European Journal of Operational Research xxx (xxxx) xxx

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

A Relative Distance-based Scalarization Scheme using Reference Levels

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ARTICLE INFO

Keywords:
Decision analysis
Data analysis
Multiple reference point methodology
Multicriteria techniques
Relative evaluation

ABSTRACT

When variables measured in different units are used to analyze a given phenomenon, it is usually necessary to scale these variables in order to bring all of them down to a common scale. This allows their subsequent aggregation into a single measurement. This is the case, for example, of the processes of constructing composite indicators from a system of simple indicators. One way to perform this scaling is through distance-based schemes, used when reference levels are available for the different variables, which allow defining different performance bands. In these cases, the scaling function is often called the achievement scalarizing function. However, the linear nature of the achievement scalarizing function employed so far implies that the achievement level of each entity is scalarized in absolute terms, that is, considering their absolute distance to the reference levels, without considering the distribution of achievement values across all considered entities. In this paper, we propose a new achievement scalarizing function based on reference levels. This new function encompasses formulations that allow for the inclusion of relative assessments. Thus, we seek to broaden the application scope of distance-based scalarizations to analyze societal aspects where it is crucial to compare the performances of the entities not only with respect to the reference levels, but also with respect to the performances achieved by other entities. Finally, since absolute or relative measures may be required for different values when scaling the same variable, we propose a more general hybrid scheme that allows the combination of both schemes.

1. Introduction

Scalarization processes bring several variables values to a common scale. Some of the most common scalarization methods are based on the distance to a reference level, measuring the position of an entity with respect to a reference level (or several ones), by means of a so-called achievement scalarizing function. This procedure overlooks the information provided by the distribution of values across all entities, which contains potentially valuable information in situations where a relative analysis of the entities' values is of interest. We propose innovating the formulation of the achievement scalarizing function to incorporate a relative evaluation of each entity's performance. We hypothesize that entities could evaluate their performance not only in absolute terms but also contextualize one's performance in the distribution of performances. That is, not only does the level of the variable matter for entities, but also how far the value is from other entities' values.

Our approach is grounded in the hypotheses of relative income and relative deprivation. Both hypotheses explore how comparisons can influence well-being. The relative income hypothesis posits that individuals' well-being is not solely determined by their absolute income levels but also by how their incomes compare to others. On the other hand, relative deprivation refers to the perception of injustice or discontent that arises when individuals perceive themselves as having less than others in terms of income or resources. Both hypotheses offer a rational explanation for what may seem like paradoxical or irrational behavior: the evaluation of an absolute performance increase may not be positive if others increase their performance to a greater extent. Therefore, relative evaluation appears crucial for understanding how entities assess their results in relation to their environment.

While these hypotheses are formulated in terms of income, they can be applied to many different variables where comparisons are relevant. We find many domains in which social-context effects are present: education (Nikolaev, 2016), overweight (Blanchflower et al., 2009), unemployment (Clark, 2003). We can think about other domains, such as

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https://doi.org/10.1016/j.ejor.2025.09.001
Received 4 February 2025; Accepted 1 September 2025
Available online 1 September 2025

presence and influence on social media platforms. Individuals may compare their online presence, followers, or engagement metrics with those of others and experience feelings of validation or inadequacy based on these comparisons. The same occurs with housing conditions or jobs. Individuals may compare their living conditions to those of others and perceive disparities in safety, comfort, and social status, which can influence their overall assessment. Even if someone holds a prestigious job, if they perceive that others have higher-status positions, they may experience feelings of inadequacy or dissatisfaction. Thus, the hypotheses of relative income and relative deprivation can be extrapolated to variables other than income. At the same time, comparisons can be made between entities other than individuals. For example, comparing values between countries can be useful in a variety of variables that reflect the state and development of a nation: Gross Domestic Product, literacy rate, life expectancy, per capita greenhouse gas emissions, etc.

To scalarize and bring down variable values to a common scale, Ruiz et al. (2020) proposed a piecewise linear function that evaluates each entity value in absolute terms in relation to a set of reference levels, which represent different levels of achievement for the variable considered. We call this function the *absolute achievement scalarizing function*. Using this function as a starting point, and in line with the perspective of relative income and relative deprivation, our paper introduces an achievement function that enables the incorporation of a relative assessment of each entity's performance level within the scalarization process, named the *relative achievement scalarizing function*. However, since the relative evaluation may not always be relevant in all contexts (for example, in purely technical performance metrics, where relative position might not provide substantial information), we introduce an innovative achievement function, referred to as the *hybrid achievement scalarizing function*, which integrates both the relative and absolute evaluations of the performance of the entities. This hybrid function allows different weights to be assigned to both relative and absolute evaluations of the entities' performance, enabling the possibility to even nullify either one as needed, so that scalarization can be adjusted to each situation.

By incorporating this relative evaluation of outcomes, researchers and policymakers can better grasp the impact of performance disparities, as well as the economic implications of relative comparisons and their influence on the overall evaluation. Clark et al. (2008) found that policies solely focused on absolute improvements may overlook the effects of relative deprivation, leading to suboptimal results. Policy evaluation requires a comprehensive understanding of the impact of interventions on different economic indicators. The incorporation of a relative evaluation of outcomes provides a nuanced and comprehensive assessment of the effectiveness of implemented policies.

In what follows, in Section 2, we introduce the classical achievement scalarizing function, which we will refer to as the absolute achievement scalarizing function, and the relative achievement scalarizing function we propose to scalarize the entities' performance in relative terms according to the distribution of the performance across entities. We also introduce the hybrid achievement scalarizing function that combines the benefits of the two former ones. In Section 3, we present an illustrative example that demonstrates the practical application of the proposed approach. It assesses the changes in the performance of countries worldwide from 2000 to 2022, focusing on Gross National Income per capita in current US dollars (GNI). We compare the insights provided by both the relative and absolute functions, highlighting the additional information each function offers and calculate hybrid evaluations with different weights for the absolute and relative approach to test for the robustness of the results. Finally, Section 4 draws some conclusions and highlights the significant advantages of the proposed achievement functions.

2. Scalarization with absolute and relative assessments

The reference point methodology was proposed for continuous multiobjective optimization problems in Wierzbicki (1980), and it is based on reference levels representing desirable values for the multiple conflicting criteria considered. In this approach, an achievement scalarizing function is usually formulated by combining the original multiobjective problem and the reference point constituted by the reference values. By minimizing the achievement scalarizing function over the feasible set, an efficient solution is obtained, which somehow constitutes the most suitable alternative to optimizing all the criteria according to the preferences of a decision maker (Miettinen, 1999; Ruiz et al., 2008; Luque et al., 2009; Greco et al., 2024).

As an extension of the reference point approach, the double reference point (DRP) preferential scheme was introduced for both continuous and discrete problems twenty years later (Wierzbicki et al., 2000). This scheme considers two reference levels, usually known as aspiration (what is desirable) and reservation (what is admissible) values for each criterion. This scheme was adapted to the construction of sustainability composite indicators in (Ruiz et al., 2011). Later, Ruiz et al. (2020) suggested a scalarization based on a multiple reference point (MRP) scheme that generalizes the previous case by considering a generic number of reference values for each criterion.

Let us assume that we have a set $I = \{1, 2, \cdots, N\}$, representing N entities (individuals, countries, …) and that a variable associated with each of these N entities is evaluated. In what follows, without loss of generality, we will assume that the variable is of the type "the more, the better" and let x_i be the value of the variable for entity i with $i \in I$. We will assume that the decision maker (a person with enough knowledge in the context domain) can give n reference levels for the variable considered, denoted as q^t with $t = 1, \ldots, n$, which somehow represent n performance levels for the variable (e.g., very poor, poor, fair, good, very good, …). Let us denote by q^0 and q^{n+1} fixed values such that $q^0 \leq \min\{x_i : i \in I\}$ and $q^{n+1} \geq \max\{x_i : i \in I\}$. Therefore, the $(n+2)$-dimensional vector $\mathbf{q} = (q^0, q^1, q^2, \cdots, q^n, q^{n+1})$ contains all the information relative to the reference levels of the variable. Note that the reference levels can be established in two ways: they can be given, as previously mentioned, by one or a group of decision makers, if they have enough knowledge about the problem and they wish to do so; or, alternatively, they can be set statistically according to a dataset. For more details, see Ruiz et al. (2020). In any case, considering these reference levels means that the subsequent scaled measurements indicate the distance of each entity from those levels. It is therefore very important that decision-making centers participate in the assignment of these levels and are aware of their meaning and that different final results may be obtained if these reference levels change.

Once the reference levels are given, the MRP scheme brings all the variables down to a common scale as explained next. To this end, we assume that a set of real values $\alpha^0, \alpha^1, \ldots, \alpha^n, \alpha^{n+1}$ is available (either provided by the decision maker or set as default values by the analyst), with $\alpha^0 < \alpha^1 < \ldots < \alpha^n < \alpha^{n+1}$. These values will define the common measurement scale to which the values of the variable for all the entities will be transformed. In practice, each α^t is the value in the common scale that a given entity has if it exactly achieves the reference value q^t, for any $t = 0, \ldots, n+1$. Obviously, the choice of the scale also has an impact on the results obtained. Typically, α^t values are equally spaced, unless decision-makers are absolutely certain they want to reward or punish values in different performance bands through these values.

In what follows, in Section 2.1, we introduce the classical MRP-based achievement scalarizing function used in Ruiz et al. (2020) to bring down the variables to a common scale, which is a piecewise linear function that evaluates each entity achievement in absolute terms just in relation to the reference levels employed (we refer to this function as *absolute achievement scalarizing function*). Next, Section 2.2 describes the *relative achievement scalarizing function* we propose to scalarize the entities' performances in relative terms according to the distribution of the performances across entities. Finally, in Section 2.3, the *hybrid achievement scalarizing function* is suggested that combines the benefits of the two former.

2.1. Absolute achievement scalarizing function

Let us consider the absolute achievement scalarizing function proposed in Ruiz et al. (2020), which we denote as $s^a(\cdot,\mathbf{q})$ and is defined as follows. For any $t = 1,\ldots,n+1$, if $x_i \in [q^{t-1},\ q^t]$, we have:

$$s^a(x_i,\mathbf{q}) = \alpha^{t-1} + \frac{\alpha^t - \alpha^{t-1}}{q^t - q^{t-1}}\ (x_i - q^{t-1}). \tag{1}$$

This is a piecewise linear function that scalarizes the values of the variable considering the ranges defined by the reference levels $q^0,\ q^1,q^2,\ \ldots,\ q^n,\ q^{n+1}$. In practice, $s^a(\cdot,\ \mathbf{q})$ takes a value between α^{t-1} and α^t if the corresponding entity performs between q^{t-1} and q^t for the variable considered, as illustrated in Fig. 1.

As demonstrated in Ruiz et al. (2020), this achievement function is invariant against positive homothety and invariant against translation of the x_i values, that is, it is invariant against affine transformations with positive slope.

2.2. Relative achievement scalarizing function

For the relative evaluation of achievements that we propose, we use the notion of relative deprivation using the framework proposed by Yitzhaki (1979). Later, Hey and Lambert (1980) extended this notion of relative deprivation to the utility space, presenting an individual-centric interpretation of relative deprivation of entity $i \in I$, referred to as $D(i)$, as follows:

$$D(i) = \frac{1}{N}\sum_{j\in I,\ x_j>x_i} (x_j - x_i), \tag{2}$$

where x_i and x_j are the values of the variable considered for two entities i and j, respectively, and N represents the total number of entities considered.

A transformation of this expression has been used by García-Pardo et al. (2021)[1] to assess the degree to which an entity with variable x_i is left behind, by means of the following function $LB(\cdot)$:

$$LB(x_i) = [1 - L(F(x_i))] - \left[\frac{x_i}{\mu}(1 - F(x_i))\right], \tag{3}$$

where $F(x_i)$ is the cumulative distribution function, $L(F(x_i))$ is the Lorenz curve[2] evaluated at x_i and μ is the mean of the variable.

Eq. (3) quantifies an entity's relative position with respect to those better positioned in the distribution of the variable analyzed. This position is expressed as the difference between a first term, which corresponds to the share of the variable held by entities performing better than entity i (i.e., $1 - L(F(x_i)) = \frac{\sum_{j\in I,\ x_j>x_i} x_j}{\sum_{j\in I} x_j}$), and a second term that represents the share of these entities if they had x_i (i.e., $\frac{x_i}{\mu}(1 - F(x_i)) = \frac{\sum_{j\in I,\ x_j>x_i} x_i}{\sum_{j\in I} x_j}$), assuming that the mean μ is constant. Moreover, given a share and proportion of entities with better achievements, the lower the relative value of the variable with respect to the mean (i.e., $\frac{x_i}{\mu}$), the more left behind.

Thus, we can say that an entity is totally left behind in a certain variable if $LB(x_i) = 1$, that is, the entity is at the bottom of the distribution of the variable. On the other hand, the entity is not left behind at all if $LB(x_i) = 0$, that is, the entity leads the distribution of performances.

Likewise, it should be noted that the extent to which an entity is left behind complements the information on the level of achievement. Thus, an entity i with a low level of achievement may not be distant from better-performing entities, and this would mean to reach a low value for $LB(x_i)$, but if the same entity was distant from better-performing entities, it would attain both a low level of achievement and a high $LB(x_i)$.

Based on the $LB(\cdot)$ function given in (3), for every $t = 1,\ldots,n+1$ we propose to consider the following function $LB^t(\cdot)$ to evaluate, in relative terms, the achievement of the entities whose values are within q^{t-1} and q^t (i.e., with $x_i \in\ [q^{t-1},\ q^t]$):

$$LB^t(x_i) = [1 - L^t(F^t(x_i))] - \left[\frac{x_i}{\mu^t}(1 - F^t(x_i))\right], \tag{4}$$

where $F^t(\cdot)$, $L^t(\cdot)$ and μ^t are the cumulative distribution, Lorenz functions and the mean, respectively, within the interval $\left[q_j^{t-1},\ q_j^t\right]$. To ensure the mathematical definition of all these functions, we need to assume that $q^0 \geq 0$ (i.e., all values are non-negative). This function, which quantifies the degree to which an entity is left behind within the interval $\left[q_j^{t-1},\ q_j^t\right]$, verifies that[3]:

$0 \leq LB^t(x_i) \leq 1,$

for all $x_i \in \left[q^{t-1},\ q^t\right]$ and every $t = 1,\ldots,n+1$.

Based on Eq. (4), we propose the new *relative achievement scalarizing function* (for a variable of type "the more, the better"), which we refer to as $s^r(\cdot,\mathbf{q})$, as follows. For any $t = 1,\ldots,n+1$, if $x_i\ \in \left[q^{t-1},\ q^t\right]$, then:

$$\begin{aligned} s^r(x_i\ ,\mathbf{q}) &= \alpha^{t-1} + \ (\alpha^t - \alpha^{t-1})\left(1 - LB^t\left(x_i - q^{t-1}\right)\right) \\ &= \alpha^t - (\alpha^t - \alpha^{t-1})LB^t\left(x_i - q^{t-1}\right). \end{aligned} \tag{5}$$

Note that, in this function, we apply the function in Eq. (4) to the transformation of the variable (i.e., to $x_i - q^{t-1}$), which consists of a change of the origin, where the function $LB^t(\cdot)$ is evaluated in the interval $[0,\ q^t - q^{t-1}]$ instead of in the interval $[q^{t-1},\ q^t]$. Observe that $LB^t(x_i - q^{t-1})$ is intended to show the degree to which an entity is left behind within the interval.

Since function $LB^t(\cdot)$ is a decreasing and convex function (see Bárcena-Martín & García-Pardo, 2022), $s^r(\cdot,\mathbf{q})$ is increasing and concave in each interval $\left[q^{t-1},\ q^t\right]$ because $s^r(\cdot,\mathbf{q})$ is an affine transformation of $LB^t(\cdot)$, where $LB^t(\cdot)$ is multiplied by a negative number and, if $LB^t(\cdot)$ is decreasing, then $s^r(\cdot,\mathbf{q})$ is increasing, and if $LB^t(\cdot)$ is convex, then $s^r(\cdot,\mathbf{q})$ is concave. Moreover, the new relative achievement scalarizing function is also increasing (monoticity) in the whole space, including at the extremes of the intervals, since it holds that:

$s^r(x_i,\mathbf{q}) \leq\ s^r(x_k,\mathbf{q})$ if $x_i\ \in \left[q^{t_1-1},\ q^{t_1}\right]$ and $x_k\ \in \left[q^{t_2-1},\ q^{t_2}\right]$, for any $t_1 < t_2$.

Fig. 2 represents with a green solid line the density function of a hypothetical variable, with a blue dashed line the function $s^a(x_i,\mathbf{q})$, and with a red dotted line the function $s^r(x_i,\mathbf{q})$. As can be seen, $s^a(\cdot,\mathbf{q})$ is an increasing piecewise linear function, with a not necessarily equal slope in each interval, while $s^r(\cdot,\mathbf{q})$ is an increasing piecewise non-linear function, which is concave in each interval.[4] Thus, the new scalarized relative achievement function is differentiable and concave at the interior points of each interval, which implies that it is quasiconcave at these interior points, but it is not assured to be either differentiable or

[1] In this work, for clarity, we denote as $LB(x_i)$ the degree to which an entity i with value x_i is left behind, which was denoted as $LB(i)$ in García-Pardo et al. (2021).

[2] The Lorenz curve at x_i provides the share of the entities with values smaller or equal to x_i as $L(F(x_i)) = \frac{\sum_{j\in I\ x_j<x_i} x_j}{\sum_{j\in I} x_j} = \frac{\sum_{j\in I\ x_j<x_i} x_j}{N\mu}$.

[3] Since $0 \leq L^t(F^t(x_i)) \leq F^t(x_i) \leq 1$, we have that $0 \leq\ LB^t(x_i) \leq 1$ for all $x_i \in \left[q^{t-1},\ q^t\right]$ and $t = 1,\ldots,n+1$, assuming that $q^0 \geq 0$ (García-Pardo et al., 2021).

[4] The degree of concavity depends on the piece of the underlying density function of the variable within the interval.

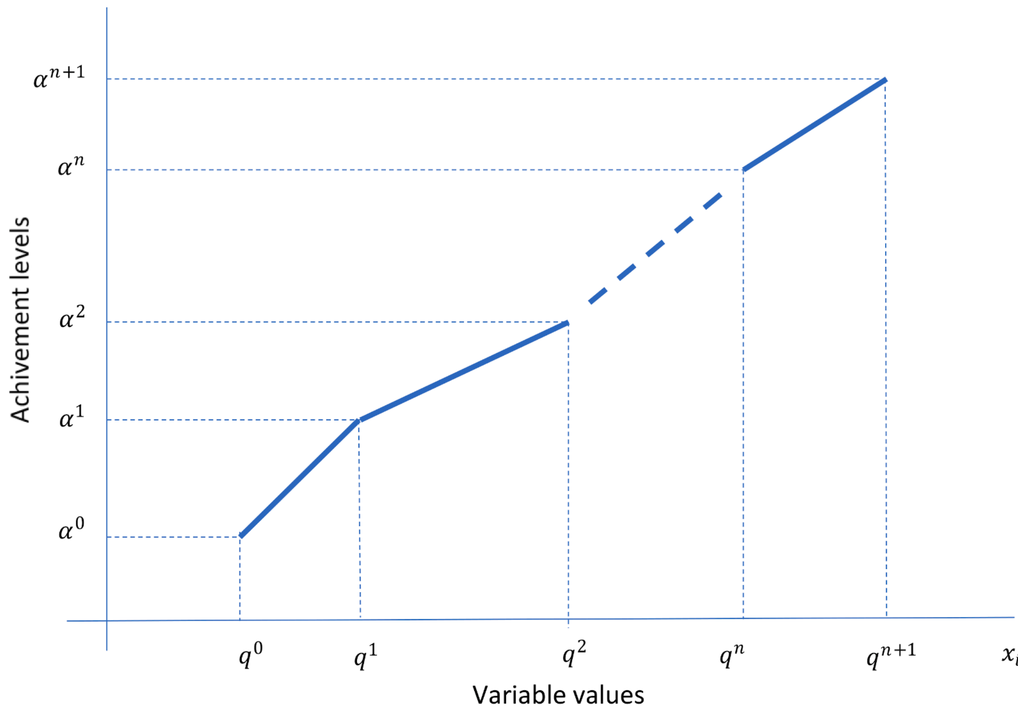

Fig. 1. Graphical idea of the absolute achievement scalarizing function. Notes: (1) $s^a(x_i, \mathbf{q})$ is represented in blue. (2) On the horizontal axis, the values of the variable are represented ($x_i,\ i \in I$), with key reference levels highlighted ($q^t, t = 0, \ldots, n+1$). On the vertical axis, the different absolute achievements are shown, along with the values ($\alpha^t, t = 0, \ldots, n+1$) that allow a common scale to be established.

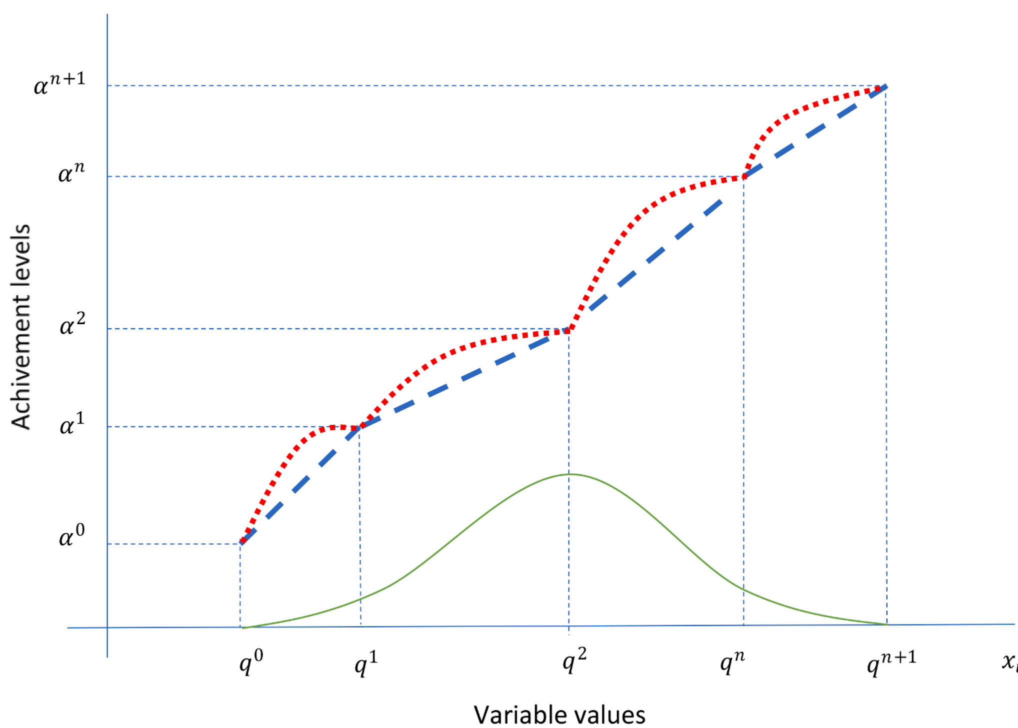

Fig. 2. Graphical comparison of the absolute and relative achievement scalarizing functions. Notes: (1) $s^a(x_i, \mathbf{q})$ is represented in blue, $s^r(x_i, \mathbf{q})$ in red and the density function of the variable in green. (2) On the horizontal axis, the values of the variable are represented, with key reference levels highlighted. On the vertical axis, the different achievements are shown, along with the values ($\alpha^t, t = 0, \ldots, n+1$) that allow a common scale to be established.

quasiconcave at the extreme points of the intervals, and therefore we cannot assure that it is concave either. Graphically, it can be seen that at the extreme points, the function is neither differentiable nor quasiconcave.

Furthermore, the relative achievement function is invariant against positive homothetic transformation, since $LB^t(x_i)$ is scale invariant with positive slope (see Bárcena-Martín & García-Pardo, 2022). Thus, for all $a > 0$:

$$s^r(a \cdot x_i, \mathbf{q}) = s^r(x_i, \mathbf{q}),$$

for all $x_i \in [q^{t-1}, q^t]$ and every $t = 1, \ldots, n+1$.

Remark 1. We observe that if a value $x_i \in [q^{t-1}, q^t]$ coincides with the lower bound of the interval, i.e., $x_i = q^{t-1}$, then $s^a(x_i, \mathbf{q}) = s^r(x_i, \mathbf{q}) = \alpha^{t-1}$. Conversely, if a value $x_j \in [q^{t-1}, q^t]$ coincides with the upper bound of the interval, i.e., $x_j = q^t$, then $s^a(x_j, \mathbf{q}) = s^r(x_j, \mathbf{q}) = \alpha^t$.

For a better understanding, we illustrate how the previous absolute and relative achievement scalarizing functions can be calculated, step by step, and we emphasize the type of information each one can provide by means of the following two toy examples.

Example 1. Let x_i be the outcomes of fifteen entities, for $i \in I = \{1, \ldots, 15\}$, named from A to O, as described in Table 1. We consider the reference levels defining $\mathbf{q} = (q^0, q^1, q^2, q^3)$, where $q^0 = 10 = \min\{x_i\}$, $q^1 = 20, q^2 = 33$ and $q^3 = 70 \geq \max\{x_i\}$. We use a default scale from 0 to 3, that is, we set $\alpha^0 = 0, \alpha^1 = 1, \alpha^2 = 2$ and $\alpha^3 = 3$, for convenience. Thus, the absolute achievement scalarizing function defined as in Eq. (1) is as follows:

$$s^a(x_i, \mathbf{q}) = \begin{cases} \dfrac{(x_i - 10)}{10} \; \textit{if} \; x_i \in [10,\ 20], \\ 1 + \dfrac{(x_i - 20)}{13} \; \textit{if} \; x_i \in [20,\ 33], \\ 2 + \dfrac{(x_i - 33)}{37} \; \textit{if} \; x_i \in [33,\ 70]. \end{cases}$$

While the relative achievement scalarizing function formulated as in Eq. (5) is:

$$s^r(x_i, \mathbf{q}) = \begin{cases} 1 - LB^1(x_i - 10) \; \textit{if} \; x_i \in [10,\ 20], \\ 2 - LB^2(x_i - 20) \; \textit{if} \; x_i \in [20,\ 33], \\ 3 - LB^3(x_i - 33) \; \textit{if} \; x_i \in [33,\ 70], \end{cases}$$

where the functions $LB^t(\cdot)$ are defined as in Eq. (4), for every $t = 1, 2, 3$.

The values obtained with the absolute and relative achievement scalarizing functions are included in Table 1 and they have been represented in Fig. 3. By Remark 1, as entity A has $x_A = 10 = q^0$, then $s^a(10, \mathbf{q}) = s^r(10, \mathbf{q}) = 0$, and since entity C has $x_C = 20 = q^1$, then $s^a(20, \mathbf{q}) = s^r(20, \mathbf{q}) = 1$.

On the other hand, for the absolute achievement function entity E has $s^a(29, \mathbf{q}) = 1.69$, which means that, within the second interval, E is 0.69 points from the lower bound and 0.31 (that is, 2 − 1.69) from the upper bound of this interval. For the relative achievement function, remember that we work with a transformation of the variable that consists of a change of the origin to the lower limit of the interval (i.e., to $x_i - q^{t-1}$). This way, entity E attains a relative value $s^r(29, \mathbf{q}) = 1.88$, which indicates that, compared to all the other entities within the interval $[20,\ 33]$, the entity E lags behind by a margin of 0.12 points (as $2 - 1.88 = 0.12$). This means that the disparity between the amount of the variable corresponding to the entities within the interval $[20,\ 33]$ outperforming E and the amount of these entities if they had $x_i - q^1 = 29 - 20 = 9$ is 0.12, in relative terms to the mean of transformation of the variable in the second interval (i.e., $\mu^t - q^{t-1}$). In essence, entities from the interval $[20,\ 33]$ outperforming entity E possess a share that is 0.12 higher than the share they would have if all of them had x_E, considering that the mean within this interval does not change. In this context, as $s^r(x_i, \mathbf{q})$ increases, the entity is less left behind, catching up to the other entities within the interval who are performing better than *i*.

Note that the absolute achievement function informs about the distance to the limits of the interval, while the relative achievement function accounts for the distribution of the achievements of the entities within the interval and informs about the distance to the variable values of the other outperforming entities within the interval, instead of a given threshold. Thus, an entity could have the same absolute achievement in two situations, because the value of the variable does not change and, therefore, the distance to the limits is the same, while the relative achievement could decrease just because the values of outperforming entities within the interval has improved, and then the entity is more left behind. This is the case of E in the following Example 2.

Example 2. Let's suppose that certain entities of Example 1 increase their outcomes. Now the new variable is $\tilde{x}_i$, as shown in Table 2, where values in bold are those that have changed. The last two columns of Table 2 indicate the amount of change in the absolute and relative achievement functions with respect to the values shown in Table 1.

We observe diverse situations. Entities A and C have maintained their outcomes, resulting in no change in either absolute or relative achievement since they occupy the extreme ends of the spectrum in the first interval defined by the reference values (i.e., $[10,\ 20]$). Entities B, D and F have increased their outcomes, but these increments are evaluated differently in absolute and relative terms. Entities B and D have increased their outcomes by 3 units, and their absolute and relative achievements have improved as well (0.30 and 0.33 increase, respectively, for entity B, and 0.23 and 0.31 increase, respectively, for entity D). Entity F has increased its outcome by 2 units, resulting in improvements of both the absolute and relative achievements, with values of 0.15 and 0.06, respectively. The outcome of entity E has not changed, leading to no change in its absolute achievement. However, there is a decrease of 0.04 units in its relative achievement because now entity E falls further behind within the interval because F has increased its outcome, implying a change in the relative position of E. Similar to entities A and C, entity G maintains its outcome, leading to no change in its absolute nor relative achievements, as it is the upper bound of the second interval. The entities in the third interval, $[33,\ 70]$, experience no change in their relative nor absolute achievements since there are no changes in their outcomes nor in the outcomes of other entities in their interval. This way, we observe the different information that the two achievement functions provide.

2.3. Hybrid achievement scalarizing function

Absolute achievement functions evaluate achievement in objective terms, while relative achievement functions consider the context and the entity's position in comparison to others. These evaluations are different but not incompatible. Combining both evaluations provides a richer and more nuanced understanding of the entities' achievements.

In some situations, focusing on absolute achievements may be more relevant. However, in other situations, relative comparisons can provide crucial additional information. In some contexts, it could be enlightening to combine the information provided by the absolute and relative achievement functions. To this aim, we also propose a new *hybrid achievement scalarizing function* defined as the linear combination of the absolute and relative achievement scalarizing functions in Sections 2.1 and 2.2:

$$s^m(x_i, \mathbf{q}) = \lambda^{t-1}\ s^a(x_i, \mathbf{q}) + (1 - \lambda^{t-1})\ s^r(x_i, \mathbf{q}), \tag{6}$$

for all $x_i \in [q^{t-1}, q^t]$, with $0 \leq \lambda^{t-1} \leq 1$ for every $t = 1, \ldots, n+1$.

Considering different values of λ in various intervals has practical significance, as we can deem one type of function more relevant than another depending on the level of the variable analysed. We can see its utility through the following example. Imagine one university wants to evaluate students' grades both absolutely and relatively, comparing them with the grades of other students. Relative assessment helps identify disparities in students' achievements that are not evident through absolute assessment. For example, let us assume that students' grades are on a scale from 0 (the worst possible grade) to 10 (the best possible one). For students with grades below a "minimum" level, say below 5, absolute evaluation might be more relevant. However, for grades between 5 and 9, a combination of absolute and relative evaluation might provide a more comprehensive view. Finally, for grades

Table 1
Values of the absolute and relative achievement functions in Example 1.

		Absolute	Relative
Entity i	x_i	$s^a(x_i, \mathbf{q})$	$s^r(x_i, \mathbf{q})$
A	10	0.00	0.00
B	12	0.20	0.33
C	20	1.00	1.00
D	21	1.08	1.13
E	29	1.69	1.88
F	30	1.77	1.94
G	32	1.92	2.00
H	35	2.05	2.15
I	40	2.19	2.46
J	42	2.24	2.57
K	43	2.27	2.62
L	47	2.38	2.76
M	52	2.51	2.90
N	55	2.59	2.95
O	60	2.73	3.00

Note: The values within the interval [20, 33] are highlighted in grey to visually facilitate distinguishing the values between the three intervals.

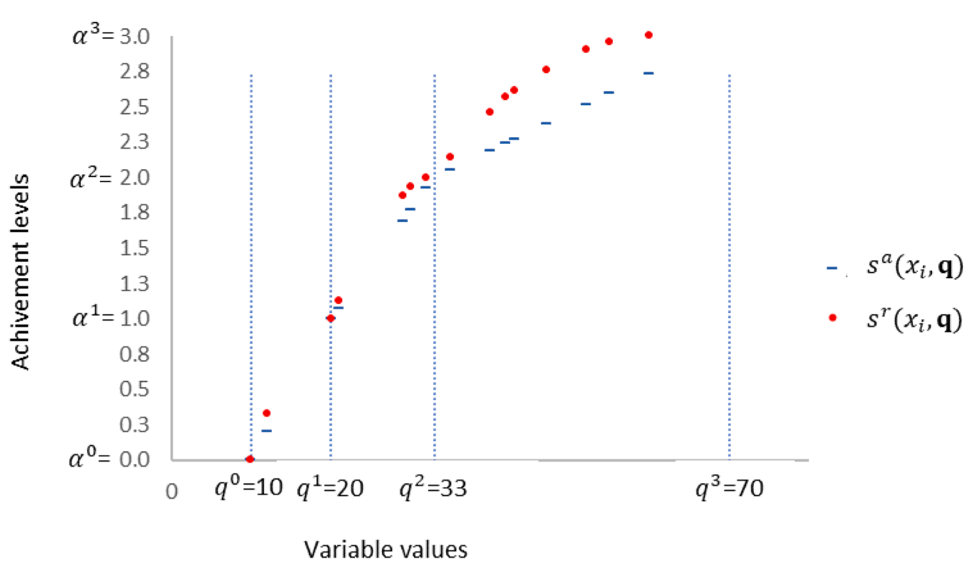

Fig. 3. Values of the absolute and relative achievement functions in Example 1.

above 9, relative evaluation might be more pertinent, as it may make sense to measure a student's score in comparison with their peers, facilitating the identification of best practices and the establishment of realistic goals.

Similarly, in environmental contexts, comparing results across countries can reveal disparities and areas requiring intervention. For instance, an improvement in a country's environmental indicator may be positive in absolute terms. However, when this result is compared to the improvements in the same indicator across other countries, we can assess whether the increment is sufficient relative to others. Thus, a

Table 2
Values of the absolute and relative achievement functions in Example 2.

			Absolute	Relative	Change in absolute function	Change in relative function
Entity i	x_i	$\tilde{x}_i$	$s^a(\tilde{x}_i, \mathbf{q})$	$s^r(\tilde{x}_i, \mathbf{q})$	Δs^a	Δs^r
A	10	10	0.00	0.00	0.00	0.00
B	**12**	**15**	**0.50**	**0.67**	**0.30**	**0.33**
C	20	20	1.00	1.00	0.00	0.00
D	**21**	**24**	**1.31**	**1.43**	**0.23**	**0.31**
E	29	29	1.69	1.84	0.00	-0.04
F	**30**	**32**	**1.92**	**2.00**	**0.15**	**0.06**
G	32	32	1.92	2.00	0.00	0.00
H	35	35	2.05	2.15	0.00	0.00
I	40	40	2.19	2.46	0.00	0.00
J	42	42	2.24	2.57	0.00	0.00
K	43	43	2.27	2.62	0.00	0.00
L	47	47	2.38	2.76	0.00	0.00
M	52	52	2.51	2.90	0.00	0.00
N	55	55	2.59	2.95	0.00	0.00
O	60	60	2.73	3.00	0.00	0.00

Note: The values within the interval [20, 33] are highlighted in grey to visually facilitate distinguishing the values between the three intervals, and the values that have experienced changes in the new variable with respect to Example 1 are marked in bold.

hybrid evaluation allows us to have more precise and relevant assessments, offering a comprehensive view of each country's performance. Consequently, it leads to better decision-making and the implementation of more effective policies.

Selecting the value of λ^t is a non-trivial task. Decision-making centers should be involved in assigning these levels, fully understanding their implications and the fact that varying these parameters can lead to different final outcomes. It is also their responsibility to determine whether an absolute or relative perspective should prevail in each performance band. Naturally, in practical applications, robustness analysis of the conclusions can also be conducted in response to changes in λ^t.

It is easy to show that the function in (6) is monotone increasing because it is the linear convex combination of two monotone increasing functions. It is also invariant to a positive linear transformation since the two functions $s^a(\cdot, \mathbf{q})$ and $s^r(\cdot, \mathbf{q})$ are invariant for this kind of transformation.

Example 3. In the context of Example 1, we consider two hybrid achievement scalarizing functions: $s^m(x_i, \mathbf{q})$ with $\lambda^0 = \lambda^1 = 0$ and $\lambda^2 = 1$; and $\tilde{s}^m(x_i, \mathbf{q})$ with $\lambda^0 = 0.9$, $\lambda^1 = 0.5$ and $\lambda^2 = 0.2$. Thus, these hybrid achievement scalarizing functions are defined as follows:

$$s^m(x_i, \mathbf{q}) = \begin{cases} \dfrac{(x_i - 10)}{10} \; if \; x_i \in [10, 20], \\ 1 + \dfrac{(x_i - 20)}{13} \; if \; x_i \in [20, 33], \\ 3 - LB(x_i - 33) \; if \; x_i \in [33, 70]. \end{cases}$$

$$\tilde{s}^m(x_i, \mathbf{q}) = \begin{cases} 0.9\dfrac{(x_i - 10)}{10} + 0.1(1 - LB(x_i - 10)) \; if \; x_i \in [10, 20], \\ 0.5\left(1 + \dfrac{(x_i - 20)}{13}\right) + 0.5(2 - LB(x_i - 20)) \; if \; x_i \in [20, 33], \\ 0.2\left(2 + \dfrac{(x_i - 33)}{37}\right) + 0.8(3 - LB(x_i - 33)) \; if \; x_i \in [33, 70]. \end{cases}$$

As it can be seen, $s^m(\cdot, \mathbf{q})$ corresponds to the absolute achievement function in the first two intervals and applies the relative achievement function in the last interval, while $\tilde{s}^m(\cdot, \mathbf{q})$ is a combination of both the absolute and the relative schemes in the three intervals. The results of both hybrid achievement functions can be seen in Table 3.

As shown in Table 3, the hybrid achievement functions combine information from the absolute and relative achievement functions in

Table 3
Values of the absolute, relative and hybrid achievement functions in Example 3.

		Absolute	Relative	Hybrid	
Entity i	x_i	$s^a(x_i, \mathbf{q})$	$s^r(x_i, \mathbf{q})$	$s^m(x_i, \mathbf{q})$	$\tilde{s}^m(x_i, \mathbf{q})$
A	10	0.00	0.00	0.00	0.00
B	12	0.20	0.33	0.20	0.21
C	20	1.00	1.00	1.00	1.00
D	21	1.08	1.13	1.08	1.10
E	29	1.69	1.88	1.69	1.78
F	30	1.77	1.94	1.77	1.85
G	32	1.92	2.00	1.92	1.96
H	35	2.05	2.15	2.15	2.13
I	40	2.19	2.46	2.46	2.41
J	42	2.24	2.57	2.57	2.51
K	43	2.27	2.62	2.62	2.55
L	47	2.38	2.76	2.76	2.69
M	52	2.51	2.90	2.90	2.82
N	55	2.59	2.95	2.95	2.88
O	60	2.73	3.00	3.00	2.95

Note: The values within the interval [20, 33] are highlighted in grey to visually facilitate distinguishing the values between the three intervals.

different ways. The specific type of combination depends on the preference of the decision maker or on the specific situation to be evaluated.

3. A practical application

In order to illustrate the behaviour and potential advantages of the relative achievement scalarizing function, in this section, we will apply the absolute, relative and hybrid functions to assess the performance and the change in it of the countries all over the world[5] in last two decades, from 2000 to 2022,[6] in terms of Gross National Income per capita in current US$ (GNI).

GNI is a measure of economic activity, and has proven to be a useful and readily available indicator that is often closely correlated with other non-monetary measures of the quality of life, such as life expectancy at birth, mortality rates of children, and enrolment rates in school. GNI measures are preferred to Gross Domestic Product as they represent national income, as opposed to the value of domestic production. GNI therefore includes income as earned by a country's residents, whether it originates from production within its borders, or from assets held abroad. GNI data is provided by the World Bank national accounts data, and OECD National Accounts data files. Specifically, the data used in this study was obtained from the World Bank's official website as of April 24, 2024, specifically from the file GNI per capita, Atlas method (current US $).

To calculate the absolute and relative achievement functions according to Eqs. (1) and (5), we need to fix the reference levels q^i and the corresponding values α^t. For the reference levels, we use the corresponding year's GNI thresholds in Table 4 that were proposed by the World Bank to classify economies into four income groups: low, lower-middle, upper-middle, and high income. Thus, we set $q^0 = 120$, $q^1 = 755$, $q^2 = 2{,}995$, $q^3 = 9{,}625$ and $q^4 = 45{,}690$ for the data in 2000, and $q^0 = 240$, $q^1 = 1{,}135$, $q^2 = 4{,}465$, $q^3 = 13{,}845$ and $q^4 = 95{,}490$ for the data in 2022. For both years, we use a common default scale from 0 to 4, that is, we set $\alpha^0 = 0$, $\alpha^1 = 1$, $\alpha^2 = 2$ and $\alpha^3 = 3$.

Table 4
GNI reference levels for each year.

	2000	2022
Low income	≤ 755	$\leq 1{,}135$
Lower-middle income	756–2,995	1,136–4,465
Upper-middle income	2,996–9,625	4,466–13,845
High income	$> 9{,}625$	$> 13{,}845$

Note: The reference levels for the year 2000 are those reported by the World Bank in its historical dataset, available in the World Bank Analytical Classifications. For the year 2022, the most recent update from the World Bank, published on the Data Blog *"World Bank Country Classifications by Income Level for 2024–2025"* on July 1, 2024, is used.
Source: World Bank GNI per capita Operational Guidelines & Analytical Classifications.

[5] We analyse 164 out of a total of 194 countries reported by the World Bank for which we have available data for years 2000 and 2022 (see Table A1 in the appendix).

[6] The final year is set in 2022 because, at the time of concluding this study, it was the year for which we maximized the number of countries analyzed.

Table 5 presents information on some selected countries notable for the evolution of their achievements in absolute and relative terms. For each selected country, we provide the absolute and relative achievements for each year and interpret some of them as examples. Subsequently, we evaluate their evolution by means of the difference between their achievements at each year, using both the absolute and the relative schemes (denoted, respectively, by Sa2022-Sa2000 and Sr2022-Sr2000). This approach allows us to highlight the relevant information captured by the relative function, which complements the information provided by the absolute function.

In Table 5, we have countries classified at different income classes. For example, Congo has absolute and relative achievement functions smaller than 1 in 2000, indicating that it is a lower income country, while Bulgaria, Islamic Republic of Iran, Jordan and Romania in 2000 are in the lower-middle income group. Botswana in 2000 is in the upper-middle income group and Romania in 2022 is in the upper income group. Specifically, the absolute achievement function for Congo in 2000 provides the following information: $s^a(560, \mathbf{q}) = 0.69$ means that, within the lower income countries, Congo is 0.69 points from the lower reference level and 0.31 from the upper reference level of the low income country classification; regarding the relative achievement function, Congo in 2000 has $s^r(560, \mathbf{q}) = 0.92$, which indicates that, compared to all the other low income countries, it lags behind by a margin of 0.08 points (as $1 - 0.92 = 0.08$). This means that the disparity between the GNI corresponding to the low-income countries outperforming Congo and the amount of these if they had the GNI of Congo is 0.08 in relative terms. In essence, low income countries outperforming Congo possess a share that is 0.08 higher than the share they would have if all of them had the same GNI than Congo. On the other hand, Congo's achievement functions in 2022 are between 1 and 2, revealing that in this year Congo is classified as lower-middle income country. Specifically, its absolute achievement function informs that Congo is 0.35 points from the lower reference level of the lower-middle income countries and 0.65 from the upper reference level. At the same time, the relative achievement function advises that lower-middle income countries outperforming Congo possess a share that is 0.42 $(2 - 1.58)$ higher than the share they would have if all of them had the GNI of Congo. All the other values can be interpreted similarly.

Note some specific situations. Bulgaria in 2022 has an absolute achievement function of 2.95, indicating it is an upper-middle country, and it is 0.05 points from the upper reference level of this category. Additionally, with a relative achievement function of 3.00, Bulgaria is the leading country in this group, as no other country within this class has a higher income. Something similar takes place for Jordan in 2022 in the lower-middle income group, with a relative achievement function of 2.00. On the other hand, Gabon is the leading country in the lower-middle income group in 2000, with a relative achievement function of 2.00, and at the same time, it is close to the upper reference level of this group (2,995), which is why its absolute achievement function value is rounded to 2.00.

Turning our attention to the changes in absolute and relative achievements, Fig. 4 presents changes in countries in the period 2000–2022. Changes in the absolute achievement functions are displayed on the horizontal axis, and changes in the relative achievement functions in the vertical axis. Selected countries with specific assessments are identified. The orange line contains the points representing countries with equal improvements in both absolute and relative terms.

Fig. 4 illustrates various scenarios. Most countries show improvements in both absolute and relative terms (those in the first quadrant). However, some countries experience declines in both aspects (those in the third quadrant). Only Argentina and Seychelles (refer to Table A1 in the appendix for detailed information) show increases in absolute achievements. Despite this, they face declines in relative terms because, although their GNI has increased, other countries have increased their GNI even more significantly. As a result, the relative achievements of Argentina and Seychelles have decreased.

At this point, it is interesting to make some comments about the differences in the absolute and relative evaluations. There are countries that show a greater increase in the absolute achievement than in relative one, specifically those below the straight line. We highlight the most extreme cases in Fig. 4. Jordan, for example, increases by 0.55 in absolute terms while it increases by 0.20 in relative terms. Bulgaria and Romania increase by 1.54 and 1.59 in absolute terms, respectively, while they increase by 1.22 and 1.26 in relative terms. These are clear examples of increases in absolute achievements that, when compared to other countries, appear more nuanced.

Conversely, we also observe the opposite situation in countries above the straight line. Gabon, Botswana, and the Democratic Republic of the Congo increase their achievements in absolute terms (0.33, 0.31, and 0.40, respectively) but these increases are smaller than their increases in relative terms (0.62, 0.60, and 0.66). This means that, although other countries have increased in absolute terms, the increases of these three countries are greater when compared to other countries.

Furthermore, there are countries with similar absolute increases but very different relative increments. For instance, the Democratic Republic of the Congo and the Islamic Republic of Iran, both show an increase of 0.40 in the absolute achievement function (see Fig. 4). The Democratic Republic of the Congo has a relative increase of 0.66 because, even though other countries have increased their GNI, it has a more marked increase in relation to other countries than the Islamic Republic of Iran that has a relative increase of 0.15. This indicates that increases that are evaluated similarly in absolute terms, when evaluated in the context of the achievements of other countries, reveal very different judgments.

Thus, we provide an alternative evaluation that allows us to assess the achievement of the countries not only in absolute terms but also in comparison to other countries, offering a complementary perspective that may be crucial in the identification of countries that are left behind.

In case it is found interesting to combine the information provided by both achievement functions (absolute and relative), the hybrid achievement function could be implemented proposing a specific value of λ^{t-1}, with $0 \leq \lambda^{t-1} \leq 1$, for every $t = 1, 2, 3, 4$.

Fig. 5 presents the values of the hybrid function for 2022 in the countries listed in Table 5, using alternative parameter values across all performance bands. Specifically, we assume that $\lambda^{t-1} = \lambda$, for every

Table 5

Values of the absolute and relative achievement functions for the selected countries in 2000 and 2022.

Country	GNI 2000	$s^a(x_i, \mathbf{q})$2000	$s^r(x_i, \mathbf{q})$2000	GNI 2022	$s^a(x_i, \mathbf{q})$2022	$s^r(x_i, \mathbf{q})$ 2022	Sa2022-Sa2000	Sr2022-Sr2000
Botswana	3,020	2.00	2.01	7,430	2.32	2.61	0.31	0.60
Bulgaria	1,660	1.40	1.78	13,350	2.95	3.00	1.54	1.22
Congo, D.R.	560	0.69	0.92	2,290	1.35	1.58	0.65	0.66
Gabon	2,990	2.00	2.00	7,530	2.33	2.62	0.33	0.62
Iran, I. R.	1,770	1.45	1.83	3,980	1.85	1.98	0.40	0.15
Jordan	1,690	1.42	1.80	4,350	1.97	2.00	0.55	0.20
Romania	1,710	1.43	1.81	15,570	3.02	3.06	1.59	1.26

Source: World Bank national accounts data, and OECD National Accounts data files and own computations.

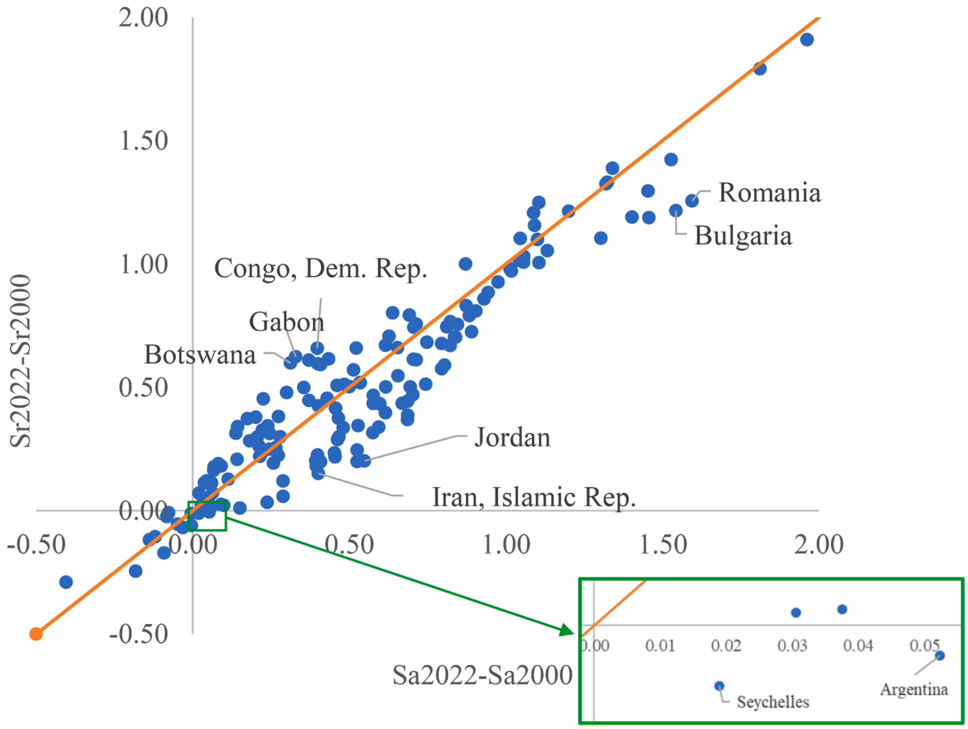

Fig. 4. Change in absolute and relative achievements in the period 2000–2022. Note: As in Table 5, we denote by Sa2022-Sa2000 (horizontal axis) the difference between the absolute achievements in 2022 and 2000, and by Sr2022-Sr2000 (vertical axis) the difference between the relative achievements attained in 2022 and 2000.

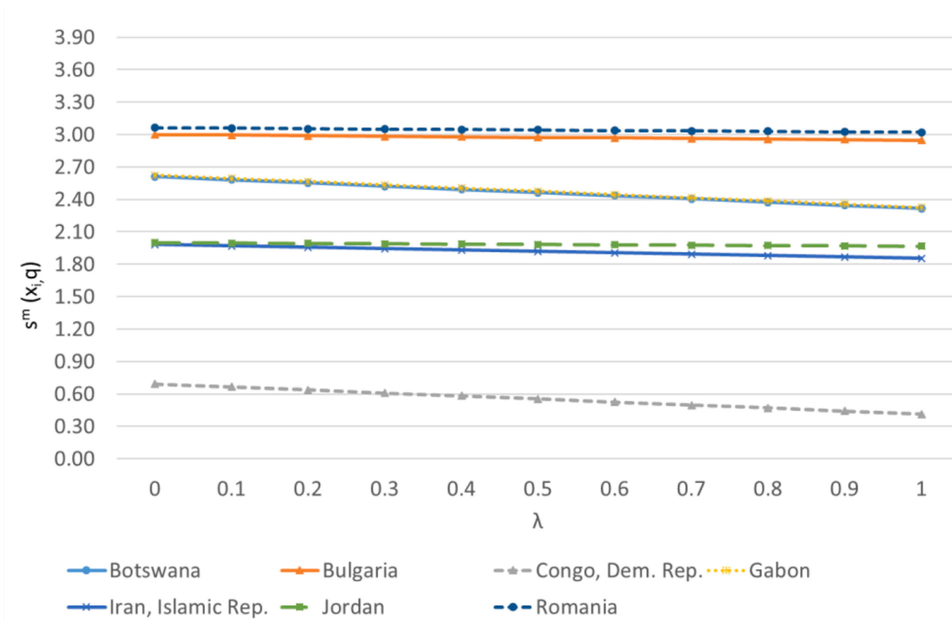

Fig. 5. Values of the hybrid achievement scalarizing function with $\lambda^{t-1} = \lambda$, for $t = 1, 2, \cdots, n+1$, for the selected countries in 2022.

$t \in T$, and we represent how the values of the hybrid function change for each entity for different values of λ, ranging from 0 to 1. As shown, Romania and Bulgaria exhibit minimal changes when moving from the relative to the absolute evaluation (i.e., when lambda increases), indicating a robust performance with respect to both approaches. Gabon and Botswana have similar relative and absolute trajectories and values, although their levels reduce as more weight is placed in the absolute perspective, indicating a better achievement when considering the distribution of values across countries in their performance band than when considering their absolute distance to the reference levels. Finally, we see that Jordan and Iran have similar relative levels (2 and 1.98), but Iran reduces its performance when more importance is given to the absolute evaluation. This is because Iran's absolute score (1.85) is lower than Jordan's (1.97).

4. Conclusions

Scalarizing methods typically assess an entity's variable value in absolute terms, in relation to a reference level. However, this scalarization can overlook valuable insights from the overall distribution of values, which may be crucial in contexts where a relative analysis is meaningful. To address this, we propose an innovative approach based on the reference point scheme that incorporates relative evaluations into the achievement scalarizing function. This approach is grounded in the hypotheses of relative income and relative deprivation, which highlight the importance of comparisons in assessing well-being. The proposed relative achievement scalarizing function allows for effective benchmarking, enabling entities to set realistic goals based on the performance of others in similar contexts. Moreover, we also suggest a hybrid achievement scalarizing function enabling to combine the advantages of both absolute and relative assessments to evaluate the entities as desired, depending on their performance level according to the intervals defined by the reference values considered.

By introducing the relative achievement function and the hybrid achievement function, which combines absolute and relative assessments, we provide a more comprehensive evaluation method. From a practical point of view, we have tested the performance of the relative achievement function by applying it to the evaluation of the progress of countries in GNI. This demonstrates that absolute achievement evaluation can be complemented by richer information that accounts for the achievements of other countries. We identify countries that make great progress in absolute terms, but when assessed in relation to the progress of others, the achievement is evaluated as of reduced magnitude or even results in a decline in relative achievement due to the greater progress of other countries. This way we can identify countries that are not progressing at the same path than other countries, and then are at risk of lagging behind, thus, providing valuable information for policymakers who may want to implement more effective actions.

We conclude that there are two important advantages of the proposed achievement functions. First, the relative achievement function provides an assessment of the performance of the entities within the context of their peers, offering a deeper understanding of an entity's performance. This helps identify where improvements are needed relative to others, and complements the absolute evaluation. Second, the hybrid achievement functions offer a flexible formulation that allows combining information from both absolute and relative achievement functions in different ways, depending on the preference of the decision maker or on the specific situation to evaluate. By considering both absolute and relative achievements, a more holistic view of progress is obtained, leading to more balanced and comprehensive decisions.

In sum, our proposed achievement functions enhance the scalarization process by integrating relative assessments, thereby offering a more complete view of entities' performance and progress. This methodology can be applied across various domains and variables, providing a versatile tool for evaluating achievements in a comparative context. This approach allows for a nuanced understanding of how entities' achievements, offering valuable insights for researchers and policymakers who may want to allocate resources more efficiently by identifying entities whose relative performance is lagging, ensuring targeted interventions.

Obviously, in a real-life case, this methodology must be complemented by a sensitivity or robustness analysis of the model's various subjective parameters. Specifically, it is interesting to analyze the robustness of the results with respect to the reference levels used, the values chosen for the common scale, as well as the weights attached to the absolute and relative scalarization in the hybrid formulation. Furthermore, in processes such as the construction of composite indicators, other parameters are often used, such as weights that determine the contribution of each simple indicator to the aggregate measure, or the degree to which each indicator can be compensated by the rest. All these parameters must be subject to a careful robustness analysis. One option for this is to perform Monte Carlo-style simulations, to allow for slight variations in the values considered and study possible deviations from the results obtained.

Furthermore, from a mathematical point of view, we could use non-monotonic variables in the formulation of the absolute scalarizing function, adapting it according to how the variable behaves in each interval (i.e., it increases or decreases). However, using this type of variable in the relative scalarizing function is not so easy and straightforward, but it could be a promising direction for future research.

Future research lines include the use of the new achievement scalarizing functions (relative and hybrid) to build composite indicators that help analyze other socio-economic phenomena by combining several variables and taking into account not only the levels of achievement of each entity but also those of all the entities considered. In addition, the comparison of such composite indicators with other synthetic measures based on absolute assessments is of interest to know the type of information each scheme can provide and how they can complement each other.

CRediT authorship contribution statement

Elena Bárcena-Martín: Writing – review & editing, Writing – original draft, Supervision, Methodology, Formal analysis, Conceptualization. **Francisca García-Pardo:** Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization. **Mariano Luque:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Funding acquisition, Formal analysis. **Ana B. Ruiz:** Writing – review & editing, Writing – original draft, Methodology, Formal analysis. **Francisco Ruiz:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Funding acquisition, Formal analysis.

Acknowledgements

This work has been supported by the Spanish Government (projects PID2022–139543OB-C42, RED2022–134540-T, and PID2024-155360NB-I00 funded by MCIN/AEI/10. 13039/501100011033) and by the Regional Government of Andalucía (projects SEJ-532, SEJ-559, and P18-RT-1566). Funding for open access charge: Universidad de Málaga / CBUA.

Appendix

Table A1

Table A1
Countries' GNI, relative and absolute achievements in 2000 and 2022.

Country	GNI 2000	Sr2000	Sa2000	GNI 2022	Sr2022	Sa2022
Albania	1100	1.3588	1.1540	6770	2.5150	2.2457
Algeria	1610	1.7571	1.3817	3920	1.9748	1.8363
Angola	360	0.6542	0.3780	1880	1.4089	1.2237
Antigua and Barbuda	10,220	3.0666	3.0262	19,050	3.1794	3.0638
Argentina	7430	2.9629	2.7073	11,590	2.9578	2.7596
Armenia	640	0.9766	0.8189	5960	2.3644	2.1594
Australia	21,300	3.6366	3.3304	60,840	3.8859	3.5756
Austria	26,810	3.8244	3.4817	55,720	3.8464	3.5129
Azerbaijan	630	0.9712	0.8031	5660	2.3024	2.1274
Bahamas, The	23,790	3.7348	3.3988	31,520	3.4893	3.2165
Bahrain	10,220	3.0666	3.0262	27,720	3.4073	3.1699
Bangladesh	430	0.7679	0.4882	2820	1.7391	1.5060
Barbados	11,060	3.1218	3.0493	19,490	3.1928	3.0691
Belarus	1380	1.6019	1.2790	7210	2.5816	2.2926
Belgium	25,890	3.8006	3.4564	53,890	3.8292	3.4905
Belize	4470	2.5518	2.2352	6630	2.4913	2.2308
Benin	470	0.8214	0.5512	1400	1.1661	1.0796
Bolivia	960	1.2256	1.0915	3490	1.8962	1.7072
Bosnia and Herzegovina	1390	1.6093	1.2835	7660	2.6398	2.3406
Botswana	3020	2.0125	2.0040	7430	2.6113	2.3161
Brazil	3910	2.3824	2.1459	8140	2.6949	2.3918
Brunei Darussalam	14,650	3.3234	3.1478	31,410	3.4870	3.2151
Bulgaria	1660	1.7838	1.4040	13,350	3.0000	2.9472
Burkina Faso	260	0.4341	0.2205	850	0.9419	0.6816
Burundi	140	0.0689	0.0315	240	0.0000	0.0000
Cabo Verde	1310	1.5457	1.2478	3950	1.9787	1.8453
Cambodia	300	0.5318	0.2835	1690	1.3222	1.1667
Cameroon	720	0.9993	0.9449	1640	1.2981	1.1517
Canada	22,740	3.6952	3.3699	53,310	3.8226	3.4834
Central African Republic	250	0.4067	0.2047	480	0.4787	0.2682
Chad	180	0.1993	0.0945	690	0.7915	0.5028
Chile	5080	2.6858	2.3325	15,360	3.0555	3.0186
China	940	1.2054	1.0826	12,850	2.9966	2.8939
Colombia	2360	1.9649	1.7165	6500	2.4677	2.2170
Comoros	730	1.0000	0.9606	1610	1.2830	1.1426
Congo, Dem. Rep.	130	0.0348	0.0157	610	0.6931	0.4134
Congo, Rep.	560	0.9184	0.6929	2290	1.5787	1.3468
Costa Rica	3560	2.2522	2.0901	12,920	2.9974	2.9014
Cote d'Ivoire	640	0.9766	0.8189	2620	1.6834	1.4459
Croatia	5300	2.7260	2.3676	19,600	3.1960	3.0705
Cyprus	14,180	3.2990	3.1349	31,520	3.4893	3.2165
Czechia	6340	2.8671	2.5335	26,100	3.3687	3.1501
Denmark	32,580	3.9267	3.6401	73,520	3.9517	3.7309
Dominica	4300	2.5060	2.2081	8430	2.7265	2.4227
Dominican Republic	2660	1.9885	1.8504	9050	2.7902	2.4888
Ecuador	1560	1.7268	1.3594	6300	2.4302	2.1956
Egypt, Arab Rep.	1370	1.5943	1.2746	4100	1.9911	1.8904
El Salvador	1880	1.8710	1.5022	4720	2.0701	2.0272
Equatorial Guinea	680	0.9920	0.8819	5240	2.2054	2.0826
Estonia	4170	2.4667	2.1874	27,120	3.3933	3.1626
Eswatini	1630	1.7681	1.3906	3750	1.9487	1.7853
Ethiopia	120	0.0000	0.0000	1020	1.0000	0.8715
Fiji	2170	1.9424	1.6317	5390	2.2417	2.0986
Finland	26,480	3.8165	3.4726	54,890	3.8392	3.5027
France	24,990	3.7737	3.4317	45,290	3.7178	3.3851
Gabon	2990	2.0000	1.9978	7530	2.6240	2.3268
Gambia, The	570	0.9278	0.7087	800	0.9040	0.6257
Georgia	790	1.0403	1.0156	5600	2.2893	2.1210
Germany	26,180	3.8087	3.4644	54,030	3.8307	3.4922
Ghana	330	0.5953	0.3307	2380	1.6094	1.3739
Grenada	4410	2.5363	2.2257	9070	2.7922	2.4909
Guatemala	1670	1.7888	1.4085	5350	2.2322	2.0944
Guinea	400	0.7217	0.4409	1190	1.0379	1.0165
Guinea-Bissau	200	0.2615	0.1260	820	0.9206	0.6480
Guyana	870	1.1303	1.0513	14,920	3.0396	3.0132
Haiti	550	0.9084	0.6772	1610	1.2830	1.1426
Honduras	1010	1.2740	1.1138	2750	1.7208	1.4850
Hungary	4620	2.5883	2.2592	19,010	3.1781	3.0633
Iceland	31,540	3.9142	3.6115	68,660	3.9311	3.6714

(continued on next page)

Table A1 (*continued*)

Country	GNI 2000	Sr2000	Sa2000	GNI 2022	Sr2022	Sa2022
India	440	0.7819	0.5039	2390	1.6126	1.3769
Indonesia	570	0.9278	0.7087	4580	2.0320	2.0123
Iran, Islamic Rep.	1770	1.8307	1.4531	3980	1.9818	1.8544
Iraq	1520	1.7006	1.3415	5270	2.2129	2.0858
Ireland	24,130	3.7463	3.4081	79,730	3.9716	3.8070
Israel	19,990	3.5792	3.2944	55,140	3.8415	3.5058
Italy	21,910	3.6621	3.3472	38,200	3.6075	3.2983
Jamaica	3290	2.1422	2.0471	5760	2.3235	2.1381
Japan	36,810	3.9668	3.7562	42,550	3.6770	3.3516
Jordan	1690	1.7982	1.4174	4350	2.0000	1.9655
Kazakhstan	1270	1.5122	1.2299	9620	2.8370	2.5496
Kenya	430	0.7679	0.4882	2170	1.5339	1.3108
Kiribati	1270	1.5122	1.2299	2810	1.7366	1.5030
Korea, Rep.	11,030	3.1199	3.0485	36,190	3.5733	3.2737
Kuwait	19,000	3.5338	3.2673	40,600	3.6466	3.3277
Kyrgyz Republic	280	0.4863	0.2520	1440	1.1890	1.0916
Lao PDR	280	0.4863	0.2520	2310	1.5858	1.3529
Latvia	3320	2.1553	2.0518	21,850	3.2588	3.0980
Lesotho	590	0.9438	0.7402	1230	1.0641	1.0285
Lithuania	3210	2.1048	2.0343	23,870	3.3119	3.1228
Luxembourg	45,690	4.0000	4.0000	89,200	3.9918	3.9230
Madagascar	280	0.4863	0.2520	510	0.5320	0.3017
Malawi	230	0.3505	0.1732	640	0.7322	0.4469
Malaysia	3490	2.2251	2.0789	11,830	2.9679	2.7852
Maldives	2050	1.9183	1.5781	10,880	2.9234	2.6839
Mali	280	0.4863	0.2520	850	0.9419	0.6816
Malta	11,380	3.1414	3.0581	32,860	3.5144	3.2329
Marshall Islands	2770	1.9942	1.8996	7270	2.5900	2.2990
Mauritania	710	0.9980	0.9291	2080	1.4968	1.2838
Mauritius	3900	2.3788	2.1443	10,360	2.8907	2.6285
Mexico	6620	2.8988	2.5782	10,820	2.9202	2.6775
Micronesia, Fed. Sts.	2130	1.9351	1.6138	4140	1.9934	1.9024
Moldova	490	0.8441	0.5827	5500	2.2669	2.1103
Mongolia	460	0.8087	0.5354	4260	1.9987	1.9384
Morocco	1570	1.7331	1.3638	3670	1.9346	1.7613
Mozambique	320	0.5746	0.3150	440	0.4028	0.2235
Myanmar	190	0.2308	0.1102	1270	1.0889	1.0405
Namibia	2210	1.9487	1.6496	5010	2.1470	2.0581
Nepal	220	0.3217	0.1575	1340	1.1310	1.0616
Netherlands	28,160	3.8540	3.5187	60,230	3.8816	3.5681
New Zealand	14,070	3.2931	3.1319	49,090	3.7716	3.4317
Nicaragua	950	1.2157	1.0871	2090	1.5011	1.2868
Niger	220	0.3217	0.1575	580	0.6481	0.3799
North Macedonia	1920	1.8846	1.5201	6660	2.4966	2.2340
Norway	36,600	3.9656	3.7504	94,540	3.9993	3.9884
Oman	6840	2.9202	2.6132	20,020	3.2082	3.0756
Pakistan	470	0.8214	0.5512	1560	1.2558	1.1276
Panama	3920	2.3858	2.1475	16,960	3.1107	3.0382
Papua New Guinea	600	0.9512	0.7559	2700	1.7069	1.4700
Paraguay	1500	1.6870	1.3326	5920	2.3564	2.1551
Peru	1960	1.8966	1.5379	6740	2.5101	2.2425
Philippines	1180	1.4321	1.1897	3950	1.9787	1.8453
Poland	4670	2.5996	2.2671	18,900	3.1746	3.0619
Portugal	12,180	3.1876	3.0800	25,950	3.3650	3.1483
Romania	1710	1.8071	1.4263	15,570	3.0629	3.0211
Russian Federation	1710	1.8071	1.4263	12,750	2.9947	2.8833
Rwanda	270	0.4609	0.2362	930	0.9799	0.7709
Saudi Arabia	7810	2.9814	2.7679	27,680	3.4063	3.1695
Senegal	670	0.9886	0.8661	1620	1.2881	1.1456
Serbia	1610	1.7571	1.3817	9290	2.8110	2.5144
Seychelles	7920	2.9849	2.7855	12,010	2.9745	2.8044
Sierra Leone	140	0.0689	0.0315	600	0.6789	0.4022
Singapore	23,680	3.7308	3.3957	67,200	3.9238	3.6535
Slovak Republic	5540	2.7635	2.4059	22,070	3.2647	3.1007
Slovenia	11,270	3.1348	3.0550	29,590	3.4483	3.1928
Solomon Islands	940	1.2054	1.0826	2210	1.5494	1.3228
South Africa	3280	2.1377	2.0455	6780	2.5166	2.2468
Spain	15,790	3.3802	3.1791	32,090	3.5002	3.2235
Sri Lanka	870	1.1303	1.0513	3610	1.9226	1.7432
St. Kitts and Nevis	8850	3.0000	2.9338	20,020	3.2082	3.0756
St. Lucia	5460	2.7519	2.3931	12,400	2.9863	2.8459
St. Vincent and the Grenadines	3380	2.1805	2.0614	9110	2.7958	2.4952
Sudan	350	0.6355	0.3622	760	0.8661	0.5810
Suriname	1940	1.8909	1.5290	4970	2.1366	2.0538
Sweden	31,610	3.9151	3.6135	63,500	3.9029	3.6082
Switzerland	44,580	3.9978	3.9695	95,490	4.0000	4.0000

(*continued on next page*)

Table A1 (*continued*)

Country	GNI 2000	Sr2000	Sa2000	GNI 2022	Sr2022	Sa2022
Tajikistan	170	0.1672	0.0787	1210	1.0512	1.0225
Tanzania	390	0.7057	0.4252	1200	1.0446	1.0195
Thailand	1980	1.9019	1.5469	7230	2.5845	2.2948
Togo	300	0.5318	0.2835	1010	0.9988	0.8603
Trinidad and Tobago	5140	2.6975	2.3421	16,190	3.0845	3.0287
Tunisia	2230	1.9513	1.6585	3830	1.9616	1.8093
Turkiye	4260	2.4944	2.2018	10,640	2.9093	2.6583
Uganda	270	0.4609	0.2362	930	0.9799	0.7709
Ukraine	680	0.9920	0.8819	4260	1.9987	1.9384
United Arab Emirates	30,050	3.8904	3.5706	49,160	3.7726	3.4325
United Kingdom	29,420	3.8791	3.5533	49,240	3.7736	3.4335
United States	35,970	3.9605	3.7332	76,770	3.9632	3.7707
Uruguay	7120	2.9429	2.6579	18,000	3.1453	3.0509
Uzbekistan	630	0.9712	0.8031	2190	1.5418	1.3168
Vanuatu	1370	1.5943	1.2746	3650	1.9308	1.7553
Viet Nam	380	0.6890	0.4094	4010	1.9845	1.8634
Zambia	350	0.6355	0.3622	1240	1.0704	1.0315
Zimbabwe	360	0.6542	0.3780	1710	1.3316	1.1727

Source: World Bank national accounts data, OECD National Accounts data files, and own computations.

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