The kernel of the Gysin homomorphism for Chow groups of zero cycles

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This is ongoing work with Rina Paucar Rojas (IMCA, Lima in Peru).

Let S be a smooth projective connected surface over an algebraically closed field k and Σ the linear system of a very ample divisor D on S. Let $d := \dim(\Sigma)$ be the dimension of Σ and

$$\varphi_\Sigma : S \hookrightarrow \mathbb{P}^d$$

the closed embedding of S into $\mathbb{P}^d$, induced by Σ.

For any closed point $t \in \Sigma \cong \mathbb{P}^{d^*}$, let C_t be the corresponding hyperplane section on S, and let

$$r_t : C_t \hookrightarrow S$$

be the closed embedding of the curve C_t into S.

Let Δ be the discriminant locus of Σ, that is,

$$\Delta := \{t \in \Sigma : C_t \text{ is singular}\}.$$

Then

$$U := \Sigma \setminus \Delta = \{t \in \Sigma : C_t \text{ is smooth}\}.$$

Let

$$r_t^* : H^1(C_t, \mathbb{Z}) \to H^3(S, \mathbb{Z})$$

be the Gysin homomorphism on cohomology groups induced by r_t, whose kernel $H^1(C_t, \mathbb{Z})_{\text{van}}$ is called the *vanishing cohomology of* C_t (see [VoiII], 3.2.3).

Let $J_t = J(C_t)$ be the Jacobian of the curve C_t and let B_t be the abelian subvariety of the abelian variety J_t corresponding to the Hodge substructure $H^1(C_t, \mathbb{Z})_{\text{van}}$ of $H^1(C_t, \mathbb{Z})$.

Let $\mathrm{CH}_0(S)_{\deg=0}$ be the Chow group of 0-cycles of degree zero on S, and for any closed point $t \in \Sigma$, let $\mathrm{CH}_0(C_t)_{\deg=0}$ be the Chow group of 0-cycles of degree zero on C_t.

For any closed point $t \in \Sigma$, let

$$r_t^* : \mathrm{CH}_0(C_t)_{\deg=0} \to \mathrm{CH}_0(S)_{\deg=0}$$

be the Gysin pushforward homomorphism on the Chow groups of degree zero 0-cycles of C_t and S, respectively, induced by r_t, whose kernel

$$G_t = \ker(r_t^*)$$

is called the *Gysin kernel* associated with the hyperplane section C_t.

Intermezzo on Cycles and Chow groups

Let X denote a smooth projective variety over an algebraically closed field k.

Definition

- An algebraic cycle of dimension r or simply an r-cycle is a finite formal linear combination

$$Z = \sum n_i Z_i,$$

where $n_i \in \mathbb{Z}$ and Z_i is a subvariety of X of dimension r.

- The group of r-cycles is denoted by $Z_r(X)$.
- Thinking in terms of codimension and if X is of dimension n we write

$$Z_r(X) = Z^{n-r}(X).$$

$Z^{n-r}(X)$ is the group of cycles of codimension $(n-r)$ on X.

Examples

Let X be a smooth projective variety of dimension n.

1. The 0-cycles on X are finite formal linear combinations

$$Z = \sum n_i P_i,$$

where $n_i \in \mathbb{Z}$ and P_i is a point on X. The group of 0-cycles is denoted by $Z_0(X)$.

2. The cycles of codimension 1 on X are finite formal linear combinations

$$Z = \sum n_i Z_i,$$

where $n_i \in Z$ and Z_i is a subvariety of codimension 1 of X. The cycles of codimension 1 are also called divisors. The group of cycles of codimension 1 is denoted by $Z^1(X)$ or by $\mathrm{Div}(X)$.

Rational and algebraic equivalence

Definition

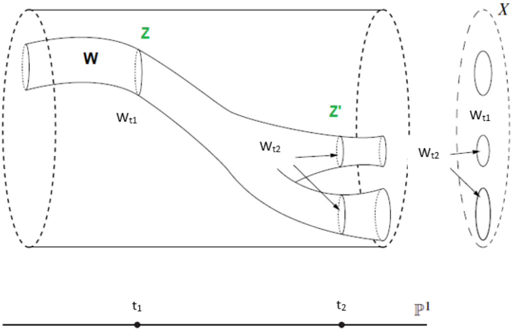
- Two r-cycles Z_1 and Z_2 on X are rationally equivalent if there exits a family of r-cycles parametrized by $\mathbb{P}^1$ interpolating between them, i.e. if there is $W \in Z_{r+1}(\mathbb{P}^1 \times X)$ not contained in any fiber $\{t\} \times X = \mathrm{pr}_1^{-1}(t), t \in \mathbb{P}^1$, such that defining by

$$W(t) := [W \cap (\{t\} \times X)]$$

  the r-cycle obtained by intersecting W with the fiber $\{t\} \times X$ over t, we have

$$Z_1 = W(t_1) \text{ and } Z_2 = W(t_2) \text{ for some } t_1, t_2 \in \mathbb{P}^1.$$

- If in the above definition we replace $\mathbb{P}^1$ by any smooth curve then we say that Z_1 and Z_2 are algebraically equivalent.



Rational equivalence between two cycles z and z' on X.

Properties

- The r-cycles rationally equivalent to 0 on X, denoted by
$$Z_r(X)_{\mathrm{rat}} = \{Z \in Z_r(X) : Z \sim_{\mathrm{rat}} 0\}$$
form a subgroup of $Z_r(X)$.
- The r-cycles algebraically equivalent to 0 on X, denoted by
$$Z_r(X)_{\mathrm{alg}} = \{Z \in Z_r(X) : Z \sim_{\mathrm{alg}} 0\}$$
form a subgroup of $Z_r(X)$.

Chow groups

Definition

The Chow group of r-cycles of X is the factor group

$$\mathrm{CH}_r(X) = Z_r(X)/Z_r(X)_{\mathrm{rat}}$$

of the group of r-cycles modulo the group of r-cycles rationally equivalent to 0, i.e. the group of rational equivalence classes of r-cycles.

Results on codimension 1-cycles or divisors

In this case one has good results.

- We have
$$\mathrm{CH}^1(X) = Z^1(X)/Z^1(X)_{\mathrm{rat}} \cong \mathrm{Pic}(X) \cong H^1_{\mathrm{Zar}}(X, \mathcal{O}_X^*),$$
where $\mathrm{Pic}(X)$ is the group of isomorphism classes of invertible sheaves on X called the Picard group, and $H^1_{\mathrm{Zar}}(X, \mathcal{O}_X^*)$ is the group of isomorphism classes of line bundles on X. One can make this group into a scheme.
- We have
$$\mathrm{NS}(X) := Z^1(X)/Z^1(X)_{\mathrm{alg}},$$
it is a finitely generated abelian group, called the Neron-Severi group of X.

The quotient group

$$A^1(X) := Z^1(X)_{\text{alg}}/Z^1(X)_{\text{rat}}$$

is the connected component of unity of the scheme $\text{Pic}(X)$ denoted by $\text{Pic}^0(X)$. In $\text{char}(k) = 0$ it has the structure of an Abelian variety called the Picard variety. $A^1(X)$ is representable and we can think of it as a torus.

In contrast to the case of cycles of codimension 1 very little is known about the above groups of cycles classes. For example, if $r > 1$

- We do not know if
$$Z^r(Z)/Z^r(Z)_{\mathrm{alg}}$$
is a finitely generated abelian group.
- In general
$$A^r(X) = Z^r(X)_{\mathrm{alg}}/Z^r(X)_{\mathrm{rat}}$$
is not representable. That is:

  In studying algebraic cycles we encounter objects which are geometric in content and simultaneously not representable!

Let $U = \Sigma \setminus \Delta$. For the formulation of the following theorem see also [VoiII], pp. 304-5.

Theorem (Paucar, 2022)

(a) *For each $t \in U$ there is an abelian variety $A_t \subset B_t$ such that*

$$G_t = \ker(r_{t^*}) = \bigcup_{countable} \textit{translates of } A_t$$

(b) *For a very general $t \in U$ (i.e. for every t in a c-open subset U_0 of U) either*

1. *$A_t = B_t$, and then $G_t = \bigcup_{countable}$ translates of B_t, or*
2. *$A_t = 0$, and then G_t is countable.*

(c) *If $alb_S : \mathrm{CH}_0(S)_{\mathrm{hom}} \to \mathrm{Alb}(S)$ is not an isomorphism, for a very general t in U, then G_t is countable.*

- The subset U_0 is countable open $\rightsquigarrow$ allows to apply for all t in U_0 in a uniform way the irreducibility of the monodromy representation on the vanishing cohomology of a smooth section (see [DK73], [D74] for the étale cohomology, [La81] for the singular cohomology and [VoiII] in a Hodge theoretical context for complex algebraic varieties).
- this is done by viewing $U = \Sigma \setminus \Delta$ as an integral algebraic scheme over k and by passing to the general fiber, i.e. for each closed point t in U_0 there exists a scheme-theoretic isomorphism to the geometric generic point $\overline{\xi}$ over k', where k' is the minimal field of definition of S [Wei62]. This induces a scheme-theoretic isomorphism κ_t between the corresponding varieties C_t and $C_{\overline{\xi}}$ over k'.

This induces an isomorphism κ_t' between A_t and $A_{\overline{\xi}}$ compatible with the isomorphism on Chow groups induced by the isomorphism κ_t. Then by [BG20] $\kappa_t'(A_t) = A_{\overline{\xi}}$ and $\kappa_t'(B_t) = B_{\overline{\xi}}$ for every k-point in U_0.

- Goal: describe the Gysin kernel G_t for the points t in $U \setminus U_0$ where the local and global monodromy representations, i.e. the action of the fundamental groups $\pi_1(V, t)$, where $V = (\Sigma \setminus \Delta) \cap D$ with D a line containing t in the dual space $\mathbb{P}^{n^*}$ such that f_D is a Lefschetz pencil for S, and $\pi_1(U, t)$ on the vanishing cohomology $H^1(C_t, k')_{\mathrm{van}}$, are not fully understood.
- Approach: construct a stratification $\{\mathcal{U}_i \subseteq U\}_{i \in I}$ of U by countable open subsets for each of which the monodromy argument applies for all $t \in \mathcal{U}_i$ in a uniform way (i.e. for a partially ordered, at most countable set I we have $\mathcal{U}_i \subseteq \mathcal{U}_j$ if $i \leq j$ and two additional conditions on I for finiteness from below and for every map $\alpha : \mathrm{Spec}(k) \to U$ the set $\{i \in I : \alpha \text{ factors through } \mathcal{U}_i\}$ has a smallest element (cf. [GL19], Def. 5.2.1.1).

We then apply a convergence argument for the stratification $\{\mathcal{U}_i\}_{i\in I}$ (cf. [GL19], Def. 5.2.2.1) such that the monodromy argument applies in a uniform way for all $t \in U$, seen as the set-theoretic directed union $U = \bigcup_{\overrightarrow{i}} \mathcal{U}_i$.

Definition (Gaitsgory - Lurie, 2019)

Let $\mathcal{X}$ be an algebraic stack. A **stratification** of $\mathcal{X}$ consists of the following data:

(a) A partially ordered set A.

(b) A collection of open substacks $\{\mathcal{U}_\alpha \subseteq \mathcal{X}\}_{\alpha \in A}$ satisfying $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$ when $\alpha \leq \beta$.

This data is required to satisfy the following conditions:

- For each index $\alpha \in A$, the set $\{\beta \in A : \beta \leq \alpha\}$ is finite.
- For every field m and every map $\eta : \mathrm{Spec}(m) \to \mathcal{X}$, the set

$$\{\alpha \in A : \eta \text{ factors through } \mathcal{U}_\alpha\}$$

  has a least element.

Let $\mathcal{X}$ be an algebraic stack equipped with a stratification $\{\mathcal{U}_\alpha\}_{\alpha\in A}$. Notation: For each $\alpha \in A$, we let $\mathcal{X}_\alpha$ denote the reduced closed substack of $\mathcal{X}$ given by the complement of $\cup_{\beta<\alpha}\mathcal{U}_\beta$. Each $\mathcal{X}_\alpha$ is a locally closed substack of $\mathcal{X}$, called the **strata** of $\mathcal{X}$.

Remark

A stratification $\{\mathcal{U}_\alpha\}_{\alpha\in A}$ is determined by the partially ordered set A together with a collection of locally closed substacks $\{\mathcal{X}_\alpha\}_{\alpha\in A}$: each $\mathcal{U}_\alpha$ is an open substack of $\mathcal{X}$ and if m is a field, then a map $\eta : \mathrm{Spec}(m) \to \mathcal{X}$ factors through $\mathcal{U}_\alpha$ iff it factors through $\mathcal{X}_\beta$ for some $\beta \leq \alpha$. So one identifies the stratification of $\mathcal{X}$ with the locally closed substacks $\{\mathcal{X}_\alpha \subseteq \mathcal{X}\}_{\alpha\in A}$ (where the partial ordering of A is understood to be implicitely specified).

Remark

*If m is a field, then for any map $\eta : \mathrm{Spec}(m) \to \mathcal{X}$ there is a **unique** index $\alpha \in A$ such that η factors through $\mathcal{X}_\alpha$. In other words, $\mathcal{X}$ is the **set-theoretic** union of the locally closed substacks $\mathcal{X}_\alpha$.*

Definition

Let $m = \mathbb{C}$ or $m = \mathbb{F}_q$ and let $\mathcal{X}$ be an algebraic stack of finite type over $\operatorname{Spec}(m)$. A stratification $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ of $\mathcal{X}$ is **convergent** if there exists a finite collection of algebraic stacks $\mathcal{T}_1, \ldots, \mathcal{T}_n$ over $\operatorname{Spec}(m)$ with the following properties:

(1) For each $\alpha \in A$ there exists an integer $i \in \{1, 2, \ldots, n\}$ and a diagram of algebraic stacks $\mathcal{T}_i \xrightarrow{f} \tilde{\mathcal{X}}_\alpha \xrightarrow{g} \mathcal{X}_\alpha$ where the map f is a fiber bundle (locally trivial with respect to the étale topology) whose fibers are affine spaces of some fixed dimension d_α and the map g is surjective, finite and radicial.

(2) The nonnegative integers d_α in (1) satisfy $\sum_{\alpha \in A} q^{-d_\alpha} < \infty$.

(3) For $1 \leq i \leq n$, the algebraic stack $\mathcal{T}_i$ can be written as a stack-theoretic quotient Y/G, where Y is an algebraic space of finite type over m and G is a linear algebraic group over m acting on Y.

Remark

In the definition, for $\mathrm{char}(m) > 0$ *the hypothesis* (2) *guarantees that the set A is at most countable. For* $\mathrm{char}(m) = 0$ *we will assume A to be at most countable.*

We have the following

Lemma (Gysin sequence)

Let X and Y be smooth quasi-projective varieties over an algebraically closed field k, let $g : Y \to X$ *be a finite radicial morphism, and let* $U \subseteq X$ *be the complement of the image of g. Then there is a canonical fiber sequence*

$$C^{*-2d}(Y;\mathbb{Z}_l(-d)) \to C^*(X,\mathbb{Z}_l) \to C^*(U;\mathbb{Z}_l),$$

where d denotes the relative dimension $\dim(X) - \dim(Y)$.

This implies a corresponding result for algebraic stacks:

Lemma

Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth algebraic stacks of constant dimension over an algebraically closed field k, let $g : \mathcal{Y} \to \mathcal{X}$ be a finite radicial morphism, and let $\mathcal{U} \subseteq \mathcal{X}$ be the open substack of $\mathcal{X}$ complementary to the image of g. Then there is a canonical fiber sequence

$$C^{*-2d}(\mathcal{Y}; \mathbb{Z}_l(-d)) \to C^*(\mathcal{X}, \mathbb{Z}_l) \to C^*(\mathcal{U}; \mathbb{Z}_l),$$

where d denotes the relative dimension $\dim(\mathcal{X}) - \dim(\mathcal{Y})$.

We have the following

Theorem (Paucar - S., 2023)

Let $U := \Sigma \setminus \Delta$. There exists a convergent stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ of U by countable open subsets for each of which the irreducibility of the monodromy representation applies for all $t \in \mathcal{U}_\alpha$ in a uniform way such that the monodromy argument applies for all $t \in U$ in a uniform way, seen as the set-theoretic directed union $U = \bigcup_{\overrightarrow{i}} \mathcal{U}_i$.

Proof.

Let $d = \dim(U)$ and let $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ be the given stratification of U. The set A is at most countable (see Remark). By adding additional elements to A and assigning to each of those additional elements the empty substack of U, we may assume that A is infinite.

By assumption the set $\{\beta \in A : \beta \leq \alpha\}$ is finite for each $\alpha \in A$, hence we can chose an enumeration

$$A = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$$

where each initial segment is a downward-closed subset of A. We can then write U as the union of an increasing sequence of open substacks

$$\mathcal{U}_0 \hookrightarrow \mathcal{U}_1 \hookrightarrow \mathcal{U}_2 \hookrightarrow \ldots$$

where $\mathcal{U}_n$ is characterized by the requirement that if m is a field, then a map $\eta : \mathrm{Spec}(m) \to \mathcal{X}$ factors through $\mathcal{U}_n$ iff it factors through one of the substacks $\mathcal{X}_{\alpha_0}, \mathcal{X}_{\alpha_1}, \ldots, \mathcal{X}_{\alpha_n}$.

By hypothesis, there exists a finite collection $\{\mathcal{T}_i\}_{1 \leq i \leq s}$ of smooth algebraic stacks over $\mathrm{Spec}(m)$, where each $\mathcal{T}_i$ has some fixed dimension d_i and for each $n \geq 0$ there exists an index $i(n) \in \{1, \ldots, s\}$ and a diagram

$$\mathcal{T}_{i(n)} \stackrel{f_n}{\to} \tilde{\mathcal{X}}_{\alpha_n} \stackrel{g_n}{\to} \mathcal{X}_{\alpha_n},$$

where g_n is a finite radicial surjection and f_n is an étale fiber bundle whose fibers are affine spaces of some fixed dimension $e(n)$. Set

$$\overline{U} = U \times_{\mathrm{Spec}(m)} \mathrm{Spec}(\overline{m})$$

$$\overline{\mathcal{U}}_n = \mathcal{U}_n \times_{\mathrm{Spec}(m)} \mathrm{Spec}(\overline{m})$$

$$\overline{\mathcal{T}}_i = \mathcal{T}_i \times_{\mathrm{Spec}(m)} \mathrm{Spec}(\overline{m})$$

The map f_n induces an isomorphism in l-adic cohomology. Applying the above Lemma to the finite radicial map

$$g_n : \overline{\mathcal{X}}_{\alpha_n} \times_{\mathrm{Spec}(m)} \mathrm{Spec}(\overline{m}) \to \overline{\mathcal{U}}_n$$

we obtain fiber sequences

$$C^{*-2e'_n}(\overline{\mathcal{T}}_{i(n)}; \mathbb{Z}_l(-e'_n)) \to C^*(\overline{\mathcal{U}}_n, \mathbb{Z}_l) \to C^*(\overline{\mathcal{U}}_{n-1}; \mathbb{Z}_l),$$

where $e'_n = e_n + d - d_{i(n)}$ denotes the relative dimension of the map $\tilde{\mathcal{X}}_{\alpha_n} \to U$. We have a canonical equivalence

$$\Theta : C^*(\overline{U}; \mathbb{Z}_l) \simeq \lim_{\overleftarrow{n}} C^*(\overline{\mathcal{U}}_n, \mathbb{Z}_l).$$

By the increasing sequence of c-open substacks

$$\mathcal{U}_0 \hookrightarrow \mathcal{U}_1 \hookrightarrow \mathcal{U}_2 \hookrightarrow$$

and using Galois descent with respect to the corresponding simplicial complex $\mathcal{N}(U)$ we obtain $U = \lim_{\substack{\rightarrow \\ n}} \mathcal{U}_n$. □

On stacks and topological spaces

Let k be an algebraically closed field, let X be an algebraic curve over k, and let G be a smooth affine group scheme over X.

Goal: understand the cohomology of the moduli stack $\mathrm{Bun}_G(X)$ of principle G-bundles on X.

Here: Let $k = \mathbb{C}$ and let G be semisimple simply connected.

Calculating the cohomology of $\mathrm{Bun}_G(X) \rightsquigarrow$ can be seen as a purely topological problem, using classical methods.

Assume additionally that the group scheme G is constant (i.e. it arises from a linear algebraic group over $\mathbb{C}$)$\rightsquigarrow$ there is a simple explicit description of the rational cohomology ring $H^*(\mathrm{Bun}_G(X); \mathbb{Q})$, due originally to Atiyah and Bott.

Let BG denote a classifying space of G, in the sense of algebraic topology: that is a quotient EG/G, where EG is a contractible space equipped with a free action of G.

BG has the universal mapping property: for any reasonably well-behaved topological space Y (e.g. any paracompact manifold), the construction

$$(f : Y \to \mathrm{BG}) \mapsto Y \times_{\mathrm{BG}} \mathrm{EG}$$

determines a bijection from the set $[Y, \mathrm{BG}]$ of homotopy classes of continuous maps from Y into BG to the set of isomorphism classes of G-bundles on Y (in the category of topological spaces).

Let M be a manifold and $\mathrm{Map}(M, \mathrm{BG}) := \{f : M \to \mathrm{BG} \mid f \text{ continuous}\}$. Regard $\mathrm{Map}(M, \mathrm{BG})$ as a topological space by equipping it with the compact open topology.

$\rightsquigarrow$ $\mathrm{Map}(M, \mathrm{BG})$ classifies G-bundles over M in the category of topological spaces: for any sufficiently nice parameter space Y identify homotopy classes of maps from Y to $\mathrm{Map}(M, \mathrm{BG})$ with isomorphism classes of principal G-bundles on the product $M \times Y$ in the category of topological spaces.

We have the following

Proposition (Gaitsgory-Lurie, 2019)

Let X be a smooth projective algebraic curve over $\mathbb{C}$ that we identify with the compact Riemann surface $X(\mathbb{C})$ of $\mathbb{C}$-valued points of X. Let $\mathrm{Bun}_G(X) = \mathrm{Bun}_{G \times_{\mathrm{Spec}(\mathbb{C})} X}(X)$ denote the moduli stack of G-bundles on X. There is a canonical homotopy equivalence

$$\mathrm{Bun}_G(X) \simeq \mathrm{Map}(X, \mathrm{BG}).$$

Remark

The moduli stack $\mathrm{Bun}_G(X)$ and the mapping space $\mathrm{Map}(X, \mathrm{BG})$ both have universal properties:
the former classifies G-bundles in the setting of algebraic geometry, and the latter classifies G-bundles in the setting of topological spaces.

The Atiyah-Bott Formula

$\rightsquigarrow$ may identify the cohomology of the moduli stack $\mathrm{Bun}_G(X)$ with the cohomology of the mapping space $\mathrm{Map}(X, \mathrm{BG})$.
$\rightsquigarrow$ give an explicit description of the rational cohomology of $\mathrm{Map}(X, \mathrm{BG})$.

Starting point: the description of the rational cohomology of BG itself:

Proposition

Let G be a simply-connected linear algebraic group over $\mathbb{C}$ and let BG denote the classifying space of G. Then the cohomology ring $H^*(\mathrm{BG}; \mathbb{Q})$ is isomorphic to a polynomial algebra on finitely many homogeneous generators $x_1, x_2, \ldots, x_r$ with even degrees $e_1, \ldots, e_r \geq 4$.

Let $V = V^*$ be a graded vector space over $\mathbb{Q}$ and let $\mathrm{Sym}^*(V)$ denote the free graded-commutative algebra generated by V.

Theorem (Atiyah-Bott)

There is an isomorphism of graded algebras

$$\mathrm{Sym}^*(H_*(X;\mathbb{Q}) \times_{\mathbb{Q}} V) \to H^*(\mathrm{Map}(X,\mathrm{BG});\mathbb{Q}).$$

More explicitly: let g be the genus of the curve X and let $e_1, e_2, \ldots, e_r$ denote the degrees of the polynomial generators of the cohomology ring $H^*(\mathrm{BG};\mathbb{Q})$. The cohomology ring $H^*(\mathrm{Map}(X;\mathrm{BG});\mathbb{Q})$ is isomorphic to a tensor product of a polynomial ring on $2r$ generators (of degrees e_i and $e_i - 2$ for $1 \leq i \leq r$) with an exterior algebra on $2gr$ generators (with $2g$ generators of each degree $e_i - 1$).

Thank you for your attention!

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