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On Differential Hopf Algebras and B_∞ Algebras

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Abstract. We establish a structure theorem analogous to the classical result of Milnor and Moore: any differential graded (not necessarily cocommutative) Hopf algebra H that is cofree as a coalgebra carries an underlying B_∞ algebra structure that restricts to the subspace of primitives, and conversely H may be recovered via a universal enveloping 2-associative differential algebra. This extends the work of Loday and Ronco (J. reine angew. Math. **592**: 123–155, 2006) where the ungraded non-differential case was treated, and only the multibrace part of the B_∞ structure was found. We show that the multibrace algebras of Loday and Ronco (J. reine angew. Math. **592**: 123–155, 2006) originate from twistings of quasi-trivial structures, complementing the work of Markl (J. Homotopy Relat. Struct. **10**, 637–667 (2015)) on the A_∞ structure underlying any algebra with a square-zero endomorphism. In this framework we can prove the multibrace and A_∞ algebras are compatible and provide the appropriate B_∞ algebra for the structure theorem.

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Contents

Introduction

We establish a structure theorem, analogous to the classical result of Milnor and Moore, for differential graded Hopf algebras and B_∞ algebras. This extends the work of Loday and the second author [12] where the ungraded non-differential case was treated: it was shown that any cofree Hopf algebra carries an underlying multibrace structure that restricts to the subspace of primitives, and conversely that any cofree Hopf algebra may be reconstructed from a multibrace algebra via a universal enveloping 2-associative algebra.

In particular, in the case of differential graded Hopf algebras, we establish an extension of the Loday–Ronco multibrace structure to a B_∞ structure.

B_∞ algebras first appeared in algebraic topology, implicitly, in work of Baues [1] and play a central role in the study of the algebraic structure carried by the Hochschild cochain complex and its singular or Tate analogue [3,4,9,13,16]. The explicit definition of a B_∞ structure, on a vector space or a chain complex A, was first formulated in [7, Section 5.2] as a differential d and a product μ on the bar construction (BA, Δ) making it a differential bialgebra. Unpacking this definition, a B_∞ algebra is equipped with families of multilinear structure maps $(m_n)_{n\geqslant 1}$ and $(m_{i,j})_{i,j\geqslant 1}$ of arities n and $i+j$ satisfying certain 'strong homotopy algebra' relations. This includes the notion of A_∞ algebra, where we just require a differential coalgebra (BA, Δ, d), that is, the structure maps m_n. It also includes the notion of multibrace algebra, where we just require a bialgebra (BA, Δ, μ), that is, the structure maps $m_{i,j}$. Multibrace algebras were termed non-differential $\mathbf{B}_\infty$ algebras in [12] and B_∞ algebras in [5].

Our second aim is provide further insight into a construction of Börjeson [2], who showed that any associative algebra with a square-zero endomorphism (not required to be a derivation) carries an underlying A_∞ structure. Markl in [14] explained the origin of this A_∞ structure as a twisting (arising from the multiplication) of the given trivial A_∞ algebra.

As we explain below, Börjeson's A_∞ structure is just the right adjoint to the tensor (co)algebra functor. Another crucial observation is that the Loday–Ronco construction fits into this framework: we discover that the underlying multibrace structure carried by any 2-associative algebra, as defined in [12, Section 3.3 and Proposition 3.4], originates from a twisting of a quasi-trivial multibrace algebra (a non-commutative quasi-shuffle algebra). See Example 2.2.5 and Theorem 3.1.2 below for details.

To keep the presentation self-contained and accessible we recall much material which should be known to the expert reader. To simplify notation and remain consistent with [2,12,14] we choose to work with the tensor coalgebra $T^c(A)$ instead of the bar construction $BA = T^c(A[1])$. Thus our definitions of A_∞ and B_∞ differ from the classical ones mentioned above. The translation (of the degrees and signs involved) is straightforward.

Outline of Results

Suppose $(V, \partial, \bullet)$ is a differential graded algebra. Then (Proposition 2.3.3) it has a quasi-trivial B_∞ structure d_∂, $\mu_\bullet$ with all structure maps zero except $m_1 = \partial$ and $m_{1,1} = \bullet$. More generally, let $d\mathcal{A}s^{1,1}$ be the category of differential graded algebras $(V, \partial, \bullet)$ with an a priori unrelated extra associative binary operation $\odot$ on V. Now (Theorem 3.1.2) we have an underlying B_∞ algebra functor

$$(\star) \qquad \begin{gathered} d\mathcal{A}s^{1,1}\text{-alg} \to B_\infty\text{-alg} \\ (V, \partial, \bullet, \odot) \mapsto (V, d_\partial^\odot, \mu_\bullet^\odot) \end{gathered}$$

obtained by twisting the quasi-trivial B_∞ structure using the extra multiplication $\odot$. Furthermore, there is a natural projection map

$$\varepsilon_V : (T^c(V), d_\partial^\odot, \mu_\bullet^\odot, \odot) \to (V, \partial, \bullet, \odot),$$

which is a $d\mathcal{A}s^{1,1}$ algebra homomorphism, where the extra multiplication $\odot$ on $T^c(V)$ is concatenation of tensors.

Any differential graded Hopf algebra $(H, \partial, \bullet, \Delta)$ which is cofree, $H = T^c(A)$, gives rise to a $d\mathcal{A}s^{1,1}$ algebra $(T^c(A), \partial, \bullet, \odot)$ where $\odot$ is concatenation of tensors, and indeed to a B_∞ algebra A. Hence $(\star)$ defines the underlying B_∞ structure on H. By induction on the arity of the structure maps (Theorem 3.1.3) we show the natural inclusion map

$$\eta_A : A \to T^c(A) = H$$

is a B_∞ algebra homomorphism, and in this sense the underlying B_∞ structure on H can be regarded as restricting to the primitives.

An analogue of the classical universal enveloping algebra can be defined as a certain quotient of the free $d\mathcal{A}s^{1,1}$ algebra:

$$\begin{gathered} B_\infty\text{-alg} \to d\mathcal{A}s^{1,1}\text{-alg} \\ U(A) = F_{d\mathcal{A}s^{1,1}}(A)/I. \end{gathered}$$

This provides a left adjoint to the underlying B_∞ algebra. By virtue of the natural transformations (η_A), (ε_V) above, an alternative construction (Proposition 3.2.1) of the left adjoint is given by

$$\begin{gathered} B_\infty\text{-alg} \to d\mathcal{A}s^{1,1}\text{-alg} \\ (A, d, \mu) \mapsto (T^c(A), d, \mu, \odot). \end{gathered}$$

We can rephrase this: our underlying B_∞ structure $(\star)$ is just the right adjoint to this essentially tautologous tensor (co)algebra functor.

Thus, $U(A)$ and $T^c(A)$ are canonically isomorphic $d\mathcal{A}s^{1,1}$ algebras and, taking into account the deconcatenation comultiplication, we can regard the universal enveloping algebra as a $d\mathcal{A}s^{1,1}$ *bialgebra*, and also as a cofree differential Hopf algebra. Now given a conilpotent $d\mathcal{A}s^{1,1}$ bialgebra we show (Theorem 3.3.3) that the underlying B_∞ structure restricts to the space of primitives and gives an adjoint equivalence of categories

$$\begin{tikzcd} B_\infty\text{-alg} \arrow[r, shift left, "U"] & d\mathcal{A}s^{1,1}\text{-bialg}_{\text{conil}} \arrow[l, shift left, "\text{Prim}"] \end{tikzcd} .$$

That is, every B_∞ algebra arises as the primitives of a $d\mathcal{A}s^{1,1}$ bialgebra.

In particular, any conilpotent $d\mathcal{A}s^{1,1}$ bialgebra is cofree, and is essentially a cofree differential Hopf algebra.

1. Coalgebras, Algebras and Bialgebras

1.1. Grading and Koszul Signs

We work with graded vector spaces V over a field $\mathbb{K}$, and denote the degree of a homogeneous element $v \in V$ by $|v|$. The tensor product of graded spaces is graded by $|a \otimes b| = |a| + |b|$. The tensor product of (homogeneous) graded maps is given by

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b),$$

which implies

$$(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|} (f \circ h) \otimes (g \circ k).$$

These signs arise from[1] the natural symmetry isomorphism of the tensor product

$$\sigma \colon V \otimes W \cong W \otimes V, \qquad \sigma(a \otimes b) = (-1)^{|a||b|} b \otimes a.$$

Given a permutation $\sigma \in \Sigma_n$ we may denote by the same name the symmetry isomorphism of an n-fold tensor product that moves the ith factor to the $\sigma(i)$th position,

$$\begin{aligned} V_1 \otimes \cdots \otimes V_n \quad &\xrightarrow{\sigma} \quad V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}, \\ \sigma(v_1 \otimes \cdots \otimes v_n) &= (-1)^{\kappa}\, v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}. \end{aligned}$$

The formula for the Koszul signs arising in this symmetry isomorphism is

$$\kappa \;=\; \operatorname{sgn}(v_1, \ldots, v_n; \sigma) \;=\; \sum_{(p,q) \in \operatorname{inv}(\sigma)} |v_p|\, |v_q| \,. \tag{1.1}$$

Here the sum is over inversions (p, q) of σ, that is, all $p < q$ with $\sigma(p) > \sigma(q)$.

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[1] The symmetry is used to define the evaluation map $\mathrm{ev} \colon \hom(V_1, W_1) \otimes \hom(V_2, W_2) \otimes V_1 \otimes V_2 \to W_1 \otimes W_2$.

1.2. Coalgebras

A *coalgebra* (or more explicitly, a counital coassociative coalgebra over $\mathbb{K}$) is a graded vector space C equipped with a comultiplication $\Delta\colon C \to C \otimes C$ and a counit $\varepsilon\colon C \to \mathbb{K}$ which satisfy the coassociativity and counit laws

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta, \qquad \lambda(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = \rho(\mathrm{id} \otimes \varepsilon)\Delta. \tag{1.2}$$

The structure maps Δ, ε have degree zero, and

$$\lambda : \mathbb{K} \otimes C \to C, \qquad \rho : C \otimes \mathbb{K} \to C \tag{1.3}$$

are the canonical isomorphisms which we often regard as identity maps and silently omit from the notation. A *coalgebra map* is a linear map that preserves the comultiplication and counit. We may write comultiplication using Sweedler notation without an explicit summation symbol,

$$\Delta(x) \;=\; x_{(1)} \otimes x_{(2)}.$$

The tensor product $C \otimes D$ of coalgebras is canonically a coalgebra, with counit $\varepsilon(x \otimes y) = \varepsilon(x)\varepsilon(y)$ and comultiplication

$$\Delta\colon C \otimes D \xrightarrow{\Delta\otimes\Delta} (C \otimes C) \otimes (D \otimes D) \xrightarrow{(23)} (C \otimes D) \otimes (C \otimes D).$$

As the two inner tensor factors are permuted, Koszul signs appear when applied to elements,

$$\Delta(x \otimes y) \;=\; (-1)^{|y_{(1)}||x_{(2)}|}(x_{(1)} \otimes y_{(1)}) \otimes (x_{(2)} \otimes y_{(2)}).$$

1.3. Reduced Coalgebras

A *unit* or *coaugmentation* for a coalgebra C is a coalgebra map $\eta\colon \mathbb{K} \to C$, or equivalently a specified element $\eta(1_{\mathbb{K}}) = 1 \in C$ which is group-like: $\Delta(1) = 1 \otimes 1$. Then $\varepsilon(1) = 1_{\mathbb{K}}$ and there is a splitting

$$(\eta\varepsilon, J)\colon C \xrightarrow{\cong} \mathbb{K}1 \oplus \overline{C}$$

where $J = \mathrm{id} - \eta\varepsilon : C \to C$ and $\overline{C} = \ker(\varepsilon) = \mathrm{im}(J)$.

For any unital coalgebra $(C, \Delta, \varepsilon, 1)$ there is a well-defined coassociative operation

$$\overline{\Delta}\colon \overline{C} \to \overline{C} \otimes \overline{C}$$

where, for $x \in \overline{C}$,

$$\overline{\Delta}(x) \;=\; \Delta(x) - x \otimes 1 - 1 \otimes x \;=\; J^{\otimes 2}\Delta(x).$$

The (non-unital, non-counital) coalgebra $(\overline{C}, \overline{\Delta})$ is called the *reduced* coalgebra. This gives an equivalence between the categories of unital counital coalgebras and their maps and of non-unital non-counital algebras and their maps.

The kernel of $\overline{\Delta}$ is the subspace $\mathrm{Prim}(C)$ of *primitive* elements of C.

Remark 1.3.1. Saying that a binary operation defined by a linear map $\mu\colon C \otimes C \to C$ is *unital* may mean, according to context, either the condition $\mu(1 \otimes 1) = 1$ or the condition $\mu(1 \otimes x) = \mu(x \otimes 1) = x$ for all $x \in C$. The latter condition will often be paraphrased as *satisfies the unit law.* In this case specifying μ is equivalent to specifying the component $\overline{C}\otimes\overline{C} \to C$ as the remaining components are the canonical isomorphisms, cf. (1.3),

$$\mu(\eta\otimes\mathrm{id})\colon \mathbb{K}\otimes C \xrightarrow{=} C, \qquad \mu(\mathrm{id}\otimes\eta)\colon C\otimes\mathbb{K} \xrightarrow{=} C.$$

1.4. The Primitive Filtration

Consider the filtration of a unital coalgebra C defined in [15, Section 3] by $F_0C = \mathbb{K}1$ and

$$F_rC \;=\; \mathbb{K}1 \oplus \{x\in\overline{C} \mid \overline{\Delta}(x)\in F_{r-1}C\otimes F_{r-1}C\} \qquad (r\geqslant 1).$$

Denote by $\Delta^{(r)}\colon C\to C^{\otimes(r+1)}$ and $\overline{\Delta}^{(r)}\colon \overline{C}\to\overline{C}^{\otimes(r+1)}$ the r-fold iterations of the coassociative comultiplications Δ and $\overline{\Delta}$. For $r=0$ these are the identity maps, and $\Delta^{(-1)}=\varepsilon : C\to\mathbb{K}$.

Definition 1.4.1. The *conilpotent radical* $R(C)$ of a unital coalgebra C is the subcoalgebra $\bigcup F_rC$.

Definition 1.4.2 (Connected and conilpotent coalgebras). A connected coalgebra (in the sense of [12]) is a unital coalgebra C such that the above filtration is exhaustive, so that $R(C)=C$. A conilpotent coalgebra is a unital coalgebra C such that for each $x\in\overline{C}$ there exists $r\in\mathbb{N}$ with $\overline{\Delta}^{(r)}(x)=0$.

It is well known that connected coalgebras are conilpotent: it was noted in [12] and can be seen in Quillen's proof of [15, Proposition 4.1]. The converse was pointed out in [6, Appendix B]. We have:

Lemma 1.4.3. *A unital coalgebra C is conilpotent if and only if it is connected. In fact,*

$$F_rC \;=\; \mathbb{K}1\oplus\ker(\overline{\Delta}^{(r)}) \;=\; \ker(J^{\otimes(r+1)}\Delta^{(r)})$$

and $R(C)$ is the largest conilpotent subcoalgebra of C.

Proof. Since $J^{\otimes(r+1)}\Delta^{(r)}(1)=0$ and $J^{\otimes(r+1)}\Delta^{(r)}(x)=\overline{\Delta}^{(r)}(x)$ for $x\in\overline{C}$ the second equality holds. Now note that taking kernels and the tensor product commute in the following sense

$$\ker(\overline{\Delta}^{(r)})\otimes\ker(\overline{\Delta}^{(r)}) \;=\; \ker(\overline{\Delta}^{(r)}\otimes\mathrm{id}) \;\cap\; \ker(\mathrm{id}\otimes\overline{\Delta}^{(r)}).$$

The first equality, therefore, follows inductively from

$$\begin{aligned}&\overline{\Delta}^{-1}\left(\ker(\overline{\Delta}^{(r)})\otimes\ker(\overline{\Delta}^{(r)})\right)\\ &\quad=\;\ker((\overline{\Delta}^{(r)}\otimes\mathrm{id})\overline{\Delta}) \;\cap\; \ker((\mathrm{id}\otimes\overline{\Delta}^{(r)})\overline{\Delta}) \;=\; \ker(\overline{\Delta}^{(r+1)}).\end{aligned}$$

□

1.5. Comodules and Coderivations

A bicomodule over a coalgebra C is a graded vector space D with a coaction $\gamma : D\to C\otimes D\otimes C$, that is, a linear map satisfying

$$(\varepsilon\otimes\mathrm{id}\otimes\varepsilon)\gamma=\mathrm{id}, \qquad (\mathrm{id}\otimes\gamma\otimes\mathrm{id})\gamma=(\Delta\otimes\mathrm{id}\otimes\Delta)\gamma.$$

Observe that the coaction γ determines left and right coactions

$$(\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon)\gamma\colon D\to C\otimes D, \qquad (\varepsilon\otimes\mathrm{id}\otimes\mathrm{id})\gamma\colon D\to D\otimes C.$$

For example, any coalgebra can be regarded as a bicomodule over itself, with left and right coactions both given by the comultiplication, and $\gamma = \Delta^{(2)}$.

A coderivation on a C-bicomodule D is a linear map $d\colon D \to C$, not necessarily of degree 0, satisfying

$$\Delta d \;=\; (\varepsilon \otimes d \otimes \mathrm{id} + \mathrm{id} \otimes d \otimes \varepsilon)\gamma,$$

where we have omitted the isomorphisms (1.3). This implies $\varepsilon d = 0$ and indeed

$$\Delta^{(k-1)} d \;=\; \sum_{\substack{i,j \geqslant 0\\ i+1+j=k}} (\Delta^{(i-1)} \otimes d \otimes \Delta^{(j-1)})\,\gamma, \qquad (k \geqslant 0).$$

1.6. Cofree Coalgebras

The *tensor coalgebra* on a graded vector space V is given by

$$T^c(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$$

with counit defined by the 0th projection $\varepsilon = p_0\colon T^c(V) \to V^{\otimes 0} = \mathbb{K}$, and comultiplication defined by deconcatenation

$$\Delta(v_1 \cdots v_k) \;=\; \sum_{i=0}^{k} v_1 \cdots v_i \;\otimes\; v_{i+1} \cdots v_k.$$

As usual when working in $T^c(V) \otimes T^c(V)$ we write $v_1 \cdots v_k \in T^c(V)$ for $v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$, and write $1 = 1_{\mathbb{K}}$ for the unit, given by the empty product, in the case $k = 0$. The reduced tensor coalgebra is given by

$$\overline{T}^c(V) = \bigoplus_{k=1}^{\infty} V^{\otimes k}, \qquad \overline{\Delta}(v_1 \cdots v_k) = \sum_{i=1}^{k-1} v_1 \cdots v_i \;\otimes\; v_{i+1} \cdots v_k.$$

The unital coalgebra $T^c(V)$ is conilpotent, its space of primitives is $V^{\otimes 1} = V$, and it has filtration

$$F_r T^c(V) \;=\; \bigoplus_{k=0}^{r} V^{\otimes k}.$$

Remark 1.6.1 (Cofreeness properties). Together with the first projection $p_1\colon T^c(V) \to V$ onto the primitives, the tensor coalgebra defines the free coalgebra in the category of conilpotent coalgebras:

a) One can write the k-th projection as $p_k = p_1^{\otimes k} \Delta^{(k-1)}\colon T^c(V) \to V^{\otimes k}$ and the identity map as

$$\mathrm{id} = \sum_{k=0}^{\infty} p_1^{\otimes k} \Delta^{(k-1)}\colon T^c(V) \to T^c(V).$$

b) Any unital coalgebra map $f\colon C \to T^c(V)$ is determined by $f_1 = p_1 f\colon C \to V$ via

$$f \;=\; \sum_{k=0}^{\infty} p_1^{\otimes k} \Delta^{(k-1)} f \;=\; \sum_{k=0}^{\infty} f_1^{\otimes k} \Delta_C^{(k-1)}\colon C \to T^c(V).$$

Conversely if C is a conilpotent coalgebra then the formula on the right hand side gives a well-defined unital coalgebra map $f = \sum f_k \colon C \to T^c(V)$ extending any linear map $f_1 \colon C \to V$ with $f_1(1) = 0$. Hence projection to the primitives gives a bijection

$$p_1^* \colon \hom_{\mathrm{conil}}(C, T^cV) \;\cong\; \hom_{\mathbb{K}}(\overline{C}, V).$$

c) Any coderivation d on a $T^c(V)$-bicomodule D is determined by $d_1 = p_1 d \colon D \to V$, via

$$d = \sum_{k=0}^{\infty} p_1^{\otimes k} \Delta^{(k-1)} d \;=\; \sum_{k=0}^{\infty} p_1^{\otimes k} \sum_{\substack{i,j \geqslant 0 \\ i+1+j=k}} (\Delta^{(i-1)} \otimes d \otimes \Delta^{(j-1)})\, \gamma$$
$$= \sum_{i,j \geqslant 0} (p_i \otimes d_1 \otimes p_j)\, \gamma$$

Conversely the formula on the right hand side gives a well-defined coderivation $d \colon D \to T^c(V)$ extending any linear map $d_1 \colon D \to V$. Hence projection to the primitives gives a bijection

$$p_1^* \colon \mathrm{coder}(D, T^cV) \;\cong\; \hom_{\mathbb{K}}(D, V).$$

A unital coalgebra is termed cofree (more precisely: cofree among conilpotent coalgebras) if it is isomorphic to a tensor coalgebra $T^c(V)$, or equivalently if it comes equipped with a projection p_1 to its space of primitives having the same cofreeness properties as above.

1.7. Bialgebras and Unital Infinitesimal Bialgebras

Recall that an *algebra* (or more explicitly, a unital associative algebra over $\mathbb{K}$) is a graded vector space B equipped with a multiplication $\mu \colon B \otimes B \to B$ and a unit $\eta \colon \mathbb{K} \to B$, $\eta(1_{\mathbb{K}}) = 1$, satisfying associativity and unit laws. A differential graded algebra is an algebra $(B, \mu, 1)$ equipped with a map $d \colon B \to B$ of degree -1 satisfying $d^2 = 0$ and $d\mu = \mu(\mathrm{id} \otimes d + d \otimes \mathrm{id})$.

An *algebra map* is a linear map preserving the unit and multiplication.

An algebra B is *augmented* or *counital* if it is equipped with an algebra map $\varepsilon \colon B \to \mathbb{K}$.

The tensor product $A \otimes B$ of algebras has a canonical algebra structure

$$A \otimes B \;\otimes\; A \otimes B \xrightarrow{(23)} A \otimes A \;\otimes\; B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B.$$

Definition 1.7.1. Let B be a graded vector space with structures of an algebra (B, μ, η) and of a coalgebra (B, Δ, ε). Then the structure $(B, \mu, \eta, \Delta, \varepsilon)$ is termed

1. a *bialgebra* if $\mu \colon B \otimes B \to B$ and $\eta \colon \mathbb{K} \to B$ are coalgebra maps, or equivalently if $\Delta \colon B \to B \otimes B$ and $\varepsilon \colon B \to \mathbb{K}$ are algebra maps,
2. a *unital infinitesimal bialgebra* if

$$\Delta(xy) \;=\; x_{(1)} \otimes x_{(2)} y \;+\; x y_{(1)} \otimes y_{(2)} \;-\; x \otimes y.$$

An equivalent definition is given in Lemma 1.7.4 and Corollary 1.7.5.

We use Sweedler notation for comultiplication and denote the multiplication μ by juxtaposition.

A bialgebra or a unital infinitesimal bialgebra is termed *conilpotent* if it is conilpotent as a unital coalgebra, and is termed *cofree* if it is cofree as a unital coalgebra.

Any conilpotent bialgebra has a canonical Hopf algebra structure, though not all Hopf algebras are conilpotent.

Example 1.7.2 (Shuffle bialgebra). The shuffle bialgebra $T^{Ш}(V)$ on a graded vector space V has underlying coalgebra $T^c(V)$ with Δ given by deconcatenation as described in section 1.6. The shuffle product $Ш\colon T^c(V)\otimes T^c(V) \to T^c(V)$ is unique coalgebra map that extends, via the cofreeness property of Remark 1.6.1 (b), the linear map

$$Ш_1 = \varepsilon\otimes p_1 + p_1\otimes\varepsilon\colon \overline{T^c(V)\otimes T^c(V)} \longrightarrow \mathbb{K}\otimes V \ \oplus\ V\otimes\mathbb{K} \longrightarrow V. \quad (1.4)$$

Observe that $Ш_1$ is zero on $\overline{T}^c(V)\otimes\overline{T}^c(V)$. The unit laws and associative law for $Ш$ follow by standard cofreeness arguments: the coalgebra maps $Ш(\eta\otimes \mathrm{id})$ and $Ш(\mathrm{id}\otimes\eta)$ are the identity since their compositions with p_1 are just p_1, and similarly the coalgebra maps $Ш(Ш\otimes\mathrm{id})$, $Ш(\mathrm{id}\otimes Ш)\colon T^c(V)^{\otimes 3}\to T^c(V)$ are equal since one can check their projections to V are equal. Observe that $x\, Ш_1\, y$ is symmetric in x, y and is non-zero only when xy is a tensor of length exactly 1. It follows that the extension $x\, Ш\, y$ is graded commutative and that each projection $Ш_k = Ш_1^{\otimes k}\,\Delta^{(k-1)}$ is non-zero only when xy is a tensor of total length exactly k. On elements we have

$$v_1\cdots v_i \ Ш\ v_{i+1}\cdots v_k \ =\ \sum_\sigma (-1)^\kappa\, v_{\sigma^{-1}(1)}\cdots v_{\sigma^{-1}(k)}. \quad (1.5)$$

Here the sum is over all $(i, k-i)$-shuffles, that is, permutations $\sigma\in\Sigma_k$ satisfying $\sigma(j)<\sigma(j+1)$ for all $j\neq i$, $1\leqslant j\leqslant k-1$, with the Koszul signs that were defined in (1.1).

The notion of unital infinitesimal bialgebra was introduced in [10,12], modifying the non-unital definition of infinitesimal bialgebras due to Joni and Rota in [8]. The unital infinitesimal relation can be expressed

$$\overline{\Delta}(xy) = \overline{\Delta}(x)(1\otimes y) + (x\otimes 1)\overline{\Delta}(y) + x\otimes y \quad (1.6)$$

and more generally one has

$$\begin{aligned}&\overline{\Delta}^{(n)}(xy)\\ &= \sum_{\substack{r,s\geqslant 0\\ r+s=n}} \big(\overline{\Delta}^{(r)}(x)\otimes 1^{\otimes s}\big)\big(1^{\otimes r}\otimes\overline{\Delta}^{(s)}(y)\big) + \sum_{\substack{r,s\geqslant 0\\ r+s=n-1}} \overline{\Delta}^{(r)}(x)\otimes\overline{\Delta}^{(s)}(y).\end{aligned} \quad (1.7)$$

Lemma 1.7.3. *Let $(W,\odot,1,\Delta,\varepsilon)$ be a unital infinitesimal bialgebra.*

1. *It $v_1, \dots, v_k \in W$ are primitive then*

$$\Delta(v_1 \odot \cdots \odot v_k) \;=\; \sum_{i=0}^{k} v_1 \odot \cdots \odot v_i \;\otimes\; v_{i+1} \odot \cdots \odot v_k.$$

2. *If $x, y \in \overline{W}$ with $\overline{\Delta}^{(i)}(x) = 0$ and $\overline{\Delta}^{(j)}(y) = 0$ then $\overline{\Delta}^{(i+j)}(x \odot y) = 0$. In particular the conilpotent radical $R(W)$ is a unital infinitesimal subbialgebra of W.*

Proof. (1): For $x = v_1 \odot \cdots \odot v_{k-1}$, and $y = v_k$ primitive, we have $\overline{\Delta}(x \odot v_k) = \overline{\Delta}(x) \odot (1 \otimes v_k) + x \otimes v_k$ from (1.6), and the result follows by induction on k. (2): This is clear from (1.7). □

In the following Lemma and Corollary we see that a unital infinitesimal bialgebra structure can also be expressed in terms of the comultiplication $\Delta\colon B \to B \otimes B$ being an algebra map or the multiplication $\mu\colon B \otimes B \to B$ being a coalgebra map if we do not use the canonical (co)multiplication on the vector space $B \otimes B$ but an alternative structure:

Lemma 1.7.4. *Let $(B, \Delta, \varepsilon, 1)$ be a unital coalgebra. Then*

$$\Delta'(x \otimes y) = (x_{(1)} \otimes 1) \otimes (x_{(2)} \otimes y) + (x \otimes y_{(1)}) \otimes (1 \otimes y_{(2)}) - (x \otimes 1) \otimes (1 \otimes y),$$

defines a unital coalgebra structure $(B^{\otimes 2}, \Delta', \varepsilon^{(2)}, 1 \otimes 1)$ with the canonical unit and counit.

Now let $(B, \odot, 1, \varepsilon)$ be a counital algebra. Then there is a counital algebra structure defined on $B^{\otimes 2}$ by $1^{\otimes 2}$, $\varepsilon^{\otimes 2}$ and $\odot'\colon B^{\otimes 2} \otimes B^{\otimes 2} \to B^{\otimes 2}$ where

$$(x_1 \otimes x_2) \odot' (y_1 \otimes y_2) = \varepsilon(y_1)\, x_1 \otimes x_2 \odot y_2 \;+\; \varepsilon(x_2)\, x_1 \odot y_1 \otimes y_2 \;-\; \varepsilon(x_2 \odot y_1)\, x_1 \otimes y_2.$$

Proof. For the first part, we check that $(\Delta' \otimes \mathrm{id})$ and $(\mathrm{id} \otimes \Delta')$ applied to $\Delta'(x \otimes y)$ both give

$$\begin{aligned}
&(x_{(1)} \otimes 1) \otimes (x_{(2)} \otimes 1) \otimes (x_{(3)} \otimes y) + (x \otimes y_{(1)}) \otimes (1 \otimes y_{(2)}) \otimes (1 \otimes y_{(3)}) \\
&\quad + (x_{(1)} \otimes 1) \otimes (x_{(2)} \otimes y_{(1)}) \otimes (1 \otimes y_{(2)}) \\
&\quad - (x \otimes 1) \otimes (1 \otimes y_{(1)}) \otimes (1 \otimes y_{(2)}) - (x_{(1)} \otimes 1) \otimes (x_{(2)} \otimes 1) \otimes (1 \otimes y)
\end{aligned}$$

so Δ' is coassociative. It is easy to verify that $(\varepsilon^{(2)} \otimes \mathrm{id})$ and $(\mathrm{id} \otimes \varepsilon^{(2)})$ applied to $\Delta'(x \otimes y)$ both give $x \otimes y$ and that $1 \otimes 1$ is grouplike.

The second part is similar. □

Corollary 1.7.5. *Suppose B is a graded vector space equipped with a unital counital algebra structure $(B, \odot, 1, \varepsilon)$ and a unital counital coalgebra structure $(B, \Delta, \varepsilon, 1)$. Then the following are equivalent:*

1. *$(B, \odot, 1, \Delta, \varepsilon)$ is a unital infinitesimal bialgebra.*
2. *The multiplication is a coalgebra map $\odot\colon (B \otimes B, \Delta') \to (B, \Delta)$.*
3. *The comultiplication is an algebra map $\Delta\colon (B, \odot) \to (B \otimes B, \odot')$.*

Example 1.7.6 (Fundamental infinitesimal bialgebra). Let V be a graded vector space and consider the unital infinitesimal bialgebra $(T^{fc}(V), \odot, 1, \Delta, \varepsilon)$ defined as follows. As a graded algebra it is the *tensor algebra* $T(V) =$

$\bigoplus_{n\geqslant 0} V^{\otimes n}$ with multiplication $\circledcirc$ defined by concatenation of tensors. This is the free algebra on V so by Corollary 1.7.5 (3) a unital infinitesimal comultiplication Δ can be specified by giving its values on the generators. In particular we can define

$$\Delta\colon (T(V),\circledcirc) \to (T(V)\otimes T(V),\circledcirc'), \qquad \Delta(v) = v\otimes 1 + 1\otimes v \quad (v\in V).$$

so that the generators are primitive. Now by Lemma 1.7.3 (1) we see that Δ coincides with the comultiplication given by deconcatenation of tensors. That is, the fundamental infinitesimal bialgebra is at once the free algebra and the cofree conilpotent coalgebra on V.

Proposition 1.7.7. *The functors* T^{fc} *and* Prim *are adjoint: for any vector space* A *and any unital infinitesimal bialgebra* W*, there is a natural bijection between linear maps* $f\colon A\to \mathrm{Prim}(W)$ *and unital infinitesimal bialgebra homomorphisms* $F\colon T^{fc}(A)\to W$*, such that* $f(v) = F(v)$ *for all* $v\in A$*. The unit of the adjunction is the natural isomorphism* $A\cong \mathrm{Prim}(T^{fc}(A))$*.*

Proof. Define F as the algebra homomorphism extending $A \xrightarrow{f} \mathrm{Prim}(W) \xrightarrow{\subseteq} W$ to the free algebra,

$$F(v_1\circledcirc\cdots\circledcirc v_k) = fv_1\circledcirc\cdots\circledcirc fv_k.$$

This is a bialgebra homomorphism by Lemma 1.7.3 (1). Conversely, restricting F to A defines the linear map f, since F sends primitives to primitives. □

For the counit, consider $f_W = \mathrm{id}\colon \mathrm{Prim}(W) \xrightarrow{=} \mathrm{Prim}(W)$ and its extension

$$F_W : T^{fc}(\mathrm{Prim}(W)) \to W. \tag{1.8}$$

Proposition 1.7.8. *The natural homomorphisms* F_W *are injective and their image is the conilpotent radical* $R(W)$*. In particular any conilpotent unital infinitesimal bialgebra is naturally isomorphic to the fundamental infinitesimal bialgebra on its primitives.*

Proof. Clearly F_W has conilpotent image and so factors

$$T^{fc}(\mathrm{Prim}(W)) = T^{fc}(\mathrm{Prim}(R(W))) \to R(W) \subseteq W.$$

Now observe that $F\colon T^{fc}(\mathrm{Prim}(R(W))) \to R(W)$ is an isomorphism, by [12, Theorem 2.6]. □

In other words:

Corollary 1.7.9. *The functor* T^{fc} *embeds the category of vector spaces as the full coreflective subcategory of unital infinitesimal bialgebras whose objects are conilpotent or, equivalently, cofree.*

2. A_∞ Algebras, Multibrace Algebras and B_∞ Algebras

2.1. A_∞ Algebras

The notion of A_∞ algebra is encoded in differential graded structures on the tensor coalgebra.

Consider the tensor coalgebra $T^c(V)$ on a graded vector space V as a bicomodule over itself.

Any coderivation $d\colon T^c(V) \to T^c(V)$, and each the projections $d_k = p_k d\colon T^c(V) \to V^{\otimes k}$, are determined by the projection to the primitives $d_1 = p_1 d\colon T^c(V) \to V$. Explicitly, if we denote

$$d_1 = \big(m_n\colon V^{\otimes n} \to V\big)_{n\geqslant 0}, \qquad d_k = \big(m_n^k\colon V^{\otimes n} \to V^{\otimes k}\big)_{n\geqslant 0},$$

then by the cofreeness property of Remark 1.6.1 (c),

$$m_n^k = \sum_{\substack{i,j\geqslant 0\\ i+1+j=k}} \mathrm{id}_V^{\otimes i} \otimes m_{n-i-j} \otimes \mathrm{id}_V^{\otimes j}\colon V^{\otimes n} \to V^{\otimes k}. \tag{2.1}$$

A coderivation d on $T^c(V)$ is termed a *differential* if it has degree -1 and satisfies $d(1) = 0$ and $d^2 = 0$. The condition $d(1) = 0$ says $m_0 = 0$ and $m_n^k = 0$ for $k > n$; we may regard a differential as a map

$$d\colon \overline{T}^c(V) \to \overline{T}^c(V) \tag{2.2}$$

determined by

$$d_1 = p_1 d = \sum_{n\geqslant 1} m_n\colon \overline{T}^c(V) \to V. \tag{2.3}$$

The condition $d^2 = 0$ therefore says

$$\sum_{\substack{i,j\geqslant 0\\ i+j<n}} m_{i+1+j}\left(\mathrm{id}_{V^{\otimes i}} \otimes m_{n-i-j} \otimes \mathrm{id}_{V^{\otimes j}}\right) = 0\colon V^{\otimes n} \to V \qquad (n \geqslant 1). \tag{2.4}$$

As the maps m_n have degree -1 we note that the usual Koszul signs appear in the summations of (2.1) or (2.4) when they are applied to elements.

Definition 2.1.1 (A_∞ algebra). An A_∞ algebra is a graded vector space V together with a differential d on $T^c(V)$, or equivalently a graded vector space V with a sequence $d_1 = (m_1, m_2, m_3, \ldots)$ of degree -1 maps $m_n\colon V^{\otimes n} \to V$ satisfying the relations (2.4).

Remark 2.1.2. This notion might more properly be called a *shifted A_∞ algebra*: an A_∞ algebra is classically defined via a differential on the bar construction rather than on the tensor coalgebra. The definitions are essentially equivalent and we adopt the latter as it simplifies degrees and signs. Let $V[1] = \mathbb{K}[1] \otimes V$, where $\mathbb{K}[1]$ is a copy of the field $\mathbb{K}$ concentrated in degree 1, and let $s\colon V \to V[1]$, $sv = 1 \otimes v$, be the canonical degree 1 suspension isomorphism. Then a classical A_∞ structure on V corresponds to a shifted A_∞ structure on $V[1]$, and V is thus equipped with a sequence of degree $n-2$ maps $s^{-1} m_n s^{\otimes n}\colon V^{\otimes n} \to V$ satisfying relations analogous to (2.4) but with additional Koszul signs that arise from rearranging the intervening suspension maps.

Example 2.1.3 (Trivial A_∞ structure). If V comes equipped with an endomorphism $\partial\colon V\to V$ of degree -1 then the map

$$d_1=(\partial,0,0,\dots)=\partial p_1\colon T^c(V)\longrightarrow V$$

extends to a unique coderivation $d=\sum d_k$ on $T^c(V)$ of degree -1. The extension satisfies $d^2=0$ if and only if $\partial^2=0$, that is, (V,∂) is a chain complex. The differential on $T^c(V)$ in this case is the *trivial A_∞ structure on V* and is given explicitly by the structure maps

$$m^k_n=0\quad(k\neq n),\qquad m^k_k=\sum_{i=0}^{k-1}\mathrm{id}_V^{\otimes i}\otimes\partial\otimes\mathrm{id}_V^{\otimes k-i-1}\colon V^{\otimes k}\to V^{\otimes k}.$$

2.2. Multibrace Algebras

The notion of multibrace algebra arises from bialgebra structures on the tensor coalgebra. If V is a graded vector space then a bialgebra structure on the tensor coalgebra $(T^c(V),1,\Delta,\varepsilon)$ is a coalgebra map $\mu\colon T^c(V)\otimes T^c(V)\to T^c(V)$ of degree zero satisfying the associative and unit laws.

As the tensor coalgebra is cofree, the coalgebra map μ and each of its projections $\mu_r=p_r\mu\colon T^c(V)\otimes T^c(V)\to V^{\otimes r}$ are determined by μ_1. Let us denote their components by

$$\mu_1=\big(m_{i,j}\colon V^{\otimes i}\otimes V^{\otimes j}\to V\big)_{i,j\geqslant 0},\qquad \mu_r=\big(m^r_{i,j}\colon V^{\otimes i}\otimes V^{\otimes j}\to V^{\otimes r}\big)_{i,j\geqslant 0}.$$

The unit law for the multiplication μ says that it is determined by its restriction to the reduced coalgebra, see Remark 1.3.1, and hence by the components $m_{i,j}$ with $i,j\geqslant 1$,

$$\begin{tikzcd}
\overline{T}^c(V)\otimes\overline{T}^c(V) \arrow[r] \arrow[dr, "\sum\limits_{i,j\geq 1} m_{i,j}"'] & T^c(V) \arrow[d, "p_1"] \\
 & V,
\end{tikzcd}\tag{2.5}$$

as the components with i or j zero are just projections to V of the identifications $\mathbb{K}\otimes T^c(V)=T^c(V)$ and $T^c(V)\otimes\mathbb{K}=T^c(V)$ respectively, cf. (1.3),

$$\begin{cases} m_{0,1}=\mathrm{id}\colon \mathbb{K}\otimes V \xrightarrow{=} V,\\ m_{0,n}=\ 0: \mathbb{K}\otimes V^{\otimes n}\longrightarrow V,\end{cases}\qquad \begin{cases} m_{1,0}=\mathrm{id}\colon V\otimes\mathbb{K}\xrightarrow{=} V.\\ m_{n,0}=\ 0: V^{\otimes n}\otimes\mathbb{K}\longrightarrow V,\end{cases}\qquad (n\neq 1).\tag{2.6}$$

The associativity law for the multiplication μ is equivalent to $\mu_1(\mu\otimes\mathrm{id})=\mu_1(\mathrm{id}\otimes\mu)$. To make this explicit we introduce some auxiliary notation.

Notation 2.2.1. Denote by $C^i_r\subset\mathbb{N}^r$ the set of sequences $\underline{i}=(i_1,\dots,i_r)$ of r nonnegative integers whose sum is i. Now each iterated comultiplication $\Delta^{(r-1)}$ on $T^c(V)\otimes T^c(V)$ has as components

$$\begin{aligned}\Delta_r^{\underline{i},\underline{j}}: V^{\otimes i}\otimes V^{\otimes j}&=(V^{\otimes i_1}\otimes\cdots\otimes V^{\otimes i_r})\otimes(V^{\otimes j_1}\otimes\cdots\otimes V^{\otimes j_r})\\ &\xrightarrow{\sigma}(V^{\otimes i_1}\otimes V^{\otimes j_1})\otimes\cdots\otimes(V^{\otimes i_r}\otimes V^{\otimes j_r})\end{aligned}$$

for all $i, j \geqslant 0$ and $(\underline{i}, \underline{j}) \in C_r^i \times C_r^j$. When $\Delta_r^{\underline{i},\underline{j}}$ is applied to elements the Koszul signs (1.1) appear.

By Remark 1.6.1 (b) we have $\mu_r = p_r\mu = \mu_1^{\otimes r}\Delta^{(r-1)}\colon T^c(V)\otimes T^c(V) \to V^{\otimes r}$, and so its components can be written as

$$m^r_{i,j} \;=\; \sum_{(\underline{i},\underline{j})\in C_r^i\times C_r^j} (m_{i_1,j_1}\otimes\cdots\otimes m_{i_r,j_r})\Delta_r^{\underline{i},\underline{j}}\colon V^{\otimes i}\otimes V^{\otimes j}\to V^{\otimes r}. \tag{2.7}$$

The components $m^{i+j}_{i,j}$ coincide with the shuffle product (1.5), and $m^r_{i,j} = 0$ if $r > i + j$.

We can now express the associativity of μ in terms of the $m_{i,j}$: for each $i, j, k \geqslant 1$ there is an equality

$$\begin{aligned} &\sum_{\substack{r\geqslant 1\\ (\underline{i},\underline{j})\in C_r^i\times C_r^j}} m_{r,k}\left((m_{i_1,j_1}\otimes\cdots\otimes m_{i_r,j_r})\Delta_r^{\underline{i},\underline{j}}\otimes \mathrm{id}_V^{\otimes k}\right)\\ (2.8)\qquad &= \sum_{\substack{s\geqslant 1\\ (\underline{j},\underline{k})\in C_s^j\times C_s^k}} m_{i,s}\left(\mathrm{id}_V^{\otimes i}\otimes(m_{j_1,k_1}\otimes\cdots\otimes m_{j_s,k_s})\Delta_s^{\underline{j},\underline{k}}\right)\end{aligned}$$

of linear maps $V^{\otimes i}\otimes V^{\otimes j}\otimes V^{\otimes k}\to V$, where terms $m_{n,0}$ and $m_{0,n}$ are given by (2.6).

Definition 2.2.2 (Multibrace algebra). A multibrace algebra is a graded vector space V with a coalgebra map $\mu\colon T^c(V)\otimes T^c(V)\to T^c(V)$ satisfying the associative and unit laws.

Equivalently, it is a graded vector space V endowed with a family of degree zero multilinear maps $m_{i,j}\colon V^{\otimes i}\otimes V^{\otimes j}\to V$, $i, j \geqslant 1$ such that the relations (2.8) hold for $i, j, k \geqslant 1$.

The definition of multibrace algebra in the ungraded world was given in [12, Definition 1.5], where it was termed (non-differential) $\mathbf{B}_\infty$ algebra. As Koszul signs do not appear in the ungraded context the symmetry isomorphisms σ and the maps $\Delta_r^{\underline{i},\underline{j}}$ involved in the relations (2.8), there termed R_{ijk}, were left implicit.

Example 2.2.3. (Trivial multibrace algebra) Consider a multibrace structure μ on V in which all components $m_{i,j}$ of μ_1 (except $m_{0,1}$ and $m_{1,0}$) are zero. Then μ_1 is the linear map $Ш_1$ considered in Example 1.7.2 (1.4), and (1.5) gives the multiplication $\mu = Ш$. That is, we can identify the trivial multibrace algebra V with the shuffle bialgebra $T^{Ш}(V)$.

Example 2.2.4 (Quasi-shuffles). If V carries a binary operation $\bullet\colon V\otimes V\to V$ then we can upgrade the shuffle bialgebra and replace (1.4) by

$$\widetilde{Ш}_1 = Ш_1 + p_1\bullet p_1\colon \overline{T^c(V)\otimes T^c(V)}\to \mathbb{K}\otimes V \;\oplus\; V\otimes\mathbb{K} \;\oplus\; V\otimes V\to V \tag{2.9}$$

This determines a unique coalgebra map $\widetilde{Ш}\colon T^c(V)\otimes T^c(V)\to T^c(V)$ which satisfies the unit law, and which is associative if and only if the binary operation $\bullet$ was. Thus we have a multibrace algebra in which $m_{1,1} = \bullet$ and all other operations $m_{i,j}$ except $m_{0,1}$ and $m_{1,0}$ are zero. If the multiplication $\bullet$ is zero we recover the previous example.

If the operation $\bullet$ is associative and graded commutative then so is $\widetilde{\text{Ш}}$ and this multibrace algebra is a *quasi-shuffle* (or *stuffle*) algebra, compare [11][2].

Suppose the multiplication $\bullet$ is unital, with unit $\eta(1_{\mathbb{K}}) = 1 \in V$. If $\widetilde{p_1}$ is the unital linear map

$$\widetilde{p_1} = \eta\varepsilon + p_1 \colon T^c(V) \to V \tag{2.10}$$

then $\widetilde{\text{Ш}}_1$ is just the restriction to $\overline{T^c(V) \otimes T^c(V)}$ of

$$\widetilde{p_1} \bullet \widetilde{p_1} \colon T^c(V) \otimes T^c(V) \to V \otimes V \to V. \tag{2.11}$$

Example 2.2.5. If V is endowed with two associative binary operations $\bullet$ and $\odot$, unrelated except that they share a common unit, it turns out that V has a unique multibrace structure $\mu_\bullet^\circ$ whose components $m_{i,j}^r \colon V^{\otimes i} \otimes V^{\otimes j} \to V^{\otimes r}$ satisfy

$$\sum_{r\geqslant 1} \odot^{(r-1)} m_{i,j}^r \; = \; \bullet(\odot^{(i-1)} \otimes \odot^{(j-1)}) \colon V^{\otimes i} \otimes V^{\otimes j} \longrightarrow V. \tag{2.12}$$

It is sufficient to specify the components $m_{i,j} = m_{i,j}^1$ which must satisfy $m_{0,0} = 0$ and, by (2.7),

$$m_{i,j} \;=\; \bullet(\odot^{(i-1)} \otimes \odot^{(j-1)}) \;-\; \sum_{r\geqslant 2} \odot^{(r-1)} \sum_{(\underline{i},\underline{j}) \in C_r^i \times C_r^j} (m_{i_1,j_1} \otimes \cdots \otimes m_{i_r,j_r}) \Delta_r^{\underline{i},\underline{j}} \tag{2.13}$$

This recursive formula to define maps $m_{i,j}$ was studied in the ungraded situation by Loday and Ronco [12, Section 3.3 and Proposition 3.4]. In Theorem 3.1.2 we will give a simple argument to show that the operations $(m_{i,j})$ so defined do indeed satisfy the multibrace axioms (2.8).

2.3. B_∞ Algebras

The notion of B_∞ algebra is encoded in differential graded bialgebra structures on the tensor coalgebra. If V is a graded vector space then a differential graded bialgebra structure on $(T^c(V), 1, \Delta, \varepsilon)$ is a differential $d \colon T^c(V) \to T^c(V)$ together with a coalgebra map $\mu \colon T^c(V) \otimes T^c(V) \to T^c(V)$ with such that $(T^c(V), \mu, d, 1)$ is a differential graded algebra. That is, a B_∞ structure is an A_∞ structure d together with a compatible multibrace structure μ.

If d and μ are given by families of multilinear maps (m_n) and $(m_{i,j})$ as above then the compatibility property is equivalent to $d_1\mu = \mu_1(d \otimes \mathrm{id} + \mathrm{id} \otimes d)$, that is, for each $i, j \geqslant 1$ there is an equality

$$\sum_{\substack{r\geqslant 1 \\ (\underline{i},\underline{j}) \in C_r^i \times C_r^j}} m_r(m_{i_1,j_1} \otimes \cdots \otimes m_{i_r,j_r}) \Delta_r^{\underline{i},\underline{j}}$$

---

[2] One should interpret [11, Proposition 1.3] and its proof with care; the diagram in the proof does not commute on elements of the form $a \otimes 1_{\mathbb{K}}$ for example.

$$= \sum_{\substack{p,q\geqslant 0\\ p+q<i}} m_{p+1+q,j}\left((\mathrm{id}_V^{\otimes p}\otimes m_{i-p-q}\otimes \mathrm{id}_V^{\otimes q})\otimes \mathrm{id}_V^{\otimes j}\right)$$
$$+ \sum_{\substack{p,q\geqslant 0\\ p+q<j}} m_{i,p+1+q}\left(\mathrm{id}_V^{\otimes i}\otimes(\mathrm{id}_V^{\otimes p}\otimes m_{j-p-q}\otimes \mathrm{id}_V^{\otimes q})\right) \tag{2.14}$$

between linear maps $V^{\otimes i}\otimes V^{\otimes j}\to V$. If one evaluates these, to write the relations as equalities between elements, then Koszul signs appear.

Definition 2.3.1. A B_∞ structure on a graded vector space V is given by a coalgebra map $\mu\colon T^c(V)\otimes T^c(V)\to T^c(V)$ satisfying the associative and unit laws, together with a differential d on the coalgebra $T^c(V)$ which is a derivation with respect to μ. Alternatively, it is given by collections of multilinear maps $\left(m_{i,j}\colon V^{\otimes i}\otimes V^{\otimes j}\to V\right)_{i,j\geqslant 1}$ of degree 0 and $(m_n\colon V^{\otimes n}\to V)_{n\geqslant 1}$ of degree -1 that satisfy the families of relations (2.4), (2.8) and (2.14).

Remark 2.3.2. This notion might more properly be called a *shifted* B_∞ algebra. The notion of B_∞ algebra is usually defined via a differential bialgebra structure on the bar construction of a graded space, rather than on the tensor coalgebra as we have done here. As mentioned in Remark 2.1.2 above, this leads to differences in degrees and signs: a B_∞ structure on V corresponds to a shifted B_∞ structure on $V[1]$ and is therefore equipped with degree $n-2$ operations $s^{-1}m_n s^{\otimes n}\colon V^{\otimes n}\to V$ and degree $i+j-1$ operations $s^{-1}m_{i,j}(s^{\otimes i}\otimes s^{\otimes j})\colon V^{\otimes i}\otimes V^{\otimes j}\to V$, satisfying relations analogous to (2.4), (2.8) and (2.14), but with some additional Koszul signs arising from commuting the operations with the suspension maps s.

The non-commutative quasi-shuffle algebra of Example 2.2.4 can be combined with the trivial A_∞ algebra of Example 2.1.3 as follows.

Proposition 2.3.3 *(Quasi-trivial B_∞ algebras). For any differential graded algebra $(V,\bullet,\partial)$ there is a B_∞ structure $\mu_\bullet, d_\partial$ on V uniquely defined by*

$$\begin{tikzcd}
\overline{T}^c(V)\otimes\overline{T}^c(V) \arrow[r, "\mu_\bullet"] \arrow[d, "p_1\otimes p_1"'] & \overline{T}^c(V) \arrow[d, "p_1"] \\
V\otimes V \arrow[r, "\bullet"] & V,
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{T}^c(V) \arrow[r, "d_\partial"] \arrow[d, "p_1"'] & \overline{T}^c(V) \arrow[d, "p_1"] \\
V \arrow[r, "\partial"] & V.
\end{tikzcd}$$

All operations $m_{i,j}$ and m_n are zero, except $m_{1,1}=\bullet$, $m_1=\partial$, and $m_{0,1}=m_{1,0}=\mathrm{id}$, see (1.3), (2.6).

Proof. The linear maps

$$d_1=\partial p_1\colon\ \overline{T}^c(V)\longrightarrow V,$$
$$\mu_1=p_1\bullet p_1\colon \overline{T}^c(V)^{\otimes 2}\to V,$$

considered in Examples 2.1.3 and 2.2.4 determine respectively a coderivation d_∂ on $T^c(V)$ and a coalgebra map $\mu_\bullet=\widetilde{Ш}\colon T^c(V)^{\otimes 2}\to T^c(V)$ satisfying the unit law. The B_∞ algebra axioms follow by uniqueness of extensions:

1. The binary operation $\mu_\bullet$ is associative if and only if $\bullet$ is associative, since $\mu_\bullet(\mu_\bullet \otimes \mathrm{id})$ and $\mu_\bullet(\mathrm{id} \otimes \mu_\bullet)$ are coalgebra maps whose projections to V are $(p_1 \bullet p_1) \bullet p_1$ and $p_1 \bullet (p_1 \bullet p_1)$.
2. $d_\partial^2 = 0$ if and only if $\partial^2 = 0$, since d_∂^2 is a coderivation satisfying $p_1 d_\partial^2 = \partial p_1 d_\partial = \partial^2 p_1$.
3. d_∂ is a derivation of $\mu_\bullet$ if and only if ∂ is a derivation of $\bullet$, since $\mu_\bullet(d_\partial \otimes \mathrm{id} + \mathrm{id} \otimes d_\partial) - d_\partial \mu_\bullet$ is a coderivation whose projection to V is $\partial p_1 \bullet p_1 + p_1 \bullet \partial p_1 - \partial(p_1 \bullet p_1)$.

Thus given a differential graded algebra $(V, \bullet, \partial)$ we have a B_∞ algebra $(V, \mu_\bullet, d_\partial)$. □

2.4. Twistings

By a *twisting* τ on a graded vector space V we mean a coalgebra automorphism of $T^c(V)$. By cofreeness a twisting τ is determined by $\tau_1 \colon T^c(V) \to V$, that is, by a sequence of multilinear maps $t_n \colon V^{\otimes n} \to V$, $n \geqslant 1$. For simplicity we will assume the component $t_1 = \mathrm{id}_V$. The inverse τ^{-1} is determined by $u_n \colon V^{\otimes n} \to V$, $n \geqslant 1$, that can be calculated recursively from $(\tau^{-1})_1 \circ \tau = p_1$, that is, $u_1 = \mathrm{id}_V$ and

$$\sum_{0<r\leqslant n} u_r \tau_1^{\otimes r} \Delta^{(r-1)}{}_{|V^{\otimes n}}$$
$$= u_n + \sum_{\substack{0<r<n \\ \underline{i} \in \overline{C_r^n}}} u_r \circ (t_{i_1} \otimes \cdots \otimes t_{i_r}) \;=\; 0 \colon V^{\otimes n} \to V \qquad (n \geqslant 2)$$

where $\overline{C_r^n} \subseteq C_r^n$ is the subset of sequences of strictly positive integers.

The twistings that interest us here will arise from associative binary operations, as follows.

Lemma 2.4.1. *Suppose V is a graded vector space equipped with an associative binary operation $\circledcirc$. Then the sequence of iterated multiplication maps $\left(\circledcirc^{(n-1)} \colon V^{\otimes n} \to V\right)_{n\geqslant 1}$ determines a twisting $\tau^\circledcirc$ whose inverse is determined by $\left((-1)^{n-1} \circledcirc^{(n-1)} \colon V^{\otimes n} \to V\right)_{n\geqslant 1}$.*

If $T^c(V)$ is considered as a free algebra, then the projection $\tau_1^\circledcirc = p_1\tau^\circledcirc \colon \overline{T}^c(V) \to V$ is a non-unital algebra homomorphism. If V has a unit and $\widetilde{p_1}$ is as defined in (2.10), then $\widetilde{p_1}\tau^\circledcirc \colon T^c(V) \to V$ is a unital algebra homomorphism.

Proof. As $\circledcirc^{(0)} = \mathrm{id}_V$, we know τ is invertible. For the inverse see for example [14, Section 2.2]. By definition the maps $p_1\tau^\circledcirc$ and $\widetilde{p_1}\tau^\circledcirc$ are given by multiplication of generators

$$\begin{tikzcd}
\overline{T}^c(V) \arrow[r, "\tau^\circledcirc"] \arrow[dr, "{(\mathrm{id},\circledcirc,\circledcirc^{(2)},\dots)}"'] & \overline{T}^c(V) \arrow[d, "p_1"] \\
 & V,
\end{tikzcd}
\qquad
\begin{tikzcd}
T^c(V) \arrow[r, "\tau^\circledcirc"] \arrow[dr, "{(\eta,\mathrm{id},\circledcirc,\circledcirc^{(2)},\dots)}"'] & T^c(V) \arrow[d, "\widetilde{p_1}"] \\
 & V.
\end{tikzcd}$$

These are just the counits of the free-forget adjunctions between vector spaces and (non-unital or unital) algebras: the homomorphisms given by extending $\mathrm{id} \colon V \to V$ to the respective free algebra. □

The A_∞ case of the following result was thoroughly investigated in [14].

Proposition 2.4.2. *Given a B_∞ algebra (V, μ, d), then for any twisting τ on V there is a twisted B_∞ algebra (V, μ^τ, d^τ) defined by*

$$\begin{tikzcd}
T^c(V) \otimes T^c(V) \arrow[r, "\mu^\tau"] \arrow[d, "\tau\otimes\tau"'] & T^c(V) \arrow[d, "\tau"] \\
T^c(V) \otimes T^c(V) \arrow[r, "\mu"] & T^c(V),
\end{tikzcd}
\qquad
\begin{tikzcd}
T^c(V) \arrow[r, "d^\tau"] \arrow[d, "\tau"'] & T^c(V) \arrow[d, "\tau"] \\
T^c(V) \arrow[r, "d"] & T^c(V).
\end{tikzcd}$$

Proof. The transfer of the original structure μ, d along the isomorphism τ gives the twisted structure $\mu^\tau = \tau^{-1}\mu\tau^{\otimes 2}$ and $d^\tau = \tau^{-1}d\tau$. Since τ is a coalgebra automorphism it is clear that μ^τ and d^τ still define a coalgebra map and a coderivation respectively and satisfy the B_∞ axioms (associativity, square zero, compatibility). $\square$

3. B_∞ Algebras and 2-Associative Differential (Bi)algebras

3.1. The Underlying B_∞ Algebra

We can now return to the construction in Example 2.2.5 of multibrace algebras from spaces with two associative operations, and generalise it to B_∞ algebras.

Definition 3.1.1. (2-associative differential algebra) A 2-associative differential algebra is a differential graded algebra $(V, \bullet, 1, \partial)$ endowed with a second associative binary operation $\circledcirc$, with the same unit 1 but not required to satisfy any other compatibility relation with $\bullet$ or with ∂. Together with their homomorphisms they form a category $d\mathcal{A}s^{1,1}$-alg.

To any 2-associative differential algebra V there is an underlying B_∞ structure on V defined by twisting the quasi-trivial structure:

Theorem 3.1.2 *(The underlying B_∞ algebra). For any 2-associative differential algebra $(V, \bullet, \circledcirc, 1, \partial)$ there is an underlying B_∞ structure $\mu^{\circledcirc}_{\bullet}, d^{\circledcirc}_{\partial}$ on V uniquely defined by*

$$\begin{tikzcd}
\overline{T}^c(V) \otimes \overline{T}^c(V) \arrow[r, "\mu^{\circledcirc}_{\bullet}"] \arrow[d, "\tau^{\circledcirc}_1\otimes\tau^{\circledcirc}_1"'] & \overline{T}^c(V) \arrow[d, "\tau^{\circledcirc}_1"] \\
V\otimes V \arrow[r, "\bullet"] & V,
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{T}^c(V) \arrow[r, "d^{\circledcirc}_{\partial}"] \arrow[d, "\tau^{\circledcirc}_1"'] & \overline{T}^c(V) \arrow[d, "\tau^{\circledcirc}_1"] \\
V \arrow[r, "\partial"] & V,
\end{tikzcd}
\tag{3.1}$$

where $\tau^{\circledcirc}$ is the twisting determined by $\circledcirc$.

Proof. Twist with $\tau^{\circledcirc}$ the quasi-trivial B_∞ structure $\mu_\bullet, d_\partial$ on V, by Propositions 2.3.3 and 2.4.2. $\square$

Homomorphisms of 2-associative differential algebras define homomorphisms of the underlying B_∞ algebras, and we have a functor

$$d\mathcal{A}s^{1,1}\text{-alg} \to B_\infty\text{-alg}. \tag{3.2}$$

There are many useful reformulations of the defining property of the underlying B_∞ structure $\mu^{\odot}_{\bullet}, d^{\odot}_{\partial}$. For example, it is the unique structure with the following equivalent properties:

- $\tau^{\odot}_1 \colon (\overline{T}^c(V), \mu^{\odot}_{\bullet}, d^{\odot}_{\partial}) \to (V, \bullet, \partial)$ is a non-unital differential graded algebra homomorphism.
- $\widetilde{p_1}\tau^{\odot} \colon (T^c(V), \mu^{\odot}_{\bullet}, d^{\odot}_{\partial}, 1_{\mathbb{K}}) \to (V, \bullet, \partial, 1)$ is a differential graded algebra homomorphism.
- there exists a (necessarily unique) 2-associative differential algebra homomorphism

$$\varepsilon_V \colon (T^c(V), \mu^{\odot}_{\bullet}, \odot, d^{\odot}_{\partial}, 1_{\mathbb{K}}) \to (V, \bullet, \odot, \partial, 1) \quad \text{such that} \quad (\varepsilon_V)_{|V} = \mathrm{id}_V, \tag{3.3}$$

  where the multiplication $\odot$ on $T^c(V)$ is concatenation. In fact $\varepsilon_V = \widetilde{p_1}\tau^{\odot}$, see Lemma 2.4.1.
- the components $m^r_{i,j} \colon V^{\otimes i} \otimes V^{\otimes j} \to V^{\otimes r}$ and $m^k_n \colon V^{\otimes n} \to V^{\otimes k}$ satsify

$$\sum_{r=1}^{i+j} \odot^{(r-1)} m^r_{i,j} = \bullet(\odot^{(i-1)} \otimes \odot^{(j-1)}), \qquad \sum_{k=1}^{n} \odot^{(k-1)}\, m^k_n = \partial\ \odot^{(n-1)}\,. \tag{3.4}$$

The first of the defining equations (3.4) is just Equation (2.12) of Example 2.2.5, and so the multibrace structure can be calculated using the recursive formula (2.13) (compare [12, Section 3.3 and Proposition 3.4]). The second says that the A_∞ structure can be calculated, using (2.1), from

$$m_n \;=\; \partial\ \odot^{(n-1)} \;-\; \sum_{k=2}^{n} \odot^{k-1} \sum_{\substack{i,j\geqslant 0\\ i+1+j=k}} \mathrm{id}_V^{\otimes i} \otimes m_{n-i-j} \otimes \mathrm{id}_V^{\otimes j}\,. \tag{3.5}$$

In fact this recursion has a very simple explicit solution (see [14, Example 1.3]) with

$$\begin{aligned} m_2(v_1 \otimes v_2) &= \partial(v_1 \odot v_2) - \partial(v_1) \odot v_2 - (-1)^{|v_1|}\, v_1 \odot \partial(v_2),\\ m_3(v_1 \otimes v_2 \otimes v_3) &= \partial(v_1 \odot v_2 \odot v_3) - \partial(v_1 \odot v_2) \odot v_3 - (-1)^{|v_1|}\, v_1 \odot \partial(v_2 \odot v_3),\\ m_n(v_1 \otimes \cdots \otimes v_n) &= \partial(v_1 \odot \ldots \odot v_n) - \partial(v_1 \odot \cdots \odot v_{n-1}) \odot v_n\\ &\quad - (-1)^{|v_1|}\, v_1 \odot \partial(v_2 \odot \cdots \odot v_n)\\ &\quad + (-1)^{|v_1|}\, v_1 \odot \partial(v_2 \odot \cdots \odot v_{n-1}) \odot v_n \text{ for all } n \geqslant 4. \end{aligned}$$

As well as the above properties of the natural projection $T^c(V) \to V$ for any 2-associative differential algebra V, we also discuss the natural inclusion $A \to T^c(A)$ for any B_∞ algebra A.

Theorem 3.1.3. *Let A be a B_∞ algebra, with structure maps given by a multiplication and a differential on the tensor coalgebra $V = T^c(A)$ that we denote $\bullet$ and ∂, respectively. Consider*

- *the 2-associative differential algebra $(V, \bullet, \odot, \partial)$,*
  (where the second associative multiplication $\odot$ on V is concatenation of tensors)

- *the underlying B_∞ structure $\mu^{\circledcirc}_{\bullet}$ and $d^{\circledcirc}_{\partial}$ on V given by Theorem 3.1.2.*

Then the inclusion map $\iota_1\colon A \to V$ is a homomorphism of B_∞ algebras.

Proof. We will show that the following relations hold,

$$(R^r_{i,j}) \qquad m^r_{i,j}(\iota_1^{\otimes i} \otimes \iota_1^{\otimes j}) = \iota_1^{\otimes r}\overline{m}^r_{i,j}, \qquad\qquad (S^k_n) \qquad m^k_n \iota_1^{\otimes n} = \iota_1^{\otimes k}\overline{m}^k_n,$$

where a bar distinguishes the operations on A from those on V. We first observe the concatenation product satisfies $\odot^{(n-1)}\iota_1^{\otimes n} = \iota_n\colon A^{\otimes n} \to V$. Applying equations (3.4) to $\iota_1^{\otimes i} \otimes \iota_1^{\otimes j}$ and to $\iota_1^{\otimes n}$ we get

$$\sum_{r=1}^{i+j} \odot^{(r-1)} m^r_{i,j}(\iota_1^{\otimes i}\otimes\iota_1^{\otimes j}) = \bullet(\iota_i\otimes\iota_j) = \sum_{r=1}^{i+j} \iota_r \overline{m}^r_{i,j}, \quad \sum_{k=1}^{n} \odot^{(k-1)} m^k_n \iota_1^{\otimes n} = \partial\iota_n = \sum_{k=1}^{n} \iota_k \overline{m}^k_n.$$

Next suppose inductively that $(R^1_{i',j'})$ holds for $i' \leqslant i$, $j' \leqslant j$ except $(i',j') = (i,j)$, and that $(S^1_{n'})$ holds for $n' < n$. Then $(R^r_{i,j})$ holds for $r \geqslant 2$ by (2.7), and (S^k_n) holds for $k \geqslant 2$ by (2.1). That is, for $r \geqslant 2$ and $k \geqslant 2$ we have

$$\odot^{(r-1)} m^r_{i,j}(\iota_1^{\otimes i} \otimes \iota_1^{\otimes j}) = \odot^{(r-1)} \iota_1^{\otimes r}\overline{m}^r_{i,j} = \iota_r \overline{m}^r_{i,j}, \ \odot^{(k-1)}\, m^k_n \iota_1^{\otimes n} = \odot^{(k-1)} \iota_1^{\otimes k}\overline{m}^k_n = \iota_k \overline{m}^k_n.$$

All terms except those for $r = 1$ and $k = 1$ in the summations above therefore cancel, leaving the relations $(R^1_{i,j})$ and (S^1_n) as required. ☐

3.2. Universal Enveloping 2-Associative Differential Algebras

The underlying B_∞ algebra functor has a left adjoint, the *universal enveloping* functor

$$U : B_\infty\text{-alg} \to d\mathcal{A}s^{1,1}\text{-alg}.$$

It has the following explicit construction. Given a B_∞ algebra A, consider the free 2-associative differential algebra $F_{d\mathcal{A}s^{1,1}}(A)$, so that composition with the inclusion $\alpha : A \to F_{d\mathcal{A}s^{1,1}}(A)$ gives a bijection between maps $\widetilde{f}\colon F_{d\mathcal{A}s^{1,1}}(A) \to V$ in $d\mathcal{A}s^{1,1}$ and linear maps $f\colon A \to V$. Observe that $\widetilde{f}\colon F_{d\mathcal{A}s^{1,1}}(A) \to V$ will always define a homomorphism of the underlying B_∞ algebras, and that $f = \widetilde{f}\alpha$ defines a B_∞ algebra homomorphism if and only if $\widetilde{f}$ vanishes on elements of the form

$$\alpha(\overline{m}_{i,j}(a \otimes b)) - m_{i,j}(\alpha^{\otimes(i+j)}(a \otimes b), a \in A^{\otimes i},\ b \in A^{\otimes j}, \tag{3.6}$$

$$\alpha(\overline{m}_n(c)) - m_n(\alpha^{\otimes n}(c)), c \in A^{\otimes n}. \tag{3.7}$$

Here $(m_{i,j}), (m_n)$ are the associated B_∞ structure maps on $F_{d\mathcal{A}s^{1,1}}(A)$ and $(\overline{m}_{i,j}), (\overline{m}_n)$ those on A. Let I be the two-sided ideal of $F_{d\mathcal{A}s^{1,1}}(A)$ generated by the above elements. Then the universal enveloping 2-associative differential algebra can be defined by the quotient

$$U(A) \;=\; F_{d\mathcal{A}s^{1,1}}(A)/I$$

since by construction we have a bijection between homomorphisms $\widetilde{f}\colon F_{d\mathcal{A}s^{1,1}}(A)/I \to V$ of 2-associative differential algebras and homomorphisms $f\colon A \to V$ of B_∞ algebras.

In view of the results of sect. 3.1 there is a simpler presentation of the universal enveloping 2-associative differential algebra:

Proposition 3.2.1. *There is a left adjoint to the underlying B_∞ algebra functor defined by*

$$T^c : B_\infty\text{-}alg \to d\mathcal{A}s^{1,1}\text{-}alg,$$

sending a B_∞ algebra A to $(T^c(A), \partial, \bullet, \odot)$, where ∂ and $\bullet$ are the B_∞ structure maps on A and $\odot$ is concatenation of tensors.

This functor is isomorphic to the universal enveloping algebra U.

Proof. Unit and counit maps for the required adjunction between T^c and the underlying B_∞ algebra functor can be defined, by inclusion and multiplication of generators,

$$\eta_A = \iota_1 \colon A \to T^c(A) \in B_\infty\text{-alg}, \qquad \varepsilon_V = \widetilde{p_1}\tau^\circ \colon T^c(V) \to V \in d\mathcal{A}s^{1,1}\text{-alg},$$

see Theorem 3.1.3 and equation (3.3). Now the fact that $\widetilde{p_1}\tau^\circ$ is right inverse to both $T^c(\iota_1)$ and ι_1 gives the triangle identities

$$\begin{tikzcd} T^c(A) \arrow[r, "T^c(\eta_A)"] \arrow[dr, "\mathrm{id}"'] & T^cT^c(A) \arrow[d, "\varepsilon_{T^c(A)}"] \\ & T^c(A), \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[dr, "\mathrm{id}"'] & T^c(V) \arrow[d, "\varepsilon_V"] \\ & V, \end{tikzcd}$$

and we indeed have an adjunction. The two left adjoints U, T^c are thus canonically isomorphic: the required natural isomorphism $\iota_1' \colon U(A) \xrightarrow{\cong} T^c(A)$ is the 2-associative differential algebra map that corresponds, under the former adjunction, to the unit $\iota_1 \colon A \to T^c(A)$ of the latter. □

3.3. The Equivalence of Categories

Definition 3.3.1 (2-associative differential bialgebra). A 2-associative differential bialgebra is a 2-associative differential algebra $(V, \partial, \bullet, \odot, 1)$ endowed with a comultiplication Δ such that $(V, \partial, \bullet, \Delta, 1)$ is a differential graded bialgebra and $(V, \odot, \Delta, 1)$ is an infinitesimal bialgebra. Together with their homomorphisms they form a category $d\mathcal{A}s^{1,1}$-bialg.

A 2-associative differential bialgebra is termed conilpotent or cofree if it is conilpotent, respectively, cofree, as a unital coalgebra.

Our aim now is to show there is an equivalence of categories

$$\begin{tikzcd} B_\infty\text{-alg} \arrow[r, shift left, "U"] & d\mathcal{A}s^{1,1}\text{-bialg}_{\mathrm{conil}} \arrow[l, shift left, "\mathrm{Prim}"] \end{tikzcd}.$$

By Proposition 3.2.1 the universal enveloping 2-associative differential algebra is just the tensor coalgebra, so the functor in one direction is clear:

Proposition 3.3.2. *The universal enveloping 2-associative differential algebra of a B_∞-algebra is naturally endowed with the structure of a cofree 2-associative differential bialgebra*

$$U(A) = (T^c(A), \partial, \bullet, \odot, \Delta, 1_{\mathbb{K}}),$$

where the comutiplication Δ is given by deconcatenation of tensors, and there is a functor

$$U\colon B_\infty\text{-}alg \longrightarrow d\mathcal{A}s^{1,1}\text{-}bialg.$$

Proof. By definition of B_∞ structures $(T^c(A), \partial, \bullet, \Delta, 1_{\mathbb{K}})$ is a differential graded bialgebra, and $(T^c(A), \odot, \Delta, 1_{\mathbb{K}})$ is a (free, cofree) unital infinitesimal bialgebra by Example 1.7.6. □

The following result is the promised analogue of the Milnor–Moore theorem, and should be compared with Corollary 1.7.9.

Theorem 3.3.3. *The underlying B_∞ algebra structure on a 2-associative differential bialgebra restricts to the primitives, and the functor*

$$\mathrm{Prim}\colon d\mathcal{A}s^{1,1}\text{-}bialg \longrightarrow B_\infty\text{-}alg$$

is right adjoint to U. The unit of the adjuntion is a natural isomorphism

$$\eta_A\colon A \xrightarrow{\cong} \mathrm{Prim}(U(A)),$$

while the counit

$$\varepsilon_V\colon U(\mathrm{Prim}(V)) \longrightarrow V$$

is a monomorphism and is an isomorphism if and only if V is conilpotent.

In particular, the functors U and Prim *define an adjoint equivalence of categories between B_∞ algebras and the full coreflective subcategory of $d\mathcal{A}s^{1,1}$-bialg given by the conilpotent objects.*

Proof. If $(V, \partial, \bullet, \odot, 1, \Delta, \varepsilon)$ is a 2-associative differential bialgebra then the conilpotent radical is a 2-associative differential subbialgebra: it is closed under ∂, $\bullet$, Δ by the usual differential bialgebra axioms and is closed under $\odot$ by Lemma 1.7.3 (2). It therefore follows from the inductive definitions (2.13) and (3.5) that it is closed under the underlying B_∞ structure maps.

It remains to show that if $(W, \partial, \bullet, \odot, 1, \Delta, \varepsilon)$ is a conilpotent 2-associative differential bialgebra then the underlying B_∞ structure restricts to the primitives. By Proposition 1.7.8 we know that W is naturally isomorphic as a unital infinitesimal bialgebra to $T^{fc}(\mathrm{Prim}(W))$, so we can assume without loss of generality that $(W, \partial, \bullet) = (T^c(\mathrm{Prim}(W)), \partial, \bullet)$. Thus $\mathrm{Prim}(W)$ has a B_∞ structure and $\mathrm{Prim}(W) \to W$ is the inclusion of a B_∞ subalgebra by Theorem 3.1.3.

Thus, for a general 2-associative differential bialgebra V, we have inclusions of B_∞ subalgebras $\mathrm{Prim}(V) = \mathrm{Prim}(R(V)) \to R(V) \to V$, and the B_∞ algebra homomorphism $A \to T^c(A) = U(A)$ of Theorem 3.1.3 gives a B_∞ algebra isomorphism onto its image,

$$\eta_A\colon A \xrightarrow{\cong} \mathrm{Prim}(U(A)).$$

The extension of the inclusion $\mathrm{Prim}(V) \to V$ of the underlying B_∞ subalgebra defines the counit

$$\varepsilon_V\colon U(\mathrm{Prim}(V)) \longrightarrow V.$$

This factors as an isomorphism onto its image, the conilpotent radical:

$$\varepsilon_V\colon U(\mathrm{Prim}(V)) = T^c(\mathrm{Prim}(R(V))) \xrightarrow{\cong} R(V) \subseteq V.$$

□

Corollary 3.3.4. *The categories of B_∞ algebras, of conilpotent 2-associative differential bialgebras and of cofree 2-associative differential bialgebras are equivalent.*

Proof. This follows from the theorem, noting that conilpotent implies cofree for $d\mathcal{A}s^{1,1}$ bialgebras since $V \cong T^c(\mathrm{Prim}(V))$. □

Remark 3.3.5. The category of cofree differential Hopf algebras $(T^c(A), d, \mu, \Delta)$ has essentially the same objects as the categories above, but more morphisms.

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