_\alpha \in \Omega^k(U_\alpha)$ is Γ_α-invariant,
2. If $V_\delta \subset V_\alpha$ and $\imath_{\delta\alpha} \colon U_\delta \to U_\alpha$ is the associated embedding, then $\imath^*_{\delta\alpha}(\eta_\alpha) = \eta_\delta$.

The space of orbifold k-forms on X is denoted by $\Omega^k(X)$. In addition, it is obvious that the wedge product of orbifold forms and the exterior differential d on X are well defined. Therefore $(\Omega^*(X), d)$ is a differential graded algebra that we call the de Rham complex of X. Its cohomology coincides with the cohomology of the space X with real coefficients, $H^*(X)$ (see [28, Proposition 2.13]).

In this paper the orbifold involved is the orbit space of a smooth manifold M under the action of $\mathbb{Z}_2 = \{\mathrm{Id}, \mathrm{j}\}$, where j is an involution. The singular locus of $X = M/\mathbb{Z}_2$ is $\mathrm{Fix}(\mathrm{j})$. In addition, let us denote by $\Omega^k(M)^{\mathbb{Z}_2}$ the space of $\mathbb{Z}_2$-invariant k-forms. Then

$$\Omega^k(X) = \Omega^k(M)^{\mathbb{Z}_2},$$

and both the wedge product and exterior derivative preserve the $\mathbb{Z}_2$-invariance. An averaging argument ensures that $H^k(X) = H^k(M)^{\mathbb{Z}_2}$.

4.2.2 G_2 structures

We now focus on G_2 structures on manifolds and orbifolds. Basic references are [23], [48], [61], [74] and [105].

Let us identify $\mathbb{R}^7$ with the imaginary part of the octonions $\mathbb{O}$. The multiplicative structure on $\mathbb{O}$ endows $\mathbb{R}^7$ with a cross product $\times$, which defines a 3-form $\varphi_0(u, v, w) = \langle u \times v, w \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbb{R}^7$. In coordinates,

$$\varphi_0 = v^{127} + v^{347} + v^{567} + v^{135} - v^{236} - v^{146} - v^{245}, \tag{4.1}$$

where $(v^1, \dots, v^7)$ is the standard basis of $(\mathbb{R}^7)^*$ and v^{ijk} stands for $v^i \wedge v^j \wedge v^k$. The stabilizer of φ_0 under the action of $\mathrm{Gl}(7, \mathbb{R})$ on $\Lambda^3(\mathbb{R}^7)^*$ is the group G_2, a simply connected 14-dimensional Lie group which is contained in $\mathrm{SO}(7)$.

Definition 4.4. Let V be a real vector space of dimension 7. A 3-form $\varphi \in \Lambda^3 V^*$ is a G_2 form on V if there is a linear isomorphism $u\colon V \to \mathbb{R}^7$ such that $u^*(\varphi_0) = \varphi$, where φ_0 is given by equation (4.1).

A G_2 structure φ determines an orientation because $\mathrm{G}_2 \subset \mathrm{SO}(7)$; the choice of a volume form vol on V compatible with the orientation determines a unique metric g_{vol} with associated unit-length volume form vol by the formula:

$$i(x)\varphi \wedge i(y)\varphi \wedge \varphi = 6 g_{\mathrm{vol}}(x,y)\mathrm{vol},$$

which ensures that the metric $u^*(g_0)$ is determined by the volume form $u^*(\mathrm{vol}_{\mathbb{R}^7})$. Note that the metric $u^*(g_0)$ does not depend on the isomorphism u with $u^*(\varphi_0) = \varphi$. We say that $g = u^*(g_0)$ is the metric associated to φ. Of course, a G_2 form φ induces a cross product $\times$ on V by the formula $\varphi(u,v,w) = g(u \times v, w)$.

The orbit of φ_0 under the action of $\mathrm{Gl}(7,\mathbb{R})$ is an open set of $\Lambda^3(\mathbb{R}^7)^*$, thus the space of G_2 forms on $\mathbb{R}^7$ is an open set.

Definition 4.5. Let M be a 7-dimensional manifold. A G_2 form on M is a 3-form $\varphi \in \Omega^3(M)$ such that for every $p \in M$ the 3-form φ_p is a G_2 form.

Let X be a 7-dimensional orbifold with atlas $\{(U_\alpha, V_\alpha, \psi_\alpha, \Gamma_\alpha)\}$. A G_2 form on X is a differential 3-form $\varphi \in \Omega^3(X)$ such that φ_α is a G_2 form on U_α.

Let φ be a G_2 form on a manifold M or an orbifold X. In both cases, φ determines a metric g and a cross product $\times$. In this case we say that (M,φ,g) or (X,φ,g) is a G_2 structure. In addition, G_2 manifolds are of course oriented. We state a well-known fact about G_2 structures (see for instance [74, Chapter 10, Section 3]).

Lemma 4.6. *There exists a universal constant m such that if (M,φ,g) is a G_2 structure and $\|\phi - \varphi\|_{C^0,g} < m$ then ϕ is a G_2 form.*

Proof. Let $(\mathbb{R}^7, \varphi_0, g_0)$ be the standard G_2 structure. Being the space of G_2 forms on $\mathbb{R}^7$ open in $\Lambda^3(\mathbb{R}^7)^*$, there exists a constant $m > 0$ such that if a 3-form ϕ_0 satisfies that $\|\phi_0 - \varphi_0\|_{g_0} < m$, then ϕ_0 is a G_2 form. We now check that m is the claimed universal constant. Let (M,φ,g) be a G_2 manifold; let ϕ such that $\|\phi_p - \varphi_p\|_{g_p} < m$ for every $p \in M$. In order to check that ϕ_p is a G_2 form, let $A\colon (T_pM, \varphi_p, g_p) \to (\mathbb{R}^7, \varphi_0, g_0)$ be an isomorphism of G_2 vector spaces, then:

$$\|A^t\phi_p - \varphi_0\|_{g_0} = \|\phi_p - \varphi_p\|_{g_p} < m$$

and therefore $A^t\phi_p$ is a G_2 form. Since A is an isomorphism, ϕ_p is also a G_2 form. □

In [48] Fernández and Gray classified G_2 structures (M,φ,g) into 16 types according to $\nabla\varphi$, where ∇ denotes the Levi-Civita connection associated to g. The motivation for such classification is the holonomy principle, stating that the holonomy of g is contained in G_2 if and only if $\nabla\varphi = 0$. In [48] they also prove that $\nabla\varphi = 0$ if and only if $d\varphi = 0$ and $d(\star\varphi) = 0$, where $\star$ denotes the Hodge star. In this paper we are interested in closed and torsion-free G_2 structures on manifolds and orbifolds:

Definition 4.7. Let (M,φ,g) or (X,φ,g) a G_2 structure on a manifold or an orbifold. We say the G_2 structure is closed if $d\varphi = 0$. If in addition $d(\star\varphi) = 0$ we say that the G_2 structure is torsion-free.

Definition 4.8. Let (X,φ) be a closed G_2 structure on a 7-dimensional orbifold. A closed G_2 resolution of (X,φ) consists of a smooth manifold endowed with a closed G_2 structure $(\widetilde{X},\phi)$ and a map $\rho\colon \widetilde{X} \to X$ such that:

1. Let $S \subset X$ be the singular locus and $E = \rho^{-1}(S)$. Then, $\rho|_{\widetilde{X}-E} \colon \widetilde{X} - E \to X - S$ is a diffeomorphism,

2. Outside a neighbourhood of E, $\rho^*(\varphi) = \phi$.

The subset E is called the exceptional locus.

G_2 involutions

Definition 4.9. Let (M, φ) be a G_2 manifold, we say that $\mathrm{j} \colon M \to M$ is a G_2 involution if $\mathrm{j}^*(\varphi) = \varphi$, $\mathrm{j}^2 = \mathrm{Id}$, and $\mathrm{j} \neq \mathrm{Id}$.

In this paper we shall focus on orbifolds that are obtained as a quotient of a closed G_2 manifold (M, φ) by the action of a G_2 involution j; that is $X = M/\mathrm{j}$. The next result states that the fixed locus L of j is a 3-dimensional submanifold.

Lemma 4.10. *The submanifold L is 3-dimensional and oriented by $\varphi|_L$. In addition, $\varphi|_L$ is the oriented unit-length volume form determined by the metric $g|_L$.*

Proof. The result is deduced from the fact that if $(\mathbb{R}^7, \varphi_0, \langle \cdot, \cdot \rangle)$ is the standard G_2 structure on $\mathbb{R}^7$ and if $\mathrm{j} \in \mathrm{G}_2$ is an involution, $\mathrm{j} \neq \mathrm{Id}$, then j is diagonalizable with eigenvalues ± 1 and $\dim(V_1) = 3$, $\dim(V_{-1}) = 4$, where $V_{\pm 1}$ denotes the eigenspace associated to the eigenvalue ± 1. In addition, $\varphi_0(v_1, v_2, v_3) = \pm 1$ if (v_1, v_2, v_3) is an orthogonal basis of V_1.

We now prove this statement; first j is diagonalizable with eigenvalues ± 1 because $\mathrm{j}^2 = \mathrm{Id}$, $\mathrm{j} \neq \mathrm{Id}$ and $\mathrm{j} \in \mathrm{SO}(7)$. Let us take a unit-length vector $v_1 \in V_1$; the vector space $W = \langle v_1 \rangle^{\perp}$ is fixed by j because $\mathrm{j} \in \mathrm{SO}(7)$, and carries in addition an SU(3) structure determined by $\omega = i(v_1)\varphi_0$, $\mathfrak{Re}\,(\Omega) = \varphi_0|_W$ (see [104]). Of course, the SU(3) structure is preserved by j. Viewed as a complex map, $\mathrm{j} \colon W \to W$ has three complex eigenvalues $\lambda_1, \lambda_2, \lambda_3$ that satisfy $\lambda_j^2 = 1$ and $\lambda_1\lambda_2\lambda_3 = 1$ because $\mathrm{j}^2 = \mathrm{Id}$ and j preserves the SU(3) structure. Being $\mathrm{j} \neq \mathrm{Id}$, we obtain that $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -1$ up to a permutation of the indices; this proves that $\dim(V_1) = 3$ and $\dim(V_{-1}) = 4$. Now observe that $\mathrm{j}(u \times v) = \mathrm{j}(u) \times \mathrm{j}(v)$, where $\times$ is the cross product on $\mathbb{R}^7$ that determines φ. Thus, let (v_1, v_2, v_3) be an orthogonal basis of V_1, then $v_1 \times v_2 \in V_1$; so necessarily, $v_1 \times v_2 = \pm v_3$ and $\varphi_0(v_1, v_2, v_3) = \pm 1$. □

Remark 4.11. If $d\varphi = 0$, Lemma 4.10 states that L is a calibrated submanifold of M in the sense of [61].

SU(2) structures

Let us identify $\mathbb{R}^4$ with $\mathbb{H}$ and SU(2) with Sp(1) as usual. The multiplication by i, j and k on the quaternions yields Sp(1)-equivariant endomorphisms I, J and K that determine invariant 2-forms by the contraction of these endomorphism with the scalar product on $\mathbb{R}^4$. In coordinates, these are:

$$\omega_1^0 = w^{12} + w^{34}, \qquad \omega_2^0 = w^{13} - w^{24}, \qquad \omega_3^0 = w^{14} + w^{23}. \tag{4.2}$$

where (w_1, w_2, w_3, w_4) denotes the standard basis of $\mathbb{R}^4$.

Definition 4.12. Let W be a real vector space of dimension 4. An SU(2) structure on W is determined by 2-forms $(\omega_1, \omega_2, \omega_3)$ such that there is a linear isomorphism $u \colon W \to \mathbb{R}^4$ with $u^*(\omega_j^0) = \omega_j$, where the forms ω_j^0 are given by equation (4.2).

An SU(2) structure on a vector space W determines a G_2 structure on $W \oplus \mathbb{R}^3$. To check this we can suppose that $(W, \omega_1, \omega_2, \omega_3) = (\mathbb{R}^4, \omega_1^0, \omega_2^0, \omega_3^0)$. Denote by (v_5, v_6, v_7) the standard basis of $\mathbb{R}^3$, then comparing with formula (4.1), we see:

$$\varphi_0 = v^{567} + \omega_1^0 \wedge v^7 + \omega_2^0 \wedge v^5 - \omega_3^0 \wedge v^6. \tag{4.3}$$

In addition if we fix on $\mathbb{R}^3$ the orientation determined by v^{567}, then W is oriented by $\frac{1}{2}(\omega_1^0)^2$.

Definition 4.13. Let N be a 4-dimensional manifold. An SU(2) structure on N consists of 2-forms $(\omega_1, \omega_2, \omega_3) \in \Omega^2(N)$ that determine an SU(2) structure on T_pN for every $p \in N$. In addition, if $d\omega_1 = d\omega_2 = d\omega_3 = 0$ we say that $(\omega_1, \omega_2, \omega_3)$ is a hyperKähler structure.

Let Y be a 4-dimensional orbifold with atlas $\{(U_\alpha, V_\alpha, \psi_\alpha, \Gamma_\alpha)\}$. An SU(2) structure on Y consists of 2-forms $(\omega_1, \omega_2, \omega_3) \in \Omega^2(Y)$ such that $(\omega_1^\alpha, \omega_2^\alpha, \omega_3^\alpha)$ is an SU(2) structure on U_α. In addition, if $d\omega_1 = d\omega_2 = d\omega_3 = 0$ we say that $(\omega_1, \omega_2, \omega_3)$ is a hyperKähler structure.

In view of Lemma 4.10 the local model of X around L is $(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{R}^3$, with $\mathbb{Z}_2 = \langle -\mathrm{Id}, \mathrm{Id} \rangle$. The standard G_2 form induces the orbifold hyperKähler SU(2) structure $(\omega_1^0, \omega_2^0, \omega_3^0)$ on $\mathbb{C}^2/\mathbb{Z}_2$. We now detail the hyperKähler resolution of $Y = \mathbb{C}^2/\mathbb{Z}_2$; this will be useful in order to construct the resolution of X in section 4.3.

The holomorphic resolution of Y is $N = \widetilde{\mathbb{C}}^2/\mathbb{Z}_2$; where $\widetilde{\mathbb{C}}^2$ is the blow-up of $\mathbb{C}^2$ at 0. That is,

$$\widetilde{\mathbb{C}}^2 = \{(z_1, z_2, \ell) \in \mathbb{C}^2 \times \mathbb{CP}^1 \text{ s.t. } (z_1, z_2) \in \ell\},$$

and the action of $-\mathrm{Id}$ lifts to $(z_1, z_2, \ell) \longmapsto (-z_1, -z_2, \ell)$. We shall call the exceptional divisor $E = \{0\} \times \mathbb{CP}^1 \subset N$. Note that there is a well-defined projection $\sigma_0 \colon N \to \mathbb{CP}^1$. Let us consider $r_0 \colon Y \to [0, \infty)$ the radial function induced from $\mathbb{C}^2$; one can check taking coordinates that r_0^2 is not smooth on N, but r_0^4 is.

Consider the blow-up map, $\chi_0 \colon N \to Y$. Then, one can check that $\chi_0^*(\omega_2^0)$ and $\chi^*(\omega_3^0)$ are non-degenerate smooth forms on N; this holds because $\omega_2^0 + i\omega_3^0 = dz_1 \wedge dz_2$ and the pullback of a holomorphic form under a holomorphic resolution is holomorphic.

A computation in coordinates shows that $\chi_0^*(\omega_1^0)$ has a pole on E. Let $a > 0$ and define $\mathsf{f}_a(x) = \mathbf{g}_a(x) + 2a\log(x)$, where $\mathbf{g}_a(x) = (x^4 + a^2)^{1/2} - a\log((x^4 + a^2)^{1/2} + a)$. Consider on $Y - E$:

$$\widehat{\omega}_1^a = -\frac{1}{4} dId\mathsf{f}_a(r_0).$$

One can check that $(\widehat{\omega}_1^a, \chi_0^*(\omega_2^0), \chi_0^*(\omega_3^0))$ is a hyperKähler structure on $N - E$; it can be extended as a hyperKähler structure on N because:

$$-\frac{1}{4} dId(\log(r_0^2)) = \sigma_0^*(\omega_{\mathbb{CP}^1}),$$

where $\omega_{\mathbb{CP}^1}$ stands for the Fubini-Study form of $\mathbb{CP}^1$.

4.2.3 Formality

In this section we review some definitions and results about formal manifolds and formal orbifolds; basic references are [40], [42], and [100].

We work with commutative differential graded algebras (in the sequel CDGAs); these consist of a pairs (A, d) where A is a commutative graded algebra $A = \oplus_{i \geq 0} A^i$ over $\mathbb{R}$, and $d \colon A^* \to A^{*+1}$ is a differential, which is a graded derivation that satisfies $d^2 = 0$. If $a \in A$ is an homogenous element, we denote its degree by $|a|$, and $\bar{a} = (-1)^{|a|} a$.

The cohomology algebra of a CDGA (A, d) is denoted by $H^*(A, d)$; it is also a CDGA with the differential being zero. If $a \in A$ is a closed element we denote its cohomology class by $[a]$. The CDGA (A, d) is said to be connected if $H^0(A, d) = \mathbb{R}$.

In our context, the main examples of CDGAs are the de Rham complex of a manifold or an orbifold. In section 4.5 we also make use of the Chevalley-Eilenberg CDGA of a Lie group G, that consists of the algebra $\Lambda^*\mathfrak{g}^*$, the differential of a 1-form is $d\alpha(x,y) = -\alpha[x,y]$, and is extended to $\Lambda^*\mathfrak{g}^*$ as a graded derivation.

Definition 4.14. A CDGA (A,d) is said to be minimal if:

1. A is free as an algebra, that is A is the free algebra ΛV over a graded vector space $V = \oplus_i V^i$.

2. There is a collection of generators $\{a_i\}_i$ indexed by a well ordered set, such that $|a_i| \leq |a_j|$ if $i < j$ and each da_j is expressed in terms of the previous a_i with $i < j$.

Morphisms between CDGAs are required to preserve the degree and to commute with the differential; a morphism of CDGAs $\kappa\colon (B,d) \to (A,d)$ is said to be a quasi-isomorphism if it induces an isomorphism on cohomology $\kappa\colon H^*(B,d) \to H^*(A,d)$.

Definition 4.15. A CDGA (B,d) is a model of the CDGA (A,d) if there exists a quasi-isomorphism $\kappa\colon (B,d) \to (A,d)$. If (B,d) is minimal we say that (B,d) is a minimal model of (A,d).

Minimal models of connected DGAs exist and are unique up to isomorphism of CDGAs. So we define the minimal model of a connected manifold or a connected orbifold as the minimal model of its associated de Rham complex.

Definition 4.16. A minimal algebra $(\Lambda V, d)$ is formal if there exists a quasi-isomorphism,

$$(\Lambda V, d) \to (H^*(\Lambda V, d), 0).$$

A manifold or an orbifold is formal if its minimal model is formal.

We now recall the definition of triple Massey products; these are objects that detect non-formality of manifolds. Let (A,d) be a CDGA and let ξ_1, ξ_2, ξ_3 be cohomology classes such that $\xi_1\xi_2 = 0$ and $\xi_2\xi_3 = 0$. Under these assumptions we can define the triple Massey product of these cohomology classes $\langle \xi_1, \xi_2, \xi_3 \rangle$. In order to provide its definition we first introduce the concept of a defining system for $\langle \xi_1, \xi_2, \xi_3 \rangle$.

Definition 4.17. A defining system for $\langle \xi_1, \xi_2, \xi_3 \rangle$ is an element $(a_1, a_2, a_3, a_{12}, a_{23})$ such that:

1. $[a_i] = \xi_i$ for $1 \leq i \leq 3$,

2. $da_{12} = \bar{a}_1 a_2$, and $da_{23} = \bar{a}_2 a_3$.

One can check that $\bar{a}_1 a_{23} + \bar{a}_{12} a_3$ is a closed $(|a_1| + |a_2| + |a_3| - 1)$-form. The triple Massey product $\langle \xi_1, \xi_2, \xi_3 \rangle$ is the set formed by the cohomology classes that defining systems determine, that is:

$$\{[\bar{a}_1 a_{23} + \bar{a}_{12} a_3] \text{ s.t. } (a_1, a_2, a_3, a_{12}, a_{23}) \text{ runs over all defining systems}\}.$$

If $0 \in \langle \xi_1, \xi_2, \xi_3 \rangle$ we say that the triple Massey product is trivial.

Theorem 4.18. *Let $(\Lambda V, d)$ be a formal minimal algebra. Let ξ_1, ξ_2, ξ_3 be cohomology classes such that the triple Massey product $\langle \xi_1, \xi_2, \xi_3 \rangle$ is defined. Then $\langle \xi_1, \xi_2, \xi_3 \rangle$ is trivial.*

As a consequence, we obtain:

Corollary 4.19. *Let* $(\Lambda V, d)$ *be the minimal model of* (A, d)*. Let* $\xi_1, \xi_2, \xi_3 \in H^*(A, d)$ *such that the triple Massey product* $\langle \xi_1, \xi_2, \xi_3 \rangle$ *is defined. If* $\langle \xi_1, \xi_2, \xi_3 \rangle$ *is not trivial then* $(\Lambda V, d)$ *is not formal.*

Proof. Suppose that $(\Lambda V, d)$ is formal and let $\kappa\colon (\Lambda V, d) \to (A, d)$ be a quasi-isomorphism. Let us take cohomology classes $\xi_1', \xi_2', \xi_3' \in H^*(\Lambda V, d)$ with $\kappa(\xi_j') = \xi_j$ then the Massey product $\langle \xi_1', \xi_2', \xi_3' \rangle$ is well-defined and there is a defining system $(a_1, a_2, a_3, a_{12}, a_{23})$ such that

$$\bar{a}_1 a_{23} + \bar{a}_{12} a_3 = d\alpha.$$

But of course $0 = \kappa[\bar{a}_1 a_{23} + \bar{a}_{12} a_3] \in \langle \xi_1, \xi_2, \xi_3 \rangle$; yielding a contradiction. □

We finally outline some aspects about finite group actions on minimal models. Let M be a compact manifold and let $\kappa\colon (\Lambda V, d) \to (\Omega(M), d)$ be its minimal model. Let Γ be a finite subgroup of $\mathrm{Diff}(M)$ acting on the left; the pullback of forms defines a right action of Γ on $(\Omega(M), d)$.

Lifting theorems for CDGAs ensure the existence of a morphism $\overline{\gamma}\colon \Lambda V \to \Lambda V$ that lifts up to homotopy the pullback by each $\gamma \in \Gamma$; that is, $\kappa \circ \overline{\gamma} \sim \gamma^* \circ \kappa$; in particular, $[\kappa(\overline{\gamma}(a))] = [\gamma^* \kappa(a)]$ if $da = 0$. This implies that $\overline{\mathrm{Id}} \sim \mathrm{Id}$ and that $\overline{\gamma\gamma'} \sim \overline{\gamma}\,\overline{\gamma}'$; therefore these liftings provide an homotopy action on ΛV. These liftings can be modified making use of group cohomology techniques (see [99, Theorem 2]) in order to endow ΛV with a right action of Γ.

Theorem 4.20. *Let* M *be a compact connected manifold and let* Γ *be a subgroup of* $\mathrm{Diff}(M)$ *acting on the left.*

There is a right action of Γ *on the minimal model* $\kappa\colon (\Lambda V, d) \to (\Omega(M), d)$ *by morphisms of CDGAs such that* $[\kappa(a\gamma)] = [\gamma^* \kappa(a)]$ *for every closed element* $a \in \Lambda V$ *and every* $\gamma \in \Gamma$*.*

If there is a right action of a finite group Γ on a CDGA (A, d) one can consider the CDGA of Γ-invariant elements (A^Γ, d). An average argument leads us to $H^*(A, d)^\Gamma = H^*(A^\Gamma, d)$. In addition, if Γ also acts on (B, d) on the right by morphisms and $\mathrm{i}\colon (A, d) \to (B, d)$ is a morphism such that $[\mathrm{i}(a\gamma)] = [(\mathrm{i}a)\gamma]$ for every closed $a \in A$ and $\gamma \in \Gamma$ one can define:

$$\underline{\mathrm{i}}\colon (A^\Gamma, d) \to (B^\Gamma, d), \qquad \underline{\mathrm{i}}a = |\Gamma|^{-1} \sum_{\gamma \in \Gamma} \mathrm{i}(a)\gamma,$$

where $|\Gamma|$ denotes the cardinal number of Γ. This satisfies that $[\underline{\mathrm{i}}(a)] = [\mathrm{i}(a)]$ for closed elements $a \in A^\Gamma$. In particular if i is a quasi-isomorphism so is $\underline{\mathrm{i}}$.

Lemma 4.21. *Let* Γ *be a finite group acting on a compact connected manifold* M *by diffeomorphisms. If* M *is formal then* M/Γ *is also formal.*

Proof. First of all, the fact that $(\Omega(M/\Gamma), d) = (\Omega(M)^\Gamma, d)$ and our previous argument ensures that $H^*(M/\Gamma) = H^*(M)^\Gamma$. Let $\kappa\colon (\Lambda V, d) \to (\Omega(M), d)$ be the minimal model of M as constructed in Theorem 4.20. The CDGA $((\Lambda V)^\Gamma, d)$ is a model for $(\Omega(M/\Gamma), d)$ because of the quasi-isomorphism $\underline{\kappa}\colon ((\Lambda V)^\Gamma, d) \to (\Omega(M)^\Gamma, d)$ defined as above. Consider $(\Lambda W, d)$ the minimal model of $(\Omega(M/\Gamma), d)$ and let $\psi\colon (\Lambda W, d) \to ((\Lambda V)^\Gamma, d)$ be a quasi isomorphism.

Being M formal one can consider a quasi-isomorphism $\mathrm{i}\colon (\Lambda V, d) \to (H^*(\Lambda V, d), 0)$ and define $\underline{\mathrm{i}}\colon ((\Lambda V)^\Gamma, d) \to (H^*(\Lambda V, d)^\Gamma, 0) = (H(\Lambda W, d), 0)$, which is also a quasi-isomorphism. Then we can construct a quasi isomorphism:

$$\underline{\mathrm{i}} \circ \psi\colon (\Lambda W, d) \to (H^*(\Lambda W, d), 0).$$

Therefore, M/Γ is formal. □

4.3 Resolution process

Let (M, φ, g) be a closed G_2 structure on a compact manifold M, let $\mathrm{j}\colon M \to M$ be a G_2 involution, and let $X = M/\mathrm{j}$. The singular locus of the closed G_2 orbifold (X, φ, g) is the set $L = \mathrm{Fix}(\mathrm{j})$, a 3-dimensional oriented manifold according to Lemma 4.10. This section is devoted to constructing a resolution $\rho\colon \widetilde{X} \to X$ under the extra assumption that L has a nowhere-vanishing closed 1-form $\theta \in \Omega^1(L)$.

This hypothesis yields a topological characterisation of L that we now outline. Let us denote by $L_1, \dots, L_r$ the connected components of L; according to Tischler's Theorem [110] each L_i is a fibre bundle over S^1 with fibre a connected surface Σ_i; that is, L_i is the mapping torus of a diffeomorphism $\psi_i \in \mathrm{Diff}(\Sigma_i)$:

$$L_i = \Sigma_i \times [0,1]/(x,0) \sim (\psi_i(x), 1).$$

Let us denote $\mathrm{q}_i\colon \Sigma_i \times [0,1] \to L_i$ the quotient map and $\mathrm{b}_i\colon L_i \to S^1$ the bundle map. The construction described in this section does not need the choice any specific nowhere-vanishing closed 1-form $\theta \in \Omega^1(L)$. However, to determine the cohomology ring of the resolution in Proposition 4.38 we need that $\theta|_{L_i} = \mathrm{b}_i^*(\theta_0)$, where θ_0 denotes the angular form on S^1. Therefore, we make this assumption from the beginning. In addition, taking into account that L_i is oriented and that $H^3(L_i) \cong \{[\alpha] \in H^2(\Sigma_i) \text{ s.t. } \psi_i^*[\alpha] = [\alpha]\}$ (see [10, Lemma 12]), we obtain that Σ_i is oriented and $\psi_i^* = \mathrm{Id}$ on $H^2(\Sigma_i)$.

The resolution process consists of replacing a neighbourhood of L with a closed G_2 manifold. The local model of the singularity is $\mathbb{R}^3 \times Y$ where $Y = \mathbb{C}^2/\mathbb{Z}_2$ as we discussed in section 4.2. The closed G_2 manifold that we introduce is the blow-up of ν/j at the zero section, where ν denotes the normal bundle of L in M. Its local model is $\mathbb{R}^3 \times N$ where $N = \widetilde{\mathbb{C}}^2/\mathbb{Z}_2$. This requires the choice of complex structure on ν/j which is determined by a choice of a unit-length vector V on L by means of the expression $I(X) = V \times X$, where $\times$ is the cross-product associated to φ. This vector field exists because L is parallelizable, but we shall choose $V = \|\theta\|^{-1}\theta^\sharp$ in order to guarantee that the G_2 form that we later define on the resolution is closed.

Before constructing a G_2 form on the resolution we study the $O(1)$ term of $\exp^*(\varphi)$ by splitting $T\nu$ into an horizontal and a vertical bundle with the aid of a connection. This allows us to obtain a formula for the $O(1)$ term that resembles the standard G_2 structure on $\mathbb{R}^3 \times Y$. Its pullback under the blow-up map has a pole at the zero section; a non-singular G_2 structure is defined on the resolution following the ideas we introduced in subsection 4.3.3 for resolving the local model. This form is not closed in general, so that we need to consider a closed approximation of it. In addition, the resolution process requires the introduction of a 1-parameter family of closed forms; small values of the parameter guarantee that these are non-degenerate and close to $\exp^*(\varphi)$ on an annulus around L after a diffeomorphism. As Remark 4.33 states, the size of the exceptional divisor decreases as the parameter tends to 0.

This section is organized as follows: in subsection 4.3.1 we introduce some notations concerning the normal bundle ν of L and we understand its second order Taylor approximation ϕ_2 in subsection 4.3.2; this is an auxiliary construction. In subsection 4.3.3 we obtain local formulas for the $O(1)$-terms and introduce the parameter t; these tools allow us to perform the resolution in subsection 4.3.4.

4.3.1 Splitting of the normal bundle

We now introduce some notations that we need for the resolution process. Let $\pi\colon \nu \to L$ be the normal bundle of L. We consider $R > 0$ such that the neighbourhood of the 0 section Z, $\nu_R = \{v_p \in \nu_p \text{ s.t. } \|v_p\| < R\}$ is diffeomorphic to a neighbourhood U of L on M via the exponential map. In this section we also denote by $\nu_s = \{v_p \in \nu_p \text{ s.t. } \|v_p\| < s\}$ for $s < R$. On ν_R we consider $\phi = (\exp)^*\varphi$, which is a closed G_2 form on ν_R. In addition, the induced involution on ν is $d\mathrm{j}(v_p) = -v_p$; but we shall also denote it by j. It shall be useful to denote the dilations by $F_t \colon \nu \to \nu$, $F_t(v_p) = tv_p$. We also define the vector field over ν, $\mathcal{R}(v_p) = \frac{d}{dt}\Big|_{t=0} e^t v_p$.

A connection ∇ on ν induces a splitting $T\nu = V \oplus H$ where $V = \ker(d\pi) \cong \pi^*\nu$ and $d\pi_{v_p} \colon H_{v_p} \to T_pL$ is an isomorphism; being $TM|_L = \nu \oplus TL$, the connection induces an isomorphism $\mathcal{T} \colon T\nu \to \pi^*(TM|_L)$. The choice of ∇ is made in subsection 4.3.4.

Note that any tensor T on $TM|_L$ defines a tensor on $\pi^*(TM|_L)$ because $\pi^*(TM|_L)_{v_p} = T_pM|_L$. Using this we define on ν:

1. A metric, $g_1 = \mathcal{T}^*(g|_L)$; that is, g_1 makes (H_{v_p}, g_1) and (T_pL, g) isometric, H_{v_p} is perpendicular to V_{v_p} and V_{v_p} isometric to ν_p.

2. A G_2 structure $\phi_1 = \mathcal{T}^*(\varphi|_L)$ with g_1 as an associated metric.

Of course, $\mathcal{T}$ is an isometry. These tensors are constant in the fibres in the following sense; under the identification $\widehat{\mathcal{T}}_{v_p} = \mathcal{T}_{0_p}^{-1} \circ \mathcal{T}_{v_p} \colon T_{v_p}\nu \to T_{0_p}\nu$ it holds that $\widehat{\mathcal{T}}_{v_p}^*(g_1) = g_1$ and $\widehat{\mathcal{T}}_{v_p}^*(\phi_1) = \phi_1$. Note also that these values coincide with $\exp^* g|_Z$ and ϕ respectively because $(d\exp)|_Z = \mathrm{Id}$. These tensors are thus independent of ∇ only on Z.

We shall also denote $W_{i,j} = \Lambda^i V^* \otimes \Lambda^j H^*$ where we understand $V^* = \mathrm{Ann}(H)$ and $H^* = \mathrm{Ann}(V)$. There are g_1-orthogonal splittings $\Lambda^k T^*\nu = \oplus_{i+j=k} W_{i,j}^k$ and given $\alpha \in \Lambda^k T^*\nu$ we denote by $[\alpha]_{i,j}$ the projection of α to $W_{i,j}$.

Observe also that one can restrict each $\beta \in \Lambda^k V^*$ to the fibre ν_p, and the restriction $\mathrm{r}_k \colon \Lambda^k T^*\nu \to \Lambda^k V^*$, $\mathrm{r}_k(\beta)_{v_p} = \beta_{v_p}|_{\nu_p}$ is an isomorphism because $T_{v_p}\nu_p = V_{v_p}$.

We now state some technical observations concerning vertical forms; proofs are computations in terms of local coordinates that we include for completeness.

Remark 4.22. Note that $H^* = \pi^*(T^*L)$ does not depend on the connection but V^* does. More precisely, in local coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, y_4) \in U \times \mathbb{R}^4$ the horizontal distribution at (x, y) is generated by:

$$\partial_{x_i} - \sum_{j=1}^{4} A_i^j(x, y)\partial_{y_j},$$

where $A_i^j(x, y) = \sum_{k=1}^4 A_{i,k}^j(x) y_k$ for some differentiable functions $A_{i,k}^j$. Then V^* is generated by:

$$\eta_j = dy_j + \sum_{i=1}^{3} A_i^j(x, y) dx_i.$$

Note also that since $A_i^j(x, ty) = tA_i^j(x, y)$ we get that $F_t^*(\eta_i) = t\eta_i$.

Lemma 4.23. *The following identities hold:*

1. $F_t^*(\phi_1) = [\phi_1]_{0,3} + t^2[\phi_1]_{2,1}$

2. $F_t^*(g_1) = g_1|_{H\otimes H} + t^2 g_1|_{V\otimes V}$

Proof. We shall prove the first equality being the second similar. Note that $\phi_1|_Z$ is a G_2 structure whose induced metric makes V perpendicular to H and $H|_Z = TZ$; thus taking into account formula (4.3) we can write in local coordinates:

$$\phi_1|_Z = f(p) dx_1 \wedge dx_2 \wedge dx_3 + \sum_{i=1}^{3} \sum_{j<k} f_{ijk}(p) dx_i \wedge dy_j \wedge dy_k.$$

Thus, $\phi_1 = [\phi_1]_{0,3} + [\phi_1]_{2,1}$, where $([\phi_1]_{0,3})_{v_p} = f(p)dx_1 \wedge dx_2 \wedge dx_3$ and $([\phi_1]_{2,1})_{v_p} = \sum_{i=1}^{3}\sum_{j<k} f_{ijk}(p)dx_i \wedge (\eta_j)_{v_p} \wedge (\eta_k)_{v_p}$. Therefore, $F_t^*([\phi]_{0,3}) = [\phi]_{0,3}$ and, according to Remark 4.22, $F_t^*[\phi]_{2,1} = t^2[\phi]_{2,1}$. □

Lemma 4.24. *1. Let $\mu \in V^*$ be a form such that $\mu = 0$ on $T\nu|_Z$. Then, $[d\mu]_{1,1} = 0$ and $[d\mu]_{0,2} = 0$ on $T\nu|_Z$.*

2. Suppose that $\alpha \in W_{1,1}$ satisfy that $\alpha = 0$ on $T\nu|_Z$. Then, $[d\alpha]_{1,2} = 0$ and $[d\alpha]_{0,3} = 0$ on $T\nu|_Z$.

Proof. For the first equality, we write in local coordinates $\mu = \sum_{i=1}^{4} f_i(x,y)\eta_i$ with $f_i(x,0) = 0$ as $\mu = 0$ on $T\nu|_Z$. Then,

$$\begin{aligned} d\mu =& \sum_{i=1}^{4}\sum_{j=1}^{3} \frac{\partial f_i}{\partial x_j}(x,y)dx_j \wedge \eta_i \\ &+ \sum_{i=1}^{4}\sum_{j=1}^{4} \frac{\partial f_i}{\partial y_j}(x,y)dy_j \wedge \eta_i + \sum_{i=1}^{4} f_i(x,y)d\eta_i. \end{aligned}$$

Since $f_i(x,0) = 0$ and $\eta_i|_{T\nu|_Z} = dy_i$ the following equalities hold on $T\nu|_Z$:

$$\begin{aligned} [d\mu]_{2,0}(x,0) =& \sum_{i=1}^{4}\sum_{j=1}^{4} \frac{\partial f_i}{\partial y_j}(x,0)dy_j \wedge dy_i, \\ [d\mu]_{1,1}(x,0) =& \sum_{i=1}^{4}\sum_{j=1}^{3} \frac{\partial f_i}{\partial x_j}(x,0)dx_j \wedge \eta_i = 0, \\ [d\mu]_{0,2}(x,0) =& 0. \end{aligned}$$

For the second, we write $\alpha = \sum_i \pi^*(\lambda_i)\wedge\mu_i$ with $\lambda_i \in \Omega^1(L)$ and $\mu_i \in V^*$ satisfying $\mu_i = 0$ on $T\nu|_Z$. Then $[d\alpha]_{1,2} = \sum_i(\pi^*(d\lambda_i)\wedge\mu_i - \pi^*(\lambda_i)\wedge[d\mu_i]_{1,1})$ and $[d\alpha]_{0,3} = -\sum_i \pi^*(\lambda_i)\wedge[d\mu_i]_{0,2}$. The claim follows from (1). □

Lemma 4.25. *Consider coordinates $(x,y) = (x_1,x_2,x_3,y_1,y_2,y_3,y_4) \in B \times \mathbb{R}^4$ of ν, with $B \subset \mathbb{R}^3$ a closed ball. Let η_j be the projection of dy_j to V^* as in Remark 4.22. Then, $\|(\eta_i)_{(x,0)}\|_{g_1} = \|(\eta_i)_{(x,y)}\|_{g_1}$ and $\|(dx_i)_{(x,0)}\|_{g_1} = \|(dx_i)_{(x,y)}\|_{g_1}$.*

There exist $C_1 > 0$, $C_2 > 0$ such that $\|[d\eta_i]_{0,2}\|_{g_1} \leq C_1 r$ and $\|[d\eta_i]_{1,1}\|_{g_1} \leq C_2$ on ν.

Proof. The first two equalities are clear taking into account that $\mathcal{T}^*(\eta_j) = \eta_j$, $\mathcal{T}^*(dx_j) = dx_j$ and that $\mathcal{T}$ is a g_1-isometry. For the third and fourth equality we first compute $d\eta_j$

$$d\eta_j = \sum_{k=1}^{4}\sum_{i,l=1}^{3} y_k \frac{\partial A_{i,k}^j(x)}{\partial x_l} dx_l \wedge dx_i + \sum_{k=1}^{4}\sum_{i=1}^{3} A_{i,k}^j(x)dy_k \wedge dx_i.$$

This implies that:

$$\begin{aligned} [d\eta_j]_{0,2} =& \sum_{k=1}^{4}\sum_{i,l=1}^{3} y_k \frac{\partial A_{i,k}^j(x)}{\partial x_l} dx_l \wedge dx_i - \sum_{k,n=1}^{4}\sum_{i,m=1}^{3} A_{i,k}^j(x)A_{m,n}^k(x)y_n dx_m \wedge dx_i, \\ [d\eta_j]_{1,1} =& \sum_{k=1}^{4}\sum_{i=1}^{3} A_{i,k}^j(x)\eta_k \wedge dx_i. \end{aligned}$$

The absolute values of the functions $A_{i,k}^j$, $\frac{\partial A_{i,k}^j}{\partial x_l}$ are bounded on B, and the g_1-norms of the terms $\eta_m \wedge dx_j$ and $dx_j \wedge dx_k$ are constant on the fibres as explained before. Taking into account that L is compact the choice of constants C_1 and C_2 becomes clear. □

4.3.2 Taylor series

We now introduce the Taylor series of ϕ and interpolate it with the seccond order approximation. This is an auxiliary tool for our resolution process.

Consider the dilation over the fibres $F_t\colon \nu \to \nu$, and define the Taylor series of $F_t^*\phi$ and F_t^*g near $t=0$ (note that $F_0^*(\phi)$ and $F_0^*(g)$ are defined on ν). That is,

$$F_t^*(\phi) \sim \sum_{k=0}^{\infty} t^{2k}\phi^{2k}, \quad F_t^* g \sim \sum_{k=0}^{\infty} t^{2k} g^{2k}.$$

Note that we only wrote even terms because both ϕ and g are j invariant and $\mathrm{j} = F_{-1}$. In addition, the equalities $F_s^*(\phi^{2k}) = s^{2k}\phi^{2k}$, $F_s^*(g^{2k}) = s^{2k}g^{2k}$ follow from $F_{ts} = F_t \circ F_s$. For $i+j=3$ and $p+q=2$ we define $\phi^{2k}_{i,j} = [\phi^{2k}]_{i,j}$, $g^{2k}_{p,q} = g^{2k}|_{V^p\otimes H^q}$; here V^p denotes the tensor product of V with itself p times.

We have the following properties:

1. $\|\phi^{2k}_{i,j}\|_{g_1} = O(r^{2k-i})$, where r is measured with respect to the metric on ν. To check it let $\|v_p\|_{g_1} = 1$; taking into account Lemma 4.23 and the fact that $F_t\colon (\nu, g_1|_{H\otimes H} + t^2 g_1|_{V\otimes V}) \to (\nu, g_1)$ is an isometry we get:

$$\begin{aligned}\|(\phi^{2k}_{i,j})_{rv_p}\|_{g_1} &= \|r^{2k}F^*_{r^{-1}}(\phi^{2k}_{i,j})_{rv_p}\|_{g_1} = r^{2k}\|(\phi^{2k}_{i,j})_{v_p}\|_{g_1|_{H\otimes H}+r^2 g_1|_{V\otimes V}} \\ &= r^{2k-i}\|(\phi^{2k}_{i,j})_{v_p}\|_{g_1}.\end{aligned}$$

2. The previous statement ensures that $\phi^{2k}_{i,j} = 0$ if $i > 2k$.

3. If $k \geq 1$, ϕ^{2k} is exact.

   Being ϕ^{2k} homogeneous of order $2k$, we have that $\mathcal{L}_{\mathcal{R}}(\phi^{2k}) = 2k\phi^{2k}$; where $\mathcal{R}(v_p) = \frac{d}{dt}\Big|_{t=0}(e^t v_p)$ is defined as above. In addition, $d\phi^{2k} = 0$ for every k because ϕ is closed. Thus, $2k\phi^{2k} = d(i(\mathcal{R})\phi^{2k})$.

Taking these properties into account we construct a G_2 form $\phi_{3,\varepsilon}$ that interpolates ϕ with the approximation $\phi_2 = \phi^0 + \phi^2$. The parameter $\varepsilon > 0$ indicates that the interpolation occurs on $r \leq \varepsilon$ and is done in such a way that $\phi_{3,\varepsilon}|_{r\leq\frac{\varepsilon}{2}} = \phi_2$. Of course, this is possible because the difference between ϕ and ϕ_2 is small near the zero section.

Proposition 4.26. *The form $\phi_2 = \phi^0 + \phi^2$ is closed and $\phi = \phi_2 + O(r)$. There exists $\varepsilon_0 > 0$ such that for each $\varepsilon < \varepsilon_0$ there exists a* j*-invariant* G_2 *form $\phi_{3,\varepsilon}$ such that $\phi_{3,\varepsilon} = \phi_2$ if $r \leq \frac{\varepsilon}{2}$ and $\phi_{3,\varepsilon} = \phi$ if $r \geq \varepsilon$.*

Proof. The first part is a consequence of the previous remark; zero order terms are $\phi^0 = \phi^0_{0,3}$ and $\phi^2_{2,1}$, thus $\phi = \phi_2 + O(r)$. In addition, ϕ_2 is closed because each ϕ^{2k} is.

Since $\phi|_Z = \phi_2|_Z$ the Poincaré Lemma for submanifolds ensures that $\phi = \phi_2 + d\xi$ for some j-invariant 2-form ξ; more precisely, $\xi_{v_x} = \int_0^1 i(\mathcal{R}_{\tau v_x})(\phi - \phi_2)d\tau$. In addition, $\|\xi\|_{g_1} = O(r^2)$ because $\xi|_Z = 0$ and $\|d\xi\|_{g_1} = \|\phi - \phi_2\|_{g_1} = O(r)$. Let ϖ be a smooth function such that $\varpi = 1$ if $x \leq \frac{1}{2}$ and $\varpi = 0$ if $x \geq 1$ and define $\varpi_\varepsilon(x) = \varpi(\frac{x}{\varepsilon})$. Then, $|\varpi'_\varepsilon| \leq \frac{C}{\varepsilon}$ so that

$$\phi_{3,\varepsilon} = \phi - d(\varpi_\varepsilon(r)\xi)$$

is a G_2 form on $r \leq \varepsilon$ if ε is small enough because it is $O(\varepsilon)$-near ϕ. The form $\phi_{3,\varepsilon}$ interpolates ϕ_2 with ϕ over the stated domains and it is j-invariant because both ϕ and $\varpi_\varepsilon(r)\xi$ are. □

4.3.3 Local formulas

The purpose of this section is making an additional preparation; we first provide a local formula for ϕ_1 that will be useful in order to construct the G_2 form of the resolution. Later we change ϕ_2 by $O(r)$ terms so that we control its local formula and we introduce the parameter t; these preparations are essential to construct a closed G_2 form on the resolution.

Formula for ϕ_1

We first write ϕ_1 and g_1 in terms of the components of the Taylor series of g and ϕ. This is an easy consequence of the homogeneus behaviour of the tensors involved:

Lemma 4.27. *The following equalities hold:*

1. $\phi_1 = \phi^0 + \phi^2_{2,1}$
2. $g_1 = g^0_{0,2} + g^2_{2,0}$

Proof. We prove the first equality, being the second similar. Using the fact that $\phi^0 = \phi^0_{0,3}$ and $\phi^2_{2,1}$ are homogeneous one can check that these are constant on the fibres. We shall do it for $\phi^2_{2,1}$, write in local coordinates (x, y):

$$\phi^2_{2,1} = \sum_{i=1}^{3} \sum_{j<k} f_{ijk}(x, y) dx_i \wedge (\eta_j)_{(x,y)} \wedge (\eta_k)_{(x,y)}.$$

Taking into account that $F_t^* \phi^2_{2,1} = t^2 \phi^2_{2,1}$ and $F_t^* \eta_i = t\eta_i$ we get $f_{ijk}(x, ty) = f_{ijk}(x, y)$. Therefore, $f_{ijk}(x, y) = f_{ijk}(x, 0)$. Since $\phi_1|_{TM|_Z} = \phi|_{TM|_Z} = (\phi^0 + \phi^2_{2,1})|_{TM|_Z}$, we obtain that $[\phi_1]_{0,3}|_{T\nu|_Z} = \phi^0|_{T\nu|_Z}$ and $[\phi_1]_{2,1}|_{T\nu|_Z} = \phi^2_{2,1}|_{T\nu|_Z}$. But these forms are constant on the fibres of the bundle $T\nu \to \nu$, so that the previous equalities hold on $T\nu$. □

We now obtain a local formula for ϕ_1. For that purpose let us define $e_1 = \|\theta\|^{-1}\theta$ and consider an orthonormal oriented frame (e_1, e_2, e_3) of TL on a neighbourhood $U \subset L$. Define also the SU(2) structure $(\omega^L_1, \omega^L_2, \omega^L_3)$ on ν by means of the equality:

$$\varphi|_L = e_1 \wedge e_2 \wedge e_3 + e_1 \wedge \omega^L_1 + e_2 \wedge \omega^L_2 - e_3 \wedge \omega^L_3.$$

More precisely, the complex structure is determined by $\omega^L_1 = i(e^\sharp_1)\varphi|_\nu$, that is $I(X) = e^\sharp_1 \times X$ where $\times$ denotes the vector product associated to $\varphi|_L$. The complex volume form is $\omega^L_2 + i\omega^L_3$; note that a counterclockwise rotation of angle σ in the plane (e_2, e_3) changes $\omega^L_2 + i\omega^L_3$ by the complex phase $e^{i\sigma}$. Using $\mathcal{T}$ we obtain:

$$\phi_1 = \pi^* e_1 \wedge \pi^* e_2 \wedge \pi^* e_3 + \pi^* e_1 \wedge \omega_1 + \pi^* e_2 \wedge \omega_2 - \pi^* e_3 \wedge \omega_3,$$

where the forms $\omega_j \in \Lambda^2 V^*$ are j-invariant and satisfy $\omega_j|_Z = \exp^*(\omega^L_j)$. For fixed $p \in L$, $(\omega_1|_{\nu_p}, \omega_2|_{\nu_p}, \omega_3|_{\nu_p})$ determines an SU(2) structure on the 4-manifold ν_p because the restriction r_2 is an isomorphism. The associated metric on $T\nu_p$ is $g_1|_{\nu_p}$ and the complex form is induced by I on ν under the canonical isomorphism.

Therefore, $\omega_1|_{\nu_p} = -\frac{1}{4} d_{\nu_p}(I[dr^2]_{\nu_p})$. In addition, since the complex volume form is $dz_1 \wedge dz_2 = \frac{1}{2} d(z_1 dz_2 - z_2 dz_1)$ there is a j-invariant 1-form $\mu \in V^*$ such that $d_{\nu_p}(\mu|_{\nu_p}) = (\omega_2 + i\omega_3)|_{\nu_p}$ and $\mu|_{T\nu|_Z} = 0$. We decompose it as $\mu = \mu_1 + i\mu_2$.

Being the restriction to the fibre r_2 a monomorphism, we obtain

$$\omega_1 = -\frac{1}{4}[d[Idr^2]_{1,0}]_{2,0}, \qquad \omega_2 + i\omega_3 = [d\mu]_{2,0},$$

here we also denoted by I the complex structure on V^* determined by the complex structure $I(X) = e^\sharp_1 \times X$ on $V = \pi^*(\nu)$, this depends on the splitting. Observe that the complex structure I on ν satisfies $j \circ I = I \circ j$ and thus, the complex structure on V^* satisfies $jI\alpha = Ij\alpha$. In particular, $I\alpha$ is j-invariant if α is.

Changing ϕ_2 by $O(r)$ terms.

First of all define the 1-parameter family

$$\phi_2^t = \phi^0 + t^2\phi^2 = F_t^*(\phi_2).$$

These forms are well-defined on ν because ϕ^0 and ϕ^2 are homogeneous. We now change this 1-parameter family by $O(r)$ terms so that we have an explicit local formula for it. Consider the exact j-invariant form:

$$\beta = -\frac{1}{4}\pi^*\theta \wedge d((\|\theta\|^{-1} \circ \pi)I[dr^2]_{1,0}) - d(\pi^*e_2 \wedge \mu_2 - \pi^*e_3 \wedge \mu_3) \in W_{2,1} \oplus W_{1,2} \oplus W_{0,3},$$

and note that $\phi_1 = \pi^*(e_1 \wedge e_2 \wedge e_3) + [\beta]_{2,1}$. In addition, β does not depend on the orthonormal oriented basis (e_2, e_3) of $\langle\theta^*\rangle^\perp$.

We now introduce a 1-parameter family of closed j-invariant forms:

$$\widehat{\phi}_2^t = \pi^*(e_1 \wedge e_2 \wedge e_3) + t^2[\beta].$$

We claim that for fixed $s > 0$ there exists $t_s > 0$ such that $\widehat{\phi}_2^t$ is a G_2 form on ν_{2s} if $t < t_s$. To check this we compare $\widehat{\phi}_2^t$ with $F_t^*\phi_1$ and use Lemma 4.6 to conclude. Denote $g_t = F_t^*(g_1)$ and observe that Lemma 4.27 implies that $F_t^*\phi_1 = \phi^0 + t^2\phi_{2,1}^2$ and $g_t = t^2 g_{2,0} + g_{0,2}$, then:

$$\|F_t^*\phi_1 - \widehat{\phi}_2^t\|_{g_t} = t\|[\beta]_{1,2}\|_{g_1} + t^2\|[\beta]_{0,3}\|_{g_1},$$

so one can bound $\|[\beta]_{1,2}\|_{g_1}$, $\|[\beta]_{0,3}\|_{g_1}$ on ν_{2s} and choose $t_s > 0$ such that for each $t < t_s$, $t\|[\beta]_{1,2}\|_{g_1} + t^2\|[\beta]_{0,3}\|_{g_1} < m$ where m is the universal constant obtained in Lemma 4.6.

We construct a G_2 form $\widehat{\phi}_{3,s}^t$ that interpolates $\widehat{\phi}_2^t$ with ϕ_2^t. The parameter $s > 0$ indicates that the interpolation occurs on the disk $r \leq s$ and we require that $\widehat{\phi}_{3,s}^t|_{r\leq\frac{s}{2}} = \widehat{\phi}_2^t$. In subsection 4.3.4 we employ large values of the parameter.

Proposition 4.28. *There is $\xi \in W_{0,2}$ such that $\|\xi\|_{g_1} = O(r^2)$ and $\phi^2 = \beta + d\xi$.*

For fixed $s > 0$ there exists $t_s' > 0$ such that for each $t < t_s'$, there is a closed j-invariant G_2 form $\widehat{\phi}_{3,s}^t$ on ν_{2s} that coincides with $\widehat{\phi}_2^t$ on $r \leq \frac{s}{2}$ and ϕ_2^t on $r \geq s$.

Proof. Write the second term of the Taylor series of ϕ as $\phi^2 = \phi_{2,1}^2 + \phi_{1,2}^2 + \phi_{0,3}^2$ and note that $\phi_{2,1}^2 = [\beta]_{2,1}$. Being β and ϕ^2 closed, we obtain $d(\phi_{1,2}^2 + \phi_{0,3}^2) = d([\beta]_{1,2} + [\beta]_{0,3})$. The Poincaré Lemma ensures that $\phi_{1,2}^2 + \phi_{0,3}^2 = [\beta]_{1,2} + [\beta]_{0,3} + d\xi$ with

$$\xi_{v_x} = \int_0^1 i(\mathcal{R}_{\tau v_x})(\phi_{1,2}^2 + \phi_{0,3}^2 - [\beta]_{1,2} - [\beta]_{0,3})d\tau = \int_0^1 i(\mathcal{R}_{\tau v_x})(\phi_{1,2}^2 - [\beta]_{1,2})d\tau.$$

Hence $\xi \in W_{0,2}$. One can check that ξ is j-invariant by taking into account that $\phi_{1,2}^2 - [\beta]_{1,2}$ is j-invariant and that $\mathcal{R}_{t\mathrm{j}(v_x)} = \mathrm{j}(\mathcal{R}_{tv_x})$.

In addition $\|\xi\|_{g_1} = O(r^2)$ because $\xi|_Z = 0$ and $\|d\xi\|_{g_1} \leq \|\phi_{1,2}^2\|_{g_1} + \|\phi_{0,3}^2\|_{g_1} + \|[\beta]_{1,2}\|_{g_1} + \|[\beta]_{0,3}\|_{g_1} = O(r)$. Here we used that $\|\phi_{1,2}^2\|_{g_1} = O(r)$, $\|\phi_{0,3}^2\|_{g_1} = O(r^2)$, and $([\beta]_{1,2} + [\beta]_{0,3}) = 0$ on $T\nu|_Z$. To obtain the last equality observe that $\beta = d\alpha$ where $\alpha \in W_{1,1}$ vanishes on $T\nu|_Z$; then use Lemma 4.24 (2).

Let ϖ be a smooth function such that $\varpi = 1$ if $x \leq \frac{1}{2}$ and $\varpi = 0$ if $x \geq 1$, and let $\varpi_s(x) = \varpi(\frac{x}{s})$. The form $\widehat{\phi}_{3,s}^t = \phi^0 + t^2\beta + t^2 d(\varpi_s(r)\xi)$ is closed and j-invariant; it coincides with $\widehat{\phi}_2^t$ on $r \leq \frac{s}{2}$ and with ϕ_2^t on $r \geq s$.

It is clear that $\widehat{\phi}_{3,s}^t$ is a G_2 form on the region $r \geq s$ for $t < t_s$; we now check that it is also a G_2 form on $r \leq s$ for some choice of t. We are going to compare $\widehat{\phi}_{3,\varepsilon}^t$ with $F_t^*\phi_1$ and use Lemma 4.6 to conclude the result.

Since $\varpi_s\xi \in W_{0,2}$ we have that $d(\varpi_s\xi) \in W_{1,2} \oplus W_{0,3}$. As a consequence if $t \leq 1$, then $\|t^2 d(\varpi_s(r)\xi)\|_{g_t} \leq t\|d(\varpi_s(r)\xi)\|_{g_1} = t(O(r^2 s^{-1}) + O(r))$ so that:

$$\begin{aligned}\|\widehat{\phi}^t_{3,s} - F_t^*\phi_1\|_{g_t} &= t(\|[\beta]_{1,2}\|_{g_1} + t\|[\beta]_{0,3}\|_{g_1} + O(r^2 s^{-1}) + O(r)) \\ &\leq t(\|[\beta]_{1,2}\|_{g_1} + \|[\beta]_{0,3}\|_{g_1} + O(r)).\end{aligned}$$

For the last equality we used that $t < 1$ and that $r \leq 2s$. Then $\widehat{\phi}^t_{3,s}$ is a G_2 form if the parameter $t < t_s$ satisfies

$$t\,(\max_{r\leq 2s}(\|[\beta]_{1,2}\|_{g_1} + \|[\beta]_{0,3}\|_{g_1} + O(r)) < m$$

where m is the constant provided by Lemma 4.6. □

4.3.4 Resolution of ν/j

The resolution process is inspired in the hyperKähler resolution $N = \widetilde{\mathbb{C}^2/\mathbb{Z}_2}$ of $Y = \mathbb{C}^2/\mathbb{Z}_2$ described in subsection 4.2.2. Consider the blow-up map $\chi_0 \colon N \to Y$ and the hyperKähler structure $(\widehat{\omega}^a_1, \chi_0^*(\omega^0_2), \chi_0^*(\omega^0_3))$ on N. Recall that $\widehat{\omega}^a_1$ denotes the extension of $-\frac{1}{4}dId\mathsf{f}_a(r_0)$, where r_0 is the radial function on $\mathbb{C}^2$ and:

$$\mathsf{f}_a(x) = \mathbf{g}_a(x) + 2a\log(x), \qquad \mathbf{g}_a(x) = (x^4 + a^2)^{1/2} - a\log((x^4+a^2)^{1/2} + a).$$

We now focus in the resolution of ν/j. For that purpose, consider the complex structure I on ν determined by the 2-form $i(e_1^\sharp)\varphi|_\nu$ and define P as the fiberwise blow-up of ν/j at 0. That is $P = P_{\mathrm{U}(2)}(\nu) \times_{\mathrm{U}(2)} N$, where $P_{\mathrm{U}(2)}(\nu)$ denotes the principal U(2)-bundle associated to ν. This construction yields projections $\chi \colon P \to \nu/\mathrm{j}$ and $\mathrm{pr} = \bar{\pi} \circ \chi$; where $\bar{\pi} \colon \nu/\mathrm{j} \to L$ denotes the map that $\pi \colon \nu \to L$ induces.

We also define $Q = \chi^{-1}(0)$; this is a $\mathbb{CP}^1$ bundle over L that can be expressed as $Q = P_{\mathrm{U}(2)}(\nu) \times_{\mathrm{U}(2)} \mathbb{CP}^1$. Note that there is a projection $\sigma_0 \colon N \to \mathbb{CP}^1$ that induces a complex line bundle $\sigma \colon P \to Q$.

A j-invariant tensor on ν descends to ν/j and its pullback by χ is smooth over $P - Q$, but it may not be smooth on P. If the tensor preserves the complex structure I on P then the pullback is smooth on P because $P = P_{U(2)}(\nu) \times_{\mathrm{U}(2)} N$. We choose ∇ such that $\nabla I = 0$, so that we can lift ∇ to P and define $TP = V' \oplus H'$; this is compatible with the splitting $T\nu = V \oplus H$ in the sense that $d\chi_p(H') = H_{\chi(p)}$, and $d\chi_p(V') = V_{\chi(p)}$ if $p \in P - Q$. In addition, $\mu_2, \mu_3, \omega_1, \omega_2, \omega_3$ induce forms on ν/j and $\chi^*(\mu_k)$, $\chi^*(\omega_k)$ are smooth for $k = \{2, 3\}$. We shall also consider $\Lambda^k T^*P = \oplus_{i+j=k}\Lambda^i(V')^* \otimes \Lambda^j(H')^*$ and define $W'_{i,j} = \Lambda^i(V')^* \otimes \Lambda^j(H')^*$. The projection of α to $W'_{i,j}$ is denoted by $[\alpha]_{i,j}$.

In order to define a G_2 structure on P we need to find a resolution of ω_1. For that purpose denote by r the pullback of the radial function on ν and define:

$$\widehat{\omega}_1 = -\frac{1}{4}d(|\theta|^{-1}I[d\mathsf{f}_{|\theta|}(r)]_{1,0}),$$

where $|\theta| = \|\theta\| \circ \mathrm{pr}$. Observe that $\mathbf{g}_{|\theta|}(r)$ is smooth on P because r^4 is. In addition, $-\frac{1}{2}dI[d(\log(r^2))]_{1,0} = \sigma^*(F_Q)$ on $P - Q$, where F_Q is the curvature of the line bundle $\sigma \colon P \to Q$. Fiberwise it coincides with the Fubini-Study form on $\mathbb{CP}^1$. Note also that $\mathrm{pr}^*\theta \wedge [\widehat{\omega}_1]_{2,0} = -\frac{1}{4}e_1 \wedge [d(I[d\mathsf{f}_{|\theta|}]_{1,0})]_{2,0}$.

We now define a G_2 form Φ^t_1 which is near $\chi^*(F_t^*\phi_1)$ on $r > 1$, this is:

$$\Phi^t_1 = \mathrm{pr}^*(e_1 \wedge e_2 \wedge e_3) + t^2[\widehat{\beta}]_{2,1},$$

where

$$\widehat{\beta} = \mathrm{pr}^*\theta \wedge \widehat{\omega}_1 - d(\mathrm{pr}^*e_2 \wedge \chi^*(\mu_2) - \mathrm{pr}^*e_3 \wedge \chi^*(\mu_3)).$$

Observe that β does not depend on the orthonormal oriented basis (e_2, e_3) of $\langle\theta^*\rangle^\perp$. In addition, the metric induced by Φ_1^1 on TP has the form $h_1 = h_{2,0} + h_{0,2}$ where $h_{2,0}$ and $h_{0,2}$ are metrics on V' and H' respectively. Observe that

In addition, the metric that Φ_1^t induces is $h_t = t^2 h_{2,0} + h_{0,2}$. We define a family of closed forms:

$$\Phi_2^t = \mathrm{pr}^*(e_1 \wedge e_2 \wedge e_3) + t^2 \widehat{\beta}.$$

Note that Φ_2^t is a G_2 structure on $\chi^{-1}(\nu_{2s})$ for some $t < t_s''$. This is ensured by Lemma 4.6 because:

$$\|\Phi_2^t - \Phi_1^t\|_{h_t} = t\|[\widehat{\beta}]_{1,2}\|_{h_1} + t^2\|[\widehat{\beta}]_{0,3}\|_{h_1},$$

and one can bound $\|[\widehat{\beta}]_{1,2}\|_{h_1}$ and $\|[\widehat{\beta}]_{0,3}\|_{h_1}$ on $\chi^{-1}(\nu_{2s})$.

The parameter t is devoted to compensate errors introduced by $\|[\widehat{\beta}]_{1,2}\|_{h_1}$ and $\|[\widehat{\beta}]_{0,3}\|_{h_1}$ that mainly come from the terms $[F_Q]_{1,1}$ and $[F_Q]_{0,2}$.

Lemma 4.29. *Let $g_1 = g_{2,0}^0 + g_{0,2}^2$ be the metric on $\nu/\mathfrak{j}$. On $r > 0$, we have the following:*

1. $\chi_*(h_{2,0}) = g_{0,2}^0$,
2. $\|\chi_*(h_{2,0}) - g_{2,0}^2\|_{g_{2,0}^2} = O(r^{-2})$.

Proof. For the first equality we consider the local $g_{0,2}^0$-orthonormal basis (e_1, e_2, e_3) of T^*L as before. Denote $\bar{\Phi}_1^1 = (\chi^{-1})^*(\Phi_1^1)$ on $r > 0$ and observe that:

$$\bar{\Phi}_1^1 = e^{123} + e^1 \wedge \tilde{\omega}_1 + e^2 \wedge \tilde{\omega}_2 - e^3 \wedge \tilde{\omega}_3,$$

where $\tilde{\omega}_1 = (\chi^{-1})^*[\widehat{\omega}_1]_{2,0}$, and $\tilde{\omega}_j = [d\mu_j]_{2,0}$ for $j \in \{2,3\}$. Observe that $(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$ is an $\mathrm{SU}(2)$ structure on V and therefore, $\tilde{\omega}_1^2 = \tilde{\omega}_2^2 = \tilde{\omega}_3^2 = 2\mathrm{vol}_V$ and $\tilde{\omega}_i \wedge \tilde{\omega}_j = 0$ if $i \neq j$. Of course, vol_V coincides with the unit-length volume form determined by $g_{0,2}^0$. To conclude, we compute $h_{0,2}$ by the formula:

$$6\chi_*(h_{0,2})(e_i^\sharp, e_j^\sharp)(e^{123} \wedge \mathrm{vol}_V) = i(e_i^\sharp)\bar{\Phi}_1^1 \wedge i(e_j^\sharp)\bar{\Phi}_1^1 \wedge \bar{\Phi}_1^1.$$

Taking into account that $(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$ is an $\mathrm{SU}(2)$ structure on V we obtain that $\chi_*(h_{0,2})(e_i, e_j) = 0$ if $i \neq j$ and $\chi_*(h_{0,2})(e_i, e_i) = 1$.

For the second equality, observe that

$$(\chi_* h_{2,0} - g_{2,0}^2)(X, Y) = -\frac{1}{4}[dI[d\bar{f}_{|\theta|}(r^2)]_{1,0}]_{2,0}(X, I(Y)),$$

where $\bar{\mathsf{f}}_a(x) = \mathsf{f}_a(\sqrt{x}) - x = a^2((x^2 + a^2)^{1/2} + x)^{-1} - a\log((x^2 + a^2)^{1/2} + a) + a\log(x)$ for $a > 0$. Note that $|\bar{f}_a'(x)| = O(x^{-1})$ and $|\bar{f}_a''(x)| = O(x^{-2})$.

From the expression of $\chi_*(h_{2,0}) - g_{2,0}^2$ it is enough to show that $\|[dI[d\bar{f}_{|\theta|}(r^2)]_{1,0}]_{2,0}\|_{g_{2,0}^2} = O(r^{-2})$. For our purposes we consider a complex $g_{2,0}^2$-unitary local trivialization $(x, y) = (x_1, x_2, x_3, y_1, y_2, y_3, y_4) \in B \times \mathbb{R}^4$ of ν; that is, $I(x, y) = (x_1, x_2, x_3, -y_2, y_1, -y_4, y_3)$ and the vectors that the trivialization determines have $g_{2,0}^2$-length one. In addition, the connection forms satisfy $I\eta_1 = -\eta_2$, $I\eta_3 = -\eta_4$; to check this one has to observe that the matrices $(A_{i,k}^j)_{k,j}$ defined in Remark 4.22 are complex linear because $\nabla I = 0$. We now observe that $I[d\bar{f}_{|\theta|}(r^2)]_{1,0} = 2\bar{f}_{|\theta|}'(r^2)(y_1\eta_2 - y_2\eta_1 + y_3\eta_4 - y_4\eta_3)$. Let us define $\eta = y_1\eta_2 - y_2\eta_1 + y_3\eta_4 - y_4\eta_3 \in V^*$ and observe that $\|\eta\|_{g_{2,0}^2} = O(r)$. Then,

$$[dI[d\bar{f}_{|\theta|}(r^2)]_{1,0}]_{2,0} = 4\bar{f}_{|\theta|}''(r^2)\left(\sum_{i=1}^{4} y_i\eta_i \wedge \eta\right) + 4\bar{f}_{|\theta|}'(r^2)(\eta_1 \wedge \eta_2 + \eta_3 \wedge \eta_4).$$

Thus, taking into account Lemma 4.25 and the estimates for $|\bar{f}_{|\theta|}'|$ and $|\bar{f}_{|\theta|}''|$, we obtain that $\|[dI[d\bar{f}_{|\theta|}(r^2)]_{1,0}]_{2,0}\|_{g_{2,0}^2} = O(r^{-2})$. □

Remark 4.30. From Lemma 4.29, one deduces that if $\alpha \in V^*$ then $\|\alpha\|_{\chi_*(h_2)} = (1 + O(r^{-1}))\|\alpha\|_{g_1}$ on $r > 0$. Therefore, if $\alpha \in W_{i,j}$ then $\|\alpha\|_{\chi_*(h_2)} = (1 + O(r^{-1}))^i\|\alpha\|_{g_1}$ on $r > 0$.

Proposition 4.31. *There exists $s_0 > 1$, such that for each $s > s_0$ one can find t_s''' such that for each $t < t_s'''$ there is a closed G_2 structure $\Phi^t_{3,s}$ such that $\Phi^t_{3,s} = \Phi^t_2$ on $r \leq \frac{s}{8}$ and $\Phi^t_{3,s} = \chi^*(\widehat{\phi}^t_{3,s})$ on $r \geq \frac{s}{4}$.*

Proof. On the anulus $\frac{s}{8} < r < \frac{s}{4}$ we have that:

$$\Phi^t_2 - \chi^*(\widehat{\phi}^t_{3,s}) = \frac{1}{4}t^2 d(\mathrm{pr}^* e_1 \wedge (I[d(\mathsf{f}_{|\theta|}(r) - r^2)]_{1,0})).$$

We now let ϖ be a smooth function such that $\varpi = 1$ if $x \leq \frac{1}{8}$ and $\varpi = 0$ if $x \geq \frac{1}{4}$ and $\varpi_s(x) = \varpi(\frac{x}{s})$; then $|\varpi'_\varepsilon| \leq \frac{C}{s}$. We define $\bar{\mathsf{f}}_a(x) = \mathsf{f}_a(\sqrt{x}) - x = a^2((x^2 + a^2)^{1/2} + x)^{-1} - a\log((x^2 + a^2)^{1/2} + a) + a\log(x)$ for $a > 0$ and

$$\xi_s = \varpi_s \mathrm{pr}^* e_1 \wedge (Id[\bar{\mathsf{f}}_\theta(r^2)]_{1,0}).$$

The form $d\xi_s$ lies in $W_{2,1} \oplus W_{1,2} \oplus W_{0,3}$. We claim that on $r > 1$:

$$\begin{aligned}
\|[d\xi_s]_{2,1}\|_{h_1} =& \|\varpi_s \mathrm{pr}^* e_1 \wedge [dI[d\bar{\mathsf{f}}_{|\theta|}(r)]_{1,0}]_{2,0} + [d\varpi_s]_{1,0} \wedge \mathrm{pr}^* e_1 \wedge I[d\bar{\mathsf{f}}_{|\theta|}(r)]_{1,0}\|_{h_1} \\
=& O(r^{-2}) + O(r^{-1}s^{-1}), \\
\|[d\xi_s]_{1,2}\|_{h_1} =& \|\varpi_s \mathrm{pr}^* e_1 \wedge [dI[d\bar{\mathsf{f}}_{|\theta|}(r)]_{1,0}]_{1,1} + \varpi_s \mathrm{pr}^*(de_1) \wedge I[d\bar{\mathsf{f}}_{|\theta|}(r)]_{1,0} \\
&+ [d\varpi_s]_{0,1} \wedge \mathrm{pr}^* e_1 \wedge I[d\bar{\mathsf{f}}_{|\theta|}(r)]_{1,0}\|_{h_1} = O(r^{-1}) + O(s^{-1}), \\
\|[d\xi_s]_{0,3}\|_{h_1} =& \|\varpi_s \mathrm{pr}^* e_1 \wedge [dI[d\bar{\mathsf{f}}_{|\theta|}(r)]_{1,0}]_{0,2}\|_{h_1} = O(1).
\end{aligned}$$

We now prove some of the estimates. For that purpose, we take into account that $\|[\alpha]_{i,j}\|_{h_1} = \|(\chi^{-1})^*[\alpha]_{i,j}\|_{\chi_*(h_1)}$ on $r > 1$. To ease notations we identify $[\alpha]_{i,j} \in W'_{i,j}$ with $(\chi^{-1})^*[\alpha]_{i,j} \in W_{i,j}$. Following this notation, the formulas that we check are:

$$\begin{aligned}
\|[d\varpi_s]_{1,0} \wedge \mathrm{pr}^* e_1 \wedge I[d\bar{\mathsf{f}}_{|\theta|}(r)]_{1,0}\|_{\chi_*(h_1)} =& O(r^{-1}s^{-1}), \\
\|[d\varpi_\sigma]_{0,1} \wedge \mathrm{pr}^* e_1 \wedge I[d\bar{\mathsf{f}}_{|\theta|}(r)]_{1,0}\|_{\chi_*(h_1)} =& O(s^{-1}).
\end{aligned}$$

These terms appear in the second and third estimates; the remaining are proved similarly. According to Remark 4.30, it is sufficient to prove the estimates on the g_1-norm. For instance, if we check $\|[d\varpi_s]_{1,0} \wedge \mathrm{pr}^* e_1 \wedge I[d\bar{\mathsf{f}}_{|\theta|}(r)]_{1,0}\|_{g_1} = O(r^{-1}s^{-1})$ on $r > 1$, then $\|[d\varpi_s]_{1,0} \wedge \mathrm{pr}^* e_1 \wedge I[d\bar{\mathsf{f}}_{|\theta|}(r)]_{1,0}\|_{\chi_*(h_1)} = (1 + O(r^{-1}))^2 O(r^{-1}s^{-1}) = O(r^{-1}s^{-1})$ on $r > 1$.

For that purpose, we consider a complex $g^2_{2,0}$-unitary local trivialization of ν:

$$(x, y) = (x_1, x_2, x_3, y_1, y_2, y_3, y_4) \in B \times \mathbb{R}^4,$$

as in the proof of Lemma 4.29. According to Lemma 4.25, we have: $\|I[d\bar{f}_{|\theta|}(r^2)]_{1,0}\|_{g_1} = 2|\bar{f}'_{|\theta|}(r^2)|\|y_1\eta_2 - y_2\eta_1 + y_3\eta_4 - y_4\eta_3\|_{g_1} = O(r^{-1})$. In addition, we compute:

$$\begin{aligned}
[d\varpi_s]_{1,0} =& \sum_{i=1}^4 \varpi'_s(r)\frac{y_i}{r}\eta_i, \\
[d\varpi_s]_{1,0} =& -\sum_{i=1}^4\sum_{j=1}^3 \varpi'_s(r)\frac{y_i}{r}A^i_j(x,y)dx_j.
\end{aligned}$$

Taking into account that $\|A^i_j(x,y)\|_{g_1} = O(r)$ we obtain that $\|[d\varpi_\sigma]_{1,0}\|_{g_1} = O(s^{-1})$, and that $\|[d\varpi_\sigma]_{0,1}\|_{g_1} = O(rs^{-1})$. A multiplication yields the desired estimates.

Our previous discussion leads to:

$$\|t^2 d\xi_s\|_{h_t} = O(r^{-2}) + O(r^{-1}s^{-1}) + t(O(r^{-1}) + O(s^{-1})) + t^2 O(1)$$

Take s_0 such that for each $0 < t < 1$ and $s > s_0$ it holds that $|O(r^{-2}) + O(r^{-1}s^{-1}) + t(O(r^{-1}) + O(s^{-1}))| < \frac{m}{4}$ on $\frac{s}{8} \le r \le \frac{s}{4}$. Let $s > s_0$ and take $t''_s < t_s$ such that $|t^2 O(1)| < \frac{m}{2}$ and $\|\Phi^t_2 - \Phi^t_1\|_{h_t} < \frac{m}{2}$ on $\chi^{-1}(\nu_{2s})$; this is possible as we argued before. Define the closed form

$$\Phi^t_{3,s} = \Phi^t_2 - \frac{t^2}{4} d\xi_s,$$

which coincides with Φ^t_2 if $r \le \frac{s}{8}$ and with $\chi^*(\widehat{\phi}^t_{3,s})$ if $r \ge \frac{s}{4}$. On the neck $\frac{s}{8} \le r \le \frac{s}{4}$ we have that:

$$\|\Phi^t_{3,s} - \Phi^t_1\|_{h_t} \le \|\Phi^t_{3,s} - \Phi^t_2\|_{h_t} + \|\Phi^t_2 - \Phi^t_1\|_{h_t} < m.$$

The statement is therefore proved. □

The map $F_t \circ \chi$ allows us to glue an annulus around the zero section on $(\nu/\mathrm{j}, \phi_2)$ and an annulus around Q on $(P, \widehat{\Phi}^t_2)$; this yields a resolution.

Theorem 4.32. *There exists a closed* G_2 *resolution* $\rho\colon \widetilde{X} \to X$*. In addition, let us denote* $D_s(Q)$ *the* s*-disk of* P *centered at* Q*; then*

$$\widetilde{X} = X - \exp(\nu_\varepsilon/\mathrm{j}) \cup_{\exp \circ F_t \circ \chi} D_s(Q)$$

for some $\varepsilon > 0$*,* $t > 0$ *and* $s > 0$*.*

Proof. Let $\varepsilon_0 < R$ and $s_0 > 0$ be the values provided by Proposition 4.26 and 4.31. Fix $s > s_0$ and choose $t < t'''_s$ with $st = \frac{\varepsilon}{4}$ for some $\varepsilon < \varepsilon_0$. The map $F_t \circ \chi$ identifies $s \le r \le 2s$ on P with $\frac{\varepsilon}{4} \le r \le \frac{\varepsilon}{2}$ on ν/j.

Consider the G_2 forms $\Phi^t_{3,s}$ on $\chi^{-1}(\nu_{2s}/\mathrm{j})$ and $\phi_{3,\varepsilon}$ on on $\nu_{2\varepsilon}/\mathrm{j}$; on the annulus $s \le r \le 2s$ of $\chi^{-1}(\nu_{2s}/\mathrm{j})$ we have that $\Phi^t_{3,s} = \chi^*(\widehat{\phi}^t_{3,s}) = \phi^t_2$ and on $\frac{\varepsilon}{4} \le r \le \frac{\varepsilon}{2}$ on ν/j we have that $\phi_{3,\varepsilon} = \phi_2$.

Being $(F_t \circ \chi)^* \phi_2 = \chi^*(\phi^t_2)$, the G_2 structure is well defined on the resolution. □

Remark 4.33. The radius of the disc $r \le 2s$ with respect to the metric h_t is $2st$. For fixed $s_0 > 0$ the map $F_t \circ \chi$ identifies $0 < r \le 2s_0$ on P with $0 < r \le 2s_0 t$ on ν; therefore if we choose $t \to 0$ then the size of the exceptional divisor decreases.

4.4 Topology of the resolution

This section is devoted to understanding the cohomology algebra of the resolution; we shall make use of real coefficients and denote by $H^*(M)$ the algebra $H^*(M, \mathbb{R})$. We start by describing $H^*(\widetilde{X})$ in terms of $H^*(X)$ and $H^*(L)$ and we then compute the induced product on it.

The fibre bundle ν is topologically trivial; this follows from the fact that every 3 manifold is parallelizable. For a proof see [75, Remark 2.14]. However, it might not be trivial as a complex bundle as we shall deduce from the computation of its total Chern class.

Let us suppose for a moment that L is connected; then L is the mapping torus of diffeomorphism $\psi\colon \Sigma \to \Sigma$, where Σ is an orientable surface of genus g. In section 4.3 we denoted by $\mathrm{q}\colon \Sigma \times [0,1] \to L$ the quotient projection, and by $\mathrm{b}\colon L \to S^1$ the bundle projection. We also chose that $\theta = \mathrm{b}^*(\theta_0)$ with θ_0 the angular form on S^1.

In Proposition 4.34 we compute the total Chern class of ν by observing first that ν admits a section and thus $\nu = \underline{\mathbb{C}} \oplus \ker\theta$; where $\underline{\mathbb{C}}$ denotes the trivial line bundle over L. Then we identify $\ker(\theta)$ with the tangent space of the fibres taking into account that $\theta = \mathrm{b}^*(\theta_0)$. A formula for $c(\nu)$ follows from these remarks.

In order to state the result it shall be useful to note that 2-forms on Σ determine closed 2-forms on L. More precisely, let us consider $\varpi\colon [0,1] \to \mathbb{R}$ a bump function with $\varpi|_{[0,1/4]} = 0$ and $\varpi|_{[3/4,1]} = 1$. Let $\beta \in \Omega^2(\Sigma)$ and let $\alpha \in \Omega^1(\Sigma)$ such that $\psi^*\beta = \beta + d\alpha$; note that this is possible because $\psi^* = \mathrm{Id}$ on $H^2(\Sigma)$. Then $\bar{\beta} = \beta + d(\varpi(t)\alpha) \in \Omega^2(\Sigma \times [0,1])$ induces a 2-form on L via the push-forward. Of course, one can show that the cohomology class of $\overline{\beta}$ does not depend on α. In addition, from the Mayer-Vietoris long exact sequence we deduce that $[\mathrm{q}_*(\bar{\beta})] \neq 0$ if $[\beta] \neq 0$.

We denote by $\omega_\Sigma \in \Omega^2(L)$ a closed 2-form induced by a volume form vol_Σ of Σ that integrates to 1 on Σ. This class represents the Poincaré dual of a circle $C \subset L$ such that $\mathrm{q}(\{p_0\} \times [0,1]) \subset C$ and $C - \mathrm{q}(\{p_0\} \times \{0\})$ is an embedded line on $\mathrm{q}(\Sigma \times \{0\})$ if it is not a point.

Proposition 4.34. *The total Chern class of ν is $c(\nu) = 1 + (2-2g)[\omega_\Sigma]$.*

Proof. Let $\times$ be the cross product on $TM|_L$ determined by φ. Consider on $E = \ker(\theta) \subset TL$ the complex structure $\mathrm{J}W = W \times e_1^\sharp$, where $e_1 = \|\theta\|^{-1}\theta$. This is well-defined because $\times$ defines a cross product on T_pL and if $\theta(X) = 0$, then $X \times e_1^\sharp \perp e_1^\sharp$. Recall also that the complex structure on ν is: $I(v) = e_1^\sharp \times v$.

We prove that there is an isomorphism of complex vector bundles:

$$\underline{\mathbb{C}} \oplus E \to \nu.$$

A nowhere-vanishing section $s\colon L \to \nu$ exists because $\dim L = 3 < 4 = \mathrm{rk}(\nu)$; we define the isomorphism $\underline{\mathbb{C}} \oplus E \to \nu$,

$$(z_1 + iz_2, W) \longmapsto z_1 s + z_2 e_1^\sharp \times s + W \times s.$$

In order to check that the isomorphism is complex linear one uses the equality [104, Lemma 2.9]:

$$u \times (v \times w) + v \times (u \times w) = g(u,w)v + g(v,w)u - 2g(u,v)w.$$

where g denotes the restriction to ν of the metric on M. In our case taking $u = e_1^\sharp$, $v = s$ and $w = W$ we obtain that $e_1^\sharp \times (W \times s) = (W \times e_1^\sharp) \times s$.

From the isomorphism we get that $c(\nu) = c(\underline{\mathbb{C}})c(E) = 1 + c_1(E)$. We now compute $c_1(E)$; note that E is the vertical distribution $d\mathrm{q}(T\Sigma \times [0,1]) \subset TM$. First consider a compactly supported 2-form $\upsilon \in \Omega^2(T\Sigma)$ representing the Thom class of the bundle $T\Sigma \to S$ that integrates to 1 over the fibres. Being the diffeomorphism $d\psi\colon T\Sigma \to T\Sigma$ volume-preserving we obtain that $(d\psi)^*\upsilon$ is also a compactly-supported 2-form that integrates to 1 over the fibres. Thus, $(d\psi)^*\upsilon = \upsilon + d\alpha$ for some compactly-supported $\alpha \in \Omega^1(T\Sigma)$. In addition let $s_0\colon \Sigma \to T\Sigma$ be the zero section; then $[s_0^*(\upsilon)] = (2-2g)[\mathrm{vol}_\Sigma]$.

The push-forward $\mathrm{q}_*(\upsilon + d(\varpi\alpha)) \in \Omega^2(E)$ of course induces the Thom class of E. Being $s[p,t] = d\mathrm{q}_{(p,t)}(s_0(p,t))$ the zero section of E we obtain:

$$c_1(E) = s^*[\mathrm{q}_*(\upsilon + d(\varpi\alpha))] = [q_*(s_0^*\upsilon + d(\varpi s_0^*\alpha))] = (2-2g)[\omega_\Sigma].$$

To obtain the last equality we have taken into account that $s_0^*(d\varpi) = 0$, $s_0^*(\psi^*\upsilon) = s_0^*\upsilon + d(s_0^*\alpha)$ and $[s_0^*(\upsilon)] = (2-2g)[\mathrm{vol}_\Sigma]$.

□

The projectivized bundle of ν coincides with Q because $\mathbb{P}(\nu) = P_{\mathrm{U}(2)}(\nu) \times_{\mathrm{U}(2)} \mathbb{CP}^1 = Q$. An obstruction-theoretic argument ensures that it is trivial:

Lemma 4.35. *The bundle $Q \to L$ is trivial.*

Proof. First recall that the spaces $\mathrm{Diff}(S^2)$ and $\mathrm{SO}(3)$ have the same homotopy type. Classifying S^2 bundles is therefore equivalent to classifying rank 3 vector bundles. In our case, denoting by $E = \ker(\theta)$ as in the proof of Proposition 4.34, if $g_{\alpha\beta} \in SO(2)$ are the transition functions of E, taking into account the diffeomorphism $\mathbb{CP}^1 \to S^2$ one can compute that the transition functions of Q are

$$h_{\alpha\beta}(x)(v_1, v_2, v_3) = (g_{\alpha\beta}(v_1, v_2), v_3)$$

Therefore, the associated rank 3 vector bundle V has transition functions $g_{\alpha\beta} \times \mathrm{Id} \in \mathrm{SO}(3)$. This is trivial if and only if Q is. We now observe that V is trivial if and only if its second Stiefel-Withney class vanishes. For that purpose consider a CW-decomposition,

$$L = \cup_{k=0}^{3} L^k.$$

Then $V|_{L^1}$ is trivial because $\mathrm{SO}(3)$ is connected. The trivialization extends to L^2 if the primary obstruction cocycle is exact; this coincides with the second Stiefel-Whitney class (see [63, Proposition 3.21]). If it vanishes, then the last obstruction cocycle lies in the cohomology group $H^3(L, \pi_2(\mathrm{SO}(3))) = 0$ and therefore the trivialization extends to L.

We now compute the second Stiefel-Whitney class of V. Regarding the transition functions $V = E \oplus \mathbb{R}$ and thus $w_2(V) = w_2(E)$. Being E a complex vector bundle, we obtain $w_2(E) = c_1(E) \pmod 2 = (2-2g)\omega_\Sigma \pmod 2 = 0$. □

Using Proposition 4.34 we re-state a well known fact. For that purpose consider the tautological bundle associated to ν:

$$\overline{P} = P_{\mathrm{U}(2)}(\nu) \times_{\mathrm{U}(2)} \widetilde{\mathbb{C}}^2.$$

Denote frames in $P_{\mathrm{U}(2)}(\nu)$ by F. There is a well-defined $\mathbb{Z}_2$ action on $\overline{P}$, determined by $[F, (z_1, z_2, \ell)] \longmapsto [F, (-z_1, -z_2, \ell)]$. The quotient $\overline{P}/\mathbb{Z}_2$ coincides with P. We denote by $\varrho\colon \overline{P} \to P$ the projection.

Proposition 4.36. *Let $\mathrm{e}(\overline{P})$ be the Euler class of the line bundle $\overline{P} \to Q$. Denote by $H^*(L)[\mathbf{x}]$ the algebra of polynomials with coeffiecients in $H^*(L)$ The map:*

$$F\colon H^*(L)[\mathbf{x}]/\langle \mathbf{x}^2 + (2-2g)[\omega_\Sigma]\mathbf{x}\rangle \to H^*(Q), \quad F(\beta) = \mathrm{pr}^*\beta, F(\mathbf{x}) = \mathrm{e}(\overline{P}),$$

is an isomorphism of algebras.

Proof. The conclussion follows from Proposition 4.34 and formula (20.7) of p. 170 in [19]. □

Recall that we denoted the projection by $\mathrm{pr}\colon P \to L$. Consider $\tau \in \Omega^2(\overline{P})$ the Thom 2-form of the line bundle $\overline{P} \to Q$ and note that we can suppose that τ is $\mathbb{Z}_2$-invariant because the involution preserves the orientation on the fibres. From Proposition 4.36 we obtain:

$$[\tau \wedge \tau] = -(2-2g)[(\varrho \circ \mathrm{pr})^*\omega_\Sigma \wedge \tau].$$

We also denote by τ the pushforward $\varrho_*\tau \in \Omega(P)$; on $H^*(P)$ it also satisfies that:

$$[\tau \wedge \tau] = -(2-2g)[\mathrm{pr}^*\omega_\Sigma \wedge \tau].$$

Of course, we can extend τ to a 2-form on $\widetilde{X}$ and it corresponds to the Poincaré dual of Q.

We now compute the cohomology of $\widetilde{X}$; for this we do not assume that L is connected and we denote by $L_1, \dots, L_r$ its connected compontents. Each L_i is the mapping torus of a diffeomorphism $\psi_i \colon \Sigma_i \to \Sigma_i$, where Σ_i is an orientable surface of genus g_i; we denote by ω_i the 2-form ω_{Σ_i} as constructed before. We also denote $Q_i = Q|_{L_i}$, $P_i = P|_{L_i}$ and τ_i the Thom form of $Q_i \subset P_i$.

Proposition 4.37. *There is a split exact sequence:*

$$0 \longrightarrow H^*(X) \xrightarrow{\pi^*} H^*(\widetilde{X}) \longrightarrow \oplus_{i=1}^r H^*(L_i) \otimes \langle \mathbf{x}_i \rangle \longrightarrow 0$$

where $\mathbf{x}_i$ has degree two.

Proof. The existence of such exact sequence is contained in the proof of [75, Proposition 6.1]; we outline it. Consider the long exact sequence of pairs (X, L) and $(\widetilde{X}, Q)$. There is a commutative diagram:

$$\begin{array}{ccccccc}
H^k(X,L) & \longrightarrow & H^k(X) & \xrightarrow{e_L^*} & \oplus_i H^k(L_i,\mathbb{R}) & \xrightarrow{D_1} & H^{k+1}(X,L) \\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\
H^k(\widetilde{X},Q) & \longrightarrow & H^k(\widetilde{X}) & \xrightarrow{e_Q^*} & \oplus_i H^k(Q_i) & \xrightarrow{D_2} & H^{k+1}(\widetilde{X},Q)
\end{array}$$

Here we denoted the inclusions $e_L \colon L \to X$ and $e_Q \colon Q \to \widetilde{X}$. The first and fourth columns are isomorphisms; these correspond to the identity map. The third column is injective with cokernel $\oplus_i H^*(Q_i)/H^*(L_i)$; this is isomorphic to $\oplus_i H^{k-2}(L_i) \otimes \langle \mathbf{x}_i \rangle$, because $Q_i = L_i \times S^2$. Thus we get a commutative diagram with exact columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
H^k(X,L) & \longrightarrow & H^k(X) & \xrightarrow{e_L^*} & \oplus_i H^k(L_i) & \xrightarrow{D_1} & H^{k+1}(X,L) \\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\
H^k(\widetilde{X},Q) & \longrightarrow & H^k(\widetilde{X}) & \xrightarrow{e_Q^*} & \oplus_i H^k(Q_i) & \xrightarrow{D_2} & H^{k+1}(\widetilde{X},Q) \\
 & & \downarrow & & \downarrow & & \\
 & & \mathrm{Coker}(\pi^*) & \xrightarrow{\bar{e}_Q} & \oplus_i H^{k-2}(L_i) \otimes \langle \mathbf{x}_i \rangle & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Of course, $\bar{e}_Q$ is the action induced by e_Q^* on the quotient. In addition, the fact that first and fourth columns are the identity implies that $\mathrm{Im}(e_L^*) = \mathrm{Im}(e_Q^*)$.

The Snake Lemma ensures that there is an exact sequence:

$$0 \to \ker(e_L^*) \to \ker(e_Q^*) \to \ker(\bar{e}_Q) \to \mathrm{Coker}(e_L^*) \to \mathrm{Coker}(e_Q^*) \to \mathrm{Coker}(\bar{e}_Q) \to 0.$$

The maps are induced by π^*, except for the connecting map $\ker(\bar{e}_Q) \to \mathrm{Coker}(e_L)$. The map $\pi^* \colon \ker(e_L^*) \to \ker(e_Q^*)$ is an isomorphism because the first column is an isomorphism and the diagram is commutative. In addition, taking into account that the fourth column is an isomorphism and that the diagram is commutative one can also check that π^* is an isomorphism between $\mathrm{Im}(D_1)$ and $\mathrm{Im}(D_2)$. Moreover:

$$\mathrm{Im}(D_1) = \oplus_i H^*(L_i)/\ker(D_1) = \oplus_i H^*(L_i)/\mathrm{Im}(e_L^*) = \mathrm{Coker}(e_L^*),$$

and the isomorphism is induced by the map that π^* induces on the quotient. Similarly, $\mathrm{Coker}(e_Q^*)$ is isomorphic to $\mathrm{Im}(D_2)$ via π^*. This means that $\ker(\bar{e}_Q) = 0 = \mathrm{Coker}(\bar{e}_Q)$ so,

$$\mathrm{Coker}(\pi^*) = \oplus_i H^{*-2}(L_i) \otimes \langle \mathbf{x}_i \rangle.$$

Consider τ_i the Poincaré dual of $Q_i \subset \widetilde{X}$ as constructed before. Then,

$$\beta \otimes \mathbf{x}_i \longmapsto \mathrm{pr}^*(\beta)\tau_i$$

is a splitting of the previous exact sequence. □

This result implies that there is an isomorphism of vector spaces between $H^*(\widetilde{X})$ and $H^*(X) \oplus \oplus_{i=1}^r H^*(L_i) \otimes \langle \mathbf{x}_i \rangle$. The algebra structure of $H^*(\widetilde{X})$ induces an algebra structure on $H^*(X) \oplus \oplus_{i=1}^r H^*(L_i) \otimes \langle \mathbf{x}_i \rangle$ that we compute in Proposition 4.38. This is necessary in order to decide whether the resolution $\widetilde{X}$ is formal or not, because the formality condition involves products of cohomology classes.

Proposition 4.38. *There is an isomorphism*

$$H^*(\widetilde{X}) = H^*(X) \bigoplus \oplus_{i=1}^r H^*(L_i) \otimes \langle \mathbf{x}_i \rangle.$$

Let $\alpha, \beta \in H^*(X)$, $\gamma_i \in H^*(L_i)$, $\gamma_j' \in H^*(L_j)$ and let $e_i \colon L_i \to X$ be the inclusion. The wedge product on $H^*(\widetilde{X})$ determines the following product on the left hand side:

1. $\alpha\beta = \alpha \wedge \beta$,
2. $\alpha(\gamma_i \otimes \mathbf{x}_i) = (e_i^*(\alpha) \wedge \gamma_i) \otimes \mathbf{x}_i$,
3. $(\gamma_i \otimes \mathbf{x}_i)(\gamma_j' \otimes \mathbf{x}_i) = 0$ *if* $i \neq j$,
4. $(\gamma_i \otimes \mathbf{x}_i)(\gamma_i' \otimes \mathbf{x}_i) = -2(\gamma_i \wedge \gamma_i')PD[L_i] - (2 - 2g_i)(\omega_i \otimes \mathbf{x}_i)$.

Proof. Let $s \colon \oplus_{i=1}^r H^*(L_i) \otimes \langle \mathbf{x}_i \rangle \to H^*(\widetilde{X})$ be the splitting map constructed in the proof of Proposition 4.37. Then, the isomorphism is determined by:

$$\mathrm{T} = (\rho^*, s) \colon H^*(X) \bigoplus \oplus_{i=1} H^*(L_i) \otimes \langle \mathbf{x}_i \rangle \to H^*(\widetilde{X}).$$

In order to obtain a formula for the product between forms η, η' we have to compute $(\mathrm{T})^{-1}(\mathrm{T}\eta \wedge \mathrm{T}\eta')$. All the statements are evident except for the last one. We only check $\mathbf{x}_i^2 = -2PD[L_i] - (2-2g_i)(\omega_i \otimes \mathbf{x}_i)$, the announced formula is deduced from this and the fact that $H^*(\widetilde{X})$ is an algebra. First of all, $\mathrm{T}\mathbf{x}_i \wedge \mathrm{T}\mathbf{x}_i = [\tau_i \wedge \tau_i]$; we now compute $\mathrm{T}^{-1}[\tau_i \wedge \tau_i]$. On the one hand taking into account the equality

$$[\tau_i \wedge \tau_i] = -(2 - 2g_i)[\mathrm{pr}^*(\omega_i) \wedge \tau_i],$$

we obtain that the restriction of $\mathrm{T}^{-1}[\tau_i \wedge \tau_i]$ to $H^*(L_i) \otimes \langle \mathbf{x}_i \rangle$ is $-(2 - 2g_i)(\omega_i \otimes \mathbf{x}_i)$. On the other hand, note first that if $x \in L_i$ then $\tau_i|_{P_x}$ is the Thom form of $Q_x \subset P_x$ because τ_i is the Thom form of $Q_i \subset P_i$. Thus:

$$\int_{P_x} \tau_i \wedge \tau_i = [Q_x][Q_x] = -2.$$

The restriction of $\mathrm{T}^{-1}[\tau_i \wedge \tau_i]$ to $H^*(X)$ has compact support around L_i and

$$\int_{\nu_x} \rho^*(\tau_i \wedge \tau_i) = \int_{\nu_x - 0} \rho^*(\tau_i \wedge \tau_i) = \int_{P_x - Q_x} \tau_i \wedge \tau_i = \int_{P_x} \tau_i \wedge \tau_i = -2.$$

The restriction in thus equal to $-2PD[L_i]$. □

4.5 Non-formal compact G_2 manifold with $b_1 = 1$

Nilpotent Lie algebras that have a closed left-invariant G_2 structure are classified in [34]; from these one obtain nilmanifolds with an invariant closed G_2 structure. Of course, excluding the 7-dimensional torus, these are non-formal and have $b_2 \geq 2$. From a $\mathbb{Z}_2$ action on a nilmanifold, in [47] authors construct a formal orbifold whose isotropy locus are 16 disjoint 3-tori; then they prove that its resolution is also formal. In this section we follow the same process to construct first a non-formal G_2 orbifold with $b_1 = 1$ from a nilmanifold; its isotropy locus consists of 16 disjoint non-formal nilmanifolds. Later we prove that its resolution is also non-formal and does not admit any torsion-free G_2 structure.

4.5.1 Orbifold with $b_1 = 1$

Let us consider the Lie algebra $\mathfrak{g}$ with structure equations

$$(0, 0, 0, 12, 23, -13, -2(16) + 2(25) + 2(26) - 2(34)),$$

and let $(e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ be the generators of $\mathfrak{g}$ that satisfy the structure equations, that is, $[e_1, e_2] = -e_4$, $[e_2, e_3] = -e_5$ and so on. Recall that the simply connected Lie group G associated to $\mathfrak{g}$ is the vector space $\mathfrak{g}$ endowed with the product $*$ determined by the Baker-Campbell-Hausdorff formula.

Remark 4.39. The Lie algebra $\mathfrak{g}$ belongs to the 1-parameter family of algebras $147E1$ listed in Gong's classification [58]; we choose the parameter $\lambda = 2$. The associated Lie group admits an invariant closed G_2 structure as proved in [34].

Define $u_1 = e_1$, $u_2 = e_2$, $u_3 = e_3$, $u_4 = \frac{1}{2}e_4$, $u_5 = \frac{1}{2}e_5$, $u_6 = \frac{1}{2}e_6$ and $u_7 = \frac{1}{6}e_7$.

Proposition 4.40. *If* $x = \sum_{k=1}^{7} \lambda_k u_k$ *and* $y = \sum_{k=1}^{7} \mu_k u_k$ *then*

$$\begin{aligned} x * y =& (\lambda_1 + \mu_1)u_1 + (\lambda_2 + \mu_2)u_2 + (\lambda_3 + \mu_3)u_3 + (\lambda_4 + \mu_4 - (\lambda_1\mu_2 - \lambda_2\mu_1))u_4 \\ &+ (\lambda_5 + \mu_5 - (\lambda_2\mu_3 - \lambda_3\mu_2))u_5 + (\lambda_6 + \mu_6 + (\lambda_1\mu_3 - \lambda_3\mu_1))u_6 \\ &+ (\lambda_7 + \mu_7 + (\lambda_1 - \mu_1 - \lambda_2 + \mu_2)(\lambda_1\mu_3 - \lambda_3\mu_1) - (\lambda_3 - \mu_3)(\lambda_1\mu_2 - \mu_2\lambda_1))u_7 \\ &+ (-(\lambda_2 - \mu_2)(\lambda_2\mu_3 - \lambda_3\mu_2) + 3(\lambda_1\mu_6 + \lambda_6\mu_1))u_7 \\ &+ (-3(\lambda_2\mu_5 - \lambda_5\mu_2) - 3(\lambda_2\mu_6 - \lambda_6\mu_2) + 3(\lambda_3\mu_4 + \lambda_4\mu_3))u_7. \end{aligned}$$

Proof. Being $\mathfrak{g}$ is 3-step, the Baker-Campbell-Hausdorff formula yields:

$$x * y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}\left([x, [x, y]] - [y, [x, y]]\right).$$

Taking into account that $u_7 \in Z(\mathfrak{g})$ and that $[u_i, [u_j, u_k]] = 0$ if $i \geq 4$ or $j \geq 4$ or $k \geq 4$, it follows:

$$\begin{aligned} x * y =& \sum_{k=1}^{7} (\lambda_k + \mu_i)u_i + \frac{1}{2} \sum_{1 \leq i < j \leq 7} (\lambda_i\mu_j - \lambda_j\mu_i)[u_i, u_j] \\ &+ \frac{1}{12} \sum_{1 \leq k \leq 3} (\lambda_k - \mu_k) \sum_{1 \leq i < j \leq 3} (\lambda_i\mu_j - \lambda_j\mu_i)[u_k, [u_i, u_j]]; \end{aligned}$$

The non-zero combinations $[u_i, u_j]$ and $[u_k, [u_i, u_j]]$ are:

$$\begin{aligned} &[u_1, u_2] = -2u_4, && [u_2, u_5] = -6u_7, && [u_3, [u_1, u_2]] = -12u_7 \\ &[u_1, u_3] = 2u_6, && [u_2, u_6] = -6u_7, && [u_1, [u_1, u_3]] = 12u_7 \\ &[u_1, u_6] = 6u_7, && [u_3, u_4] = 6u_7, && [u_2, [u_1, u_3]] = -12u_7, \\ &[u_2, u_3] = -2u_5, && && [u_2, [u_2, u_3]] = 12u_7. \end{aligned}$$

The announced formula easily follows from this. □

Proposition 4.40 ensures that

$$\Gamma = \Big\{ \sum_{i=1}^{7} n_i u_i, \text{ s.t. } n_i \in \mathbb{Z} \Big\},$$

is a discrete subgroup of G, which is of course co-compact. Indeed, a straightforward computation gives a fundamental domain for the left action of Γ on G:

Proposition 4.41. *A fundamental domain for the left action of* Γ *on* G *is*

$$\mathcal{D} = \Big\{ \sum_{i=1}^{7} t_i u_i, \text{ s.t. } 0 \leq t_i \leq 1 \Big\}.$$

According to [34, Lemma 5], the group G admits an invariant closed G_2 structure determined by:

$$\varphi = v^{127} + v^{347} + v^{567} + v^{135} - v^{236} - v^{146} - v^{245}.$$

where:

- $v^1 = \sqrt{3}(2e^1 + e^5 - e^2 + e^6)$;
- $v^2 = 3e^2 - e^5 + e^6$;
- $v^3 = e^3 + 2e^4$;
- $v^4 = \sqrt{3}(e^3 + e^7)$;
- $v^5 = \sqrt{2}(e^6 - e^5)$;
- $v^6 = \sqrt{6}(e^5 + e^6)$,
- $v_7 = 2\sqrt{2}(e^4 - e^3)$.

Consider $M = G/\Gamma$; points of M will be denoted by $[x]$, for some $x \in G$. The nilmanifold M inherits a closed G_2 structure that we also denote by φ. We now define an involution j on M such that $\mathrm{j}^*\varphi = \varphi$. For that purpose it is sufficient to define an order 2 isomorphism $\mathrm{j}\colon G \to G$ of G with $\mathrm{j}^*\varphi = \varphi$, and $\mathrm{j}\Gamma = \Gamma$. The desired map is:

$$\mathrm{j}(e_k) = e_k, \quad k \in 3,4,7, \qquad \mathrm{j}(e_k) = -e_k, \quad k \in \{1,2,5,6\}.$$

Looking at the structure constants of G it becomes clear that j is an automorphism of $\mathfrak{g}$. The Baker-Campbell-Hausdorff formula ensures that j is an homomorphism. In addition, it is clear that $\mathrm{j}(\Gamma) \subset \Gamma$. Finally, one can easily deduce that $\mathrm{j}^*(\varphi) = \varphi$.

We define the orbifold $X = M/\mathrm{j}$, which has a closed G_2 structure determined by φ. We now study its singular locus:

Proposition 4.42. *The isotropy locus has* 16 *connected components; these are all diffeomorphic and their universal covering is the Heisenberg group. Let us define* $H_0 = \{\lambda_3 u_3 + \lambda_4 u_4 + \lambda_7 u_7, \text{ s.t. } \lambda_j \in \mathbb{R}\}$ *and* $\mathcal{E} = \{\varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_5 u_5 + \varepsilon_6 u_6, \text{ s.t. } \varepsilon_j \in \{0, \frac{1}{2}\}\}$. *The* 16 *connected components of the isotropy locus are:*

$$H_\varepsilon = [L_\varepsilon H_0], \quad \varepsilon \in \mathcal{E},$$

where L_ε *denotes the left translation on* G *by the element* $\varepsilon \in \mathcal{E}$.

Proof. It is clear that H_0 is a connected component of $\mathrm{Fix}(\mathrm{j})$ that contains 0, which is the unit of G. Being j an homomorphism, we conclude that H_0 is a subgroup of G. It is thus sufficient to prove that the Lie algebra $\mathfrak{h}$ of H_0 is the Heisenberg algebra. This is of course true because $\mathfrak{h} = \langle e_3, e_4, e_7 \rangle$ with $[e_3, e_4] = e_7$ and $[e_j, e_7] = 0$ for $j \in \{3,4\}$.

Let $\mathcal{K} = \{1, 2, 5, 6\}$ and consider $x = \sum_{k\in\mathcal{K}} \lambda_k u_k \in \mathcal{D}$, that is, $\lambda_k \in [0, 1]$. We now check that if $\gamma * x = \mathrm{j}(x)$ for some $\gamma \in \Gamma$ then $[x] \in H_\varepsilon$ for some $\varepsilon \in \mathcal{E}$. Let us denote $\gamma = \sum_{k=1}^{7} n_k u_k$; taking into account Proposition 4.40 one obtains:

$$\begin{aligned}\gamma * x =&(n_1 + \lambda_1)u_1 + (n_2 + \lambda_2)u_2 + n_3u_3 + (n_4 - n_1\lambda_2 + n_2\lambda_1)u_4\\ &+ (n_5 + \lambda_5 + n_3\lambda_2)u_5 + (n_6 + \lambda_6 - n_3\lambda_1)u_6 + \lambda' u_7,\end{aligned}$$

for some $\lambda' \in \mathbb{R}$. The equation $\mathrm{j}(x) = \gamma * x$ yields immediately to $2\lambda_j = -n_j$ for $j = \{1, 2\}$ and $n_3 = 0$. Taking this into account, $n_4 - n_1\lambda_2 + n_2\lambda_1 = n_4$, $n_5 + \lambda_5 + n_3\lambda_2 = n_5 + \lambda_5$, $n_6 + a_6 - n_3\lambda_1 = n_6 + \lambda_6$ and thus $n_4 = 0$, $2\lambda_5 = -n_5$ and $2\lambda_6 = -n_6$. Thus, $x = -\frac{1}{2}\sum_{k\in\mathcal{K}} n_k u_k$, so that $x \in H_\varepsilon$ for some $\varepsilon \in \mathcal{E}$.

We now let $[y]$ be an isotropy point; one can write: $y = x_1 * x_2$; with $x_1 = \sum_{k\in\mathcal{K}} \lambda_k u_k$ and $x_2 = \sum_{k\notin\mathcal{K}} \mu_k u_k \in H_0$. The choice becomes clear from the equality:

$$\begin{aligned}x_1 * x_2 =&\lambda_1u_1 + \lambda_2u_2 + \mu_3u_3 + \mu_4u_4 + (\lambda_5 - \lambda_2\mu_3)u_5 + (\lambda_6 + \alpha_1\mu_3)u_6\\ &+ (\mu_7 + (\lambda_1 - \lambda_2)(\lambda_1\mu_3) + \lambda_2\mu_3)u_7,\end{aligned}$$

that is of course deduced from Proposition 4.40.

Using this decomposition we obtain the equality $\gamma * x_1x_2 = \mathrm{j}(y) = \mathrm{j}(x_1)x_2$ that implies $\mathrm{j}(x_1) = \gamma x_1$. Take $x_1' \in \mathcal{E}$ with $x_1 = \gamma' x_1'$, then $[y] = [\gamma' x_1' x_2] = [x_1' x_2] \in [L_{x_1'} H_0]$. □

4.5.2 Non-formality of the resolution

We start by computing the real cohomology algebra of the orbifold. Nomizu's theorem [98] ensures that $(\Lambda^*\mathfrak{g}^*, d)$ is the minimal model of M. Taking into account that $H^*(X) = H^*(M)^{\mathbb{Z}_2}$ we obtain that $((\Lambda^*\mathfrak{g}^*)^{\mathbb{Z}_2}, d)$ is a model for X. The cohomology of X is:

$$\begin{aligned}H^1(X) =&\langle [e^3] \rangle,\\ H^2(X) =&\langle [e^{25}], [e^{15} - e^{26}], [e^{15} - e^{34}] \rangle,\\ H^3(X) =&\langle [e^{235}], [e^{135}], [e^{356}], [e^{124}], [e^{146}], [e^{245}], [e^{127} + 2e^{145}],\\ &[e^{125} + e^{167} - e^{257} - 2e^{456} - e^{347}] \rangle.\end{aligned}$$

We now prove that X is not formal.

Proposition 4.43. *The triple Massey product* $\langle [e^3], [e^{15} - e^{26}], [e^3] \rangle$ *of* $((\Lambda^*\mathfrak{g}^*)^{\mathbb{Z}_2}, d)$ *is not trivial. Therefore, X is not formal.*

Proof. First of all, one can check that that space of exact 3-forms of $((\Lambda\mathfrak{g})^{\mathbb{Z}_2}, d)$ is:

$$B^3((\Lambda^*\mathfrak{g}^*)^{\mathbb{Z}_2}, d) = \langle e^{123}, e^{135} - e^{236}, -e^{136} + e^{235} + e^{236}, e^{127} - 2e^{146} + 2e^{245} + 2e^{246} \rangle.$$

and the space of closed 2-forms is:

$$Z^2((\Lambda^*\mathfrak{g}^*)^{\mathbb{Z}_2}, d) = \langle e^{12}, -e^{16} + e^{25} + e^{26} - e^{34}, e^{25}, e^{15} - e^{26}, e^{15} - e^{34} \rangle.$$

Let us take $\xi_1 = [e^3] = \xi_3$, $\xi_2 = [e^{15} - e^{26}]$; the representatives of these cohomology classes are $\alpha_1 = \alpha_3 = e^3$ and $\alpha_2 = e^{15} - e^{26} + dx$ for some $x \in (\mathfrak{g}^*)^{\mathbb{Z}_2}$; our previous computations ensure that the Massey product $\langle \xi_1, \xi_2, \xi_3 \rangle$ is well defined. More precisely, $\bar{\alpha}_1 \wedge \alpha_2 = d(-e^{56} + e^3x + \beta_1)$ and $\bar{\alpha}_2 \wedge \alpha_3 = d(e^{56} - e^3x + \beta_2)$, where β_1 and β_2 are closed forms. Defining systems for $\langle \xi_1, \xi_2, \xi_3 \rangle$ are $(e^3, e^{15} - e^{26} + dx, e^3, -e^{56} + e^3x + \beta_1, e^{56} - e^3x + \beta_2)$ and the triple Massey product is

$$\langle \xi_1, \xi_2, \xi_3 \rangle = \{[2e^{356} + e^3\beta] \text{ s.t. } d\beta = 0\}.$$

The zero cohomology class is not an element of this set due to our previous computations. Corollary 4.19 ensures that X is not formal. □

Let $\rho\colon \tilde{X} \to X$ be the closed G_2 resolution constructed in Theorem 4.32. Lifting this triple Massey product to $\widetilde{X}$ we prove that $\widetilde{X}$ is not formal.

Proposition 4.44. *The resolution $\widetilde{X}$ is not formal.*

Proof. Let $(\Lambda V, d)$ be the minimal model of $\widetilde{X}$ with $V = \oplus_{i=1}^{7} V^i$, and let $\kappa\colon \Lambda V \to \Omega(\widetilde{X})$ be a quasi-isomorphism. From Proposition 4.38 we deduce that $H^1(\widetilde{X}) = \langle \rho^*(e^3) \rangle$ and that:

$$H^2(\widetilde{X}) = \langle \rho^*(e^{25}), \rho^*(e^{15} - e^{26}), \rho^*(e^{15} - e^{34}), \tau_1, \ldots, \tau_{16} \rangle.$$

In addition, $\rho^*(e^3 \wedge (e^{15} - e^{26})) = d\rho^*(e^{56})$ and $\rho^*[e^{235}]$ and $\rho^*[e^{135}]$ are linearly independent on $H^3(\widetilde{X}, \mathbb{R})$. Then, according to Proposition 4.38 one can choose:

$$\begin{aligned} V^1 =& \langle a \rangle, \\ V^2 =& \langle b_1, b_2, b_3, y_1, \ldots, y_{16}, n \rangle. \end{aligned}$$

with $da = 0$, $db_j = dy_j = 0$ and $dn = ab_2$ and the map κ is:

$$\begin{aligned} \kappa(a) =& \rho^*(e^3), & \kappa(b_2) =& \rho^*(e^{15} - e^{26}), & \kappa(n) =& \rho^*(e^{56}), \\ \kappa(b_1) =& \rho^*(e^{25}), & \kappa(b_3) =& \rho^*(e^{15} - e^{34}), & \kappa(y_j) =& \tau_j. \end{aligned}$$

We now define a Massey product. Let us take $\xi_1 = [a] = \xi_3$, $\xi_2 = [b_2]$; the representatives of these cohomology classes are $\alpha_1 = \alpha_3 = a$ and $\alpha_2 = b_2$. Then $\bar{\alpha}_1 \wedge \alpha_2 = d(-n + \beta_1 + \omega_1)$ and $\bar{\alpha}_2 \wedge \alpha_3 = d(n + \beta_2 + \omega_2)$ with $\beta_1, \beta_2 \in \langle b_1, b_2, b_3 \rangle$ and $\omega_1, \omega_2 \in \langle y_1, \ldots, y_{16} \rangle$. Therefore, defining systems of $\langle \xi_1, \xi_2, \xi_3 \rangle$ are $(a, b_2, a, -n + \beta_1 + \omega_1, n + \beta_2 + \omega_2)$ and the Massey product is the set

$$\{[2an + a\beta + a\omega] \text{ s.t. } \beta \in \langle b_1, b_2, b_3 \rangle, \quad \omega \in \langle y_1, \ldots, y_{16} \rangle\}.$$

We now observe that $[2an + a\beta + a\omega] = 0$ in $H^*(\Lambda V, d)$ if and only if $\omega = 0$ and $[\kappa(2an + a\beta)] = 0$. This is because $[\kappa(a\omega)] = [\rho^*(e^3) \wedge \kappa(\omega)] = 0$ if and only if $\omega = 0$, and if $[\omega] \neq 0$, the elements $[\kappa(a\omega)]$ and $[\kappa(2an + a\beta)]$ are linearly independent.

In addition, $\kappa(2an + a\beta) = \rho^*(2e^{356} + e^3 \wedge \beta')$, with $\beta' \in \langle e^{25}, e^{15} - e^{26}, e^{15} - e^{34} \rangle$. Taking into account Proposition 4.38 $[\kappa(2an + a\beta] = 0$ if and only if $[2e^{356} + e^3 \wedge \beta'] = 0$ on X. But $[2e^{356} + e^3 \wedge \beta'] \neq 0$ as shown in Proposition 4.43. □

There is another non-trivial triple Massey product that comes from the isotropy locus. In order to describe it we have to construct the subspace V^3 of our minimal model; it is a direct sum $V^3 = C \oplus N$; such that $dC = 0$ and there are not closed elements on N. To construct C one takes a basis of the space $H^3(\widetilde{X})/H^1(\widetilde{X})H^2(\widetilde{X})$; for instance:

$$\begin{aligned} &\langle \rho^*[e^{346}], \rho^*[e^{124}], \rho^*[e^{146}], \rho^*[e^{245}], \rho^*[e^{127} + 2e^{145}], \\ &\rho^*[e^{125} + e^{167} - e^{257} - 2e^{456} - e^{347}] \rangle \oplus \langle \{[e^4 \otimes \mathbf{x}_i]\}_{i=1}^{16} \rangle, \end{aligned}$$

Let $C = \langle c_1, \ldots, c_6, z_1, \ldots z_{16} \rangle$ with $dC = 0$ and define $\kappa(c_1) = \rho^*(e^{346}), \kappa(c_2) = \rho^*(e^{124}), \ldots,$ $\kappa(c_6) = \rho^*(e^{125} + e^{167} - e^{257} - 2e^{456} - e^{347})$ and $\kappa(z_i) = e^4 \otimes \mathbf{x}_i$.

With this notation, the triple Massey product coming from the singular locus

$$\langle [a], [z_j], -[a] \rangle$$

is not trivial.

Proposition 4.45. *The fundamental group of $\widetilde{X}$ is $\pi_1(\widetilde{X}) = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_6$.*

Proof. Let us denote $\pi\colon M \to X$ the quotient projection. In order to compute $\pi_1(X)$ we first observe that $\pi_1(M)$ is isomorphic to Γ due to the exact sequence $0 \to \pi_1(G) \to \pi_1(M) \to \Gamma \to 0$. Of course, each generator $u_i \in \Gamma$ is identified with the homotopy class f_i determined by the image of the path from 0 to u_i under the quotient map $q\colon G \to M$. Denote by $[\cdot,\cdot]$ the commutator of two elements on $\pi_1(M)$; then the product structure on Γ determines that the non-zero commutators are:

$$[f_1, f_2] = f_4^{-2}, \qquad [f_1, f_2] = f_5^{-2}, \qquad [f_2, f_5] = f_7^{-6}, \qquad [f_3, f_4] = f_7^{6}.$$
$$[f_1, f_3] = f_6^{2}, \qquad [f_1, f_6] = f_7^{6}, \qquad [f_2, f_6] = f_7^{-6},$$

Taking into account [21, Corollary 6.3] the map $\pi_*\colon \pi_1(M) \to \pi_1(X)$ is surjective; we now analyze $\pi_*(f_j)$. First of all, under the projection π the image of the loop f_1 is the same as the path from 0 to $\frac{1}{2}x_1$ followed by the same path in the reverse direction; this is of course contractible and thus $\pi_*(f_1) = 0$; in the same manner $\pi_*(f_2) = \pi_*(f_5) = \pi_*(f_6) = 0$. Taking into account commutator relations this implies that $\pi_*(f_4^2) = 0$, $\pi_*(f_7^6) = 0$ and that $\pi_*(f_3)$, $\pi_*(f_4)$, $\pi_*(f_7)$ commute. Thus, $\pi_1(X) = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_6$.

We now prove that the resolution process does not alter the fundamental group. For each $\varepsilon \in \mathcal{E}$ consider a small tubular neighbourhood B^ε of H_ε and suppose additionally that B^ε are pairwise disjoint. Take $D^\varepsilon \subset B^\varepsilon$ a smaller tubular neighbourhood of H_ε. Define U a connected open set containing $\cup_\varepsilon B^\varepsilon$ that is homotopy equivalent to $\bigvee_\varepsilon H_\varepsilon$ and $V = X - \cup_\varepsilon D^\varepsilon$.

Seifert-Van Kampen theorem states that $\pi_1(X)$ is the amalgamated product of $\pi_1(V)$ and $\pi_1(U)$ via $\pi_1(U \cap V)$. Define $\widetilde{U} = \rho^{-1}(U)$, $\widetilde{V} = \rho^{-1}(V)$; note that $\widetilde{V}$ and V are diffeomorphic via ρ; in addition, $\rho_*\colon \pi_1(\widetilde{U}) \to \pi_1(U)$ is an isomorphism because $\widetilde{U}$ is homotopy equivalent to $\bigvee_\varepsilon H_\varepsilon \times S^2$. This observation and a further application of the Seifert-Van Kampen theorem ensures that $\pi_1(\widetilde{X}) = \pi_1(X)$. □

Proposition 4.46. *The manifold $\widetilde{X}$ does not admit torsion-free* G_2 *structures.*

Proof. Suppose that $\widetilde{X}$ admits a torsion-free G_2 structure. Since g is Ricci flat and $b_1 = 1$, [18] ensures that there is a finite covering $N \times S^1 \to \widetilde{X}$; with N a compact simply connected 6-dimensional manifold. Note that the covering is regular because $\pi_1(\widetilde{X})$ is abelian; thus $(N \times S^1)/H = \widetilde{X}$, where H denotes the deck group of the covering.

The manifold N is formal because it is simply-connected and 6-dimensional (see [49, Theorem 3.2]); therefore $N \times S^1$ is formal (see [49, Lemma 2.11]). Lemma 4.21 allows us to conclude that $(N \times S^1)/H = \widetilde{X}$ is formal; yielding a contradiction. □

Remark 4.47. We can also prove Proposition 4.46 by making use of the topological obstruction of torsion-free G_2 structures obtained in [29]. Suppose that $\widetilde{X}$ has a torsion-free G_2 structure, then [29, Theorem 4.10] guarantees the existence of CDGAs (A, d) and (B, d) with the differential $d\colon B^k \to B^{k+1}$ being zero except for $k = 3$, and quasi-isomorphisms:

$$(\Omega(\widetilde{X}), d) \longleftarrow (A, d) \longrightarrow (B, d).$$

This implies [29, Corollary 4.13] that non-zero triple Massey products $\langle \xi_1, \xi_2, \xi_3 \rangle$ on $(\Omega(\widetilde{X}), d)$ satisfy that $|\xi_1| + |\xi_2| = 4$ and $|\xi_2| + |\xi_3| = 4$. Let (A', d) be the minimal model of (A, d), then one can obtain quasi-isomorphisms:

$$(\Lambda V, d) \longleftarrow (A', d) \longrightarrow (B, d).$$

The same conclusion holds for non-zero Massey products on $(\Lambda V, d)$. This contradicts the fact that there is a non-zero Massey product $\langle \xi_1, \xi_2, \xi_3 \rangle$ on $(\Lambda V, d)$ with $|\xi_1| = |\xi_3| = 1$ and $|\xi_2| = 2$ as it is obtained in the proof of Proposition 4.44. Therefore $\widetilde{X}$ does not have a torsion-free G_2 structure.

Remark 4.48. There exists a finite covering $Y \to \widetilde{X}$ such that $\pi_1(Y) = \mathbb{Z}$ because $\pi_1(\widetilde{X}) = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_6$. The manifold Y is also non-formal as a consequence of Lemma 4.21 and of course, it has first Betti number $b_1 = 1$ and admits a closed G_2 structure. Arguing as in the proof of Proposition 4.46 one can conclude that Y does not admit any torsion-free G_2 structure.

Resumen en español

Esta tesis se compone de cuatro artículos que he redactado en el transcurso de mi doctorado, dos de ellos en colaboración con otros autores. Estos trabajos abordan varios problemas en el área de las estructuras geométricas, tales como el estudio de las estructuras Spin(7) y la construcción de variedades compactas con estructura simpléctica y de tipo G_2 cerrada. En el caso de las variedades con estructura G_2, prestamos especial atención a dos propiedades topológicas: formalidad y primer número de Betti. Las técnicas que empleamos son básicamente, teoría de espinores, estructuras invariantes por la izquierda en nilvariedades y resolución de orbifolds. Dedicamos este resumen a presentar el estado del arte de estos temas y a exponer los resultados principales de la tesis.

Dentro de la geometría Riemanniana, la teoría de holonomía motiva el estudio de las estructuras geométricas no integrables. El *grupo de holonomía* $\mathrm{Hol}(g)$ de una variedad Riemanniana (M, g) es un invariante que cuantifica cómo cambia cada vector de T_pM tras su transporte paralelo a lo largo cada uno de los lazos con punto base p. Tras su definición, el interés por determinar los grupos de holonomía que pueden tener las variedades Riemannianas simplemente conexas, completas e irreducibles creció rápidamente. La hipótesis de que M sea simplemente conexa garantiza que $\mathrm{Hol}(g)$ sea un subgrupo de Lie conexo de $\mathrm{SO}(n)$. Bajo ésta, la completitud de (M, g) junto con su irreducibilidad descartan que $\mathrm{Hol}(g)$ sea un producto. En concreto, el teorema de descomposición de de Rham [39] demuestra que toda variedad Riemanniana (M, g) simplemente conexa y completa es un producto Riemanniano $(M_1, g_1) \times \ldots \times (M_\ell, g_\ell)$ tal que la acción de $\mathrm{Hol}(g_i)$ en $T_{p_i}M_i$ es irreducible. Cartan caracterizó los grupos de holonomía de las variedades simétricas en [26, 27] utilizando la teoría de grupos de Lie como herramienta. Más tarde, Berger aborda el caso de las variedades no simétricas en [17], obteniendo:

Teorema 1. *Sea (M, g) una variedad Riemanniana simplemente conexa, completa, irreducible y no simétrica de dimensión n. Ocurre exactamente uno de los siguientes casos:*

$$\mathrm{Hol}(g) = \mathrm{SO}(n),$$

$$\mathrm{Hol}(g) = \mathrm{U}(m) \subset \mathrm{SO}(2m) \text{ con } n = 2m \text{ y } m \geq 2,$$

$$\mathrm{Hol}(g) = \mathrm{SU}(m) \subset \mathrm{SO}(2m) \text{ con } n = 2m \text{ y } m \geq 2,$$

$$\mathrm{Hol}(g) = \mathrm{Sp}(k) \subset \mathrm{SO}(4k) \text{ con } n = 4k \text{ y } k \geq 2,$$

$$\mathrm{Hol}(g) = \mathrm{Sp}(k) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4k) \text{ con } n = 4k \text{ and } k \geq 2,$$

$$\mathrm{Hol}(g) = \mathrm{G}_2 \subset \mathrm{SO}(7) \text{ con } n = 7,$$

$$\mathrm{Hol}(g) = \mathrm{Spin}(7) \subset \mathrm{SO}(8) \text{ con } n = 8.$$

En conjunto, los grupos $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(k)$, y $\mathrm{Sp}(k)\cdot\mathrm{Sp}(1)$ se conocen como *grupos de holonomía especial*. Mientras que G_2 y $\mathrm{Spin}(7)$ son los *grupos de holonomía excepcional*. Es interesante destacar que los grupos de holonomía del teorema de Berger están relacionados con las álgebras de división reales. Los grupos $\mathrm{U}(m)$ y $\mathrm{SU}(m)$ se asocian a las variedades *Kähler* y *Calabi-Yau*; éstas son complejas desde el punto de vista de la geometría diferencial. Los grupos $\mathrm{Sp}(k)$ y $\mathrm{Sp}(k)\cdot\mathrm{Sp}(1)$ se asocian a los cuaterniones y corresponden a las variedades *hyperKähler* y *quaternionic-Kähler*. Los grupos G_2 y $\mathrm{Spin}(7)$ son símplemente conexos y están relacionados con los octoniones. El producto octoniónico en $\mathbb{R}^8 = \mathbb{O}$ determina un *producto vectorial triple* $\times$, esto es, una aplicación multilineal $\mathbb{R}^8\times\mathbb{R}^8\times\mathbb{R}^8\to\mathbb{R}^8$ tal que el producto $u\times v\times w$ tiene norma $\|u\wedge v\wedge w\|$ y es perpendicular a los vectores u, v y w. La contracción de $\times$ con el producto escalar proporciona la 4-forma $\Omega_0(u,v,w,z)=\langle u\times v\times w, z\rangle$, que tiene la siguiente expresión respecto de la base canónica $(e_0,\dots,e_7)$:

$$\begin{aligned}\Omega_0 = {} & e^{0123}-e^{0145}-e^{0167}-e^{0246}+e^{0257}-e^{0347}-e^{0356}\\ & +e^{4567}-e^{2367}-e^{2345}-e^{1357}+e^{1346}-e^{1256}-e^{1247}.\end{aligned}$$

Denotemos $\mathbb{R}^8=\mathbb{R}(e_0)\times\mathbb{R}^7$; el producto triple de $\mathbb{R}^8$ determina un producto vectorial en $\mathbb{R}^7$ mediante la expresión $u\times' v = e_0\times u\times v$. De manera equivalente, la 4-forma Ω_0 determina una 3-forma $\varphi_0 = i(e_0)\Omega_0$ en $\mathbb{R}^7$. $\mathrm{Spin}(7)$ es el subgrupo de $\mathrm{SO}(8)$ que preserva el producto vectorial triple de $\mathbb{R}^8$, o sea, $\mathrm{Stab}(\Omega_0)$, y G_2 es el subgrupo de $\mathrm{SO}(7)$ que preserva $\times'$, o sea, $\mathrm{Stab}(\varphi_0)$. Naturalmente, $\mathrm{G}_2\subset\mathrm{Spin}(7)$.

La demostración del teorema de Berger es algebraica y en el momento de su publicación no se conocían ejemplos de métricas completas con holonomía G_2 o $\mathrm{Spin}(7)$. Bryant y Salamon [24] construyeron ejemplos de este tipo en 1898. A partir de esta lista también surge el problema de construir variedades Riemannianas compactas con holonomía $\mathrm{SU}(m)$, $\mathrm{Sp}(k)$, $\mathrm{Sp}(k)\cdot\mathrm{Sp}(1)$, G_2, y $\mathrm{Spin}(7)$. Estas construcciones requieren profundos teoremas del área del análisis; por ejemplo, la construcción de variedades compactas de holonomía $\mathrm{SU}(m)$ y $\mathrm{Sp}(k)$ emplea el teorema de Yau, que demuestra la conjetura de Calabi e implica que toda variedad Kähler compacta con fibrado canónico trivial admite una métrica Calabi-Yau. Las variedades compactas con holonomía G_2 y $\mathrm{Spin}(7)$ fueron las últimas en aparecer allá por 1996. Más adelante en este resumen revisaremos la construcción de Joyce desarrollada en [71, 72, 73].

El *principio de holonomía* permite interpretar la condición $\mathrm{Hol}(g)\subset\mathrm{G}$ como una combinación de dos obstrucciones, una topológica y otra analítica ([74, Lema 2.5.2]).

Proposición 2. *Sea (M,g) una variedad Riemanniana, sea $p\in M$ y sea $\mathrm{Hol}(g)$ el grupo de holonomía con punto base p. Entonces,*

1. *Si T es un tensor paralelo de M entonces $\mathrm{Hol}(g)\subset\mathrm{Stab}(\mathrm{T}_p)$.*
2. *Si S es un tensor de $\mathbb{R}^n$ tal que $\mathrm{Hol}(g)\subset\mathrm{Stab}(\mathrm{S})$, existe un tensor paralelo T de M tal que $\mathrm{T}_p=\mathrm{S}$.*

La dificultad para encontrar ejemplos con holonomía especial y excepcional, junto con el principio de holonomía, motivaron el estudio de las *estructuras geométricas* asociadas a grupos de Lie $\mathrm{G}\subset\mathrm{SO}(n)$. Una G estructura en una variedad Riemanniana (M,g) es una reducción del fibrado $\mathrm{SO}(n)$ principal de M al grupo G. Esta noción es equivalente a la existencia de tensores $\{\mathrm{T}_i\}$ con estabilizador común G. Por este motivo denotamos por $(M^n, g, \{\mathrm{T}_i\})$ a una G estructura en una variedad Riemanniana de dimensión n. Centrémonos en el caso de los grupos $\mathrm{U}(m)$, $\mathrm{SU}(m)$, G_2 y $\mathrm{Spin}(7)$:

1. (M^{2m}, g, J) es una estructura $\mathrm{U}(m)$ o una estructura casi hermítica si J es una estructura casi compleja compatible con g. Es decir, para cada $p \in M^{2m}$ existe una isometría $f_p\colon (T_pM^{2m}, g_p) \to (\mathbb{C}^m, \langle\cdot,\cdot\rangle)$ tal que $f_p \circ J_p \circ f_p^{-1}(v) = \mathbf{i}v$ para cada $v \in \mathbb{C}^m$. En este caso, definimos la 2-forma $\omega(v, w) = g(Jv, w)$.

2. (M^{2m}, g, J, Θ) es una estructura $\mathrm{SU}(m)$ si (M^{2m}, g, J) es una estructura $\mathrm{U}(m)$ y las aplicaciones $\{f_p\}_{p\in M}$ también verifican $f_p^*(dz_1 \wedge \cdots \wedge dz_m) = \Theta_p$.

3. (M^7, g, φ) es una estructura G_2 si φ es una 3-forma tal que para cada $p \in M^7$ existe una isometría $f_p\colon (T_pM^7, g_p) \to (\mathbb{R}^7, \langle\cdot,\cdot\rangle)$ tal que $f_p^*\varphi_0 = \varphi_p$.

4. (M^8, g, Ω) es una estructura $\mathrm{Spin}(7)$ si Ω es una 4-forma tal que para cada $p \in M^8$ existe una isometría $f_p\colon (T_pM^8, g_p) \to (\mathbb{R}^8, \langle\cdot,\cdot\rangle)$ tal que $f_p^*\Omega_0 = \Omega_p$.

Asimismo, las estructuras geométricas permiten estudiar situaciones geométricas que el grupo de holonomía no puede distinguir. Este es el caso de las estructuras $\mathrm{U}(m)$ y $\mathrm{SU}(m)$ en variedades de dimensión impar $(2m + 1)$. Las primeras se llaman estructuras casi contacto métricas y están relacionadas con la geometría de contacto.

En general, se obtienen propiedades geométricas interesantes cuando los tensores que definen la estructura geométrica verifican ciertas ecuaciones en derivadas paricales. Éstas son normalmente más fáciles de resolver que la condición $\mathrm{Hol}(g) \subset \mathrm{G}$. Los ejemplos incluyen las variedades casi Kähler y las estructuras hermíticas, que son variedades simplécticas y complejas desde el punto de vista de la geometría diferencial. Una estructura $\mathrm{U}(m)$ es casi Kähler si $d\omega = 0$ y hermítica si el tensor de Nijenhuis N_J se anula. Esto motivó a Gray y a Hervella a comenzar un programa de clasificación de G estructuras en [59], que aborda el caso de las estructuras casi hermíticas. La *torsión intrínseca* Γ es el objeto que permite clasificar las G estructuras. Ésta es una sección de un fibrado $\mathcal{W}$ sobre M con fibra $\mathbb{R}^n \otimes \mathfrak{g}^\perp$; donde $\mathfrak{g}$ denota el álgebra de Lie de $\mathrm{G} \subset \mathrm{SO}(n)$ vista como un subespacio de $\Lambda^2\mathbb{R}^n = \mathfrak{so}(n)$, donde tomamos su complemento ortogonal. El G módulo $\mathbb{R}^n \otimes \mathfrak{g}^\perp$ se descompone como suma directa de subespacios invariantes irreducibles, determinando una descomposición $\mathcal{W} = \oplus_{i\in I}\mathcal{W}_i$. Las clases *no integrables* se definen por $\Gamma \in \oplus_{i\in J}\mathcal{W}_i$ para algún $J \subset I$, $J \neq \emptyset$; el caso *paralelo* corresponde a $\Gamma = 0$, condición que equivale a $\mathrm{Hol}(g) \subset \mathrm{G}$.

Estas clases se describen normalmente en términos de la derivada covariante o la derivada exterior de los tensores que definen la estructura. Centrémonos en el caso de las estructuras G_2, obtenidas por Fernández y Gray en [48] y posteriormente reformuladas por Bryant en [23]. Las clases de estructuras G_2 están determinadas por $d\varphi$ y $d \star \varphi$; más precisamente, las *formas de torsion* $\tau_k \in \Omega^k(M)$ verifican:

$$\begin{aligned} d\varphi &= \tau_0 \star \varphi + 3\,\tau_1 \wedge \varphi + \star\tau_3, \\ d \star \varphi &= 4\,\tau_1 \wedge \star\varphi + \tau_2 \wedge \varphi, \end{aligned}$$

y además τ_2 y τ_3 satisfacen: $\tau_2 \wedge \star\varphi = 0$, $\tau_3 \wedge \star\varphi = 0$ y $\tau_3 \wedge \varphi = 0$. Naturalmente, estas ecuaciones se deducen de la descomposición de los espacios $\Lambda^4(\mathbb{R}^7)^*$ y $\Lambda^5(\mathbb{R}^7)^*$ en subespacios G_2 invariantes irreducibles. La 1-forma θ se conoce por el nombre de *forma de Lee* de la estructura. Las *clases puras* son aquellas en las que se anulan todas las formas de torsión salvo una; los casos más estudiados son las clases *nearly parallel*, caracterizada por $d\varphi = \tau_0 \star \varphi$, *cerrada*, definida por $d\varphi = 0$, y *localmente conformemente paralela*, descrita por $d\varphi = 3\tau_1 \wedge \varphi$ y $d\star\varphi = 4\tau_1 \wedge \star\varphi$. Las estructuras *cocerradas* son aquellas que verifican $d\star\varphi = 0$; Crowley y Nördstrom demostraron en [37] que existen en cualquier variedad compacta con una estructura G_2. La prueba emplea el h-principio de Gromov [60]. Ejemplos explícitos son las hipersuperficies una variedad con una estructura Spin(7) paralela, tal como $\mathbb{R}^8$, dotadas de la estructura G_2 inducida por la estructura Spin(7). Si la hipersuperficie es totalmente umbílica, como la esfera $S^7 \subset \mathbb{R}^8$, la estructura es nearly parallel.

Fue Wang quien exploró por primera vez la conexión entre el grupo de holonomía y la teoría de espinores. El teorema de Wang [112] enuncia que una variedad Riemanniana completa simplemente conexa e irreducible tiene un espinor paralelo si y solo si su grupo de holonomía es simplemente conexo, o sea, si es uno de los siguientes: $\mathrm{SU}(m)$, $\mathrm{Sp}(k)$, G_2, $\mathrm{Spin}(7)$. En cuanto a estructuras geométricas, si el grupo de estructura G es símplemente conexo, entonces la variedad es spin y está dotada de una cierta cantidad de espinores nunca nulos.

La teoría de espinores tiene sus orígenes en la búsqueda, llevada a cabo por Dirac, de un operador de ondas $\not{D}$ acorde con la teoría de la relatividad. El objetivo era, básicamente, encontrar la raíz cuadrada del operador Laplaciano en $\mathbb{R}^n$. Este cálculo llevó a Dirac a introducir el *álgebra de Clifford* Cl_n de $\mathbb{R}^n$: el $\mathbb{R}$-álgebra con unidad generado por $\mathbb{R}^n$, cocientado por las relaciones $v \cdot v = -|v|^2 \cdot 1$. El operador $\not{D}$ es el operador de Dirac; la introducción de [54] recoge una exposición detallada de su desarrollo. Uno de los principales logros de la teoría de espinores es el toerema del índice de Atiyah-Singer, que relaciona el índice del operador de Dirac con un invariante topológico: el $\hat{A}$-género. La teoría de espinores juega un papel importante en diferentes situaciones geométricas: proporciona todos los campos de vectores linealmente independientes en las esferas, e interviene tanto en la existencia de métricas con curvatura escalar positiva como en el carácter entero de ciertas clases características.

El recubridor universal $\mathrm{Ad}\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ se construye a partir del álgebra de Clifford: $\mathrm{Spin}(n)$ es el subgrupo multiplicativo de $\mathrm{Cl}_n - \{0\}$,

$$\mathrm{Spin}(n) = \{v_1 \cdots v_{2k} \text{ tales que } 2k \leq n, |v_j| = 1\},$$

y la aplicación recubridora es la conjugación $\mathrm{Ad}(g)(x) = gxg^{-1}$. El formalismo espinoral en $\mathbb{R}^n$ consiste en un Cl_n módulo ireducible Δ_n que proviene de un isomorfismo $\rho\colon \mathrm{Cl}_n \to \mathbf{k}(m)$ o $\rho\colon \mathrm{Cl}_n \to \mathbf{k}(m) \oplus \mathbf{k}(m)$; donde hemos denotado por $\mathbf{k}(m)$ al álgebra de matrices de dimensión m sobre el cuerpo (o álgebra de división) $\mathbf{k} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Durante un tiempo se encontraron problemas para generalizar el formalismo espinorial en variedades orientables. Éstos fueron solventados a través de la noción de *estructura spin*. Las variedades que admiten una estructura spin se llaman *variedades spin* y son aquellas que tienen segunda clase de Stiefel-Whitney nula.

Sea (M, g) una variedad Riemanniana orientada de dimensión n; denotemos por $\mathrm{P}_{\mathrm{SO}}(M)$ su fibrado $\mathrm{SO}(n)$ principal. Una estructura spin consiste en un fibrado principal $\mathrm{p}\colon \mathrm{P}_{\mathrm{Spin}}(M) \to \mathrm{P}_{\mathrm{SO}}(M)$ compatible con la aplicación $\mathrm{Ad}\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$, esto es, $\mathrm{p}(\gamma y) = \mathrm{Ad}(\gamma)\mathrm{p}(y)$ si $\gamma \in \mathrm{Spin}(n)$, $y \in \mathrm{P}_{\mathrm{Spin}}(M)$. El *fibrado espinorial* de una variedad spin (M, g) es:

$$\Sigma(M) = \mathrm{P}_{\mathrm{Spin}}(M) \times_{\rho'} \Delta_n,$$

donde $\rho'\colon \mathrm{Spin}(n) \to \mathrm{End}(\Delta_n)$ es la restricción de una respresentación irreducible $\mathrm{Cl}_n \to \mathrm{End}(\Delta_n)$. La particularidad de las secciones de este fibrado, los *espinores*, es que pueden ser multiplicados tanto por vectores como por formas; este producto existe porque la representación ρ' extiende a $\mathrm{Cl}_n \to \mathrm{End}(\Delta_n)$. Además, la conexión de Levi-Civita levanta al fibrado espinoral, permitiendo definir ecuaciones en derivadas parciales sobre espinores sin introducir información adicional. Este es el caso de la condición *armónica*, determinada por la anulación del *operador de Dirac*. Éste es un operador autoadjunto de primer orden, que tiene la siguiente expresión en términos de una base ortonormal local $(e_1, \ldots, e_n)$:

$$\not{D}\eta = \sum_{i=1}^{n} e_i \nabla_{e_i} \eta.$$

Friedrich demostró en [53] que el primer autovalor λ del operador de Dirac está relacionado con la curvatura escalar a través de la desigualdad $\lambda^2 \geq \frac{n}{4(n-1)} \min_{p \in M}\{\mathrm{scal}_p\}$, y que la igualdad se alcanza en presencia de un *espinor Killing*. Los espinores Killing son aquellos que verifican $\nabla_X \eta = \mu X \eta$, y si bien ya se estudiaban en el área de la relatividad general, aparecieron por primera vez dentro de la geometría Riemanniana en el artículo [53]. La relación entre espinores armónicos y estructuras geométricas será explorada más adelante en este resumen, dado que comprende parte del trabajo desarrollado en el Capítulo 2. Los espinores Killing nunca nulos determinan las estructuras G_2 nearly parallel y las estructuras SU(3) *nearly Kähler*; las últimas están caracterizadas por las condiciones $d\omega = 3\mathfrak{Re}(\Theta)$ y $d\mathfrak{Im}(\Theta) = -2\omega^2$. Las esferas S^7 y S^6 equipadas con su métrica estándar son ejemplos de tales estructuras.

La presencia de ciertas estructuras geométricas dan lugar a propiedades de curvatura especiales. El tensor de cuarvatura Riemanniano $\mathcal{R}$ verifica puntualmente $\mathcal{R} \in \mathrm{Sym}^2(\mathfrak{hol}(g))$; esta relación restringe la forma del tensor Ricci cuando la variedad tiene holonomía especial o excepcional. El tensor de Ricci de una variedad Kähler está determinado por la 1-forma de Ricci y es nulo cuando el grupo de holonomía es simplemente conexo. Asimismo, las variedades con holonomía $\mathrm{Sp}(k) \cdot \mathrm{Sp}(1)$ son Einstein. En el caso de las G estructuras, el tensor de Ricci está determinado por las formas de torsión tal y como expresan [23] y [69] para estructuras G_2 y Spin(7). Un ejemplo ilustrativo son las G estructuras determinadas por espinores Killing; en este caso la métrica asociada es Einstein. Esta propiedad es consecuencia de la fórmula que relaciona el tensor de Ricci con la derivada covariante del espinor en [54, p. 64]. En el caso de las variedades con grupo de holonomía simplemente conexo, se demuestra que el tensor de Ricci es nulo combinando esta fórmula con el teorema de Wang.

La interacción entre la holonomía de una variedad Riemanniana y sus propiedades cohomológicas es bien conocida en el caso de las variedades compactas Kähler. Éstas son formales y su álgebra de cohomología admite una descomposición de Hodge y verifica la propiedad dura de Lefschetz. La fórmula de Weitzenböck para el Laplaciano permite generalizar la descomposición de Hodge en variedades Riemannianas compactas (M, g) con holonomía contenida en un grupo $G \subset SO(n)$ de la lista de Berger. El espacio de las formas armónicas $\mathcal{H}^k(M, \mathbb{R})$ admite una descomposición en suma directa de subespacios determinados por las componentes irreducibles de la respresentación de G en $\Lambda^k(\mathbb{R}^n)^*$. Ésta permite definir los *números de Betti refinados*. De manera explícita, sea $\Lambda^k(\mathbb{R}^n)^* = \oplus_{i \in I} \Lambda_i^k$ la descomposición en subespacios G invariantes e irreducibles. Descompongamos $\Omega^k(M) = \oplus_{i \in I} \Omega_i^k(M)$, la fórmula de Weitzenböck garantiza que el Laplaciano actúa en cada uno de los subfibrados $\Omega_i^k(M)$. En consecuencia,

$$\mathcal{H}^k(M) = \oplus_{i \in I} \mathcal{H}_i^k(M).$$

Además, si dos representaciones Λ_i^k y Λ_j^l son isomorfas entonces $\mathcal{H}_i^k(M) \cong \mathcal{H}_j^l(M)$. Los números de Betti refinados son $b_i^k = \dim(\mathcal{H}_i^k(M))$. Cuando el grupo de holonomía es igual a G se obtienen más obstrucciones, por ejemplo, las variedades con holonomía G_2 y Spin(7) tienen $b_1 = 0$.

Deligne, Griffiths, Morgan, y Sullivan demostraron en [40] que las variedades Kähler compactas son formales. Este resultado es consecuencia del Lema $\partial\bar{\partial}$. La noción de formalidad proviene del área de la *homotopía racional* fundada por Sullivan en [107]. Su objeto de estudio es la parte libre de torsión de los grupos de homotopía de orden superior $\pi_k(M)\otimes\mathbb{Q}$, $k\geq 2$ e introduce nociones algebraicas como la de *álgebra diferencial conmutativa graduada* (ADCG) y su *modelo minimal*. El modelo minimal de una ADCG $(\mathcal{A},d)$ es una ADCG minimal (véase la definición 4.14) $(\mathcal{M},d)$ y un homomorfismo $\Psi\colon(\mathcal{M},d)\to(\mathcal{A},d)$ que induce un isomorfismo entre sus grupos de cohomología.

Sea M un complejo simplicial conexo de tipo finito y sea $(\mathcal{A}_{PL}(M),d)$ el ADCG de las formas racionales polinómicas. Una k-forma racional polinómica está determinada por una k-forma en cada símplex σ de M cuyos coeficientes son polinomios sobre $\mathbb{Q}$, de manera que $\omega_\sigma=\omega_{\sigma'}|_\sigma$ cuando $\sigma\subset\partial\sigma'$. El teorema PL de de Rham garantiza que la cohomología de $(\mathcal{A}_{PL}(M),d)$ es $H^*(M,\mathbb{Q})$. El invariante introducido por Sullivan es el *modelo minimal de* M, que es el modelo minimal del ADCG $(\mathcal{A}_{PL}(M),d)$. Éste siempre existe y es único salvo isomorfismo. La relación entre los grupos de homotopía racionales y los modelos minimales se establece en [107, Teorema 10.1]:

Teorema 3. *Sea M un complejo simplicial conexo de tipo finito y nilpotente, y sea $(\mathcal{M},d)$ su modelo minimal. El grupo de homotopía racional $\pi_k(M)\otimes\mathbb{Q}$ con $k\geq 2$ es el espacio dual del subespacio de grado k en $\mathcal{M}$*

La hipótesis de que M sea *nilpotente* requiere que $\pi_1(M)$ sea nilpotente y que actúe en $\pi_k(M)$ como un homomorfismo nilpotente. Si el modelo minimal de $(\mathcal{A}_{PL}(M),d)$ coincide con el modelo minimal de $(H^*(M,\mathbb{Q}),d=0)$ decimos que M es *formal*. El cálculo del modelo minimal es un proceso formal, hecho que explica el nombre de la propiedad: los grupos de homotopía racional de los espacios formales se obtienen de manera formal a partir de los grupos de cohomología racional.

Cuando M es una variedad, su modelo minimal real se obtiene a partir del complejo de de Rham $(\Omega^*(M),d)$. En la práctica, el cálculo del modelo minimal puede ser difícil; el concepto de *s-formalidad* se emplea normalmente para decidir si una variedad es formal o no. De modo breve, esta propiedad depende de los generadores del modelo minimal de grado menor o igual que s. El teorema de dualidad de Poincaré permite probar en [49] que una variedad compacta orientable de dimensión $2n$ o $2n-1$ es formal si y solo si es $(n-1)$-formal. Además, los *productos de Massey* se utilizan frecuentemente para probar que una variedad es no formal. Su definición y relación con el concepto de formalidad se puede leer en [100, Sección 1.6].

El resultado de [40] implica que las variedades compactas con holonomía $\mathrm{SU}(m)$ y $\mathrm{Sp}(k)$ son formales. Las variedades compactas con holonomía contenida en $\mathrm{Sp}(k)\cdot\mathrm{Sp}(1)$ y curvatura escalar positiva son también formales [4]; en la prueba se utiliza la formalidad de las variedades compactas Kähler. Aún no se ha demostrado ni descartado que las variedades con holonomía excepcional sean formales. Se han obtenido resultados parciales en [29], [38] y [76]. Los resultados en [29] y [76] se basan en una idea de Verbitsky en [111], donde define un operador diferencial $\mathcal{L}_\omega$ en una variedad Kähler (M,g,J) para dar una prueba alternativa de la formalidad de las variedades Kähler. Este operador está bien definido en las variedades Riemannianas dotadas de una k-forma paralela; el estudio de los operadores $\mathcal{L}_\varphi$, $\mathcal{L}_{\star\varphi}$ o $\mathcal{L}_\Omega$ definidos por φ, $\star\varphi$ o Ω cuando la holonomía está contenida en G_2 o $\mathrm{Spin}(7)$ ha resultado fructífera pero no responde la pregunta. Además, el artículo [38] estudia el caso de 7-variedades; entre otros resultados, los autores prueban que si existiese una variedad compacta no formal con holonomía G_2 tendría $b_2\geq 4$.

La búsqueda de variedades compactas con clases concretas de estructuras geométricas comienza normalmente en *nilvariedades y solvariedades*. Éstas son cocientes compactos de un grupo de Lie G por un retículo Γ; el grupo de Lie es nilpotente en el primer caso y resoluble en el segundo. Las nilvariedades y solvariedades son especiales desde el punto de vista topológico: son asféricas con $\pi_1(\Gamma\backslash G) = \Gamma$ y su primer número de Betti es mayor o igual que 2 en nilvariedades y mayor o igual que 1 en solvariedades. El teorema de Nomizu [98] establece que el modelo minimal real de una nilvariedad $\Gamma\backslash G$ es el ADCG de Chevalley-Eilenberg $(\Lambda\mathfrak{g}^*, d)$, cuya diferencial está determinada por $d\alpha(X,Y) = \alpha[X,Y]$ si $\alpha \in \mathfrak{g}^*$. Hemos denotado por $\mathfrak{g}$ el álgebra de Lie de G. La situación es distinta en caso de solvariedades; el teorema de Hattori [64] enuncia que el ADCG de Chevalley-Eilenberg es un modelo para una subclase de solvariedades, pero puede no ser el modelo minimal. Esta es la subclase de las *solvariedades completamente resolubles*, aquellas en las que los endomorfismos $\mathrm{ad}(X)\colon \mathfrak{g} \to \mathfrak{g}$, $X \in \mathfrak{g}$, solo tienen autovalores reales. Además, las nilvariedades no abelianas son no formales, mientras que las solvariedades pueden ser tanto formales como no formales.

Normalmente dotamos a estos espacios de estructuras geométricas invariantes por la izquierda, inducidas por el grupo de Lie. La curvatura de las métricas asociadas a éstas es especial: tal como se prueba en [91], las métricas son planas o tienen curvatura escalar estrictamente negativa. Además las métricas no planas nunca son de tipo Einstein. En cuanto a las estructuras geométricas, las ecuaciones en derivadas parciales que definen cada clase se transforman en un sistema de ecuaciones que involucran las constantes de estructura del álgebra de Lie. Este enfoque simplifica el problema y es la razón por la que hablamos de estructuras geométricas en álgebras de Lie nilpotentes y resolubles. Las álgebras de Lie de dimensión menor o igual que 7 están clasificadas, véase [14] y [58]; apoyándose en dicha clasificación numerosos artículos tratan de determinar las álgebras de Lie nilpotentes que admiten una G estructura particular.

El comportamiento las estructuras geométricas en nilvariedades y solvariedades es amplio pero limitado. Un ejemplo ilustrativo es el caso de la variedad de Kodaira-Thurston, una nilvariedad de dimensión 4. Ésta fue la primera variedad simpléctica sin estructuras Kähler conocida. Naturalmente, el carácter no formal de las nilvariedades no abelianas impide que éstas sean Kähler. Además, las solvariedades completamente resolubles no abelianas no admiten métricas con holonomía contenida en G_2 o Spin(7). De acuerdo con el teorema de Cheeger-Gromoll, si este fuera el caso de una solvariedad completamente resoluble $(\Gamma\backslash G, g)$ entonces su recubridor universal sería un producto $\mathbb{R}^k \times N$ donde $k = b_1(\Gamma\backslash G)$ y N es una variedad compacta simplemente conexa. Como el recubridor universal de $\Gamma\backslash G$ es G, isomorfo a $\mathbb{R}^7$ o a $\mathbb{R}^8$, tenemos que $b_1(\Lambda\mathfrak{g}^*, d) = 7, 8$; por tanto, el grupo G es abeliano. De modo similar, algunas clases de G estructuras no ocurren en nilvariedades y solvariedades. Este es el caso de aquellas que inducen métricas de curvatura escalar positiva, tales como las estructuras SU(3) nearly Kähler y las G_2 nearly parallel. Lo mismo ocurre en una subclase de estructuras localmente conformemente paralela (LCP) de tipo G_2 o Spin(7). La segunda está definida por la ecuación $d\Omega = \theta \wedge \Omega$; la 1-forma θ también se llama forma de Lee. Cuando la forma de Lee es cocerrada, la curvatura escalar es positiva. De hecho, las variedades LCP con forma de Lee nunca nula verifican un teorema de estructura [70]. Éstas son mapping torus de una variedad N con recubridor universal compacto. La variedad N posee una estructura SU(3) nearly Kähler en el caso de G_2 y una estructura G_2 nearly parallel en el caso de Spin(7). De esta caracterización se sigue que las solvariedades no admiten estructuras LCP invariantes por la izquierda.

Las propiedades topológicas de las nilvariedades y las solvariedades son limitadas. Las técnicas de resolución de orbifolds se ofrecen como alternativa para construir ejemplos compactos con propiedades topológicas diferentes. Este es la idea que se sigue en [11] y [50] para construir variedades simplécticas simplemente conexas. Las acciones de grupos finitos en nilvariedades no son difíciles de construir. Cuando la acción preserva alguna G estructura invariante por la izquierda, el espacio cociente de la nilvariedad por la acción determina un orbifold dotado de una G estructura. La desingularización, si es posible, proporciona una variedad con dicha G estructura y con propiedades topológicas diferentes. Más adelante en este resumen detallaremos este procedimiento.

Procedemos ahora a exponer los resultados principales de cada capítulo. Dividiremos la discusión en dos partes: el estudio de las estructuras Spin(7) desde el punto de vista de la teoría de espinores y la resolución de orbifolds simplécticos y G_2. Los artículos que avalan la publicación de esta tesis por compendio son [85, 86, 87]. Los artículos [85] y [87] corresponden a la segunda parte de la tesis y están respectivamente contenidos en los Capítulos 4 y 3. El artículo [86] se incluye dentro de la primera parte y corresponde al Capítulo 1. El trabajo [12] está siendo revisado para su publicación y complementa el trabajo desarrollado en el artículo [86]. Por tanto, su exposición es relevante para desarrollar el estado del arte de la tesis. Para hacer la exposición más clara presentamos su contenido en el Capítulo 2, en lugar de desarrollarlo en la parte de introducción a la tesis.

Un enfoque espinorial de las estructuras Spin(7) y estructuras geométricas definidas por espinores

Desde que Fernández clasificase las estructuras Spin(7) no integrables en [43], pocos trabajos se han dedicado a su estudio. Una de las razones es que todavía quedan muchos problemas abiertos acerca de las estructuras G_2. Además, la casificación de las estructuras Spin(7) es pequeña: solo hay 4 clases, frente a los 16 tipos de estructuras G_2 y $U(m)$. Una propiedad distintiva de la geometría Spin(7) es que las propiedades de ser paralela y cerrada son equivalentes para la 4-forma que define la estructura. Las clases están determinadas por $d\Omega$; el espacio $\Lambda^5(\mathbb{R}^8)^*$ descompone como suma directa de dos subespacios Spin(7) invariantes y por tanto las clases no integrables puras son:

1. Localmente conformemente paralelas, si $d\Omega = \theta \wedge \Omega$ donde θ es una 1-forma cerrada.
2. Balanced, si $(\star d\Omega) \wedge \Omega = 0$.

En el Capítulo 1 empleamos el enfoque espinorial para rescribir la clasificación de las estructuras Spin(7) en términos de la derivada covariante del espinor que define la estructura. Este estudio nos motiva a diseñar un método para construir estructuras Spin(7) de tipo balanced en el Capítulo 2, que sugiere definir una nueva clase de estructuras geométricas: *las estructuras spin-harmonic.*

Clasificación espinorial de las estructuras Spin(7)

Dedicamos el Capítulo 1 al estudio de las estructuras Spin(7) desde el punto de vista de la teoría de espinores. Este trabajo continúa el formalismo espinorial desarrollado en [1] para estructuras SU(3) y G_2, y complementa el artículo [69], que investiga algunas propiedades de las estructuras Spin(7) a través de la geomería espinorial. Además, este enfoque nos permite recuperar los resultados de [83, 84] sobre estructuras G_2 en hipersuperficies de variedades con estructura Spin(7) y la construcción de estructuras Spin(7) en fibrados S^1-principales sobre variedades G_2. Este marco conceptual prueba ser útil en la construcción de ejemplos de estructuras Spin(7) de tipo balanced y localmente conformemente balanced.

En la primera parte de este capítulo, reescribimos la clasificación de las estructuras Spin(7) en términos de espinores. Antes de enunciar los resultados, describimos algunos conceptos necesarios para comprenderlos. El álgebra Cl_8 es isomorfa a $\mathbb{R}(16)$, y por tanto la representación espinorial es $\Delta_8 = \mathbb{R}^{16}$. Este espacio se descompone como suma directa de dos subespacios $\Delta_\pm$ de dimensión 8, que se conocen como el subespacio positivo y negativo. Los espacios $\Delta_\pm$ son los autoespacios asociados al endomorfismo determinado por la multiplicación por el elemento de volumen $e_0 \cdots e_8 \in \mathrm{Cl}_8$, y por tanto son invariantes bajo la acción del grupo Spin(8). El estabilizador de un espinor no nulo del subespacio positivo o negativo bajo la acción del grupo Spin(8) es isomorfo a Spin(7); las imágenes de dichos grupos a través de la aplicación recubridora $\mathrm{Ad}\colon \mathrm{Spin}(8) \to \mathrm{SO}(8)$ no son conjugadas en SO(8), pero sí lo son en O(8).

Sea (M, g) una variedad spin de dimensión 8; la igualdad $\Delta_8 = \Delta_+ \oplus \Delta_-$ induce una descomposición del fibrado espinorial $\Sigma(M) = \Sigma^+(M) \oplus \Sigma^-(M)$. Tal y como se enuncia en la Proposición 1.8, un espinor de norma unidad η en $\Sigma^+(M)$ da lugar a una estructura Spin(7) a través de la expresión:

$$\Omega(W, X, Y, Z) = \frac{1}{2}((-WXYZ + WZYX)\eta, \eta).$$

Además, la Proposición 1.13 demuestra que la derivada covariante de η y la torsión intrínseca de la estructura Spin(7) contienen la misma información. La relación entre estos objetos nos permite demostrar el Teorema 1.21, que pasamos a enunciar:

Teorema A (Teorema 1.21)**.** *La estructura* Spin(7) *determinada por un espinor η es,*

1. *Paralela si $\nabla \eta = 0$.*
2. *Balanced si $\not{D}\eta = 0$.*
3. *Localmente conformemente paralela si existe un campo vectorial $V \in \mathfrak{X}(M)$ tal que $\nabla_X \eta = \frac{2}{7}(X^* \wedge V^*)\eta$. En tal caso, $\not{D}\eta = V\eta$.*

El operador de Dirac juega un papel central en la clasificación dado que determina la *forma de Lee*, definida por la igualdad $\theta = -\frac{1}{7} \star (\star(d\Omega) \wedge \Omega)$. En términos del Teorema 1.21, $\theta = \frac{8}{7} V^*$ (véase la Proposición 1.23). Si bien la existencia de una estructura Spin(7) balanced en una variedad es una condición geométrica, ésta proporciona una solución a la ecuación de Dirac que es interesante desde el punto de vista analítico.

Nuestro planteamiento es diferente al propuesto por [1] para el estudio de las estructuras SU(3) y G_2. Sea ϕ el espinor que determina la G estructura, entonces $\nabla_X \phi = \frac{1}{2}\Gamma(X)\phi$; donde Γ ddenota la torsión intrínseca de la G estructura, y $\mathrm{G} \in \{\mathrm{SU}(3), \mathrm{G}_2, \mathrm{Spin}(7)\}$. Sea (N, g, J, Θ) una estructura SU(3), existen $\gamma \in \Omega^1(N)$ y $\mathcal{S}_N \in \mathrm{End}(TN)$ tales que $\Gamma = i(\mathcal{S}_N)\mathfrak{Re}\,(\Theta) - \frac{2}{3}\gamma \otimes \omega$, donde $(i(\mathcal{S}_N)\mathfrak{Re}\,(\Theta))(X, Y, Z) = \mathfrak{Re}\,(\Theta)(\mathcal{S}_N(X), Y, Z)$. Sea (Q, g, φ) una estructura G_2, existe $S_Q \in \mathrm{End}(TQ)$ tal que $\Gamma = -\frac{2}{3} i(\mathcal{S}_Q)\varphi$. Estas igualdades son ciertas dado que $\mathfrak{su}(3)^\perp = \langle \omega \rangle \oplus i(\mathbb{R}^6)\mathfrak{Re}\,(\Theta)$, y$\mathfrak{g}_2^\perp = i(\mathbb{R}^7)\varphi$. Sean ϕ_N y ϕ_Q los espinores que determinan la estructura geométrica en N y Q. De acuerdo con [1, Lemas 2.2 y 2.3] se cumple:

$$\nabla_X \phi_N = \frac{1}{2}\Gamma(X)\phi_N = \mathcal{S}_N(X)\phi_N + \gamma(X)\mathrm{j}(\phi_N),$$
$$\nabla_X \phi = \frac{1}{2}\Gamma(X)\phi_Q = \mathcal{S}_Q(X)\phi.$$

donde j es una estructura compleja en $\Sigma(N)$ que anticonmuta con el producto de Clifford por campos vectoriales de N(véase la subsección 2.2.2). Dado que $\mathfrak{spin}(7)^{\perp}$ no contiene a $\mathbb{R}^8$ como subrepresentación, (véase la subsección 1.2.3 para una descripción explícita), no podemos seguir la misma estrategia en nuestro caso. De hecho, $\nabla_X \eta \in \Sigma^+(M)$ y $\mathcal{S}(X)\eta \in \Sigma^-(M)$. Por este motivo, trabajamos directamente con la ecuación $\nabla_X \eta = \frac{1}{2}\Gamma(X)\eta$.

En este capítulo introducimos la noción de *distribuciones* G_2: una distribución de dimensión 7 coorientada con una estructura G_2 en una variedad Spin(7). Este formalismo unifica distintas situaciones geométricas que involucran estructuras G_2 y Spin(7), tales como hipersuperficies G_2 de variedades Spin(7), productos warped de variedades G_2 con $\mathbb{R}$ y fibrados S^1-principales con base una variedad G_2; algunos de éstos habían sido estudiados por Martín-Cabrera en [83, 84]. Por ejemplo, una hipersuperficie Q de (M, g, Ω) tiene una estructura G_2 inducida por $\varphi = i(N)\Omega$, donde N es un campo vectorial normal de norma unidad. Tal como establece el Teorema 1.39, la clase de φ como estructura G_2 depende tanto del tipo de Ω como estructura Spin(7), como de las propiedades Riemannianas del embebimiento de Q en M. La idea clave de esta parte, que también será explotada en el Capítulo 2, es la siguiente: el espinor que determina la estructura Spin(7) de la variedad ambiente también induce una estructura G_2 en la distribución. Esto es, un único objeto codifica toda la información geométrica.

El formalismo de las distribuciones G_2 nos permite estudiar las estructuras Spin(7) invariantes por la izquierda en grupos de Lie cuasi-abelianos. El estudio de las estructuras G_2 en álgebras de Lie cuasi abelianas de dimensión 7 ha sido fructífero; esto nos motiva a llevar a cabo un estudio análogo en el caso de Spin(7). En [51] el autor determina qué álgebras de Lie admiten una estructura G_2 cocerrada. Además en [52] el autor construye variedades de cohomogeneidad 1 con holonomía SU(4) resolviendo las ecuaciones de Hitchin partiendo de las estructuras G_2 obtenidas en [51].

Estos grupos de Lie resolubles son productos semidirectos $\mathbb{R} \ltimes_{\epsilon} \mathbb{R}^7$ donde $\varepsilon = \exp(\mathrm{ad}(\mathcal{E}))$ con $\mathcal{E} \in \mathbb{R}(7)$. Una estructura Spin(7) invariante por la izquierda en $\mathbb{R} \ltimes_{\epsilon} \mathbb{R}^7$ determina una estructura G_2 paralela en cada hipersuperficie $\{t\} \times \mathbb{R}^7$. La clase de la estructura Spin(7) depende únicamente del endomorfismo $\mathcal{E}$, como probamos en el Teorema 1.49. Las clases puras se obtienen imponiendo condiciones a los autovalores complejos de la parte antisimétrica de $\mathcal{E}$. Además, la traza de $\mathcal{E}$ determina la componente de la forma de Lee que es paralela a dt. Este estudio nos permite obtener ejemplos compactos cuando encontramos retículos. Dado que no existen solvariedades con una estructura LCP invariante por la izquierda, buscaremos ejemplos de tipo balanced. De hecho, en la sección 1.8 proporcionamos el primer ejemplo de una estructura Spin(7) balanceada con $b_1 = 2$ que no es un producto $S^1 \times N^7$.

Nuestros resultados nos permiten abordar problemas de clasificación de estructuras Spin(7) en álgebras de Lie nilpotentes cuasi-abelianas, que son 14 salvo isomorfismo. Determinamos aquellas que admiten una estructura Spin(7) balanced o una estructura *estrictamente localmente conformemente balanced*. Las últimas se introdujeron en el contexto de la teoría de supergravedad y verifican que la forma de Lee es cerrada y no nula. Nuestro análisis concluye el siguiente resultado:

Teorema B (Teorema 1.4)**.** *Sea L_3 el álgebra de Lie del grupo de Heisenberg de dimensión* 3*, sea L_4 el único álgebra de Lie nilpotente e indescomponible de dimensión* 4*, y sea A_j el álgebra abeliana j-dimensional.*

1. *Toda estructura* Spin(7) *invariante en A_8 es paralela.*

2. Las álgebras de Lie $\mathfrak{g} = A_5 \oplus L_3$ y $\mathfrak{g} = A_3 \oplus L_4$ admiten una estructura estrictamente localmente conformemente balanced pero no admiten estructuras balanced.

3. El resto de álgebras de Lie cuasi abelianas nilpotentes admiten tanto una estructura balanced como una estructura estrictamente localmente conformemente balanced.

Estructuras spin-harmonic y nilvariedades

El objetivo del Capítulo 2 es contruir estructuras Spin(7) de tipo balanced en nilvariedades de dimensión 8. Emplemos las ecuaciones espinoriales obtenidas en el Capítulo 1. Nuestro enfoque nos lleva a estudiar una nueva clase de estructuras geométricas en variedades de dimensión baja: las estructuras *spin-harmonic*.

El primer ejemplo compacto de una estructura balanced [46] es el producto de una nilvariedad de dimensión 5 con un 3-toro. Posteriormente se obtienen más ejemplos compactos gracias a los trabajos [83, 84], éstos incluyen productos $N \times S^1$ donde (N, g, φ) tiene una estructura G_2 cerrada o *puramente cocalibrada*, esto es, que verifica $\tau_i = 0$ si $i \neq 3$. En este capítulo trabajamos con nilvariedades Riemannianas de la forma $(N^6 \times T^2, g_6 + g_2)$, donde (N^6, g_6) es una nilvariedad de dimensión 6 y (T^2, g_2) es un toro plano; asumimos también que la estructura Spin(7) es invariante en la dirección de T^2. La razón de nuestra simplificación radica en que las álgebras nilpotentes de dimensión 8 no están clasificadas y la lista en dimensión 7 es muy extensa. Analizamos de manera separada el caso en que $N^6 = N^5 \times S^1$ y $g_6 = g_5 + g_1$, donde g_1 es la métrica plana de S^1. Nuestro estudio nos permite recuperar la estructura Spin(7) de [46].

La estructura Spin(7) de $N^6 \times T^2$ induce una estructura SU(3) en N^6 o una estructura SU(2) en N^5 cuando $N^6 = N^5 \times S^1$. Tal como se obtiene en [35], las formas $(\alpha, \omega_1, \omega_2, \omega_3) \in \Omega^1(N^5) \times \Omega^2(N^5)^3$ determinan una estructura SU(2) si:

1. $\omega_i \wedge \omega_j = 0$ for $i \neq j$, $\omega_1^2 = \omega_2^2 = \omega_3^2$ y $\alpha \wedge \omega_1^2 \neq 0$,

2. Si $i(X)\omega_1 = i(Y)\omega_2$, entonces $\omega_3(X, Y) \geq 0$.

La condición balanced induce condiciones sobre las estructuras SU(3) o SU(2); éstas no son clases recogidas en la clasifiicación de las estructuras SU(3) o SU(2) tal como se prueba en [1, Teorema 3.7] y el Corolario 2.39. Dado que las ecuaciones en términos de las formas que definen las estructuras son complicadas, empleamos el enfoque espinorial desarrollado en el Capítulo 1 y que consiste en encontrar espinores armónicos en $N^k \times T^{8-k}$ con $k \in \{5, 6\}$. Dividimos nuestra búsqueda en tres pasos: reducción dimensional, elección de una estructura spin y obtención de una fórmula para el operador de Dirac en términos de las ecuaciones de estructura.

La reducción dimensional consiste en relacionar el espinor armónico de $N^k \times T^{8-k}$ que determina la estructura Spin(7) con un espinor en N^k. El fibrado espinoral de N^k resulta ser el pullback mediante la inclusión al producto $N^k \times T^{8-k}$ del fibrado $\Sigma^+(N^k \times T^{8-k})$; esto se deduce de las igualdades $\mathrm{Cl}_5 = \mathbb{C}(4)$ y $\mathrm{Cl}_6 = \mathbb{R}(8)$. Como consecuencia de nuestras hipótesis, existe un único modo de definir un espinor $\eta' \in \Sigma(N^k)$ partiendo de un espinor $\eta \in \Sigma^+(N^k \times T^{8-k})$; el espinor η es armónico si y solo si η' lo es. Motivados por esta reducción dimensional, definimos una estructura *spin-harmonic* como la estructura geométrica determinada por un espinor armónico de norma unidad. Las ecuaciones en términos de las formas que definen la estructura se recogen en [1] en los casos de G_2 y SU(3); y en la sección 2.4 del Capítulo 2 en el caso de SU(2).

Después nos centramos en los espinores que determinan estructuras geométricas invariantes por la izquierda: dotamos a la variedad de su fibrado espinorial trivial y elegimos espinores constantes. Esto es, las propiedades geométricas están determinadas por el álgebra de Lie y no dependen del retículo. Finalmente obtenemos una fórmula para el operador de Dirac de un espinor de este tipo en términos de las constantes de estructura del álgebra de Lie:

Proposición C (Proposición 2.41)**.** *Supongamos que* $(e_1, \dots, e_n)$ *es una base ortonormal y sea* ϕ *un espinor invariante por la izquierda en un álgebra de Lie resoluble. Entonces,*

$$4\not{D}\phi = -\sum_{i=1}^{n}(e^i \wedge de^i + i(e_i)de^i)\phi.$$

Resolveremos la ecuación $\not{D}\phi = 0$ de manera directa cuando tratamos con nilvariedades de dimensión 6. La estrategia en dimensión 5 es diferente y consiste en calcular el cuadrado del operador de Dirac $\not{D}^2$. Este planteamiento en dimensión 5 nos permite determinar todas las métricas invariantes por la izquierda que admiten espinores armónicos de este tipo. De acuerdo con la Proposición 2.50 tenemos:

$$\not{D}^2\phi = \mu\phi + v\mathrm{j}_1\phi.$$

La contante $\mu > 0$ y el campo invariante $v \in \mathfrak{X}(N^5)$ están determinados por la métrica y las ecuaciones de estructura del álgebra de Lie. De esta fórmula deducimos que las métricas que admiten estructuras spin-harmonic están caracterizadas por la condición $\|v\| = \mu$. Además, el espacio de espinores armónicos invariantes tiene dimensión 4. Cuando existe una estructura spin-harmonic, el vector v tiene una interpretación geométrica: la 1-forma obtenida a través del endomorfismo musical v^* es proporcional a α. El siguiente teorema resume nuestros resultados:

Teorema D (Teoremas 2.53, 2.58, Subsección 2.6.3 y Proposición 2.59)**.** *Sea* N^k *una nilvariedad de dimensión* k *y sea* $\mathfrak{n}$ *el álgebra de Lie de su recubridor universal. Supongamos además que* $\mathfrak{n}$ *no es abeliana.*

1. *Si* $k = 5$ *y* N^5 *admite estructuras spin-harmonic invariantes por la izquierda entonces* $\mathfrak{n} = \mathrm{L}_{5,j}$, $j = 1, 2, 3, 4, 6$.

2. *Si* $k = 6$ *y* N^6 *no admite estructuras spin-harmonic invariantes por la izquierda entonces* $\mathfrak{n}$ *es* $\mathrm{L}_3 \oplus \mathrm{A}_3$ *o* $\mathrm{L}_4 \oplus \mathrm{A}_2$.

3. *Las álgebras de Lie* $L_3 \oplus A_5$ *y* $L_4 \oplus A_4$ *no admiten estructuras* Spin(7) *balanced.*

Los resultados del Capítulo 2 sugieren que hay *muchas* estructuras Spin(7) balanced. Este fenómeno está relacionado con un resultado de Hitchin en [67] que establece que toda variedad spin de dimensión 8 admite un espinor armónico. Sin embargo, este espinor no determinaría una estructura Spin(7) balanced si se anulara en algún punto. Además, la ecuación $\not{D}\eta = 0$ está sobredeterminada; ambos hechos nos llevan a pensar que podría investigarse la existencia de un h-principio en el sentido de Gromov para este tipo de estructuras.

Orbifolds con estructuras geométricas y sus resoluciones

Los orbifolds fueron introducidos por Satake en [106] y se han mostrado útiles en numerosos contextos geométricos. Los orbifolds están modelados localmente en $\mathbb{R}^n/\Gamma$ donde Γ es un subgrupo finito de $\mathrm{O}(n)$. Por tanto, tienen *singularidades* que en el modelo local son los puntos fijos de alguna isometría de Γ distinta de la identidad. Muchos de los objetos empleados en geometría Riemanniana también son útiles en el contexto de los orbifolds: métricas, formas, fibrados y operadores.

En esta tesis construimos resoluciones de orbifolds con una estructura simpléctica o de tipo G_2 cerrada para así obtener variedades con tales estructuras geométricas y diferentes propiedades topológicas. Los orbifolds de los que partimos son normalmente el cociente global de una variedad por un grupo finito de difeomorfismos que preservan la estructura geométrica. Algunas propiedades topológicas de la resolución, tal como el grupo fundamental o los grupos de cohomología, pueden ser deducidos de las propiedades del orbifold y del lugar singular; vease por ejemplo la proposición 4.38.

Este procedimiento permitió a Joyce construir variedades compactas con holonomía G_2 y Spin(7). Sus resultados combinan técnicas de resolución de orbifolds y resultados de existencia analíticos. Estos orbifolds son cocientes de un 7 u 8 toro plano bajo la acción de un grupo de isometrías que preserva la estructura. El orbifold, bajo ciertas hipótesis, puede ser resuelto y dotado de una familia 1-paramétrica de estructuras geométricas cuya torsión tiende a 0; este procedimiento requiere técnicas de geometría algebraica. Los Teoremas 11.6.1 y 13.6.1. de [74] garantizan la existencia de una estructura sin torsión. En ambos casos, la acción del grupo se construye de modo que el grupo fundamental del orbifold sea finito; en el caso de Spin(7) también requiere que su $\hat{A}$-género sea 1. Estas propiedades topológicas garantizan que el grupo de holonomía de las variedades construidas sea precisamente G_2 o Spin(7).

Resolución de orbifolds simplécticos de dimensión 4.

En el Capítulo 3 demostramos que los orbifolds simplécticos de dimensión 4 pueden resolverse; empleamos técnicas que provienen del área de la geometría algebraica, en la línea de los artículos [11], [50] y [93].

Desde el punto de vista de la geometría diferencial, los teoremas clásicos en geometría simpléctica se adaptan al contexto de los orbifolds; en [93] encontramos una exposición clara y precisa de estos resultados. Un ejemplo es la existencia de cartas de Darboux, que son de la forma (U, ω_0) donde $U \subset \mathbb{C}^m/\Gamma$; el grupo de isotropía Γ es un subgrupo de $\mathrm{U}(m)$ y ω_0 es la forma simpléctica estándar de $\mathbb{C}^n$. Otros ejemplos incluyen la construcción de una estructura casi compleja en el fibrado normal de una singularidad. El primer contrajemplo [50] de la *conjetura de Thurston-Weinstein* en dimensión 8 es un logro notable de las técnicas de resolución de orbifolds simplécticos. Esta conjetura establecía que una variedad simpléctica simplemente conexa de dimensión mayor o igual que 8 es necesariamente formal. Su falsedad en dimensión ≥ 10 fue demostrada en [5]. Otro ejemplo destacado es la construcción de una variedad de dimensión 6 que no es Kähler pero es a su vez compleja y simpléctica [11].

El procedimento empleado en [11, 50] es ad-hoc y aprovecha técnicas procedentes de la resolución algebraica de singularidades. Estas técnicas ya habían sido utilizadas antes para la desingularizar orbifolds simplécticos cuyas singularidades son *puntos aislados* [28]. Procedemos a discutir brevemente su estrategia; en esta situación el único punto fijo de cada elemento distinto de la identidad es 0. Por tanto, podemos reemplazar un entorno del 0 en el orbifold por un entorno del divisor excepcional en la resolución proyectiva de la singularidad cociente $\mathbb{C}^m/\Gamma$, que existe apelando a los teoremas clásicos de Hironaka [65, 66]. La forma simpléctica se construye interpolando la forma Kähler de la resolución con ω_0 mediante el *proceso de inflación* introducido por Thurston en [108].

Aún no se ha probado que cada orbifold simpléctico admita una resolución simpléctica. Tal como se expone en la introducción del Capítulo 3, existen casos especiales en las que la desingularización si es posible. En el Capítulo 3 demostramos:

Teorema E (Teorema 3.26)**.** *Sea* (X, ω) *un orbifold simpléctico compacto de dimensión* 4*. Existe una variedad simpléctica* $(\tilde{X}, \tilde{\omega})$ *y una aplicación diferenciable* $\pi : (\tilde{X}, \tilde{\omega}) \to (X, \omega)$ *que es un simplectomorfismo excepto un entorno pequeño del lugar de isotropía de* X*.*

Este teorema fue probado previamente por Chen en [30] empleando técnicas propias de geometría simpléctica tales como rellenos simplécticos de variedades de contacto y reducciones simplécticas. Nuestro método es diferente y sigue las ideas de [28] y su generalización [93]. El artículo [93] trata el caso de los orbifolds con *isotropía homogénea*, que son aquellos en los que los lugares de isotropía no se intersecan unos con otros. La desingularización tiene lugar en el fibrado normal, que tiene una singularidad compleja en la fibra; para garantizar que la resolución en distintas fibras sean compatibles, los autores necesitan la resolución algebraica de [41] en lugar de los teoremas clásicos de Hironaka. La propiedad distintiva de la resolución construida en [41] es su equivarianza bajo la acción de grupos.

La ventaja de los orbifolds simplécticos de dimensión 4, en comparación con los de dimensión superior, reside en que la configuración de sus singularidades es más simple. Esto se sigue del hecho de que los elementos de U(2) distintos de la identidad fijan el origen o una linea compleja. Aparte de las singularidades aisladas, definimos los conjuntos de singularidades Σ^* y Σ^1 mediante una carta de Darboux (U, ω_0) con $U \subset \mathbb{C}^2/\Gamma$:

1. $x \in \Sigma^*$ si existe una linea compleja $L \subset \mathbb{C}^2$ tal que para todo elemento $1 \neq \gamma \in \Gamma$ se cumple $\mathrm{Fix}(\gamma) = L$.

2. $x \in \Sigma^1$ si existen al menos dos lineas complejas $L_1, L_2 \subset \mathbb{C}^2$ y $\gamma_1, \gamma_2 \in \Gamma$ tales que $L_1 = \mathrm{Fix}(\gamma_1)$ y $L_2 = \mathrm{Fix}(\gamma_2)$.

Las componentes conexas de Σ^* son superficies y las de Σ^1 son los puntos de intersección de los cierres de las componentes conexas de Σ^*. Lo desafiante de la resolución es compatibilizar las resoluciones de diferentes superficies singulares de Σ^* cuyos cierres se intersecan en puntos de Σ^1. Los puntos de Σ^* tienen entornos contenidos en $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_m)$, que es topológicamente una variedad. Hay distintos modos resolver el modelo local, pero elegimos dotar al cociente de estructura de variedad compleja y cambiar la forma simpléctica mediante una perturbación. Para pasar del modelo local al caso general, construimos el fibrado normal de la singularidad e introducimos una conexion. Además, los modelos locales entorno a puntos $x \in \Sigma^1$ se pueden cambiar por otro modelo local en el que x es una singularidad aislada. Para probarlo, en primer lugar argumentamos que $\mathbb{C}^2/\Gamma = (\mathbb{C}^2/\Gamma')/(\Gamma/\Gamma')$, donde Γ' es el subgrupo normal de Γ formado por los elementos que fijan alguna linea compleja. Posteriormente, un resultados clásico de teoría invariante de grupos nos permite afirmar que $\mathbb{C}^2/\Gamma'$ es una variedad compleja. Finalmente observamos que Γ/Γ' actúa libremente en $(\mathbb{C}^2 - \{0\})/\Gamma'$.

Esta discusión nos lleva a diseñar una estrategia en cuatro pasos para resolver los orbifolds simplécticos de dimensión 4 sin singularidades aisladas. En primer lugar, definimos un atlas de variedad en $X - \Sigma^1$ y una 2-forma cerrada ω' que es 0 en un entorno perforado de Σ^1 y es simpléctica fuera del mismo. El teorema de extensión de Riemann nos permite extender este atlas a X de modo que las singularidades del nuevo atlas son aisladas. Más tarde construimos una forma simpléctica partiendo de ω'. Finalmente, resolvemos las singularidades aisladas empleando el método descrito en [28].

Una variedad compacta no formal con $b_1 = 1$ dotada de una estructura G_2 cerrada

En el Capítulo 4 construimos una variedad compacta no formal con $b_1 = 1$ dotada de una estructura G_2 cerrada y probamos que esta no admite ninguna estructura G_2 paralela. Este es el primer ejemplo conocido de tales características. La construcción sigue algunas ideas del artículo [47] y requiere el desarrollo de técnicas de resolución de orbifolds G_2 cerrados, inspirados en el artículo [75].

El problema de determinar las propiedades topológicas de las variedades compactas con una estructura G_2 cerrada que no admiten ninguna estructura G_2 paralela está lejos de ser entendido. Tal como se expone en la introducción al Capítulo 4, antes de este trabajo los ejemplos conocidos con $b_1 = 1$ eran formales [47], [81]. Sin embargo, no había razones para descartar la existencia de un ejemplo no formal con $b_1 = 1$. De hecho, los ejemplos en [34] son nilvariedades y por tanto no formales con $b_1 \geq 2$. Merece la pena mencionar que aún no se ha construido ningún ejemplo con $b_1 = 0$. Tal como anunciabamos antes, el teorema principal del Capítulo 4 es el siguiente:

Teorema F (Proposiciones 4.44, 4.46)**.** *Existe una variedad compacta no formal M con $b_1 = 1$ dotada de una estructura G_2 cerrada que admite ninguna estructura G_2 paralela.*

Nuestra construcción, al igual que la realizada en el artículo [47] emplea técnicas de resolución de orbifolds. Definimos un orbifold X con una estructura G_2 cerrada a través del cociente de una nilvariedad N bajo la acción del grupo $\mathbb{Z}_2$. Esta acción preserva la estructura G_2 de N, que es la obtenida en [34]. La resolución M de X es no formal; de hecho X tampoco lo es, dado que la acción de $\mathbb{Z}_2$ preserva un producto de Massey no nulo en N. El producto de Massey no nulo de X levanta a M por pullback. Además, para garantizar que $b_1(M) = 1$ construimos la acción de manera que $b_1(X) = 1$ dado que el primer número de Betti no cambia tras el proceso de resolución (véase la Proposición 4.38). El lugar singular del orbifold se compone de 16 copias disjuntas de la variedad de Heisenberg de dimensión 3; hasta donde sabemos, esta es la primera vez que tal configuración ocurre.

Para desingularizar nuestro orbifold, desarrollamos un método de resolución de orbifolds con una estructura G_2 cerrada. Éste se inspira en el trabajo de Joyce y Karigiannis en [75], en el que resuelven orbifolds X definidos como el cociente de una variedad N con holonomía contenida en G_2 bajo la acción de el grupo $\mathbb{Z}_2$; la holonomía de la resolución también está contenida en G_2. Hasta la fecha, éste y el trabajo fundacional de Joyce [71, 72], son los únicos que abordan la resolución de orbifolds con estructura G_2 paralela. El torema de resolución de [75] funciona en el caso en que el lugar singular L de la acción, que tiene dimensión 3, tenga una 1-forma armónica nunca nula. La estrategia que siguen es parecida a la empleada por Joyce en [71, 72] y está descrita en la introducción al Capítulo 4; centrémonos en algunos detalles.

El fibrado normal a L en N tiene una estructura compleja determinada por una 1-forma nunca nula; el fibrado normal a L en X tiene por tanto fibra $\mathbb{C}^2/\mathbb{Z}_2$, cuya resolución algebraica es el *espacio de Eguchi-Hanson* (véase la subsección 4.2.2). Se asume que la 1-forma sea cerrada para garantizar que las formas G_2 definidas sean cerradas, y que sea cocerrada para asegurar que la torsión de éstas sea pequeña. El teorema que probamos en el Capítulo 4 es el siguiente:

Teorema G (Teorema 4.32)**.** *Sea (M, φ, g) una estructura G_2 cerrada en una variedad compacta. Supongamos que $\mathrm{j}\colon M \to M$ es una involución tal que $\mathrm{j}^*\varphi = \varphi$ y consideremos el* *orbifold* $X = M/\mathrm{j}$. *Sea* $L = \mathrm{Fix}(\mathrm{j})$ *el lugar singular de* X *y supongamos que existe una* 1-*forma cerrada nunca nula* $\theta \in \Omega^1(L)$. *Entonces existe una variedad compacta dotada de una estructura* G_2 *cerrada* $(\widetilde{X}, \widetilde{\varphi}, \widetilde{g})$ *y una aplicación* $\rho\colon \widetilde{X} \to X$ *tales que:*

1. *La aplicación* $\rho\colon \widetilde{X} - \rho^{-1}(L) \to X - L$ *es un difeomorfismo.*

2. *Existe un entorno* U *de* L *tal que* $\rho^*(\varphi) = \widetilde{\varphi}$ *en* $\widetilde{X} - \rho^{-1}(U)$.

Dado que en nuestro trabajo no estimamos la torsión, las hipótesis de nuestro teorema son mas laxas que las de [75]. En nuestro caso, la 1-forma nunca nula del lugar singular ha de ser cerrada en lugar de armónica; esta condición significa que cada componente conexa del lugar singular es un mapping torus sobre una superficie. Asimismo, aunque empleamos la misma estrategia para probar la existencia de la resolución, algunas partes técnicas se simplifican o evitan.

Finalmente, los dos argumentos que proporcionamos para probar que la variedad M construida en este capítulo no admite ninguna métrica con holonomía contenida en G_2 se basan en la formalidad. La variedad M no verifica la *obstrucción de casi formalidad* obtenida en [29], que describimos brevemente. El álgebra de de Rham de una variedad con holonomía contenida en G_2 es cuasi-isomorfo a un ADCG con todas las diferenciales 0 salvo en grado 3. El álgebra se construye mediante el operador diferencial $\mathcal{L}_\varphi$ introducido anteriormente. Esto implica que los productos de Massey son nulos salvo quizá aquellos $\langle[\alpha], [\beta], [\gamma]\rangle$ tales que $|\alpha| + |\beta| = 4$ y $|\beta| + |\gamma| = 4$, donde $|\alpha|$ denota el grado de α. La variedad M no es casi formal porque tiene un producto de Massey no nulo $\langle[\alpha], [\beta], [\gamma]\rangle$ tal que $|\alpha| = |\gamma| = 1$ y $|\beta| = 2$.

References

[1] I. Agricola, S. G. Chiossi, T. Friedrich, and J. Höll, *Spinorial description of* SU(3)*- and* G_2*-manifolds*, J. Geom. Phys. **98** (2015), 535–555.

[2] I. Agricola and T. Friedrich, *Geometric structures of vectorial type*, J. Geom. Phys. **56** (2006), 2403–2414.

[3] B. Amann and C. Bär, *The Dirac operator on nilmanifolds and collapsing circle bundles*, Ann. Global Anal. Geom. **16** (1998), no. 3, 221–253.

[4] M. Amann and V. Kapovitch, *On fibrations with formal elliptic fibers*, Adv. Math. **231** (2012), 2048–2068.

[5] I. K. Babenko and I. A. Taimanov, *On non-formal simply connected symplectic manifolds*, Sib. Math. J. **41** (2000), no. 2, 204–217.

[6] L. Bagaglini, *Invariant cocalibrated* G_2*-structures on nilmanifolds*, (2015), arXiv:1509.01971.

[7] C. Bär, *Real Killing spinors and holonomy*, Comm. Math. Phys. **154** (1993), no. 3, 509–521.

[8] ______, *On nodal sets for Dirac and Laplace operators*, Comm. Math. Phys. **188** (1997), no. 3, 709–721.

[9] H. Baum, T. Friedrich, R. Grunewald, and I. Kath, *Twistors and Killing spinors on Riemannian manifolds*, Teubner-Texte zur Mathematik, vol. 124, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1991.

[10] G. Bazzoni, M. Fernández, and V. Muñoz, *Non-formal co-symplectic manifolds*, Trans. Amer. Math. Soc. **367** (2015), 4459–4481.

[11] ______, *A* 6*-dimensional simply connected complex and symplectic manifold with no Kähler metric*, J. Symplectic Geom. **16** (2018), no. 4, 1001–1020.

[12] G. Bazzoni, L. Martín-Merchán, and V. Muñoz, *Spin-harmonic structures and nilmanifolds*, arXiv:1904.01462[math.DG] (2019).

[13] G. Bazzoni and V. Muñoz, *Classification of minimal algebras over any field up to dimension 6*, Trans. Amer. Math. Soc. **364** (2012), no. 2, 1007–1028.

[14] G. Bazzoni and V. Muñoz, *Manifolds which are complex and symplectic but not Kähler*, Essays in Mathematics and its applications: In Honor of Vladimir Arnold, Springer, 2016, pp. 49–69.

[15] L. Bedulli and L. Vezzoni, *The Ricci tensor of SU(3)-manifolds*, J. Geom. Phys. **57** (2007), no. 4, 1125–1146.

[16] ———, *Torsion of* SU(2)*-structures and Ricci curvature in dimension 5*, Differ. Geom. Appl. **27** (2009), no. 1, 85–99.

[17] M. Berger, *Sur les groupes d'holonomie homogénes de variétés à conexion affine et des variétés Riemanniennese*, Bull. Soc. Math. France **83** (1955), 279–330.

[18] S. Bochner, *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. **52** (1946), no. 9, 776–797.

[19] R. Bott and W. Tu, *Differential forms in algebraic topology*, Springer-Verlag, New York, 1982.

[20] C. Boyer and K. Galicki, *Sasakian geometry*, Oxford University Press, 2007.

[21] G. Bredon, *Introduction to compact transformation groups*, Academic Press, New York and London, 1972.

[22] R. Bryant, *Metrics with exceptional holonomy*, Ann. Math. **126** (1987), 525–576.

[23] ———, *Some remarks on* G_2*-structures*, Proc. Gökova Geom.-Topol. Conf. 2005, Gökova Geometry/Topology Conference (GGT), Gökova, 2006, pp. 75–109.

[24] R. Bryant and S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989), no. 3, 829–850.

[25] A. Cannas da Silva, *Lectures on symplectic geometry*, Springer, 2001.

[26] E. Cartan, *Sur une classe remarquable d'espaces de Riemann I*, Bull. Soc. Math. **54** (1926), 214–264.

[27] ———, *Sur une classe remarquable d'espaces de Riemann II*, Bull. Soc. Math. **55** (1926), 114–134.

[28] G. Cavalcanti, M. Fernández, and V. Muñoz, *Symplectic resolutions, Lefschetz property and formality*, Adv. Math. **218** (2008), no. 2, 576–599.

[29] K. Chan, S. Karigiannis, and C. Tsang, *The* $\mathcal{L}_B$*-cohomology on compact torsion-free* G_2 *manifolds and an application to 'almost' formality*, Ann. Global Anal. Geom. **55** (2018), 325–369.

[30] W. Chen, *Resolving symplectic orbifolds with applications to finite group actions*, J. Gökova Geom.-Topol. (2018), 1–39.

[31] C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. (1955), 778–782.

[32] S. Chiossi and S. Salamon, *The intrinsic torsion of* SU(3) *and* G_2 *structures*, Differential geometry, Valencia, 2001, World Sci. Publ., River Edge, NJ, 2002, pp. 115–133.

[33] R. Cleyton and S. Ivanov, *On the geometry of closed* G_2*-structures*, Commun. Math. Phys. **270** (2007), 56–63.

[34] D. Conti and M. Fernández, *Nilmanifolds with a calibrated* G_2*-structure*, Differ. Geom. Appl. **29** (2011), 493–506.

[35] D. Conti and S. Salamon, *Generalized Killing spinors in dimension 5*, Trans. Amer. Math. Soc. **359** (2007), no. 11, 5319–5343.

[36] A. Corti, M. Haskins, J. Nordström, and Pacini T., G_2*-manifolds and associative submanifolds via semi-Fano 3-folds*, Duke Math. J. **164** (2015), no. 10, 1971–2092.

[37] D. Crowley and J. Nordström, *New invariants of* G_2*-structures*, Geom. Topol. **19** (2015), no. 5, 2949–2992.

[38] ———, *The rational homotopy type of* $(n-1)$*-connected manifolds of dimension up to* $5n-3$, J. Topol. **13** (2020), no. 2, 539–575.

[39] G. de Rham, *Sur la reductibilié d'un espace de Riemann.*, Comment. Math. Helv. **26** (1952), 328–334.

[40] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifold*, Invent. Math **29** (1975), 245–274.

[41] S. Encinas and O. Villamayor, *Real homotopy theory of Kähler manifold*, Progr. in Math. **181** (2000), 147–227.

[42] Y. Félix, J. Oprea, and D. Tanré, *Algebraic models in geometry*, Oxford University Press, Oxford, 2008.

[43] M. Fernández, *A classification of Riemannian manifolds with structure group* Spin(7), Ann. Mat. Pura Appl. (4) **143** (1986), 101–122.

[44] ———, *An example of a compact calibrated manifold associated with the exceptional lie group* G_2, J. Differ. Geom. **26** (1987), 367–370.

[45] ———, *A family of compact solvable* G_2*-calibrated manifolds*, Tôhoku Math. J. **39** (1987), 287–289.

[46] ———, *A new example of compact Riemannian manifold with structure group* Spin(7), Port. Math. **44** (1987), no. 2, 161–165.

[47] M. Fernández, A. Fino, A. Kovalev, and V. Muñoz, *A compact* G_2*-calibrated manifold with first Betti number* $b_1 = 1$, Adv. Math. **381** (2021), 107623.

[48] M. Fernández and A. Gray, *Riemannian manifolds with structure group* G_2, Ann. di Mat. Pura ed Appl. **32** (1982), 19–45.

[49] M. Fernández and V. Muñoz, *Formality of Donaldson submanifolds*, Math. Z. **250** (2005), 149–175.

[50] ———, *An 8-dimensional non-formal simply connected symplectic manifold*, Ann. Math. **167** (2008), no. 2, 1045–1054.

[51] M. Freibert, *Cocalibrated structures on Lie algebras with a codimension one abelian ideal*, Ann. Glob. Anal. Geom. **42** (2012), no. 4, 537–563.

[52] ———, SU(4)*-holonomy via the left-invariant hypo and hitchin flow*, Ann. Mat. Pura Appl. **197** (2018), no. 4, 1051 – 1087.

[53] T. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkruemmung*, Math. Nachr. **97** (1980), 117–146.

[54] ______, *Dirac operators in Riemannian geometry*, Graduate Studies in Mathematics, vol. 25, American Mathematical Society, Providence, RI, 2000.

[55] ______, *On types of non-integrable geometries*, Rend. Circ. Mat. Palermo **71** (2003), 99–113.

[56] T. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2002), 303–336.

[57] L. Godinho, *Blowing-up symplectic orbifolds*, Ann. Glob. Anal. Geom. **20** (2001), 117–162.

[58] M. Gong, *Classification of nilpotent lie algebras of dimension* 7 *(over algebraically closed fields and* $\mathbb{R}$), Ph.D. thesis, University of Waterloo, Ontario, 1998.

[59] A. Gray and L. M. Hervella, *The sixteen classes of almost hermitian manifolds and their linear invariants*, Ann. di Mat. Pura ed Appl. **123** (1980), 35–58.

[60] M. Gromov, *Partial differential relations*, Springer-Verlag, Berlin Heidelberg, 1987.

[61] R. Harvey and H.B Lawson, *Calibrated geometries*, Acta Math. **148** (1982), no. 3, 47–157.

[62] K. Hasegawa, *Minimal models of nilmanifolds*, Proc. Amer. Math. Soc. **106** (1989), no. 1, 65–71.

[63] A. Hatcher, *Vector bundles and K-theory*, 2007.

[64] A. Hattori, *Spectral sequence in the de rham cohomology of fibre bundles*, J. Fac. Sci. Univ. Tokyo **8** (1960), 289–331.

[65] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero, I*, Ann. Math. **79** (1964), no. 2, 109–203.

[66] ______, *Resolution of singularities of an algebraic variety over a field of characteristic zero, II*, Ann. Math. **79** (1964), no. 2, 205–326.

[67] N. Hitchin, *Harmonic spinors*, Adv. Math. **14** (1974), 1–55.

[68] ______, *The geometry of three-forms in six dimensions*, J. Differ. Geom. **55** (2000), no. 3, 547–576.

[69] S. Ivanov, *Connections with torsion, parallel spinors and geometry of Spin(7) manifolds*, Math. Res. Lett. **11** (2004), no. 2-3, 171–186.

[70] S. Ivanov, M. Parton, and P. Piccinni, *Locally conformal parallel g2 and spin(7) manifolds*, Math. Res. Lett. **13** (2006), no. 2-3, 167–177.

[71] D. D. Joyce, *Compact riemannian 7-manifolds with holonomy* G_2*, I*, J. Differ. Geom. **43** (1996), 291–328.

[72] ______, *Compact riemannian 7-manifolds with holonomy* G_2*, II*, J. Differ. Geom. **43** (1996), 329–375.

[73] ______, *Compact riemannian 8-manifolds with holonomy* Spin(7), Invent. Math. **123** (1996), 507–552.

[74] ———, *Compact manifolds with special holonomy*, Oxford Science Publications, Oxford, 2008.

[75] D. D. Joyce and S. Karigiannis, *A new construction of compact torsion-free* G_2*-manifolds by gluing families of Eguchi-Hanson spaces*, J. Differ. Geom **117** (2021), 255–343.

[76] K. Kawai, H.V. Lê, and L. Schwachhöfer, *Frölicher-Nijenhuis cohomology on* G_2 *and* Spin(7)*-manifolds*, Internat. J. Math. **29** (2018), no. 12.

[77] A. Kovalev, *Twisted connected sums and special Riemannian holonomy*, J. Reine Angew. Math. **595** (2003), 125–160.

[78] A. Kovalev and N. Lee, $K3$ *surfaces with non-symplectic involution and compact irreducible* G_2*-manifolds*, Math. Proc. Camb. Phil. Soc **151** (2011), 193–218.

[79] H. Lawson Jr, H.B. and M.L. Michelsohn, *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989.

[80] A.I. Malcev, *On a class of homogeneous spaces*, Amer. Math. Soc. Transl. Ser. 1 **9** (1962), 276–307.

[81] V. Manero, *Compact solvmanifolds with calibrated and cocalibrated* G_2*-structures*, Manuscripta Math. **162** (2020), 315–339.

[82] F. Martín-Cabrera, *On Riemannian manifolds with* Spin(7)*-structure*, Publ. Math. Debrecen **46** (1995), no. 3-4, 271–283.

[83] ———, Spin(7) *structures in principal fibre bundles over Riemannian manifolds with* G_2*-structure*, Rend. Circ. Mat. Palermo **44** (1995), no. 3, 235–247.

[84] ———, *Orientable hypersurfaces of Riemannian manifolds with* Spin(7)*-structure*, F. Acta Math. Hungarica **76** (1997), no. 2, 249–272.

[85] L. Martín-Merchán, *A compact non-formal closed* G_2 *manifold with* $b_1 = 1$, To appear in Math. Nachr.

[86] ———, *Spinorial classification of* Spin(7) *structures*, Ann. Sc. Norm. Super. Pisa **21** (2020), 873–910.

[87] L. Martín-Merchán and J. Rojo, *Resolution of 4-dimensional symplectic orbifolds*, Ann. Glob. Anal. Geom. **59** (2021), 385–416.

[88] J. McCarthy and J. Wolfson, *Symplectic gluing along hypersurfaces and resolution of isolated orbifold singularities*, Invent. Math. **119** (1995), 129–154.

[89] D. McDuff, *Examples of symplectic simply connected manifolds with no Kähler structure*, J. Differ. Geom. **20** (1984), 267–277.

[90] D. McDuff and S. Salamon, *Introduction to symplectic topology*, Oxford Mathematical Monographs, 1998.

[91] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. **21** (1976), 293–329.

[92] A. Moroianu and U. Semmelmann, *Parallel spinors and holonomy groups*, J. Math. Phys. **41** (2000), no. 4, 2395–2402.

[93] V. Muñoz and J. Rojo, *Symplectic resolution of orbifolds with homogeneous isotropy*, Geom. Dedicata **204** (2020), no. 1, 339–363.

[94] V. Muñoz, J. Rojo, and A. Tralle, *Homology Smale-Barden manifolds with K-contact but not Sasakian structures*, Int. Math. Res. Not. **2020** (2020), no. 21, 7397–7432.

[95] V. Muñoz and A. Tralle, *Simply-connected K-contact and sasakian manifolds of dimension* 7, Math. Z. **281** (2015), no. 1, 457–470.

[96] K. Niederkrueger and F. Pasquotto, *Desingularization of orbifolds obtained from symplectic reduction at generic coadjoint orbits*, Int. Math. Res. Not. **23** (2009), 4463–4479.

[97] ______, *Resolution of symplectic cyclic orbifold singularities*, J. Symplect. Geom. **7** (2009), 337–355.

[98] K. Nomizu, *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*, Ann. of Math. **59** (1954), 531–538.

[99] J. Oprea, *Lifting homotopy actions in rational homotopy*, J. Pure Appl. Algebra **32** (1984), no. 2, 177–190.

[100] J. Oprea and A. Tralle, *Sympectic manifolds with no Kähler stuctures*, Springer-Verlag, Berlin Heidelberg, 1997.

[101] D. Prill, *Local classification of quotients of complex manifolds by discontinuous groups*, Duke Math. J. **34** (1967), no. 2, 375–386.

[102] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag Berlin Heidelberg New-York, 1972.

[103] Y. Rudyak and Tralle A., *On Thom spaces, Massey products and non-formal symplectic manifolds*, Int. Math. Res. Not.. **2000** (2000), no. 10, 495–513.

[104] D. A. Salamon and T. Walpuski, *Notes on the octonions*, Proc. Gökova Geom.-Topol. Conf. 2016, Gökova Geometry/Topology Conference (GGT), Gökova, 2017, pp. 1–85.

[105] S. Salamon, *Riemannian geometry and holonomy groups*, vol. 201, Longman Scientific & Technical, 1989.

[106] I. Satake, *On a generalization of the notion of manifold*, Proc. Natl. Acad. Sci. USA **42** (1956), no. 6, 359–363.

[107] D. Sullivan, *Infinitesimal computations in topology*, Publ. Math. IHÉS **47** (1977), no. 269, 269–331.

[108] W. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. **55** (1976), no. 2, 467–468.

[109] ______, *Three-dimensional geometry and topology vol. 1.*, Princeton University Press, 1997.

[110] D. Tischler, *On fibering certain foliated manifolds over* S^1, Topology **9** (1970), 153–154.

[111] M. Verbitsky, *Manifolds with parallel differential forms and Kähler identities for* G_2 *manifolds*, J.Geom. Phys. **61** (2011), 1001–1016.

[112] M.Y. Wang, *Parallel spinors and parallel forms*, Ann. Global Anal. Geom. **7** (1989), 59–68.