

**The center and the Lie  
structure  
of a Leavitt path algebra**  
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**CIMPA research school**  
**Recent trends on Non-commutative algebras**

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## Abstract

These notes contain all the explanations and references corresponding to the course I delivered during the CIMPA research school in Pune (India) in June 2017 (see the complete program of the school here: <https://iiserpunecimpa17.wordpress.com/program/>).

The purpose of the course is to explain the essentials on Leavitt path algebras in order to get the tools needed to determine the center of a Leavitt path algebra. First when the graph is row finite; then, in the general setting (for arbitrary graphs). We start in Chapter 1 by the very beginning of the theory of Leavitt path algebras, explain Conditions (L) and (K) and characterize them in graph terms and also in ring-theoretical terms. In Chapter 2 we characterize the simplicity of Leavitt path algebras and determine the graph pieces involved in every graded ideal. We determine the structure of certain graded ideals, concretely: the ideal generated by line points (which is precisely the socle of the Leavitt path algebra), the ideal generated by vertices in cycles without exits (which is the ideal generated by the primitive non-minimal idempotents of the algebra), and the ideal generated by extreme cycles (which is a direct sum of graded ideals each of which is purely infinite simple). The end of the chapter refers to decomposable Leavitt path algebras. For this part, as well for the structure of the center of an arbitrary graph, we use the Steinberg algebra associated to the groupoid related to a graph. We give not here all the details about groupoids and Steinberg algebras because the course of Professor Lisa O. Clark is devoted to that end. In Chapter 3 we explore the center of a Leavitt path algebra and give its structure. Finally, in chapter 4, we give a taste of the work that has been done concerning the study of the simplicity related to the Lie structure of a Leavitt path algebra.

We refer the reader to the book [2], the first one devoted to Leavitt path algebras.

Pune, India, June 2017.




# Chapter 1

## Preliminary results

We introduce the main object of this course: Leavitt path algebras.

A *directed graph* is a 4-tuple  $E = (E^0, E^1, r_E, s_E)$  consisting of two disjoint sets  $E^0, E^1$  and two maps  $r_E, s_E : E^1 \rightarrow E^0$ . The elements of  $E^0$  are called the *vertices* of  $E$  and the elements of  $E^1$  the edges of  $E$  while for  $e \in E^1$ ,  $r_E(e)$  and  $s_E(e)$  are called the *range* and the *source* of  $e$ , respectively. If there is no confusion with respect to the graph we are considering, we simply write  $r(e)$  and  $s(e)$ .



**Graph (Directed)**

$$E = (E^0, E^1, r, s)$$

vertices  $\swarrow$   $\searrow$  edges

$$r : E^1 \rightarrow E^0 \quad \text{range}$$
$$s : E^1 \rightarrow E^0 \quad \text{source}$$

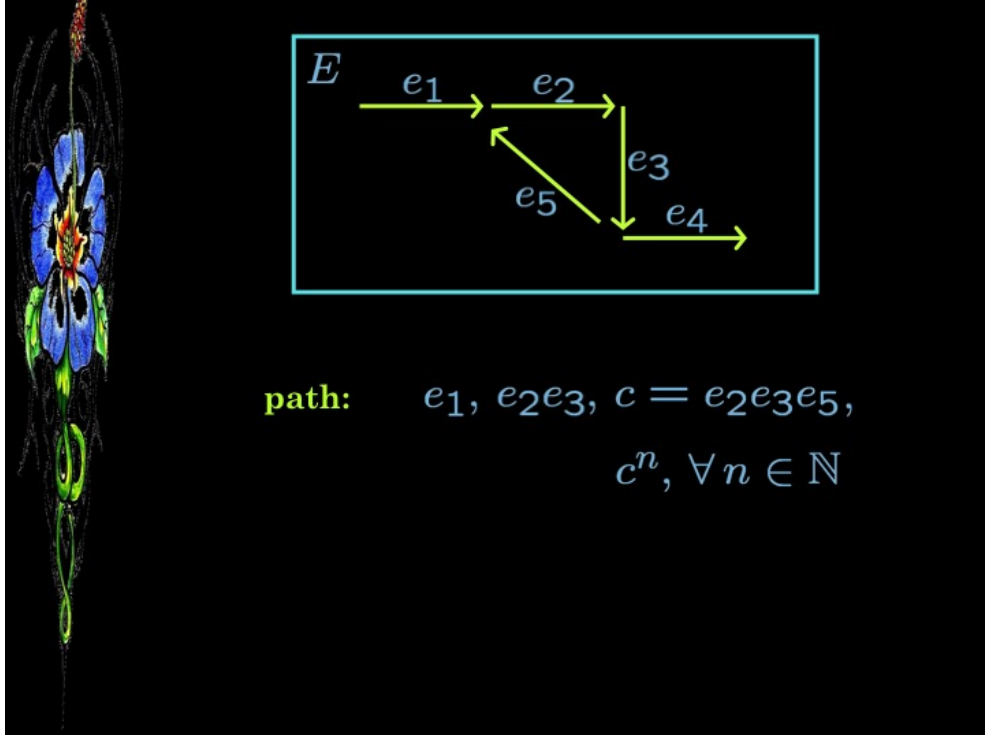
**Example**

$$\begin{array}{ccc} \bullet & \xrightarrow{e} & \bullet \\ u & & v \end{array} \quad \begin{array}{l} s(e) = u \\ r(e) = v \end{array}$$

Given a (directed) graph  $E$  and a field  $K$ , the *path  $K$ -algebra* of  $E$ , denoted by  $KE$ , is defined as the free associative  $K$ -algebra generated by the set of paths of  $E$  with relations:

$$(V) \quad vw = \delta_{v,w}v \text{ for all } v, w \in E^0.$$

$$(E1) \quad s(e)e = er(e) = e \text{ for all } e \in E^1.$$



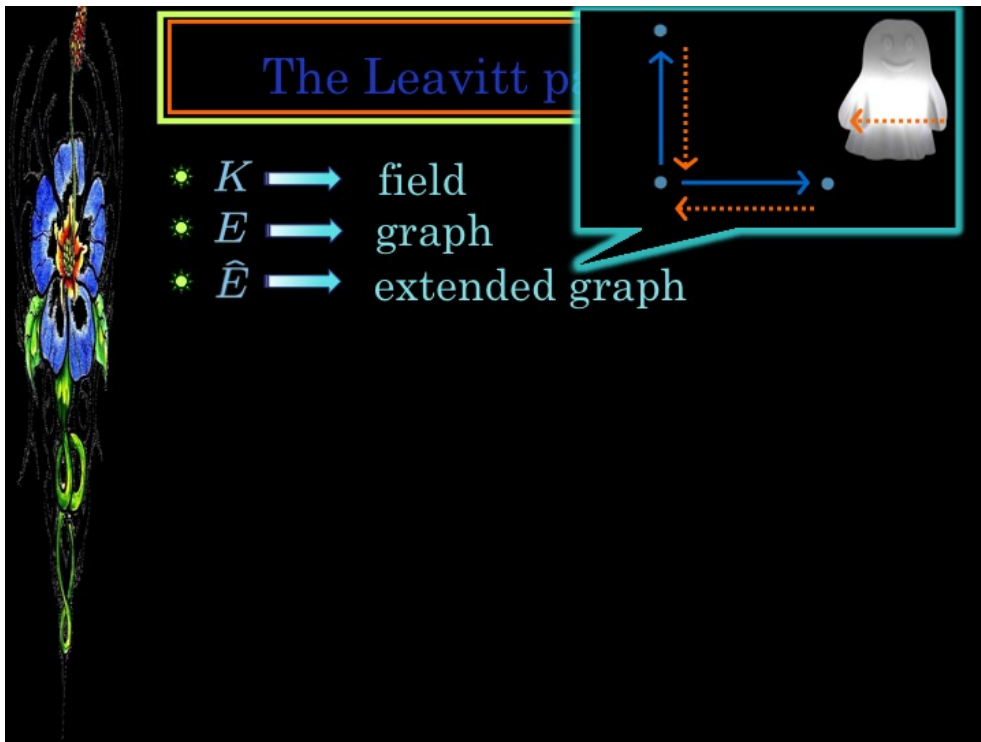
If  $s^{-1}(v)$  is a finite set for every  $v \in E^0$ , then the graph is called *row-finite*. If  $E^0$  is finite and  $E$  is row-finite, then  $E^1$  must necessarily be finite as well; in this case we say simply that  $E$  is *finite*.

A vertex which emits no edges is called a *sink*. A vertex  $v$  is called an *infinite emitter* if  $s^{-1}(v)$  is an infinite set, and a *regular vertex* otherwise. The set of infinite emitters will be denoted by  $E_{inf}^0$  while  $\text{Reg}(E)$  will denote the set of regular vertices.

The *extended graph* of  $E$  is defined as the new graph  $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r_{\widehat{E}}, s_{\widehat{E}})$ , where  $(E^1)^* = \{e_i^* \mid e_i \in E^1\}$  and the functions  $r_{\widehat{E}}$  and  $s_{\widehat{E}}$  are defined as

$$r_{\widehat{E}|_{E^1}} = r, \quad s_{\widehat{E}|_{E^1}} = s, \quad r_{\widehat{E}}(e_i^*) = s(e_i), \quad \text{and} \quad s_{\widehat{E}}(e_i^*) = r(e_i).$$

The elements of  $E^1$  will be called *real edges*, while for  $e \in E^1$  we will call  $e^*$  a *ghost edge*.



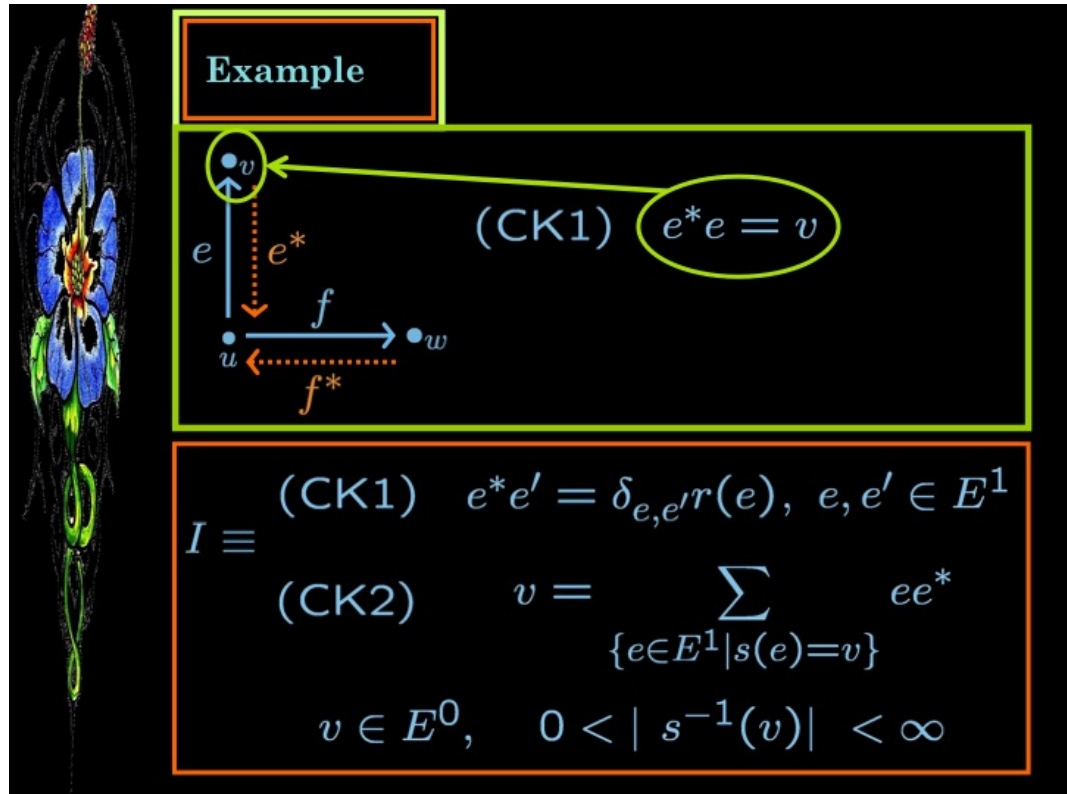
## 1.1 Leavitt path algebras.

The *Leavitt path algebra* of  $E$  with coefficients in  $K$ , denoted  $L_K(E)$ , is the quotient of the path algebra  $K\widehat{E}$  by the ideal of  $K\widehat{E}$  generated by the relations:

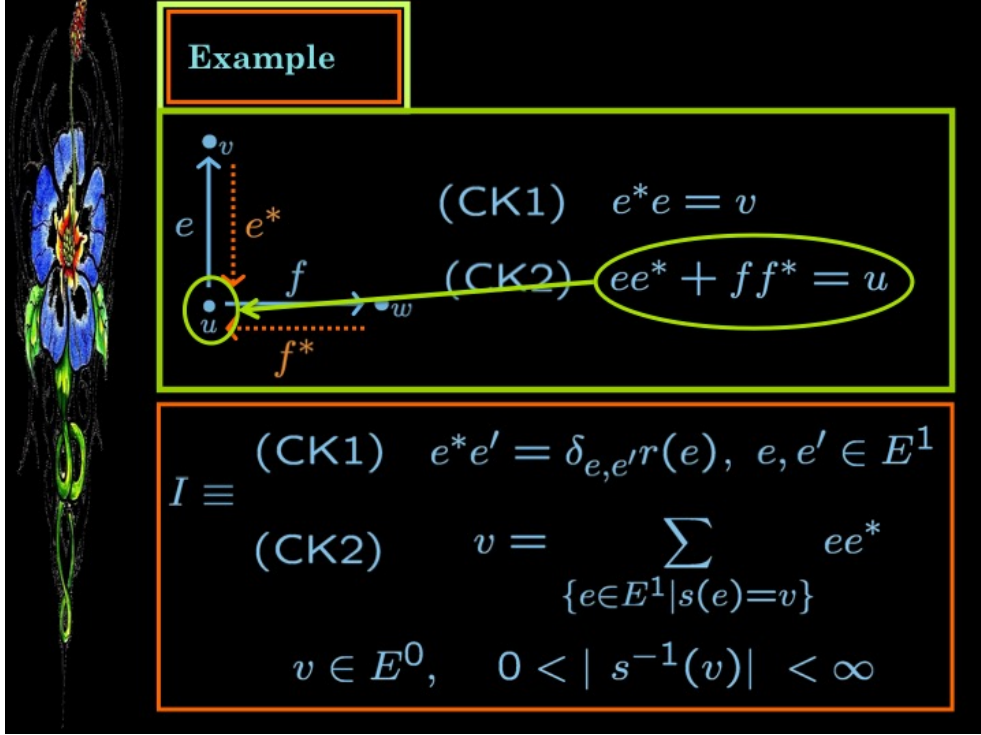
$$(CK1) \quad e^*e' = \delta_{e,e'}r(e) \quad \text{for all } e, e' \in E^1.$$

$$(CK2) \quad v = \sum_{\{e \in E^1 \mid s(e)=v\}} ee^* \quad \text{for every } v \in \text{Reg}(E).$$

Here we can see how Condition (CK1) works.



Condition (CK2) can be seen in the following example.



Observe that in  $K\widehat{E}$  the relations (V) and (E1) remain valid and that the following is also satisfied:

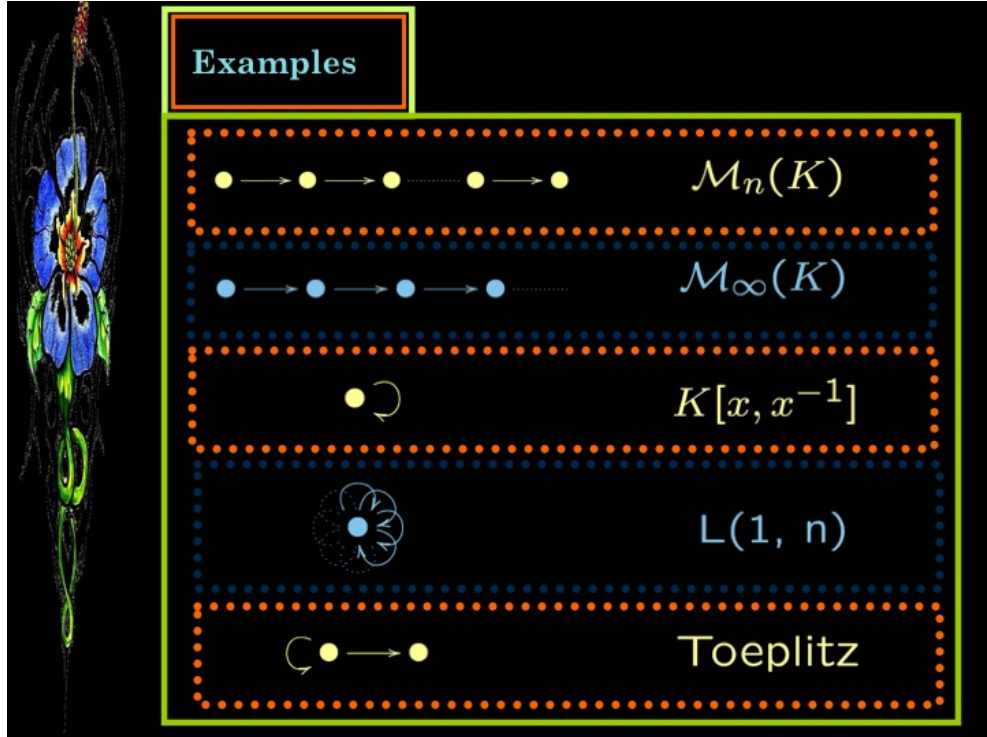
$$(E2) \quad r(e)e^* = e^*s(e) = e^* \text{ for all } e \in E^1.$$

Note that if  $E$  is a finite graph, then  $L_K(E)$  is unital with  $\sum_{v \in E^0} v = 1_{L_K(E)}$ ; otherwise,  $L_K(E)$  is a ring with a set of local units consisting of sums of distinct vertices (for a ring  $R$  the assertion  $R$  has local units means that each finite subset of  $R$  is contained in a *corner* of  $R$ , that is, a subring of the form  $eRe$  where  $e$  is an idempotent of  $R$ ). Note that since every Leavitt path algebra  $L_K(E)$  has local units, it is the directed union of its corners.

A *path*  $\mu$  in a graph  $E$  is a finite sequence of edges  $\mu = e_1 \dots e_n$  such that  $r(e_i) = s(e_{i+1})$  for  $i = 1, \dots, n-1$ . In this case,  $s(\mu) := s(e_1)$  and  $r(\mu) := r(e_n)$  are the *source* and *range* of  $\mu$ , respectively, and  $n$  is the *length* of  $\mu$ . We also say that  $\mu$  is a *path from*  $s(e_1)$  *to*  $r(e_n)$  and denote by  $\mu^0$  the set of its vertices, i.e.,  $\mu^0 := \{s(e_1), r(e_1), \dots, r(e_n)\}$ . By  $\mu^1$  we denote the set of edges appearing in  $\mu$ , i.e.,  $\mu^1 := \{e_1, \dots, e_n\}$ .

We view the elements of  $E^0$  as paths of length 0. The set of all paths of a graph  $E$  is denoted by  $\text{Path}(E)$ .

There are many examples of known algebras that can be seen as Leavitt path algebras. Here we see some.



## 1.2 The grading.

The Leavitt path algebra  $L_K(E)$  is a  $\mathbb{Z}$ -graded  $K$ -algebra, spanned as a  $K$ -vector space by  $\{\alpha\beta^* \mid \alpha, \beta \in \text{Path}(E)\}$ . In particular, for each  $n \in \mathbb{Z}$ , the degree  $n$  component  $L_K(E)_n$  is spanned by the set

$$\{\alpha\beta^* \mid \alpha, \beta \in \text{Path}(E) \text{ and } \text{length}(\alpha) - \text{length}(\beta) = n\}.$$

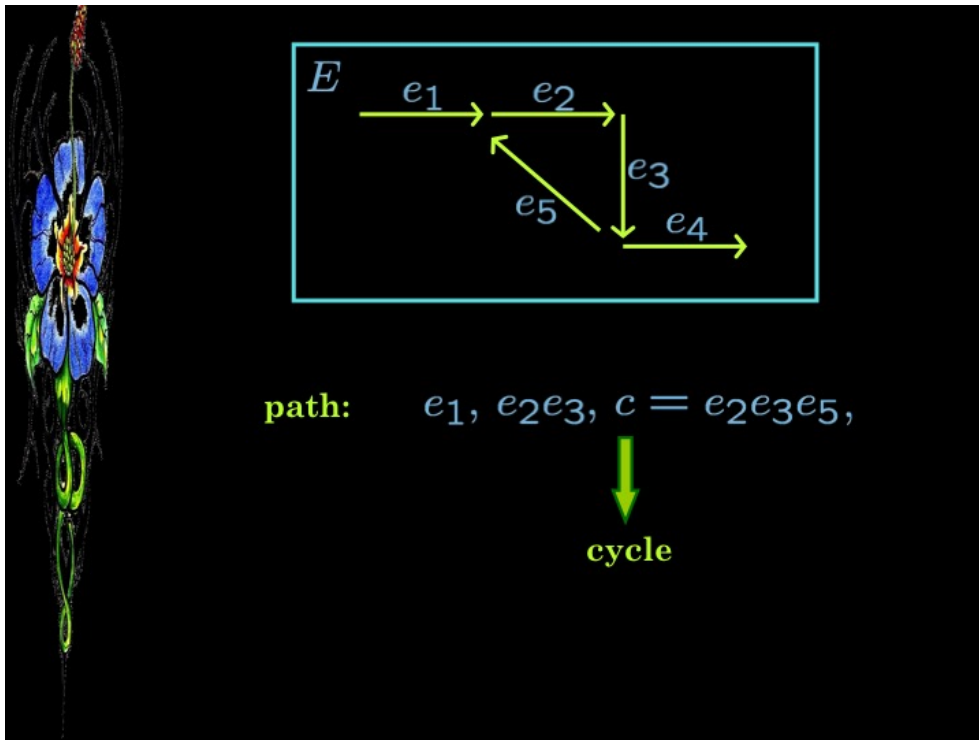
Denote by  $h(L_K(E))$  the set of all homogeneous elements in  $L_K(E)$ , that is,

$$h(L_K(E)) := \cup_{n \in \mathbb{Z}} L_K(E)_n.$$

## 1.3 Cycles.

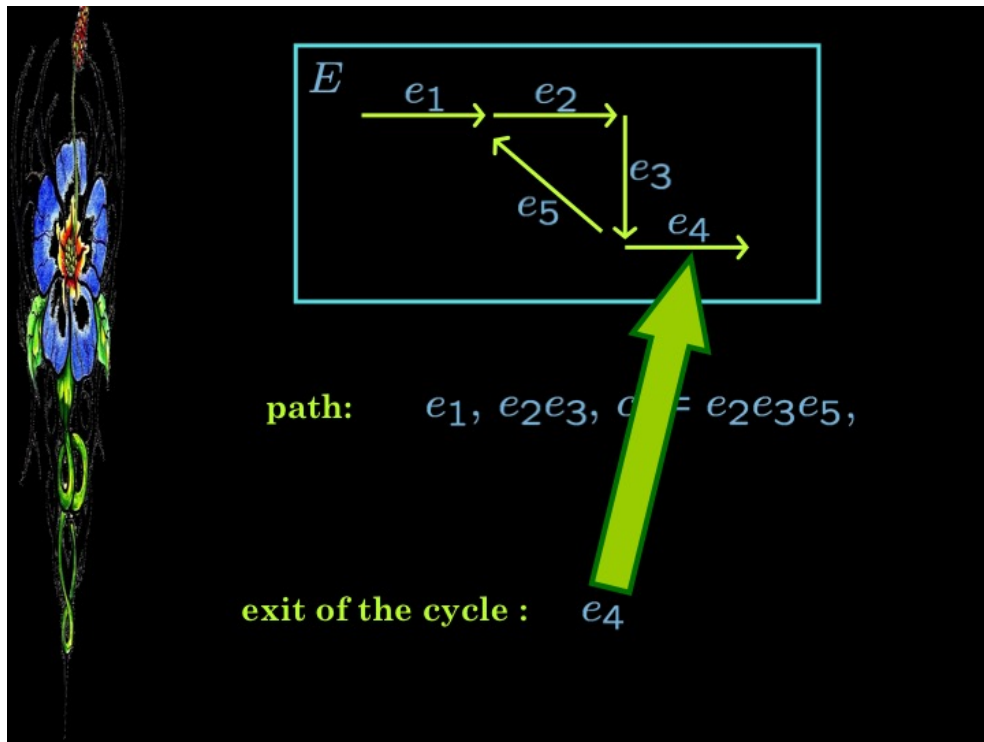
If  $\mu$  is a path in  $E$ , and if  $v = s(\mu) = r(\mu)$ , then  $\mu$  is called a *closed path based at  $v$* . If  $s(\mu) = r(\mu)$  and  $s(e_i) \neq s(e_j)$  for every  $i \neq j$ , then  $\mu$

is called a *cycle*. A graph which contains no cycles is called *acyclic*. For  $\mu = e_1 \dots e_n \in \text{Path}(E)$  we write  $\mu^*$  for the element  $e_n^* \dots e_1^*$  of  $L_K(E)$ .



An edge  $e$  is an *exit* for a path  $\mu = e_1 \dots e_n$  if there exists  $i \in \{1, \dots, n\}$  such that  $s(e) = s(e_i)$  and  $e \neq e_i$ . We say that  $E$  satisfies *Condition (L)* if every cycle in  $E$  has an exit.

Let  $X$  be a subset of  $E^0$ . A *path in  $X$*  is a path  $\alpha$  in  $E$  with  $\alpha^0 \subseteq X$ . We say that a path  $\alpha$  in  $X$  has an *exit in  $X$*  if there exists  $e \in E^1$  which is an exit for  $\alpha$  and such that  $r(e) \in X$ .



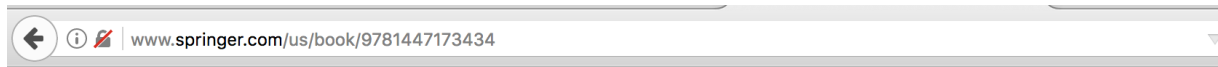
We denote by  $P_c(E)$  ( $P_c$  if there is no confusion about the graph) the set of vertices of a graph  $E$  lying in cycles without exits, and decompose it as:

$$P_c(E) = P_c(E)^+ \sqcup P_c(E)^-,$$

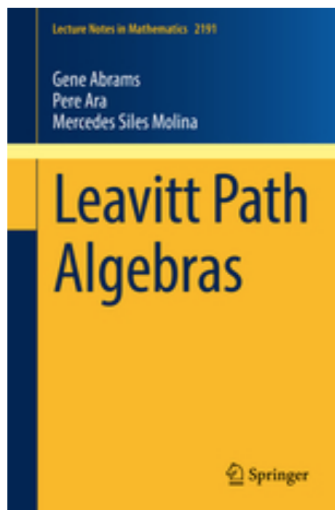
where  $P_c(E)^+$  are those elements in  $P_c(E)$  for which there exists a cycle without exits  $c$  such that the number of paths ending at a vertex of  $c^0$  and not containing all the edges of  $c$  is infinite,  $P_c(E)^- = P_c(E) \setminus P_c(E)^+$  and  $\sqcup$  denotes the disjoint union. If it is clear from the context the graph we are referring to, we will write simply  $P_c^+$  or  $P_c^-$ .

## 1.4 The book [2].

Let me explain you the contents of the book.



Lecture Notes in Mathematics



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# Leavitt Path Algebras

Authors: **Abrams**, Gene, **Ara**, Pere, **Siles Molina**, Mercedes

The first book solely devoted to Leavitt path algebras

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## 1.5 Conditions (L) and (K).

As it has been said in other courses of this CIMPA research school, Condition (K) is equivalent for the Leavitt path algebra to have every ideal graded, that is, every ideal is generated by the idempotents it contains; concretely, every ideal  $I$  of a Leavitt path algebra  $L_K(E)$  where  $E$  satisfies Condition (K) is generated by  $I \cap E^0$ . And this is equivalent for the Leavitt path algebra to have the exchange property.

As for condition (L), it can be characterized using ring-theoretical properties of the Leavitt path algebra. concretely, using primitive idempotents. Let us recall very briefly this notion.

**Proposition 1.1.** *Let  $e$  be an idempotent in a ring  $R$  (not necessarily unital). The following conditions are equivalent:*

- (i)  $eR$  is an indecomposable right  $R$ -module (equivalently,  $Re$  is an indecomposable left  $R$ -module).
- (ii)  $eRe$  is a ring without nontrivial idempotents.
- (iii)  $e$  has no decomposition into  $a + b$ , where  $a, b$  are nonzero orthogonal idempotents in  $R$ .

*Proof.* As in [26, Proposition 21.8]. □

**Definition 1.2.** Following [26], if an idempotent  $0 \neq e \in R$  satisfies any of these conditions, we say that  $e$  is a *primitive idempotent*.

### Examples 1.3.

- In Leavitt path algebras, Condition (CK2) says that every vertex not being a sink and being the source of finitely many edges, can be decomposed as the sum of non-trivial orthogonal idempotents.
- Every line point is a primitive minimal idempotent.
- Every vertex in a cycle without exits is a primitive idempotent which is not minimal.

The primitive vertices in Leavitt path algebras are characterized.

### Proposition 1.4. ([13, Proposition 5.3])

*Let  $E$  be an arbitrary graph and let  $v \in E^0$ . Then  $v$  is a primitive idempotent of  $L_K(E)$  if and only if its tree  $T(v)$  has no bifurcations.*

**Theorem 1.5.** ([13, Theorem 5.8]) *Let  $E$  be an arbitrary graph. The following are equivalent:*

- (i)  $E$  satisfies Condition (L).
- (ii)  $L_K(E)$  has no non-minimal primitive idempotents.

## 1.6 Ordering paths.

Given paths  $\alpha, \beta$ , we say  $\alpha \leq \beta$  if  $\beta = \alpha\alpha'$  for some path  $\alpha'$ .



# Chapter 2

## Ideals

In this chapter we will speak mainly about graded ideals of Leavitt path algebras. Of course, in an arbitrary Leavitt path algebra not every ideal has to be graded. This is the case, for example, of the Leavitt path algebra associated to the graph  $E$  having one vertex, say  $v$ , and one edge, say  $e$ . It is isomorphic to the Laurent polynomial ring  $K[x, x^{-1}]$  and here not every ideal is graded. For example,  $\langle 1 + x \rangle$  is not a graded ideal. This, translated to  $L_K(E)$  means that  $\langle v + e \rangle$  is not a graded ideal.

### 2.1 Simplicity of a Leavitt path algebra

Before stating some results on graded ideals, we will recall here the characterization of simple Leavitt path algebras (those algebras  $A$  whose unique ideals are 0 and  $A$ ).

Recall that for a graph  $E$ , the set  $\mathcal{H}_E$  consists of all hereditary and saturated subsets of vertices of  $E$ . Also, we recall that a graph  $E$  is said to satisfy *Condition (L)* if every cycle has an exit.

**Theorem 2.1. (The Simplicity Theorem)** *Let  $E$  be an arbitrary graph and  $K$  any field. Then the Leavitt path algebra  $L_K(E)$  is simple if and only if  $E$  satisfies the following conditions:*

(i)  $\mathcal{H}_E = \{\emptyset, E^0\}$ .

(ii)  $E$  satisfies *Condition (L)*

The Dichotomy Principle for Leavitt path algebras (see, for example [2, Theorem 3.1.11]) states that a simple Leavitt path algebra is locally matricial, or it is a purely infinite simple Leavitt path algebra (*locally matricial* means

that it is a direct limit of matricial algebras; a *matricial algebra* is a finite direct product of full finite dimensional matrix algebras over the field  $K$ ).

We recall here a characterization of purely infinite rings and the theorem characterizing purely infinite simple Leavitt path algebras.

For the definition that follows, see e.g. [10, Definitions 1.2]. Let  $R$  be a ring. An idempotent  $e$  in  $R$  is said to be *infinite* if there exist orthogonal idempotents  $f, g \in R$  such that  $e = f + g$ ,  $g \neq 0$ , and  $Re \cong Rf$  as left  $R$ -modules. In other words,  $e$  is infinite if  $Re$  is isomorphic to a proper direct summand of itself. In such a situation we say that  $Re$  is a *directly infinite module*.

**Theorem 2.2. (The Purely Infinite Simplicity Theorem;** see, e.g. [2, Theorem 3.1.10.]). *Let  $E$  be an arbitrary graph and  $K$  any field. Then the Leavitt path algebra  $L_K(E)$  is purely infinite simple if and only if  $E$  satisfies the following conditions:*

- (i)  $\mathcal{H}_E = \{\emptyset, E^0\}$ ,
- (ii)  $E$  satisfies Condition (L), and
- (iii) every vertex in  $E^0$  connects to a cycle.

## 2.2 Hereditary and saturated sets of vertices

We define a relation  $\geq$  on  $E^0$  by setting  $v \geq w$  if there exists a path in  $E$  from  $v$  to  $w$ . A subset  $H$  of  $E^0$  is called *hereditary* if  $v \geq w$  and  $v \in H$  imply  $w \in H$ . A hereditary set is *saturated* if every regular vertex which feeds into  $H$  and only into  $H$  is again in  $H$ , that is, if  $s^{-1}(v) \neq \emptyset$  is finite and  $r(s^{-1}(v)) \subseteq H$  imply  $v \in H$ . Denote by  $\mathcal{H}_E$  the set of hereditary saturated subsets of  $E^0$ .

Hereditary and saturated subsets of vertices play an important role in the theory of Leavitt path algebras. In fact, they are closely related to graded ideals of Leavitt path algebras (as was highlighted for the first time in [11]), and also to general ideals since every ideal  $I$  in a Leavitt path algebra  $L_K(E)$  contains a graded part: the ideal generated by  $I \cap E^0$  (see [2, Theorem 2.8.6]).

Let  $X$  be a subset of vertices in  $E^0$ . Denote by  $I(X)$  the ideal of  $L_K(E)$  generated by  $X$ . Then,  $I(X)$  is a graded ideal. The reason is that it is generated by elements of degree zero. Moreover, if  $E$  is a row-finite graph every graded ideal  $J$  of  $L_K(E)$  is  $I(H)$  for  $H$  a hereditary and saturated subset of  $E^0$ ; concretely,  $H = J \cap E^0$  (see [18, Lemma 2.1 and Remark 2.2]). Although not every hereditary subset has to be saturated, hereditary subsets

(much more easy to get) give important information about the Leavitt path algebra.

Whenever  $X$  is a set of vertices of a graph  $E$ , the *saturated closure* of  $X$  is defined as  $\cup_{i \in \mathbb{N}} \Lambda_i(X)$ , where  $\Lambda_0(X) = X$  and by recurrence  $\Lambda_i(X) = \Lambda_{i-1}(X) \cup \{v \in \text{Reg}(E) \mid r(s^{-1}(v)) \in \Lambda_{i-1}(X)\}$ . In particular, for a hereditary subset of vertices, say  $H$ , this saturated closure is hereditary and saturated and is denoted by  $\overline{H}$ .

For  $X$  a subset of vertices in a graph  $E$ , the *hereditary closure* of  $X$  is defined as the minimum hereditary subset of  $E^0$  containing  $X$ . It always exists because is just the intersection of all hereditary subsets of  $E^0$  which contains  $X$ . The *hereditary and saturated closure* of a set of vertices is defined as the saturated closure of the hereditary closure.

**Lema 2.3.** *Let  $E$  be a graph and  $K$  a field. Let  $H$  be a hereditary subset of  $E^0$ . Then*

$$I(H) = \left\{ \sum_{i=1}^n k_i \gamma_i \lambda_i^* \mid n \geq 1, k_i \in K^\times, \gamma_i, \lambda_i \in \text{Path}(E), r(\gamma_i) = r(\lambda_i) \in H \right\}.$$

Moreover, if  $\overline{H}$  denotes the saturated closure of  $H$ , then  $I(H) = I(\overline{H})$ .

*Proof.* Let  $I$  be the set in the second part of the identity in the statement. To see that  $I$  is an ideal of  $L_K(E)$  we show that for every element  $\alpha\beta^*$ , where  $r(\alpha) = r(\beta) = u \in H$ , and for every  $a, b \in L_K(E)$ , we have  $a\alpha u \beta^* b \in I$ . It is enough to prove that  $\gamma\lambda^* u \mu\eta^* \in I$  for every  $\gamma, \lambda, \mu, \eta \in \text{Path}(E)$  and  $u \in H$ .

If  $\gamma\lambda^* u \mu\eta^* = 0$  we are done. Suppose otherwise that  $\gamma\lambda^* u \mu\eta^* \neq 0$ . Then  $\gamma\lambda^* u \mu\eta^* = \gamma\mu'\eta^*$  if  $\mu = \lambda\mu'$ , or  $\gamma\lambda^* u \mu\eta^* = \gamma(\lambda')^*\eta^*$  if  $\lambda = \mu\lambda'$ . Note that  $u = s(\mu)$  and  $H$  hereditary imply  $r(\mu) \in H$ , therefore,  $r(\mu') = r(\mu) \in H$  in the first case, and  $r(\lambda') = r(\mu) \in H$  in the second case, which imply  $\gamma\lambda^* u \mu\eta^* \in I$  in both cases. This shows that  $I$  is an ideal of  $L_K(E)$ ; as it contains  $H$  and must be contained in every ideal containing  $H$ , it must coincide with  $I(H)$ .

Now we prove  $I(H) = I(\overline{H})$ . Clearly  $I(H) \subseteq I(\overline{H})$ . The converse can be proved by induction (see [2, Lemma 2.4.1]).  $\square$

Results that will be very useful are the following.

**Lema 2.4.** *Let  $E$  be an arbitrary graph and let  $H_1, H_2$  be non empty hereditary subsets of vertices of  $E$ . Then:*

- (i)  $\Lambda_m(H_i)$  is hereditary for every  $m \in \mathbb{N}$ .
- (ii)  $\overline{H_1} \cap \overline{H_2} = \overline{H_1 \cap H_2}$ .

*Proof.* (i). For  $m = 0$  the result is trivial. Suppose the result true for  $m - 1$  and let us show it for  $m$ . Take  $u \in \Lambda_m(H_i)$  and let  $e \in E^1$  be such that  $s(e) = u$ . Then  $r(e) \in r(s^{-1}(u)) \subseteq \Lambda_{m-1}(H_i) \subseteq \Lambda_m(H_i)$ . This implies the result.

(ii). It is immediate to see  $\overline{H_1 \cap H_2} \subseteq \overline{H_1} \cap \overline{H_2}$ . For the converse we will prove:  $\Lambda_m(H_1) \cap \Lambda_m(H_2) = \Lambda_m(H_1 \cap H_2)$ . Note that the first observation implies  $\Lambda_m(H_1) \cap \Lambda_m(H_2) \supseteq \Lambda_m(H_1 \cap H_2)$ . For the converse containment, use induction. If  $m = 0$  then the result is trivially true. Suppose our assertion is true for  $m - 1$  and show it for  $m$ . If  $u \in \Lambda_m(H_1) \cap \Lambda_m(H_2)$  then  $u \in \Lambda_{m-1}(H_1)$  or  $r(s^{-1}(u)) \subseteq \Lambda_{m-1}(H_1)$ . In the first case, and since  $\Lambda_{m-1}(H_1)$  is hereditary (by (i)) we have also  $r(s^{-1}(u)) \subseteq \Lambda_{m-1}(H_1)$ . Analogously we prove  $r(s^{-1}(u)) \subseteq \Lambda_{m-1}(H_2)$ . This means  $r(s^{-1}(u)) \subseteq \Lambda_{m-1}(H_1) \cap \Lambda_{m-1}(H_2) = \Lambda_{m-1}(H_1 \cap H_2)$  (by the induction hypothesis) and so  $u \in \Lambda_m(H_1 \cap H_2)$ .  $\square$

**Lema 2.5.** *Let  $E$  be a graph and  $H$  a hereditary subset of  $E^0$ . Then, for every  $v \in \overline{H}$  there exists a finite number of paths  $\alpha_1, \dots, \alpha_n$  satisfying  $r(\alpha_i) \in H$  and  $v = \sum_{i=1}^n \alpha_i \alpha_i^*$ .*

*Proof.* For  $v$  in  $H$  we get immediately the result. Take  $v$  in  $\Lambda_1(H)$  not being a sink. Then  $v = \sum_{f \in s^{-1}(v)} f f^*$  and we have the claim. Suppose the result true for every  $u \in \Lambda_{i-1}(H)$  and take  $v \in \Lambda_i(H)$ . Then, for every  $f \in s^{-1}(v)$ , since  $r(f) \in r(s^{-1}(v)) \subseteq \Lambda_{i-1}(H)$ , by the induction hypothesis, there exists a finite number of paths  $\beta_1^f, \dots, \beta_m^f$  in  $\text{Path}(E)$  such that  $r(f) = \sum_i \beta_i^f (\beta_i^f)^*$  and  $r(\beta_i^f) \in H$ . Hence  $v = \sum_{f \in s^{-1}(v)} f f^* = \sum_f f (\sum_i \beta_i^f (\beta_i^f)^*) f^* = \sum_{f,i} f \beta_i^f (\beta_i^f)^* f^*$  and we have finished.  $\square$

The result that follows was stated for the first time in [15, Proposition 3.1] (although in that paper the statement is slightly different); it has proved to be very useful in many different contexts, for example in order to get the Uniqueness Theorems (see [15, Theorem 3.5]), to prove that every Leavitt path algebra is semisimple, etc. In this notes we also find another context where it can be used.

Except otherwise stated,  $E$  will denote an arbitrary graph and  $K$  an arbitrary field. As usual, we will use the notation  $K^\times$  for  $K \setminus \{0\}$ .

**Theorem 2.6. (The Reduction Theorem.)** (MSM; Aranda Pino; Martín Barquero; Martín González)

*For every nonzero element  $a$  in a Leavitt path algebra  $L_K(E)$ , there exist  $\alpha, \beta \in \text{Path}(E)$  such that:*

- (i)  $0 \neq \alpha^* a \beta = kv$  for some  $k \in K^\times$  and  $v \in E^0$ , or

- (ii)  $0 \neq \alpha^*a\beta = p(c, c^*)$ , where  $c$  is a cycle without exits in  $E$  and  $p(c, c^*)$  denotes the evaluation of a polynomial  $p(x, x^{-1}) \in K[x, x^{-1}]$  at  $c$ .

Some immediate consequences of this theorem are the Uniqueness Theorems for Leavitt path algebras, that can be proved by taking into account the Reduction Theorem and that the kernel of a (graded) homomorphism is a (graded) ideal.

**Remark 2.7.** The Reduction Theorem is also valid when considering a commutative and unital ring instead of a field.

Of special interest will be the following result.

**Corollary 2.8.** *Let  $a$  be a nonzero homogeneous element in a Leavitt path algebra  $L_K(E)$ . Then, there exist  $\alpha, \beta \in \text{Path}(E)$ ,  $k \in K^\times$  and  $v \in E^0$ , such that  $0 \neq \alpha^*a\beta = kv$ .*

*Proof.* Let  $\alpha, \beta \in \text{Path}(E)$  be such that  $0 \neq \alpha^*a\beta$  is as in cases (i) or (ii) in Theorem 2.6. In the first case, we have finished. In the second one, use the grading to obtain that in fact  $p(c, c^*)$  in (ii) has to be a monomial, that is,  $\alpha^*a\beta = kc^m$  for some  $k \in K^\times$  and a certain integer  $m$ . If  $m = 0$ , there is nothing more to do. If  $m > 0$ , then  $(c^*)^m\alpha^*a\beta = kr(c)$  and we are done. If  $m < 0$ , then  $\alpha^*a\beta c^{-m} = kr(c)$  and the proof is complete.  $\square$

Another useful result which derives from Theorem 2.6 is:

**Corollary 2.9.** *Let  $H$  be a non-empty hereditary subset of a graph  $E$ . Then, for every nonzero homogeneous  $a \in I(H)$  there exist  $\alpha, \beta \in \text{Path}(E)$  such that  $\alpha^*a\beta = kv$  for some  $k \in K^\times$  and  $v \in H$ .*

*Proof.* Given the nonzero element  $a \in I(H)$  apply Corollary 2.8 and choose  $\lambda, \mu \in \text{Path}(E)$  such that  $\lambda^*a\mu = kw$  for some  $k \in K^\times$  and  $w \in E^0$ . Observe that  $w \in I(H)$ . Use Lemma 2.3 to write  $w = \sum_{i=1}^m k_i \lambda_i \mu_i^*$  with  $k_i \in K^\times$ ,  $\lambda_i, \mu_i \in \text{Path}(E)$ ,  $r(\lambda_i) = r(\mu_i) \in H$ , and suppose  $\lambda_i \mu_i^* \neq \lambda_j \mu_j^*$  for every  $i \neq j$ . Then for  $v = r(\mu_1)$ ,  $\alpha = \lambda \mu_1$  and  $\beta = \mu \mu_1$  we have  $\alpha^*a\beta = \mu_1^* \lambda^* a \mu \mu_1 = k \mu_1^* w \mu_1 = k \mu_1^* \mu_1 = kr(\mu_1) = kv$ , which is nonzero and satisfies  $v \in H$ .  $\square$

**Proposition 2.10.** *Let  $\{H_i\}_{i \in \Lambda}$  be a family of hereditary subsets of a graph  $E$  such that  $H_i \cap H_j = \emptyset$  for every  $i \neq j$ . Then:*

$$I(\overline{\cup_{i \in \Lambda} H_i}) = I(\cup_{i \in \Lambda} H_i) = \oplus_{i \in \Lambda} I(H_i) = \oplus_{i \in \Lambda} I(\overline{H_i}).$$

*Proof.* The union of any family of hereditary subsets is again hereditary, hence  $H := \cup_{i \in \Lambda} H_i$  is a hereditary subset of  $E^0$ . By Lemma 2.3 every

element  $a$  in  $I(H)$  can be written as  $a = \sum_{l=1}^n k_l \alpha_l \beta_l^*$ , where  $k_l \in K^\times$ ,  $\alpha_l, \beta_l \in \text{Path}(E)$  and  $r(\alpha_l) = r(\beta_l) \in H$ . Separate the vertices appearing as ranges of the  $\alpha_l$ 's depending on the  $H_i$ 's they belong to, and apply again Lemma 2.3. This gives  $a \in \sum i \in \Lambda I(H_i) \subseteq I(H)$  since  $H_i \subseteq H$  and so  $\sum i \in \Lambda I(H_i) = I(H)$ .

Now we prove that the sum of the  $I(H_i)$ 's is direct. If this is not the case, since we are dealing with graded ideals, we may suppose that there exists a homogeneous element  $0 \neq a \in I(H_j) \cap \sum_{j \neq i \in \Lambda} I(H_i)$  for some  $j \in \Lambda$ ; by Corollary 2.9 there exist  $\alpha, \beta \in \text{Path}(E)$  and  $k \in K^\times$  such that  $0 \neq k^{-1} \alpha^* a \beta = w \in H_j$ . Observe that  $w$  also belongs to  $I(\cup_{j \neq i \in \Lambda} H_i)$ .

Write  $w = \sum_{l=1}^n k_l \alpha_l \beta_l^*$ , with  $k_l \in K$ ,  $\alpha_l, \beta_l \in \text{Path}(E)$ ,  $r(\alpha_l) = r(\beta_l) \in \cup_{j \neq i \in \Lambda} H_i$ , and assume that every summand is non-zero. Then  $0 \neq r(\beta_1) = \beta_1^* \beta_1 = \beta_1^* w \beta_1 \in \cup_{j \neq i \in \Lambda} H_i$ . On the other hand,  $s(\alpha_1) = w \in H_j$  implies (since  $H_j$  is a hereditary set)  $r(\alpha_1) \in H_j$ ; therefore,  $r(\alpha_1) = r(\beta_1) \in H_j \cap (\cup_{j \neq i \in \Lambda} H_i)$ , a contradiction.

To conclude the proof we point out that the first and last identities follow from [18, Lemma 2.1].  $\square$

A similar relation can be established for the ideal generated by the intersection of a family of hereditary subsets.

**Lema 2.11.** *Let  $\{H_i\}_{i \in \Lambda}$  be a family of hereditary subsets of an arbitrary graph  $E$ . Then:*

- (i)  $I(\cap_{i \in \Lambda} \overline{H_i}) = \cap_{i \in \Lambda} I(H_i)$ .
- (ii) *If  $\Lambda$  is finite, then  $I(\cap_{i \in \Lambda} H_i) = \cap_{i \in \Lambda} I(H_i)$ .*

*Proof.* (i). By the isomorphism given in [2, Theorem 2.4.13] among hereditary and saturated subsets of vertices and a special type of graded ideals,  $I(\cap_{i \in \Lambda} \overline{H_i}) = \cap_{i \in \Lambda} I(\overline{H_i}) = \cap_{i \in \Lambda} I(H_i)$ .

(ii). When  $\Lambda$  is finite, then  $\cap_{i \in \Lambda} \overline{H_i} = \overline{\cap_{i \in \Lambda} H_i}$  (use Lemma 2.4), and consequently

$$I(\cap_{i \in \Lambda} H_i) = I(\overline{\cap_{i \in \Lambda} H_i}) = \cap_{i \in \Lambda} I(\overline{H_i}) = \cap_{i \in \Lambda} I(H_i).$$

$\square$

Before stating the result that will allow us to work with connected graphs, we need to recall that the ideal generated by a hereditary and saturated subset in a Leavitt path algebra is isomorphic to a Leavitt path algebra.

Let  $E$  be a graph. For every non empty hereditary subset  $H$  of  $E^0$ , define

$$F_E(H) = \{\alpha = e_1 \dots e_n \mid e_i \in E^1, s(e_1) \in E^0 \setminus H, r(e_i) \in E^0 \setminus H \text{ for } i < n, r(e_n) \in H\}.$$

Denote by  $\overline{F}_E(H)$  another copy of  $F_E(H)$ . For  $\alpha \in F_E(H)$ , we write  $\overline{\alpha}$  to denote a copy of  $\alpha$  in  $\overline{F}_E(H)$ . Then, we define the graph

$${}_H E = ({}_H E^0, {}_H E^1, s', r')$$

as follows:

1.  ${}_H E^0 = ({}_H E)^0 = H \cup F_E(H)$ .
2.  ${}_H E^1 = ({}_H E)^1 = \{e \in E^1 \mid s(e) \in H\} \cup \overline{F}_E(H)$ .
3. For every  $e \in E^1$  with  $s(e) \in H$ ,  $s'(e) = s(e)$  and  $r'(e) = r(e)$ .
4. For every  $\overline{\alpha} \in \overline{F}_E(H)$ ,  $s'(\overline{\alpha}) = \alpha$  and  $r'(\overline{\alpha}) = r(\alpha)$ .

The following result was first proved in [18, Lemma 5.2] and then in [12, Lemma 1.2] for row-finite graphs, although the result is valid in general (see [2, Theorem 2.4.22]).

**Lema 2.12.** *Let  $E$  be an arbitrary graph and  $K$  any field. For a hereditary subset  $H \subseteq E^0$ , the ideal  $I(H)$  is isomorphic to the Leavitt path algebra  $L_K({}_H E)$ . Concretely, there is an isomorphism (which is not graded)  $\varphi : L_K({}_H E) \rightarrow I(H)$  acting as follows:*

$$\varphi(v) = v \text{ for every } v \in H,$$

$$\varphi(\alpha) = \alpha\alpha^* \text{ for every } \alpha \in F_E(H),$$

$$\varphi(e) = e \text{ and } \varphi(e^*) = e^* \text{ for every } e \in E^1 \text{ such that } s(e) \in H.$$

$$\varphi(\overline{\alpha}) = \alpha \text{ and } \varphi(\overline{\alpha}^*) = \alpha^* \text{ for every } \overline{\alpha} \in \overline{F}_E(H).$$

When we build the Leavitt path algebra of a graph  $E$ , we consider paths not only in  $E$ , but in the extended graph  $\widehat{E}$ ; this means that when we think of a connected graph we have in mind ghost paths too. For this reason we say that a graph  $E$  is *connected* if  $\widehat{E}$  is a connected graph in the usual sense, that is, if given any two vertices  $u, v \in E^0$  there exist  $h_1, \dots, h_m \in E^1 \cup (E^1)^*$  such that  $\eta := h_1 \dots h_m$  is a path in  $K\widehat{E}$  (in particular it is non-zero) such that  $s(\eta) = u$  and  $r(\eta) = v$ .

The *connected components* of a graph  $E$  are the graphs  $\{E_i\}_{i \in \Lambda}$  such that  $E$  is the disjoint union  $E = \sqcup_{i \in \Lambda} E_i$ , where every  $E_i$  is connected.

**Corollary 2.13.** *Let  $E$  be a graph and suppose  $E = \sqcup_{i \in \Lambda} E_i$ , where each  $E_i$  is a connected component of  $E$ . Then  $L_K(E) \cong \oplus_{i \in \Lambda} L_K(E_i)$ .*

*Proof.* By Proposition 2.10,  $L_K(E) = \bigoplus_{i \in \Lambda} I(E_i^0)$  and by Lemma 2.12,  $I(E_i^0)$  is isomorphic to the Leavitt path algebra  $L_K(E_i^0 E)$ . Since the graph  $E_i^0 E$  is just  $E_i$ , the result follows.  $\square$

By means of this corollary, and for our purposes, from now on we will restrict our attention to Leavitt path algebras of connected graphs.

Another application of Proposition 2.10 is stated below. Concretely, we will see that essentiality of graded ideals generated by hereditary and saturated subsets in a Leavitt path algebra (which is equivalent to density as one-sided ideals) can be expressed in terms of properties of the underlying graph.

**Proposition 2.14.** *Let  $H$  be a hereditary subset of a graph  $E$ . Then  $I(H)$  is a dense (left/right) ideal if and only if every vertex of  $E^0$  connects to a vertex in  $H$ .*

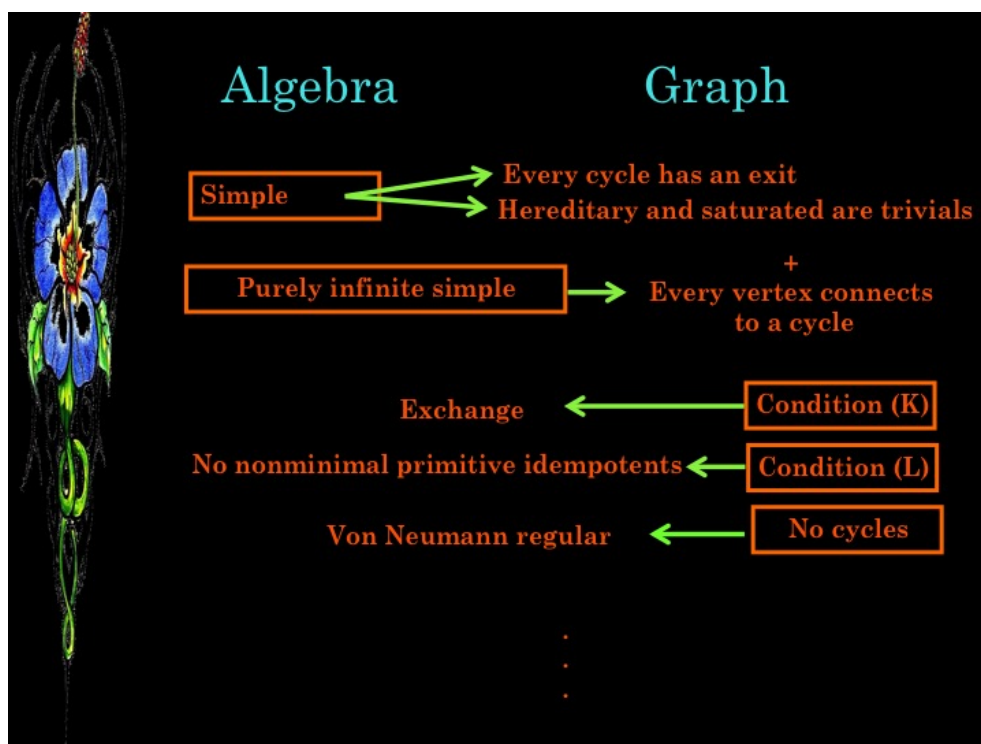
*Proof.* We first remark that since every Leavitt path algebra is left nonsingular (see [30, Proposition 4.1]), the notions of dense left/right ideal and that of essential are equivalent in this context (by [27, (8.7) Proposition]), and by [27, (14.1) Proposition],  $I(H)$  is essential as a left/right ideal if and only if it is essential as an ideal. Moreover, as  $I(H)$  is a graded ideal, by [28, 2.3.5 Proposition] essentiality and graded essentiality of  $I(H)$  are equivalent. Hence, we will show that  $I(H)$  is a graded essential ideal if and only if every vertex of  $E^0$  connects to a vertex in  $H$ .

Suppose first that  $I(H)$  is a graded essential ideal of  $L_K(E)$ . Let  $v \in E^0$ . If  $H \cap T(v) = \emptyset$ , then Lemma 2.11 would imply  $I(H) \cap I(T(v)) = 0$ , but this cannot happen as  $I(H)$  is a graded essential ideal. Hence  $H \cap T(v) \neq \emptyset$ . This implies that  $v$  connects to a vertex in  $H$ .

Now we prove the converse, i.e., that  $I(H)$  is an essential graded ideal. Let  $J$  be a nonzero graded ideal and pick a nonzero homogeneous element  $x = uxv \in J$ , where  $u, v \in E^0$ . Since the Leavitt path algebra  $L_K(E)$  is an algebra of right quotients of  $KE$ , by [29, Proposition 2.2] (which is valid even for non necessarily row-finite graphs) there exists  $\mu \in \text{Path}(E)$  such that  $0 \neq x\mu \in KE$ . Denote by  $w$  the range of  $\mu$ . By the hypothesis  $w$  connects to a vertex in  $H$ , hence there exists  $\lambda \in \text{Path}(E)$  such that  $w = s(\lambda)$  and  $r(\lambda) \in H$ . If  $x\mu\lambda = 0$  then  $x\mu \in uL_K(E)w \cap KE$  would satisfy  $\lambda \in \text{Path}(E) \cap \text{ran}(x\mu) = \emptyset$ , by [23, Lemma 1], a contradiction, hence  $0 \neq x\mu\lambda \in I(H) \cap J$  which shows our claim.  $\square$

## 2.3 The ideal generated by line points.

There is a strong connection among properties of a Leavitt path algebra and properties of the graph. This is one of the advantages of considering these kind of algebras: you can visualize algebraic properties and can build algebras satisfying certain conditions. Here we show some of these connections.



Let  $E$  be a graph. For  $v \in E^0$ , the *tree* of  $v$ , written  $T(v)$ , denotes the set  $\{w \in E^0 \mid v \geq w\}$ .

A vertex  $v \in E^0$  is called a *bifurcation* vertex (or it is said that *there is a bifurcation at  $v$* ) if  $s_E^{-1}(v)$  contains at least two edges of  $E$ .

A vertex  $u \in E^0$  is called a *line point* if there are neither bifurcations nor cycles at any vertex of  $T(u)$ . Obviously, every sink is a line point.

The set of line points of the graph  $E$  will be denoted by  $P_l(E)$ . It is always a hereditary subset of  $E^0$  although it is not necessarily saturated.

It is shown in [15, 16] that every line point generates a minimal left ideal, hence it is in the socle. It is not difficult to see (apply, for example [15, Proposition 2.6]) that:

**Lema 2.15.** *Let  $E$  be an arbitrary graph. Let  $v$  be a line point. Then:*

$$vL_K(E)v \cong K.$$

Moreover,

**Lema 2.16.** *Let  $E$  be an arbitrary graph and let  $v$  be a line point. Then  $I(v) \cong M_\Lambda(K)$ , where  $\Lambda$  denotes a set whose cardinal coincides with the cardinal of all paths ending at  $v$ .*

It is possible to decompose  $P_l(E) = \bigsqcup_{i \in \Lambda} H_i$ , where every  $H_i$  is a hereditary subset and  $I(H_i) = I(v_i)$  for some line point  $v_i \in H_i$ . Then, using Proposition 2.10 we get

$$I(P_l(E)) = \bigoplus_{i \in \Lambda} M_{\Lambda_i}(K).$$

Moreover (see [2, Theorem 2.6.14]):

**Theorem 2.17.** *Let  $E$  be an arbitrary graph and  $K$  any field. Decompose  $P_l(E) = \bigsqcup_{i \in \Gamma} H_i$  as it has been explained. Then*

$$\text{Soc}(L_K(E)) = I(P_l(E)) \cong \bigoplus_{i \in \Lambda} M_{\Lambda_i}(K),$$

where for every  $i \in \Lambda$ , if  $v_i$  is an arbitrary element of  $H_i$  then  $I(v_i) \cong M_{\Lambda_i}(K)$ .

## 2.4 The ideal generated by vertices in cycles without exits.

A similar pattern as for line points can be applied to vertices in cycles without exits.

The following result is [16, Lemma 1.5].

**Lema 2.18.** *Let  $E$  be an arbitrary graph. Let  $v$  be a vertex in  $E^0$  such that there exists a cycle without exits  $c$  based at  $v$ . Then:*

$$vL_K(E)v = \left\{ \sum_{i=-m}^n k_i c^i \mid k_i \in K; m, n \in \mathbb{N} \right\} \cong K[x, x^{-1}],$$

where  $\cong$  denotes a graded isomorphism of  $K$ -algebras, and considering (by abuse of notation)  $c^0 = w$  and  $c^{-t} = (c^*)^t$ , for any  $t \geq 1$ .

*Proof.* First, it is easy to see that if  $c = e_1 \dots e_n$  is a cycle without exits based at  $v$  and  $u \in T(v)$ , then  $s(f) = s(g) = u$ , for  $f, g \in E^1$ , implies  $f = g$ . Moreover, if  $r(h) = r(j) = w \in T(v)$ , with  $h, j \in E^1$ , and  $s(h), s(j) \in T(v)$

then  $h = j$ . We have also that if  $\mu \in E^*$  and  $s(\mu) = u \in T(v)$  then there exists  $k \in \mathbb{N}^*$ ,  $1 \leq k \leq n$  verifying  $\mu = e_k \mu'$  and  $s(e_k) = u$ .

Let  $x \in vL_K(E)v$  be given by  $x = \sum_{i=1}^p k_i \alpha_i \beta_i^* + \delta v$ , with  $s(\alpha_i) = r(\beta_i^*) = s(\beta_i) = v$  and  $\alpha_i, \beta_i \in E^*$ . Consider  $A = \{\alpha \in E^* : s(\alpha) = v\}$ ; we prove now that if  $\alpha \in A$ ,  $\deg(\alpha) = mn + q$ ,  $m, q \in \mathbb{N}$  with  $0 \leq q < n$ , then  $\alpha = e^m e_1 \dots e_q$ . We proceed by induction on  $\deg(\alpha)$ . If  $\deg(\alpha) = 1$  and  $s(\alpha) = s(e_1)$  then  $\alpha = e_1$ . Suppose now that the result holds for any  $\beta \in A$  with  $\deg(\beta) \leq sn + t$  and consider any  $\alpha \in A$ , with  $\deg(\alpha) = sn + t + 1$ . We can write  $\alpha = \alpha' f$  with  $\alpha' \in A$ ,  $f \in E^1$  and  $\deg(\alpha') = sn + t$ , so by the induction hypothesis  $\alpha' = c^s e_1 \dots e_t$ . Since  $s(f) = r(e_t) = s(e_{t+1})$  implies  $f = e_{t+1}$ , then  $\alpha = \alpha' f = c^s e_1 \dots e_{t+1}$ .

We shall show that the elements  $\alpha_i \beta_i^*$  are in the desired form, i.e.,  $c^d$  with  $d \in \mathbb{Z}$ . Indeed, if  $\deg(\alpha_i) = \deg(\beta_i)$  and  $\alpha_i \beta_i^* \neq 0$ , we have  $\alpha_i \beta_i^* = c^p e_1 \dots e_k e_k^* \dots e_1^* c^{-p} = v$  by (4). On the other hand  $\deg(\alpha_i) > \deg(\beta_i)$  and  $\alpha_i \beta_i^* \neq 0$  imply  $\alpha_i \beta_i^* = c^{d+q} e_1 \dots e_k e_k^* \dots e_1^* c^{-q} = c^d$ ,  $d \in \mathbb{N}^*$ . In a similar way, from  $\deg(\alpha_i) < \deg(\beta_i)$  and  $\alpha_i \beta_i^* \neq 0$  it follows that  $\alpha_i \beta_i^* = c^q e_1 \dots e_k e_k^* \dots e_1^* c^{-q-d} = c^{-d}$ ,  $d \in \mathbb{N}^*$ . Define  $\varphi: K[x, x^{-1}] \rightarrow L_K(E)$  by  $\varphi(1) = v$ ,  $\varphi(x) = c$  and  $\varphi(x^{-1}) = c^*$ . It is a straightforward routine to check that  $\varphi$  is a graded monomorphism with image  $vL_K(E)v$ , so that  $vL_K(E)v$  is graded isomorphic to  $K[x, x^{-1}]$  as a graded  $K$ -algebra.  $\square$

The structure of the ideal generated by any vertex in a cycle without exits is known. See, for example [2, Lemma 2.17.1].

**Lema 2.19.** *For any vertex  $v \in P_c(E)$ , let  $\Lambda_v$  denote the (possibly infinite) set of paths in  $E$  which end at  $v$ , but which do not contain all the edges of  $c$ , where  $c$  is the cycle without exits such that  $s(c) = v$ . Then:*

$$I(c^0) = I(v) \cong M_{\Lambda_v}(K[x, x^{-1}]).$$

*Proof.* It is easy to see that  $I(c^0) = I(v)$ . To prove the isomorphism in the statement, consider the family

$$\mathcal{B} := \{\mu c^k \eta^* \mid \mu, \eta \in \Lambda_v, k \in \mathbb{Z}\},$$

where as usual  $c^0$  denotes  $v$  and  $c^k$  denotes  $(c^*)^{-k}$  for  $k < 0$ . Then  $\mathcal{B}$  is a  $K$ -linearly independent set which generates  $I(v)$  (see [2, Lemma 2.17.1]). The map  $\varphi: I(v) \rightarrow M_{\Lambda_v}(K[x, x^{-1}])$  given by

$$\varphi(\mu c^k \eta^*) = x^k e_{\mu, \eta}$$

where  $x^k e_{\mu, \eta}$  denotes the element of  $M_{\Lambda_v}(K[x, x^{-1}])$  which is  $x^k$  in the  $(\mu, \eta)$  entry, and zero otherwise is a  $K$ -algebra isomorphism.  $\square$

A consequence is the structure of the ideal generated by vertices in cycles without exits (see [2, Theorem 2.7.3]).

**Theorem 2.20.** *Let  $E$  be an arbitrary graph and  $K$  any field. Then:*

$$I(P_c(E)) \cong \bigoplus_{i \in \Upsilon} M_{\Lambda_i}(K[x, x^{-1}]),$$

where  $\{c_i\}_{i \in \Upsilon}$  is the set of different cycles without exits in  $E$  and  $\Lambda_i$  is the set of paths in  $E$  which end at the base  $v_i$  of the cycle  $c_i$ , but do not contain all the edges of  $c_i$ .

The importance of the ideal  $I(P_c(E))$  relays also in that every ideal non containing vertices is contained in it (see [18] for a proof of this fact for row-finite graphs).

## 2.5 The ideal generated by extreme cycles.

Now we introduce the notion of extreme cycle. Roughly speaking it is a cycle such that every path starting at a vertex of the cycle comes back to it. The ideal generated by extreme cycles will be proved to be a direct sum of purely infinite simple Leavitt path algebras.

**Definitions 2.21.** Let  $E$  be a graph and  $c$  a cycle in  $E$ . We say that  $c$  is an *extreme cycle* if  $c$  has exits and for every path  $\lambda$  starting at a vertex in  $c^0$  there exists  $\mu \in \text{Path}(E)$  such that  $0 \neq \lambda\mu$  and  $r(\lambda\mu) \in c^0$ . We will denote by  $P_{ec}(E)$  the set of vertices which belong to extreme cycles.

**Definitions 2.22.** Let  $X'_{ec}$  be the set of all extreme cycles in a graph  $E$ . We define in  $X'_{ec}$  the following relation: given  $c, d \in X'_{ec}$ , we write  $c \sim d$  whenever  $c$  and  $d$  are connected, that is,  $T(c^0) \cap d^0 \neq \emptyset$ , equivalently,  $T(d^0) \cap c^0 \neq \emptyset$ . It is not difficult to see that  $\sim$  is an equivalence relation. Denote the set of all equivalence classes by  $X_{ec} = X'_{ec} / \sim$ . When we want to emphasize the graph we are considering we will write  $X'_{ec}(E)$  and  $X_{ec}(E)$  for  $X'_{ec}$  and  $X_{ec}$ , respectively.

For any  $c \in X'_{ec}$ , let  $\tilde{c}$  denote the class of  $c$  and let use  $\tilde{c}^0$  to represent the set of all vertices which are in the cycles belonging to  $\tilde{c}$ .

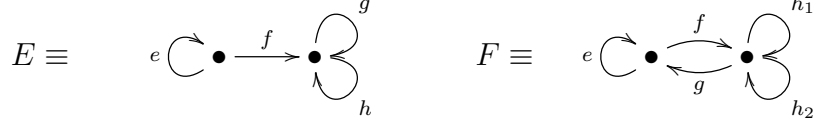
The following will be of use.

**Remark 2.23.** For a graph  $E$  and using the notation described above:

- (i) For any  $c \in X'_{ec}$ ,  $\tilde{c}^0 = T(c^0)$ , hence  $\tilde{c}^0$  is a hereditary subset.
- (ii) Given  $c, d \in X'_{ec}$ ,  $\tilde{c} \neq \tilde{d}$  if and only if  $\tilde{c}^0 \cap \tilde{d}^0 = \emptyset$ .

(iii) For  $c, d \in X'_{ec}$ ,  $c \sim d$  if and only if  $T(c) = T(d)$ .

**Examples 2.24.** Consider the following graphs



Then  $X'_{ec}(E) = \{g, h\}$ ,  $X_{ec}(E) = \{\tilde{g}\}$ ,  $X'_{ec}(F) = \{e, fg, gf, h_1, h_2\}$  and  $X_{ec}(F) = \{\tilde{e}\}$ .

Now we will analyze the structure of the ideal generated by  $P_{ec}(E)$ .

**Lema 2.25.** Let  $E$  be an arbitrary graph and  $K$  any field. For every cycle  $c$  such that  $c \in X'_{ec}$ , the ideal  $I(\tilde{c}^0)$  is isomorphic to a purely infinite simple Leavitt path algebra. Concretely, to  $L_K({}_H E)$ , where  $H = \tilde{c}^0$ .

*Proof.* Use Lemma 2.12 to have  $I(\tilde{c}^0)$  isomorphic to the Leavitt path algebra  $L_K({}_H E)$  and let us show that this Leavitt path algebra is purely infinite and simple. For that, we will use the characterization [4, Theorem 4.3], which is valid for arbitrary graphs as can be proved by using the techniques described in [25] or in [13] in order to translate certain characterizations of Leavitt path algebras from the row-finite case to the case of an arbitrary Leavitt path algebra.

Every vertex of  ${}_H E$  connects to a cycle: take  $v \in {}_H E^0$ ; if  $v \in H$  then it connects to  $c$ , otherwise there exists a path  $\mu \in \text{Path}(E)$  such that  $s(\mu) = v$  and  $r(\mu) \in H$ . Since  $r(\mu)$  connects to the cycle  $c$  in  $E$ , the vertex  $v$  also connects to  $c$  in  ${}_H E$ .

Every cycle in  ${}_H E$  has an exit: this follows because any cycle in this graph comes from a cycle  $d$  in  $H$  and, by construction,  $\tilde{d} = \tilde{c}$ ; this means that  $d$  connects to  $c$  and hence it has an exit which is an exit in  ${}_H E$ .

The only hereditary and saturated subsets of  ${}_H E^0$  are  $\emptyset$  and  ${}_H E^0$ : let  $H' \in \mathcal{H}_{{}_H E}$  be non empty and consider  $v \in H'$ ; if  $v \in H$  then  $H \subseteq H'$ ; since  $H'$  is saturated,  $H' = {}_H E^0$ ; if  $v \notin H$  then there exists  $f \in {}_H E^1$  such that  $v = s(f)$  and  $r(f) \in H$ ; this implies  $H \subseteq H'$ ; now, apply the construction of  ${}_H E$  and that  $H'$  is saturated to get  $H' = {}_H E^0$ .  $\square$

**Proposition 2.26.** Let  $E$  be any graph and  $K$  any field. Then  $I(P_{ec}(E)) = \bigoplus_{\tilde{c} \in X_{ec}} I(\tilde{c}^0)$  and every  $I(\tilde{c}^0)$  is isomorphic to a Leavitt path algebra which is purely infinite simple.

*Proof.* The hereditary set  $P_{ec}(E)$  can be decomposed as:  $P_{ec}(E) = \sqcup_{\tilde{c} \in X_{ec}} \tilde{c}^0$ . By the Remark 2.23 and the Proposition 2.10,  $I(P_{ec}(E)) = I(\sqcup_{\tilde{c} \in X_{ec}} \tilde{c}^0) = \bigoplus_{\tilde{c} \in X_{ec}} I(\tilde{c}^0)$ . Finally, observe that every  $I(\tilde{c}^0)$  is isomorphic to a purely infinite simple Leavitt path algebra by Lemma 2.25.  $\square$

**Lema 2.27.** *For any graph  $E$  the hereditary sets  $P_l(E)$ ,  $P_c(E)$  and  $P_{ec}(E)$  are pairwise disjoint. Moreover, the ideal generated by their union is  $I(P_l(E)) \oplus I(P_c(E)) \oplus I(P_{ec}(E))$ .*

*Proof.* By the definition of  $P_l(E)$ ,  $P_c(E)$  and  $P_{ec}(E)$ , they are pairwise disjoint. To get the result, apply Proposition 2.10.  $\square$

The following ideal, will be of use in order to establish the center of a Leavitt path algebra, so we name it here.

**Definition 2.28.** *For a graph  $E$  we define*

$$I_{lce} := I(P_l(E)) \oplus I(P_c(E)) \oplus I(P_{ec}(E)).$$

**Theorem 2.29.** *Let  $E$  be an arbitrary graph and  $K$  any field. Then:*

- (i)  $I_{lce} \cong (\bigoplus_{i \in \Lambda_1} M_{m_i}(K)) \oplus (\bigoplus_{j \in \Lambda_2} M_{n_j}(K[x, x^{-1}])) \oplus (\bigoplus_{l \in \Lambda_3} I(\tilde{c}_{n_l}^0))$ , where  $\Lambda_1$  is the index set of the sinks of  $E$ ,  $\Lambda_2$  is the index set of the cycles without exits in  $E$  and  $\Lambda_3$  indexes  $X_{ec}(E)$ .
- (ii) If  $|E^0| < \infty$  then  $I_{lce}$  is a dense ideal of  $L_K(E)$ .

*Proof.* For our purposes we may suppose that the graph  $E$  is connected because if  $E = \sqcup_{i \in \Lambda} E_i$  is the decomposition of  $E$  into its connected components and we show the claims for every connected component, then the result will be true for  $E$  by virtue of Corollary 2.13.

(i). We know that  $I(P_l(E))$  is the socle of the Leavitt path algebra  $L_K(E)$  (see [16, Theorem 5.2] and [8, Theorem 1.10]), moreover  $I(P_l(E)) \cong \bigoplus_{i \in \Lambda_1} M_{m_i}(K)$ , where  $\Lambda_1$  ranges over the sinks of  $E$  and for any  $i \in \Lambda_1$ , if  $s_i$  is a sink (finite or not), then  $m_i$  is the cardinal of the set of paths ending at  $s_i$ .

The structure of  $I(P_c(E))$  is also known: by [5, Proposition 3.5] (which also works for arbitrary graphs)  $I(P_c(E)) \cong \bigoplus_{j \in \Lambda_2} M_{n_j}(K[x, x^{-1}])$ , where the cardinal of  $\Lambda_2$  is the cardinal of the set of cycles without exits and  $n_j$  is the cardinal of the set of paths ending at a given cycle without exits  $c_j$  and not containing the edges appearing in the cycle.

Finally, the structure of the third summand in  $I_{lce}$  follows by Proposition 2.26.

(ii). Take a vertex  $v \in E^0$ . Since  $E^0$  is finite then  $v$  connects to a line point, to a cycle without exists or to an extreme cycle. This means that every vertex of  $E$  connects to the hereditary set  $H := P_l(E) \cup P_c(E) \cup P_{ec}(E)$ . By Proposition 2.14 this means that  $I(H)$  is a dense ideal of  $L_K(E)$  and by Lemma 2.27 it coincides with  $I_{lce}$ .  $\square$

**Remark 2.30.** Following a similar reasoning as in the proof of Theorem 2.29 it can be shown that any ideal of  $L_K(E)$  generated by vertices in  $P$  is isomorphic to

$$(\oplus_{i \in \Lambda_1} M_{m_i}(K)) \oplus (\oplus_{j \in \Lambda_2} M_{n_j}(K[x, x^{-1}])) \oplus (\oplus_{l \in \Lambda_3} I(\tilde{c}_{n_l}^0))$$

for some  $\Lambda_1, \Lambda_2, \Lambda_3$ .

We specialize this result in the case of a prime Leavitt path algebra.

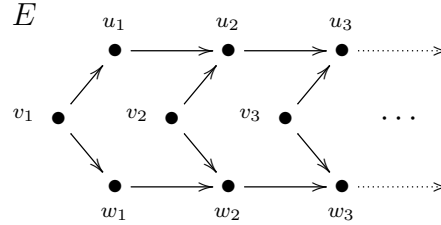
**Corollary 2.31.** *Let  $E$  be a graph with a finite number of vertices such that  $L_K(E)$  is a prime algebra. Then, we have the following three mutually exclusive cases:*

- (i) *There is a unique sink  $v$  in  $E$  and every vertex of  $E$  connects to  $v$ . In this case  $I(P_l(E)) = I(T(v)) \cong M_m(K)$ , where  $m$  is the cardinal of the set of paths ending at  $v$ , or*
- (ii) *there is a unique cycle without exists  $c$  and every vertex of  $E$  connects to it. In this case  $I(P_c(E)) = I(c^0) \cong M_n(K[x, x^{-1}])$ , where  $n$  is the cardinal of the number of paths ending at  $c$  and not containing all the edges of  $c$ , or*
- (iii)  *$X_{ec}(E) = \{\tilde{c}\}$ , where  $c$  is an extreme cycle and every vertex of  $E$  connects to  $c$ . In this case  $I(P_{ec}(E)) = I(c^0)$  is a purely infinite simple unital ring.*

## 2.6 Decomposable Leavitt path algebras

We have seeing that whenever  $E^0 = \sqcup H_i$  then  $L_K(E) = \oplus I(H_i)$ . In particular, the Leavitt path algebra  $L_K(E)$  can be decomposed as a direct sum of ideals. There are other cases in which a Leavitt path algebra can be decomposed. In a paper jointly written with Lisa Orloff Clark, Dolores Martín Barquero and Cándido Martín González (see [21]), we characterize those Leavitt path algebras that are decomposable. We use the Steinberg algebra, but I'm not going to go into the details of these algebras. concretely, we show:

**Example 2.32.** Let  $E$  be the following graph:



Let  $X = \{u_n \mid n \in \mathbb{N}\}$  and  $Y = \{w_n \mid n \in \mathbb{N}\}$ . Then the only hereditary and saturated sets are  $X, Y, \emptyset$  and  $E^0$ . Note that  $L_K(E)$  is decomposable as  $L_K(E) = I(X) \oplus I(Y)$ .

**Definition 2.33.** ([17, Definition 4.1]) Let  $E$  be an arbitrary graph and  $H$  a hereditary and saturated set of vertices. A path  $\lambda = \lambda_1 \dots \lambda_{|\lambda|}$ , where  $\lambda_i \in E^1$ , is said to be  $H$ -compatible if  $r(\lambda) \in H$  and  $s(\lambda_{|\lambda|}) \notin H \cup B_H$ .

The key point for decomposability of a Leavitt path algebra is precisely compatibility of hereditary and saturated sets.

**Theorem 2.34.** (MSM; Lisa O. Clark; Dolores Martín Barquero; Cándido Martín González)

Let  $K$  be a field and  $E$  an arbitrary graph. Then  $L_K(E)$  is decomposable if and only if there exist two nontrivial hereditary and saturated subsets  $H_1$  and  $H_2$  such that  $H_1 \cap H_2 = \emptyset$  and for every  $v \in E^0 \setminus (H_1 \cup H_2)$ , there exists at least one but finitely many paths starting at  $v$  which are either  $H_1$ -compatible or  $H_2$ -compatible.

# Chapter 3

## The center of a Leavitt path algebra

### 3.1 Introduction to the center

In this chapter we will completely describe the center of the Leavitt path algebra. We start by the description and the proof of the results concerning row-finite graphs (results belonging to [23, 24]) and then we state the results in the general case, which were proved by using the Steinberg algebra associated to the groupoid related to a graph. This work is [22].

Here are some of the reasons because of which we are interested in the center of a Leavitt path algebra.

**Why are we interested in the center?**

- ★ Zero dimensional cohomology groups of algebras
- ★ One dimensional cohomology groups of algebras
- ★ Derivations  $\longleftrightarrow$  Center
- ★ Intrinsic value
- ★ Other connexions

$A$  simple  $\begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} \times \\ \times \end{matrix} \begin{matrix} \rightarrow \\ \rightarrow \end{matrix} A^- \text{ simple} \quad \boxed{Z(A) \triangleright A^-}$

$\begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} \times \\ \times \end{matrix} \begin{matrix} \rightarrow \\ \rightarrow \end{matrix} [A, A] \text{ simple} \quad \boxed{Z([A, A]) \triangleright [A, A]}$

Relative to the simplicity of the Lie algebra related to an associative algebra, we have:

**Why are we interested in the center?**

- ✦ Zero dimensional cohomology groups of algebras
- ✦ One dimensional cohomology groups of algebras
- ✦ Derivations  $\longleftrightarrow$  Center
- ✦ Intrinsic value
- ✦ Other connexions

A simple  $\longrightarrow$   $[A, A] / Z([A, A])$  simple

- ✦ General philosophy in LPA's world

As for the general philosophy in Leavitt path algebras, we will see that the center can be computed by looking at the graph. That is, again, algebraic properties can be read from the graph.

The key pieces for the description of the center are: the set of line points, the vertices in cycles without exits and the vertices in extreme cycles, jointly with an equivalence relation defined on  $E^0$  whose set of classes is indexed in a subset of  $P := P_l(E) \cup P_c(E) \cup P_{ec}(E)$ .

### 3.2 The extended centroid of the Leavitt path algebra of a finite graph

The extended centroid of the ideal generated by  $P$  will coincide with the center of the Martindale symmetric ring of quotients of the Leavitt path algebra, notion that plays an important role.

Recall that for an associative algebra  $A$ , the *center* of  $A$ , denoted  $Z(A)$ , is defined by:

$$Z(A) := \{x \in A \mid [x, a] = 0 \text{ for every } a \in A\},$$

where  $[a, b] := ab - ba$  and juxtaposition stands for the product in the algebra  $A$ .

For a semiprime algebra  $A$ , the *extended centroid* of  $A$ , denoted by  $\mathbb{C}(A)$  is defined as:

$$\mathbb{C}(A) = Z(Q_s(A)) = Z(Q_{max}^l(A)) = Z(Q_{max}^r(A)),$$

where  $Q_s(A)$ ,  $Q_{max}^l(A)$  and  $Q_{max}^r(A)$  are the Martindale symmetric ring of quotients of  $A$ , the maximal left ring of quotients of  $A$  and the maximal right ring of quotients of  $A$ , respectively (see [27, (14.18) Definition]).

**Lema 3.1.** *Let  $A$  be a unital simple algebra. Then  $Z(A) = \mathbb{C}(A)$ .*

*Proof.* We see first  $Z(A) \subseteq \mathbb{C}(A)$ . Suppose this containment is not true. Then there exists  $x \in Z(A)$  and  $q \in Q_s(A)$  such that  $xq - qx \neq 0$ . Use that  $Q_s(A)$  is a right ring of quotients of  $A$  to find  $a \in A$  such that  $0 \neq (xq - qx)a$  and  $qa \in A$ . Then  $0 \neq x(qa) - q(xa) = (qa)x - q(ax) = 0$ , a contradiction.

To prove  $\mathbb{C}(A) \subseteq Z(A)$ , consider  $q \in \mathbb{C}(A) \setminus \{0\}$ . By [27, (14.22) Corollary],  $\mathbb{C}(A)$  is a field, hence there exist  $q^{-1} \in \mathbb{C}(A)$ . Use again that  $Q_s(A)$  is a right ring of quotients of  $A$  to find  $x \in A$  such that  $0 \neq q^{-1}x \in A$ . Since  $A$  is simple and unital  $Aq^{-1}xA = A$ , in particular there exists a finite number of elements  $a_1, \dots, a_m, b_1, \dots, b_m \in A$  such that  $1 = \sum_{i=1}^m a_i q^{-1} x b_i$ . Multiply this identity by  $q$  and use that  $q$  is in the center of  $Q_s(A)$  to get  $q = \sum_{i=1}^m a_i x b_i \in A$  as desired.  $\square$

**Theorem 3.2.** (MSM; Corrales García; Martín Barquero, Martín González; Solanilla Hernández)

*Let  $E$  be graph such that  $|E^0| < \infty$  and consider  $I := I_{lce}$ . Then the extended centroid of  $L_K(E)$  coincides with the extended centroid of  $I$ ,  $\mathbb{C}(I)$ ; moreover,*

$$\mathbb{C}(L_K(E)) = \mathbb{C}(I) \cong \left( \bigoplus_{i=1}^m K \right) \oplus \left( \bigoplus_{j=1}^n K[x, x^{-1}] \right) \oplus \left( \bigoplus_{i=1}^{n'} K \right),$$

*where  $m$  is the number of sinks,  $n$  is the number of cycles without exits and  $n'$  is the number of equivalence classes of extreme cycles. If  $P_l(E)$ ,  $P_c(E)$  or  $P_{ec}(E)$  are empty, then the ideals they generate are zero and the corresponding summands in  $\mathbb{C}(I)$  do not appear.*

*Proof.* As in the proof of Theorem 2.29 we may suppose that our graph is connected. Apply Theorem 2.29 (ii) and [27, (14.14) Theorem] to obtain  $Q_s(L_K(E)) = Q_s(I)$ . Then by [29, Lemma 1. 3 (i)]  $\mathcal{C}(L_K(E)) = \mathcal{C}(I) = \mathcal{C}(I(P_l(E))) \oplus \mathcal{C}(I(P_c(E))) \oplus \mathcal{C}(I(P_{ec}(E)))$ .

By Theorem 2.29 (i), Lemma 3.1 and Proposition 2.26,  $\mathcal{C}(I(P_{ec}(E))) = \bigoplus_{l=1}^{n'} \mathcal{C}(I(\tilde{c}_l^0))$ , where  $\tilde{c}_l \in X_{ec}$  and  $n' = |X_{ec}|$ . Use [14, Theorem 4.2] to obtain  $\mathcal{C}(I(\tilde{c}_l^0)) \cong K$  for every  $l$ .

To finish, use the three pieces of information in the paragraphs before to obtain  $\mathcal{C}(L_K(E)) = \mathcal{C}(I) = (\bigoplus_{i=1}^n K) \oplus (\bigoplus_{j=1}^n K[x, x^{-1}]) \oplus (\bigoplus_{l=1}^{n'} K)$  and we get the claim in the statement.  $\square$

### 3.3 The center

The following remarks and results will be useful to study the center of a Leavitt path algebra. Their proofs are straightforward.

**Remark 3.3.** If  $A$  is an algebra and  $\{I_i\}$  is a set of ideals of  $A$  whose sum is direct, then  $Z(\bigoplus_i I_i) = \bigoplus_i (Z(I_i))$ .

**Lema 3.4.** Let  $G$  be an abelian group. The center of a  $G$ -graded algebra  $A$  is  $G$ -graded, that is, if  $x \in Z(A)$  and  $x = \sum_{g \in G} x_g$  is the decomposition of  $x$  into its homogenous components, then  $x_g \in Z(A)$  for every  $g \in G$ .

*Proof.* Indeed, for every  $y = \sum_{h \in G} y_h$  in  $A$ ,  $0 = [x, y_h] = [\sum_{g \in G} x_g, y_h] = \sum_{g \in G} [x_g, y_h]$ ; using the grading on  $A$  and that  $G$  is abelian (to be sure that  $x_g y_h$  and  $y_h x_g$  are in the same homogeneous component) we get  $[x_g, y_h] = 0$  for any  $g \in G$ . Hence  $0 = \sum_{h \in G} [x_g, y_h] = [x_g, \sum_{h \in G} y_h] = [x_g, y]$  which means  $x_g \in Z(A)$ .  $\square$

**Notation 3.5.** The homogeneous component of degree  $g$  in the center of a  $G$ -graded algebra  $A$  will be denoted by  $Z_g(A)$ .

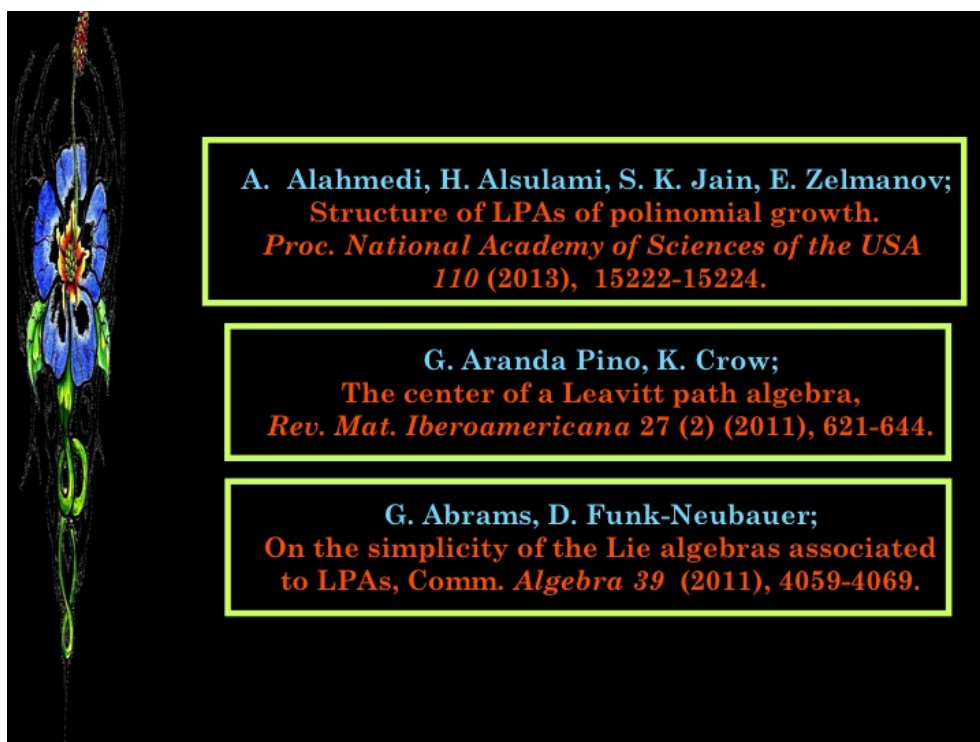
**Lema 3.6.** Let  $I$  be an ideal of an algebra  $A$ . If for every  $y \in Z(I)$  there exist  $n \in \mathbb{N}$  and  $\{a_i, b_i\}_{i=1}^n \subseteq Z(I)$  such that  $y = \sum_{i=1}^n a_i b_i$ , then  $Z(I) = I \cap Z(A)$ .


*Proof.* It is clear that  $I \cap Z(A) \subseteq Z(I)$ . To show  $Z(I) \subseteq Z(A)$ , take  $y \in Z(I)$  and  $x \in A$ . Write  $y = \sum_{i=1}^n a_i b_i$ , for  $a_i, b_i \in Z(I)$ . Then  $yx = \sum_{i=1}^n a_i b_i x = \sum_{i=1}^n a_i (b_i x) = \sum_{i=1}^n (b_i x) a_i = \sum_{i=1}^n b_i (x a_i) = \sum_{i=1}^n (x a_i) b_i = xy$ .  $\square$

**Corollary 3.7.** Let  $I$  be an ideal of a Leavitt path algebra  $L_K(E)$  such that for every  $y \in Z(I)$  there exist  $a, b \in Z(I)$  such that  $y = ab$ . Then  $Z(I) = I \cap Z(L_K(E))$ . This happens, in particular, for every ideal of  $L_K(E)$  generated by vertices in  $P$ .

*Proof.* The first statement follows immediately from Lemma 3.6 For  $I$  generated by vertices in  $P$ , by Remark 2.30,  $I$  is a direct sum of ideals of  $L_K(E)$  which are isomorphic to  $M_n(K)$ ,  $M_n(K[x, x^{-1}])$  or  $J$ , for  $J$  purely infinite and simple. In this last case, by [23, Theorem 3.6],  $Z(J)$  is 0 or isomorphic to  $K$ ; in the other cases, the centers are zero or isomorphic to  $K$ , or to  $K[x, x^{-1}]$ . In all of these situations our hypothesis on the ideal is satisfied.  $\square$

Here are some previous papers related to the study of the center and of the derivations of a Leavitt path algebra.






*Simple*  
*Locally artinian*  
*Locally noetherian*

to the center of a LPA

**G. Aranda Pino, K. Crow;**  
**The center of a Leavitt path algebra,**  
*Rev. Mat. Iberoamericana* 27 (2) (2011), 621-644.

**G. Abrams, D. Funk-Neubauer;**  
**On the simplicity of the Lie algebras associated**  
**to LPAs, Comm. Algebra** 39 (2011), 4059-4069.



*Simplicity of*  
 $[M_m(L_K(n))^- , M_m(L_K(n))^-]$

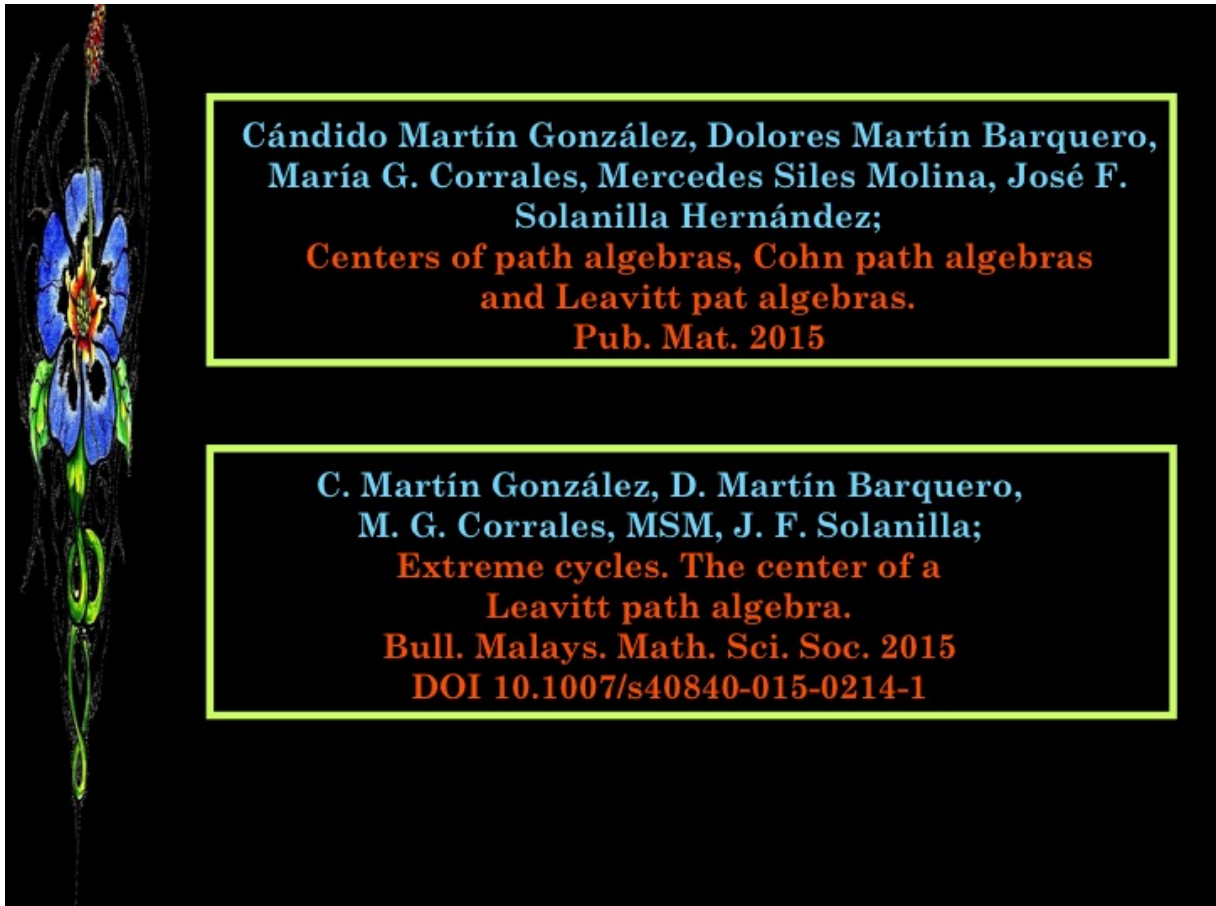
Previous works re

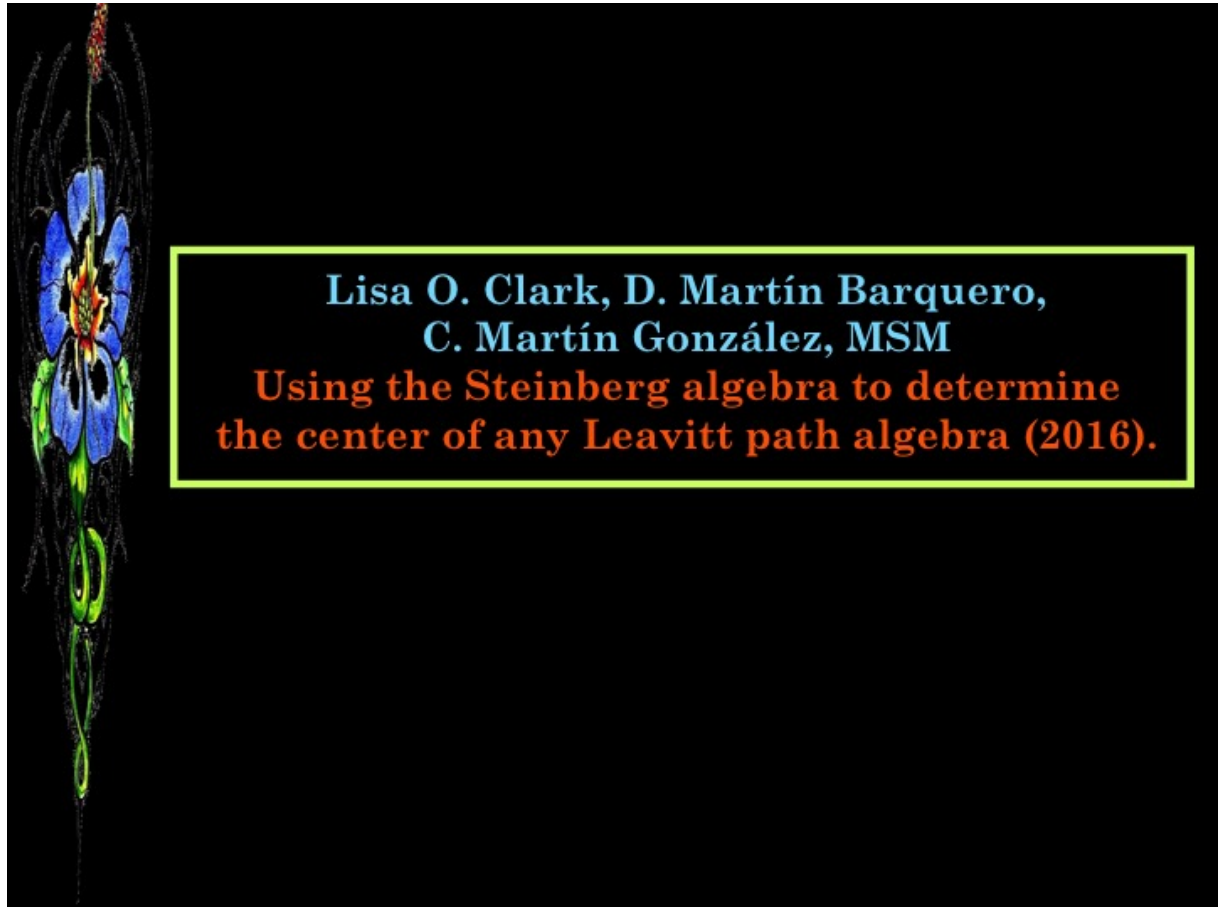
**G. Aranda Pino, K. Crow;**  
**The center of a Leavitt path algebra,**  
*Rev. Mat. Iberoamericana* 27 (2) (2011), 621-644.

**G. Abrams, D. Funk-Neubauer;**  
**On the simplicity of the Lie algebras associated**  
**to LPAs, Comm. Algebra** 39 (2011), 4059-4069.

As for our contribution, we determine the center of some Cohn and graph


path algebras, as well as the center of the Leavitt path algebra associated to a row finite graphs in these papers ([23, 24]):




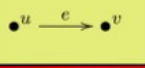

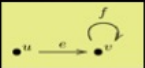


The following definition was motivated by our analysis of the centers of different Leavitt path algebras.

Here are some examples.



### Computing the center

Graph	Center	Basis
	K	$\{ u \}$
	K	$\{ u + v \}$
	$K[x, x^{-1}]$	$\{ v, c^n \}$
	$K[x, x^{-1}]$	$\{ u+v, ef^n e^* \}$

### Computing the center

$\mathbb{K} \oplus \mathbb{K}[x, x^{-1}]$

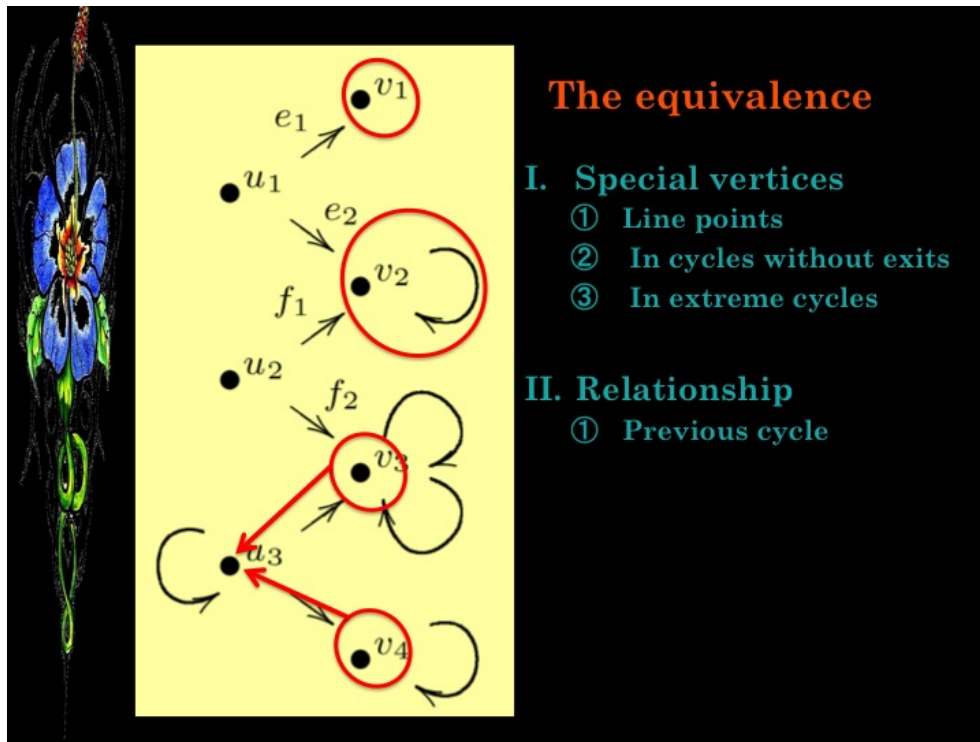
$\mathbb{K}$

$\mathbb{K} \oplus \mathbb{K}[x, x^{-1}] \oplus \mathbb{K}$

$a_{[v_1]} = v_1 + e_1 e_1^*$

$a_{[v_2]} = v_2 + e_2 e_2^* + f_1 f_1^*$

$a_{[v_3]} = a_{[v_4]} = u_3 + v_3 + v_4 + f_2 f_2^*$



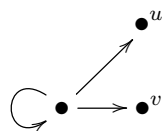
**Definitions 3.8.** Let  $u, v \in E^0$ . The element  $\lambda_{u,v}$  will denote a path such that  $s(\lambda_{u,v}) = u$  and  $r(\lambda_{u,v}) = v$ .

In  $E^0$  we define the following relation: given  $u, v \in E^0$  we write  $u \sim^1 v$  if and only if  $u = v$  or:

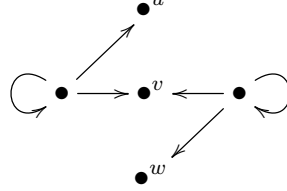
- (i)  $u \leq v$  or  $v \leq u$  and there are no bifurcations at any vertex in  $T(u)$  and  $T(v)$ .
- (ii) there exist a cycle  $c$ , a vertex  $w \in c^0$  and  $\lambda_{w,u}, \lambda_{w,v} \in Path(E)$ .

This relation  $\sim^1$  is reflexive and symmetric. Consider the transitive closure of  $\sim^1$ ; we shall denote it by  $\sim$ . The notation  $[v]$  will stand for the class of a vertex  $v$ .

**Example 3.9.** The vertices  $u$  and  $v$  in the following graph are related:



**Remark 3.10.** The following is an example of a graph which illustrates why do we need to consider the transitive closure of the relation  $\sim^1$  in Definition 3.8. Note that  $u \sim^1 v$ ,  $v \sim^1 w$  and  $u \not\sim^1 w$ ; however,  $u \sim w$ .



In what follows we are going to describe the zero component of the center of a Leavitt path algebra  $L_K(E)$  associated to a row-finite graph.

**Notation 3.11.** Let  $E$  be an arbitrary graph. Consider  $P = P_l \cup P_c \cup P_{ec}$  and define  $X = P / \sim$ . Decompose  $P = P_f \sqcup P_\infty$ , where  $P_f$  are those elements  $v$  of  $P$  such that

- (i)  $|\overline{[v]}| < \infty$ , and
- (ii)  $|F_E(\overline{[v]})| < \infty$

In the same vein we decompose  $X = X_f \sqcup X_\infty$ , where

$$X_f = \{[u] \in X \mid \text{for all } v \in [u], v \in P_f\}$$

and

$$X_\infty = \{[u] \in X \mid \text{for some } v \in [u], v \in P_\infty\}.$$

Finally, we decompose  $X_f = X_f^l \cup X_f^c \cup X_f^{ec}$ , where each of these subsets consists of equivalence classes induced by elements which are in  $P_f \cap P_l$ , in  $P_f \cap P_c$  and in  $P_f \cap P_{ec}$ , respectively. Note that if  $u$  and  $v$  are vertices in  $P_f$ , then  $u \sim v$  if and only if  $u, v \in P_l$ ,  $u, v \in P_c$  or  $u, v \in P_{ec}$ .

When we want to emphasize the graph  $E$  we are considering, we will write  $P_l(E)$ ,  $X(E)$ , etc.

**Lema 3.12.** *Let  $E$  be an arbitrary graph and  $u, v \in P$ . Then:*

- (i)  $[u]$  is a hereditary set.
- (ii) If  $u \not\sim v$  then  $\overline{[u]} \cap \overline{[v]} = \emptyset$ .

*Proof.* (i). Let  $w \in [u]$  and consider  $w' \in r(s^{-1}(w))$ . Since  $w \sim u$  there exists a finite set  $\{v_1 \dots v_n\}$  of vertices such that  $w = v_1 \sim^1 v_2 \sim^1 \dots \sim^1 v_n = u$ . Note that  $w' \sim^1 v_2$  and so  $w' \in [u]$ .

(ii). By the hypothesis  $[u] \cap [v] = \emptyset$ . Use (i) and Lemma 2.4 to get the result.  $\square$

**Definition 3.13.** Let  $E$  be a graph and  $K$  be any field. An element  $a \in L_K(E)$  which can be written as

$$a = \sum_{[v] \in X_f} k_{[v]} a_{[v]}, \quad \text{where } k_{[v]} \in K^\times \quad \text{and} \quad a_{[v]} = \sum_{u \in \overline{[v]}} u + \sum_{\alpha \in F_E(\overline{[v]})} \alpha \alpha^*,$$

will be said to be written in the *standard form*.

We will prove that every element in the zero component of the center of a Leavitt path algebra can be written in the standard form.

**Lema 3.14.** *Let  $E$  be an arbitrary graph and  $K$  be any field. Consider  $[v] \in X$ . For every  $u \in \overline{[v]}$  and  $\alpha, \beta \in F_E(\overline{[v]})$  we have:*

- (i) *If  $s(\alpha) = s(\beta)$ , then  $\alpha^* \beta \neq 0$  if and only if  $\alpha = \beta$ .*
- (ii) *If  $s(\alpha) \neq s(\beta)$  then  $\alpha^* \beta = 0$ .*
- (iii)  *$u\alpha = 0$ .*

*Proof.* (i). If  $\alpha^* \beta \neq 0$  then  $\alpha = \beta\gamma$  or  $\beta = \alpha\delta$  for some  $\gamma, \delta \in \text{Path}(E)$ . By the definition of  $F_E(\overline{[v]})$ , necessarily  $\alpha = \beta$ .

(ii). This case follows immediately.

(iii). For  $u$  and  $\alpha$  as in the statement,  $u\alpha \neq 0$  implies  $u = s(\alpha)$ , but this is not possible as  $\alpha \in F_E(\overline{[v]})$ .  $\square$

**Notation 3.15.** For a graph  $E$  we denote by  $P_e$  the set of vertices in cycles with exits.

**Lema 3.16.** *Let  $E$  be an arbitrary graph. Then, for every  $a \in Z_0$  and  $v \in P \cup P_e$  there exists  $k_v \in K$  such that if  $u \in \overline{[v]}$  then  $uau = k_v u$ .*

*Proof.* Suppose first  $u = v$ . If  $v \in P_c \cup P_e$  the result follows by [23, Corollary 7]. If  $v \in P_l$ , by the proof of [8, Proposition 1.8]  $vL_K(E)v = Kv$ , hence  $av = va \in Kv$ .

Now, suppose  $u \in [v]$ . Since the relation  $\sim$  is given as the transitive closure of  $\sim^1$ , we may assume first that there exists a vertex  $w$  in a cycle, and paths  $\lambda, \mu$  satisfying  $\lambda = w\lambda u$  and  $\mu = w\mu v$ . By the paragraph before there exists an element  $k \in K$  such that  $waw = kw$ . Then  $uau = ua = \lambda^* \lambda a = \lambda^* a \lambda = \lambda^* (kw) \lambda = k \lambda^* \lambda = ku$  and analogously we show  $va = kv$ . Repeating this argument a finite number of steps we reach our claim.

It is easy to see that when  $u$  is in the hereditary closure of  $[v]$  we also have the result.

Finally, take  $u \in \overline{[v]}$ . We may assume that  $u$  is not a sink (this case has been studied yet). By Lemma 3.12 (i) and Lemma 2.5 we may write  $u = \sum_{i=1}^n \alpha_i \alpha_i^*$ , where the  $\alpha_i$ 's are paths and  $r(\alpha_i)$  is in the hereditary closure of  $[v]$ . Then  $au = a \sum_{i=1}^n \alpha_i \alpha_i^* = \sum_{i=1}^n \alpha_i a \alpha_i^* = \sum_{i=1}^n \alpha_i k r(\alpha_i) \alpha_i^* = k \sum_{i=1}^n \alpha_i \alpha_i^* = ku$ .  $\square$

**Proposition 3.17.** *Let  $E$  be a row-finite graph and consider a nonzero homogeneous element  $a$  of degree zero in  $Z(L_K(E))$ . Then:*

- (i)  $av = 0$  for every  $v \in P_\infty$ .
- (ii)  $a = \sum_{[v] \in X_f} k_{[v]} a_{[v]}$  where  $a_{[v]} = \sum_{u \in \overline{[v]}} u + \sum_{\alpha \in F_E(\overline{[v]})} \alpha \alpha^*$

**The zero component of the center of the LPA of a rfg**

**Theorem**

$$a = \sum_{[v] \in X_f} k_{[v]} a_{[v]}$$

$$a_{[v]} = \sum_{u \in \overline{[v]}} u + \sum_{\alpha \in F_E(\overline{[v]})} \alpha \alpha^*$$

Saturated closure of the elements in the class of  $v$ 
Paths arriving for the first time at  $[v]$

*Proof.* (i). Suppose first  $|\overline{[v]}| = \infty$ . If  $av \neq 0$ , by Lemma 3.16 we have  $au \neq 0$  for every  $u \in \overline{[v]}$ , which is an infinite set; this is not possible, so  $av = 0$  and we have finished. Now, assume  $|\overline{[v]}| < \infty$ . Since  $v \in P_\infty$ , necessarily  $|F_E(\overline{[v]})| = \infty$ , then we have an infinite collection of paths  $\{\alpha_n\}_{n \in \mathbb{N}} \subseteq F_E(\overline{[v]})$  with  $\alpha_m \neq \alpha_n$  for  $m \neq n$ . Next we prove that  $as(\alpha_n) \neq 0$  for each  $n$ . First, note that  $r(\alpha_n) \in \overline{[v]}$ ; since  $av \neq 0$  then  $ar(\alpha_n) \neq 0$  by Lemma 3.16, therefore  $0 \neq ar(\alpha_n) = a\alpha_n^* \alpha_n = \alpha_n^* as(\alpha_n) \alpha_n$  which implies our claim.

Since there is only a finite number of vertices  $u \in E^0$  such that  $au \neq 0$ , the set  $\{s(\alpha_n)\}_{n \in \mathbb{N}}$  must be finite. Moreover, the set  $\overline{[v]}$ , which contains all the

ranges of the paths  $\alpha_n$  is finite. Since  $|\{\alpha_n^0\}_{n \in \mathbb{N}}| < \infty$  (because every vertex in  $\alpha_n^0$  does not annihilate  $a$  by Lemma 3.16),  $v \notin P_f$  and no vertex in  $\alpha_n^0 \setminus r(\alpha_n)$  is an infinite emitter, we may conclude that there is a path  $\alpha_m = \gamma_1 \cdots \gamma_r \cdots \gamma_t$ , with  $\gamma_i \in \text{Path}(E)$  such that  $l(\gamma_r) \geq 1$  and  $s(\gamma_r) = r(\gamma_r)$ , hence  $\alpha_m$  contains a cycle based at  $s(\gamma_r)$ , implying  $s(\gamma_r) \in [v]$ . This is a contradiction with the fact  $\alpha_m \in F_E(\overline{[v]})$  and (i) has been proved.

(ii). Write  $a = \sum au$ , with  $u \in E^0$  and  $au \neq 0$ . If  $u \sim v$ , with  $v \in P \cup P_e$ , then  $au = k_v u$  by Lemma 3.16 and  $k_v \in K^\times$ . If  $u \not\sim v$  for any  $v \in P \cup P_e$  then, as it is not a sink, by (CK2), we may write  $u = \sum_{s(e)=u} ee^*$ . Then  $au = a \sum_{s(e)=u} ee^* = \sum_{s(e)=u} eae^*$ . Take  $e$  in this summand. If  $r(e) \in P \cup P_e$  then  $ar(e) = k_{r(e)} r(e)$  where  $k_{r(e)} \in K$  and we get a summand as in the statement. Otherwise we apply (CK2) to  $r(e)$  and write  $r(e) = \sum_{s(f)=r(e)} ff^*$ ; then  $aee^* = ae \sum_{s(f)=r(e)} ff^* e^*$ . Every summand  $aeff^*e^*$  with  $r(f) \in P \cup P_e$  is  $k_{r(ef)} e f f^* e^*$  by Lemma 3.16, which is a summand as in the statement. For every nonzero summand not being in this case we apply again (CK2). This process stops because otherwise we would have an infinite path  $e_1 e_2 \dots$  with  $e_i \in E^0$  such that  $s(e_i) \neq s(e_j)$  for every  $i \neq j$  and  $ae_i \neq 0$  for every  $i$ , which is not possible as the number of vertices not annihilating  $a$  has to be finite. Note that the path  $e_1 e_2 \dots e_n$  we arrive at is, by construction, an element in  $F_E(\overline{[v]})$ . We remark that  $v \in P_f$  by (i).

Take  $v \in E^0$  such that  $[v] \in X_f$  and  $av \neq 0$ ; by Lemma 3.16 we have  $av = k_v v$  for some  $k_v \in K$ . Note that  $k_v \neq 0$ . For any  $u \in \overline{[v]}$ , Lemma 3.16 shows that  $au = k_v u \neq 0$ . If  $\beta \in F_E(\overline{[v]})$ , then by Lemma 3.14  $\beta^* a = k_{r(\beta)} \beta^* \beta \beta^*$ . Since  $\beta^* a = a \beta^* = ar(\beta) \beta^* = k_v \beta^*$ , we get  $k_{r(\beta)} = k_v \neq 0$ . This shows that  $a$  can be written as a linear combination of elements of the form

$$(\dagger) \quad \sum_{u \in \overline{[v]}} u + \sum_{\alpha \in F_E(\overline{[v]})} \alpha \alpha^*,$$

where  $[v] \in X_f$ . □

**Theorem 3.18.** (MSM; Corrales García; Martín Barquero, Martín González; Solanilla Hernández)

Let  $E$  be a row-finite graph. For every class  $[v] \in X_f$ , denote by  $a_{[v]} = \sum_{u \in \overline{[v]}} u + \sum_{\alpha \in F_E(\overline{[v]})} \alpha \alpha^*$ . Then

$$\mathcal{B}_0 = \{a_{[v]} \mid [v] \in X_f\}$$

is a basis of the zero component of the center of  $L_K(E)$ .

*Proof.* By Proposition 3.17 every element in the zero component of the center is a linear combination of the elements of  $\mathcal{B}_0$ . Lemmas 3.12 and 3.14 imply that  $\mathcal{B}_0$  is a set of linearly independent elements.

In what follows we show that every  $a_{[v]}$  in  $\mathcal{B}_0$  is in the center of  $L_K(E)$ . To start, consider a vertex  $w$  in  $E^0$ . It is not difficult to see  $wa_{[v]} = a_{[v]}w$  since  $a_{[v]}$  is in  $\bigoplus_{i=1}^n u_i L_K(E) u_i$  for a certain finite family of vertices  $\{u_i\}_{i=1}^n$ .

Consider an edge  $e$  in  $E$  and denote by  $A := \overline{[v]} \cup F_E(\overline{[v]})$ . We claim:

$$(\dagger) \quad r(e) \in A \quad \text{if and only if} \quad s(e) \in A \quad \text{or} \quad e \in A.$$

Assume  $r(e) \in A$ . If  $s(e) \notin A$  then  $s(e) \notin \overline{[v]}$  and since  $r(e) \in \overline{[v]}$  we conclude that  $e \in F_E(\overline{[v]}) \subseteq A$ . Reciprocally, if  $s(e) \in A$  then  $r(e) \in A$  because  $\overline{[v]}$  is hereditary (Lemma 3.12 (i)). Finally, if  $e \in A$  then  $e \in F_E(\overline{[v]})$  which implies  $r(e) \in A$ .

Now we see that  $a_{[v]}$  commutes with  $e$ . Suppose first that  $r(e) \in A$ . Then  $ea_{[v]} = e$  by Lemma 3.14. On the other hand, if  $s(e) \in A$ , then  $a_{[v]}e = e + 0 = e$  by Lemma 3.14; if  $s(e) \notin A$ , then  $a_{[v]}e = 0 + e = e$  by Lemma 3.14. Now, suppose  $r(e) \notin A$ . Then, by  $(\dagger)$  we have  $s(e), e \notin A$ . For each  $\alpha$  we have  $e\alpha = 0$  or  $e\alpha \neq 0$  and, in this last case,  $e\alpha \in A$  since  $s(e) \notin A$ , by hypothesis, and  $r(e) = s(\alpha) \notin A$ .

Applying the results explained above,

$$ea_{[v]} = e \sum_{\alpha \in F_E(\overline{[v]})} \alpha\alpha^* = e \sum_{\alpha \in F_E(\overline{[v]}) \alpha^*e=0} \alpha\alpha^* + e \sum_{\alpha \in F_E(\overline{[v]})} e\alpha\alpha^*e^*.$$

We claim that the second summand must be zero. Indeed, suppose  $e^2\alpha \neq 0$ ; then,  $s(e) \sim w$  for a vertex  $w \in [v]$ ; this implies  $s(e) \in [v]$ , a contradiction since we assume  $s(e) \notin A$ .

Therefore

$$ea_{[v]} = \sum_{\alpha \in F_E(\overline{[v]}) \alpha^*e=0} e\alpha\alpha^* \tag{3.1}$$

In what follows we compute  $a_{[v]}e$ .

$$\begin{aligned} a_{[v]}e &= \left( \sum_{\alpha \in F_E(\overline{[v]}) \alpha^*e=0} \alpha\alpha^* \right) e + \left( \sum_{\alpha \in F_E(\overline{[v]})} e\alpha\alpha^*e^* \right) e = \sum_{\alpha \in F_E(\overline{[v]})} e\alpha\alpha^*e^*e \\ &= \sum_{\alpha \in F_E(\overline{[v]})} e\alpha\alpha^* = \sum_{\alpha \in F_E(\overline{[v]}) \alpha^*e=0} e\alpha\alpha^* + \sum_{\alpha \in F_E(\overline{[v]}) \alpha^*e \neq 0} e\alpha\alpha^*. \end{aligned} \tag{3.2}$$

We claim that the second summand is zero. The reason is again that for  $\alpha$  in the second summand,  $e\alpha$  must be zero because  $\alpha$  starts by  $e$  and

$\alpha \in F_E(\overline{[v]})$ . Hence

$$a_{[v]}e = \sum_{\alpha \in F_E(\overline{[v]})} e\alpha\alpha^* = ea_{[v]}$$

by (3.1).  $\square$

**Notation 3.19.** Let  $E$  be a graph. For a cycle  $c = e_1 \dots e_n$  in  $E$  and  $u_i = s(e_i)$  we will write  $c_{u_i}$  to denote the cycle  $e_i \dots e_{i-1}$ . Let  $\mathcal{S}$  be the set of all those cycles without exits  $c$  such that there is a finite number of paths ending at  $c$  and not containing  $c$ .

Note that if  $c$  is a cycle in  $\mathcal{S}$  and  $u, v \in c^0$  with  $u \neq v$ , then  $c_u$  and  $c_v$  will be different elements in  $\mathcal{S}$ .

**Proposition 3.20.** *Let  $E$  be an arbitrary graph and  $K$  any field. Consider a homogeneous element  $a$  in  $Z(L_K(E))$  with  $\deg(a) > 0$ . Then  $au = 0$  for all  $u \in P_l \cup P_e$ . If  $u \in P_c$  then  $au = k_u c_u^r$ , where  $k_u \in K$ ,  $c_u$  is a cycle without exits and  $r \in \mathbb{N}$ . Moreover, if  $u \in P_c^+$ , then  $k_u = 0$ .*

*Proof.* If  $u \in P_l$ , then  $au = uau \in uL_K(E)u = Ku$ , by [16, Proposition 4.7 and Remark 4.8], hence there exists  $\lambda \in K$  such that  $au = \lambda u$ . Then  $\lambda$  must be zero as the degree of  $\lambda u$  is zero and  $\deg(a) > 0$ .

Now, take  $u \in P_e$  and let  $d$  be a cycle such that  $u = s(d)$ .

We will use partially a reasoning that appears in [23, Theorem 6]; we include it here for the sake of completeness.

A generator system for  $uL_K(E)u$  is  $A \cup B$ , for

$$A = \{d^n(d^m)^* \mid n, m \geq 0\}$$

and

$$B = \{d^n\alpha\beta^*(d^m)^* \mid n, m \geq 0, \alpha, \beta \in \text{Path}(E), \\ s(\alpha) = u = s(\beta), d \not\prec \alpha, \beta, \alpha^1 \cup \beta^1 \not\prec d^1\}.$$

For  $n = 0$  we understand  $d^n = u$ . Note that given  $n, m \geq 0$  and  $d^n\alpha\beta^*(d^m)^* \in B$ , there exists a suitable  $r \in \mathbb{N}$  such that  $(d^r)^*d^n\alpha\beta^*(d^m)^*d^r = 0$ . This gives us that if we define the map  $S : L_K(E) \rightarrow L_K(E)$  by  $S(x) = d^*xd$ , for every  $b \in B$  there is an  $n \in \mathbb{N}$  satisfying  $S^n(b) = 0$ . Note that  $au$  is a fixed point for  $S$ . A consequence of this reasoning is that  $au \in \text{span}(A)$ . Write  $au = \sum_n k_n d^n(d^m)^*$ , for  $m = n - \deg(a)$ , where  $k_n \in K$ .

Then, for some  $l \in \mathbb{N}$  we have  $au = S^l(au) = \sum_n k_n d^{\deg(a)}$ . Since  $au$  commutes with every element in  $uL_K(E)u$ , the same should happen to  $d^{\deg(a)}$ , but this is not true as it does not commute with  $d^*$ , giving  $au = 0$ .

Next we show the second part of the statement. For  $u \in P_c$ ,  $au = uau \in uL_K(E)u \cong K[x, x^{-1}]$  (by [13, Proposition 2.3]). Since  $\deg(a) > 0$ ,  $au = k_u c_u^r$  for some  $r \in \mathbb{N} \setminus \{0\}$ ,  $c_u$  a cycle without exits and  $k_u \in K$ .

To finish, consider  $u \in P_c^+$  and write  $au = k_u c^r$ , for  $c$  a cycle without exits and  $k_u \in K$ . Then, two cases can happen. First, assume that there exists a cycle  $d$  such that  $v := s(d) \geq u$  and take a path  $\alpha$  satisfying  $\alpha = v\alpha u$ . Then, as we have proved before,  $av = 0$  and so  $k_u c^r = au = a\alpha^* \alpha = \alpha^* a\alpha = \alpha^* av\alpha = 0$ ; this implies  $k_u = 0$  as required. In the second case there exists infinitely many paths  $\gamma_n$ , for  $n \in \mathbb{N}$  ending at  $v$ . Since  $E$  is row-finite (and we are assuming that we are not in the first case, so there are no cycles involved),  $|\{s(\gamma_n)\}| = \infty$ . Then there exists  $m \in \mathbb{N}$  such that  $as(\gamma_m) = 0$  and so  $av = a\gamma_m^* \gamma_m = \gamma_m^* as(\gamma_m) \gamma_m = 0$  as desired.  $\square$

**Lema 3.21.** *Let  $E$  be a row-finite graph and  $K$  any field. For any homogeneous element  $a$  in  $Z(L_K(E))$  with  $\deg(a) > 0$  we have  $a = \sum a\alpha\alpha^*$ , where  $\alpha \in F_E(c^0)$  for some cycle without exits  $c$  and  $r(\alpha) \in P_c^-$ .*

*Proof.* Let  $u$  be in  $E^0$  such that  $au \neq 0$ . Define

$$T_1(u) = \{v \in T(u) \mid \exists e \in E^1, \text{ such that } s(e) = u, r(e) = v\}.$$

Let  $T_n(u)$  be

$$T_n(u) = \{v \in T(u) \setminus \left( \bigcup_{i < n} T_i(u) \right) \mid \exists \alpha \in \text{Path}(E), l(\alpha) = n, \\ \text{such that } s(\alpha) = u, r(\alpha) = v\}.$$

Given  $0 \neq au$ , write  $au = a \sum_{s^{-1}(u)=e} ee^* = \sum_{s^{-1}(u)=e} eae^*$  (note that  $u$  cannot be a sink and so we may use (CK2)). By Proposition 3.20, the possibly nonzero summands are those  $eae^*$  such that  $r(e) \in T_1(u) \setminus (P_l \cup P_e \cup P_c^+)$ . Let

$$S_1 := \{aee^* \text{ such that } r(e) \in P_c^-\}.$$

Now, to every element  $aee^*$  with  $r(e) \in T_1(u) \setminus (P_l \cup P_e \cup P_c)$ , apply Condition (CK2). Then  $aee^* = a(e \sum_{s^{-1}(r(e))=f} ff^*)e^* = (e \sum_{s^{-1}(r(e))=f} faf^*)e^*$ .

Again by Proposition 3.20 the nonzero summands are those  $efaf^*e^*$  such that  $r(f) \in T_2(u) \setminus (P_l \cup P_e \cup P_c^+)$ . Let

$$S_2 := \{aeff^*e^* \text{ such that } r(f) \in P_c^-\}.$$

This reasoning must stop in a finite number of steps, say  $m$ , because otherwise we would have infinitely many edges  $e_1, e_2, \dots$  such that  $ae_1 \dots e_n \neq 0$  for every  $n$ . Since for every vertex  $w$  in a cycle  $aw = 0$ , the ranges of these paths are all different. Therefore, there would be infinitely many vertices not annihilating  $a$ , a contradiction. Note that, by construction,  $a = \sum_{x \in \cup_{i=1}^m S_i} x$  and so  $a$  can be written as in the statement, where  $\alpha \in F_E(c^0)$  for some cycle without exits  $c$ .  $\square$

**Lema 3.22.** *Let  $E$  be an arbitrary graph and consider  $c_u, d_w \in \mathcal{S}$ ,  $\alpha \in F_E(c_u^0)$ ,  $\beta \in F_E(d_w^0)$  and  $n, m \in \mathbb{Z}$ .*

(i) *If  $w = u$ , then  $\alpha^* \beta \neq 0$  if and only if  $\alpha = \beta$ .*

(ii) *If  $w \neq u$  then  $c_u^m \alpha^* \beta d_w^n = 0$ .*

*Proof.* (i). If  $\alpha^* \beta \neq 0$  then  $\alpha = \beta\gamma$  or  $\beta = \alpha\delta$  for some  $\gamma, \delta \in \text{Path}(E)$ . In the first case, since  $r(\beta) \in c_u^0$ , by the very definition of  $F_E(c_u^0)$  and taking into account  $\alpha \in F_E(c_u^0)$  we get  $\alpha = \beta$ . In the second case we get analogously the same conclusion. The converse is obvious.

(ii). Take  $w \neq u$  and assume  $\alpha^* \beta \neq 0$ ; this implies as before  $\alpha = \beta\gamma$  or  $\beta = \alpha\delta$  for some  $\gamma, \delta \in \text{Path}(E)$ . In the first case,  $r(\beta) \in d_w^0$ ; since  $d_w$  has no exits, then  $\gamma^0 \subseteq d_w^0$ . On the other hand,  $r(\alpha) = r(\gamma) \in c_u^0 \cap d_w^0$ , therefore  $c_u^0 = d_w^0$  and arguing as in (i) we conclude  $\alpha = \beta$ . This implies  $c_u^m \alpha^* \beta d_w^n = c_u^m d_w^n = 0$  since  $w \neq u$ .

The other possibility yields the same conclusion.  $\square$

The following result will relate elements in the  $n$  component of the center to elements in the 0 component.

**Lema 3.23.** *Let  $E$  an arbitrary graph and  $K$  any field. Let  $a$  be a nonzero element in  $Z(L_K(E))$ . Then  $aa^* \neq 0$ .*

*Proof.* By Theorem 2.6, there are  $\alpha, \beta \in \text{Path}(E)$  such that  $0 \neq \alpha^* a \beta = kv$  for some  $k \in K^\times$ ,  $v \in E^0$ , or  $0 \neq \alpha^* a \beta = p(c, c^*)$ , where  $p(c, c^*)$  is a polynomial in a cycle without exits  $c$ .

In the first case  $0 \neq k^2 v = (kv)(kv^*) = \alpha^* a \beta \beta^* a^* \alpha = \alpha^* \beta \beta^* a a^* \alpha$ , since  $a$  is in the center of  $L_K(E)$ , and so  $aa^*$  must be nonzero.

In the second case,  $0 \neq p(c, c^*)p(c, c^*)^*$  since  $p(c, c^*) \in r(c)L_K(E)r(c)$  which is isomorphic to the Laurent polynomial algebra  $K[x, x^{-1}]$  (see [13, Proposition 2.3]), hence  $0 \neq \alpha^*a\beta\beta^*a^*\alpha = \alpha^*\beta\beta^*aa^*\alpha$ , and so  $aa^*$  must be nonzero.  $\square$

**Theorem 3.24.** (MSM; Corrales García; Martín Barquero, Martín González; Solanilla Hernández)

Let  $E$  be a row-finite graph. Then, the set

$$\mathcal{B}_n = \left\{ \sum_{\substack{m \cdot l(c) = n \\ \alpha \in F_E(c^0) \cup \{c^0\} \\ u \in c^0}} \alpha c_u^m \alpha^* \mid c \in \mathcal{S} \right\}$$

is a basis of  $Z_n(L_K(E))$  with  $n \in \mathbb{Z} \setminus \{0\}$ .

*Proof.* We will assume  $n > 0$ . The case  $n < 0$  follows by using the involution in the Leavitt path algebra.

Let  $a \in Z_n(L_K(E))$ . For every  $u \in E^0$  such that  $au \neq 0$ , Lemma 3.21 and Proposition 3.20 imply

$$au = a \sum \alpha \alpha^* = \sum \alpha ar(\alpha) \alpha^* = \sum \alpha k_\alpha c_\alpha^{m_\alpha} \alpha^* = \sum k_\alpha \alpha c_\alpha^{m_\alpha} \alpha^*$$

where  $\alpha \in F_E(c^0) \cup \{u\}$ ,  $r(\alpha) \in P_c^-$  and  $s(\alpha) = u$ .

Hence

$$a = \sum k_\alpha \alpha c_v^m \alpha^*, \quad \text{where the } c_v \text{'s are elements of } \mathcal{S}, \quad (3.3)$$

$$\alpha \in F_E(c^0) \cup \{v\} \text{ and } r(\alpha) \in P_c^-.$$

Now we see that  $k_\alpha = k_\beta$  for every  $\alpha, \beta \in F_E(c^0)$  appearing in the expression before.

We first note that  $\alpha c^r \beta^*$  is a nonzero element in  $L_K(E)$  for every  $r \in \mathbb{N}$ . Since  $a\alpha\beta^* = \alpha\beta^*a$ , using Lemma 3.22 we get  $k_\alpha \alpha c_u^m \beta^* = k_\beta \alpha c_u^m \beta^*$  and so  $k_\alpha = k_\beta$ .

In what follows we prove that if  $c_u$  appears in (3.3) then for every  $v \in c_u^0$  and for every  $\beta \in F_E([c_u^0])$  we have that  $c_v$  and  $\beta c_v \beta^*$  also appear.

Apply (3.3) and Lemmas 3.23 and 3.14 to get

$$0 \neq aa^* = \left( \sum k_\alpha \alpha c_v^m \alpha^* \right) \left( \sum k_\alpha \alpha c_v^{-m} \alpha^* \right) = \sum k_\alpha^2 \alpha v \alpha^*.$$

By Theorem 3.18,  $v$  must appear in this summand and also every  $\beta \in F_E([c_v^0])$ . This shows that the set  $\mathcal{B}_n$  generates the  $n$ -component of the center.

In what follows we see that the elements of  $\mathcal{B}_n$  are linearly independent. In fact, we show that elements of the form  $\alpha c_u^m \alpha^*$  are linearly independent. To this end, let  $k_\alpha \in K$  and suppose  $\sum k_\alpha \alpha c_u^m \alpha^* = 0$ , where all the summands are different. Choose a nonzero  $\beta c_v^{m_v} \beta^*$ . Then  $0 = \beta^* \sum k_\alpha \alpha c_u^m \alpha^* =$  (by Lemma 3.22)  $\beta^* k_\beta \beta c_v^{m_v} \beta^* = k_\beta c_v^{m_v} \beta^*$ ; this implies  $k_\beta = 0$  and our claim has been proved.

Finally we see that  $\mathcal{B}_n \subseteq Z(L_K(E))$ . Fix  $c_u \in \mathcal{S}$  and denote by  $A_n = \{\alpha c_u^m \alpha^* \mid m.l(c_u) = n, \alpha \in F_E([c_u^0]) \cup \{u\}\}$ , i.e.,  $A_n$  contains all the summands of  $a$  in (3.3). We see that for every  $\beta \in \text{Path}(E)$ ,  $A_n \beta = \beta A_n$  and  $A_n \beta^* = \beta^* A_n$  for all  $n \in \mathbb{Z}$ ; this will prove our statement.

So, take  $\alpha c_u^m \alpha^* \in A_n$ . Then  $\alpha c_u^m \alpha^* \beta$  is non zero if and only if  $\beta = \alpha$  (by Lemma 3.22); in this case,  $\alpha c_u^m \alpha^* \beta = \beta c_u^m \in \beta A_n$ . Suppose  $\alpha c_u^m \alpha^* \beta = 0$ ; if  $\beta \alpha c_u^m \alpha^* = 0$  then we have shown that  $0 \in \beta A_n$ ; otherwise  $\beta \alpha c_u^m \alpha^* \neq 0$ ; this implies  $r(\beta) = s(\alpha)$ . Now we distinguish two cases: first,  $\alpha = u$ . Note that  $c_u^m \beta = 0$  implies  $\beta \neq c_u \gamma$ , for any  $\gamma \in \text{Path}(E)$  and so  $\beta c_u^m \beta^* \in A_n$ . Then  $0 = \beta \beta c_u^m \beta^* \in \beta A_n$  (because  $r(\beta) = u$ ). If  $\alpha$  is not a vertex, then  $r(\beta) \neq u$  hence  $0 = \beta c_u^m \in \beta A_n$ . This shows  $A_n \beta \subseteq \beta A_n$ .

For the converse, note that  $\beta \alpha c_u^m \alpha^*$  is nonzero if and only if  $\beta \alpha$  is a nonzero element in  $F_E([c_u^0])$ , for  $w = s(\beta)$ ; in this case  $\beta \alpha c_u^m \alpha^* \beta^* \in A_n$  and so  $\beta \alpha c_u^m \alpha^* = \beta \alpha c_u^m \alpha^* \beta^* \beta \in A_n \beta$ . Now, suppose  $\beta \alpha c_u^m \alpha^* = 0$ . Then, multiplying on the right hand side by  $\alpha (c_u^m)^*$  and on the left hand side by  $\beta^*$  we get  $r(\beta) \alpha = 0$ . If  $c_u^m \beta = 0$  we have shown  $0 = c_u^m \beta \in A_n \beta$ . If  $c_u^m \beta \neq 0$ , as  $c_u$  has no exits, then  $\beta = c_u^s \lambda$ , where  $\lambda$  is such that  $c_u$  starts by  $\lambda$ . If  $l(c_u) = 1$ , then  $\beta = c_u^s$  for some  $s \in \mathbb{N}$ ; we see that this implies  $\alpha c_u^m \alpha^* \beta = 0$ . Suppose otherwise  $\alpha c_u^m \alpha^* \beta \neq 0$ . Then  $\alpha^* \beta \neq 0$  and so  $\alpha \geq \beta$  or  $\beta \geq \alpha$ . The first case is not possible (note that neither  $c_u^1 \subseteq \alpha^1$  nor  $\alpha = u$ ), hence  $\beta \geq \alpha$ , implying  $s(\alpha) = u$ . This happens only when  $\alpha = u$ , but this is not possible as  $0 = \beta \alpha c_u^m \alpha^* = c_u^{s+m}$  is a contradiction. Now, suppose  $l(c) > 1$ . Let  $e$  be the last edge in  $c_u$ . Then  $e c_u^m e^* \beta$ , which is an element in  $A_n \beta$ , is zero (because the first edge of  $\beta$  is the first one of  $c_u$ , which is not  $e$ ). This proves  $\beta A_n \subseteq A_n \beta$  and therefore  $A_n \beta = \beta A_n$ . Applying the involution we have  $\beta^* A_{-n} = A_{-n} \beta^*$  for all  $n \in \mathbb{Z}$ , as required.  $\square$

**Definitions 3.25.** Let  $E$  be an arbitrary graph. Denote by  $P_{b\infty}$  the set of all vertices in  $E$  such that  $T(v)$  has infinite bifurcations. Define

$$H_f := \bigsqcup_{[v] \in X_f} \overline{[v]} \quad \text{and} \quad H_\infty := \left( \bigsqcup_{[v] \in X_\infty} \overline{[v]} \right) \cup P_{b\infty}.$$

**Proposition 3.26.** *Let  $E$  be a row-finite graph and  $K$  be any field. Then*

$$L_K(E) = I(H_f) \oplus I(H_\infty).$$

*Proof.* We will show  $v \in I(H_f \cup H_\infty)$  for every  $v \in E^0$ . If  $v \in H_f \cup H_\infty$  we have finished. Suppose that this is not the case. In particular, this means  $s^{-1}(v) \neq \emptyset$ . Write  $v = \sum_{e \in s^{-1}(v)} ee^*$ . If for every  $e$  in this sum  $r(e) \in H_f \cup H_\infty$ , then we have finished. If for some  $e$ ,  $r(e)$  is a sink, we have finished; if  $r(e)$  is a vertex in a cycle, taking into account that every vertex of  $E^0$  connects to  $H_f \cup H_\infty$  then  $r(e) \sim u$  for some  $u \in H_f \cup H_\infty$ . Note that  $u \notin P_{b_\infty}$  as otherwise  $v \in P_{b_\infty}$ , and we are assuming  $v \notin H_f \cup H_\infty$ , therefore  $r(e) \in \cup_{w \in X} [w]$ . For those  $e \in s^{-1}(v)$  in the remaining cases we apply again Condition (CK2) to  $u_1 := r(e)$  and write  $u_1 = \sum_{f \in s^{-1}(u_1)} ff^*$ . Now we proceed in the same way concerning  $r(f)$ . This process must stop because otherwise there would be infinitely many bifurcations and so  $v \in P_{b_\infty}$ , a contradiction.

Apply Proposition 2.10 to the disjoint hereditary subsets  $H_f \cup H_\infty$  to get the result.  $\square$

**Theorem 3.27. (Structure Theorem for the center of a row-finite graph).** (MSM; Corrales García; Martín Barquero, Martín González; Solanilla Hernández)

*Let  $E$  be a row-finite graph. Then:*

$$Z(L_K(E)) \cong K^{|X \setminus X_f^c|} \oplus K[x, x^{-1}]^{|X_f^c|}$$

*More concretely,*

$$Z(L_K(E)) \cong K^{|X_f^l|} \oplus K^{|X_f^{gc}|} \oplus K[x, x^{-1}]^{|X_f^c|}.$$

**Structure of the center  
(row-finite graphs)**

**Theorem**

$$Z(L_K(E)) \cong K^{|X_f^l|} \oplus K^{|X_f^{ec}|} \oplus K[x, x^{-1}]^{|X_f^c|}$$

Classes in line points
Classes in extreme cycles
Classes in cycles without exits

*Proof.* By Proposition 3.26,  $L_K(E) = I(H_f) \oplus I(H_\infty)$ . Since  $H_f := \sqcup_{[v] \in X_f} \overline{[v]}$  and  $H_\infty := \left( \sqcup_{[v] \in X_\infty} \overline{[v]} \right) \cup P_{b_\infty}$ , Proposition 2.10 implies

$$L_K(E) = \left( \bigoplus_{[v] \in X_f} I(\overline{[v]}) \right) \oplus I(H_\infty),$$

hence

$$Z(L_K(E)) = \left( \bigoplus_{[v] \in X_f} Z(I(\overline{[v]})) \right) \oplus Z(I(H_\infty)).$$

By Theorems 3.18 and 3.24 we know  $Z(L_K(E)) \subseteq \bigoplus_{[v] \in X_f} Z(I(\overline{[v]}))$ ; therefore

$$Z(L_K(E)) = \bigoplus_{[v] \in X_f} Z(I(\overline{[v]})).$$

We claim that  $Z(I(\overline{[v]}))$  is isomorphic to  $K$  if  $v \in P_l \cup P_{ec}$  or isomorphic to  $K[x, x^{-1}]$  in case  $v \in P_c$ .

Indeed, take  $x \in Z(I(\overline{[v]}))$ . By the theorems of the basis of the center we may write  $x = \sum_w x_w$ , where  $[w] \in X_f$  and

$$x_w = k_{w,0} a_{[w]} + \sum_n k_{w,n} \left( \sum_{\substack{m \cdot l(c_w) = n \\ \alpha \in F_E(c_w^0) \cup \{c_w^0\} \\ u \in c_w^0}} \alpha c_u^m \alpha^* \right),$$

where  $a_{[w]}$  is as in Theorem 3.18,  $k_{w,0}, k_{w,n} \in K$  and  $k_{w,n}$  is zero for almost every  $n \in \mathbb{Z}$  and  $w \in P_c$ . Note that  $x_w \in I(\overline{[w]})$  and that  $\{I(\overline{[w]})\}_{[w] \in X_f}$  is an independent set (in the sense that its sum is direct). This means that, necessarily,  $x = x_v$ .

If  $v \in P_l \cup P_{ec}$ , then  $x = k_v^0 a_{[v]}$  and we have an isomorphism of  $K$ -algebras spaces between  $Z(I(\overline{[v]}))$  and  $K$ . If  $v \in P_c$  then the following map:

$$a_{[v]} \mapsto 1 \quad \text{and} \quad \sum_{\substack{m \cdot l(c_w) = n \\ \alpha \in F_E(c_w^0) \cup \{c_w^0\} \\ u \in c_w^0}} \alpha c_u^m \alpha^* \mapsto x^m$$

describes an isomorphism of  $K$ -algebras between  $Z(I(\overline{[v]}))$  and  $K[x, x^{-1}]$ . This finishes the proof.  $\square$

### 3.4 The center of an arbitrary Leavitt path algebra

**Theorem 3.28.** (MSM; Clark; Martín Barquero, Martín González)

Let  $E$  be an arbitrary graph and  $R$  a commutative unital ring. Then:

- (i) The zero component of the center of  $L_R(E)$  is a free  $R$ -module. One basis is given by:

$$\left\{ \sum_{v \in H} v + \sum_{\alpha \in F'_E(H)} \alpha \alpha^* + \sum_{\substack{\alpha \in \text{Path}(E) \\ r(\alpha) \in B_H}} (\alpha \alpha^* - \sum_{e \in F_\alpha} \alpha e e^* \alpha^*) \mid (H, B_H) \in \mathcal{T}_E^{cm} \right\}. \quad (3.4)$$

- (ii) When the  $n$ -component of the center of  $L_R(E)$  is nonzero it is a free  $R$ -module and one basis is given by:

$$\bigsqcup_{\substack{[c] \in [C_{ne}^f] \\ m|c| = n}} \left\{ \sum_{\substack{d \in [c] \\ \alpha \in F_E(c^0) \cup \{s(d)\} \\ r(\alpha) = s(d)}} \alpha d^m \alpha^* \right\}. \quad (3.5)$$

To finish, we explain the concepts involved in the theorem. In particular, compact subsets of  $G_E^{(0)}$  will play an crucial role.

Let  $H$  be a hereditary and saturated subset of  $E^0$ , and let  $v \in E^0$ . We recall that  $v$  is a *breaking vertex* of  $H$  if  $v$  belongs to the set

$$B_H := \{v \in E^0 \setminus H \mid v \in \text{Inf}(E) \text{ and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$

(See, for example, [2, Definitions 2.4.3]). In words,  $B_H$  consists of those vertices which are infinite emitters, which do not belong to  $H$ , and for which the ranges of the edges they emit are all, except for a finite (but nonzero) number, inside  $H$ . For  $\alpha \in \text{Path}(E)$  such that  $r(\alpha) \in B_H$  write

$$F_\alpha := \{e \in s^{-1}(r(\alpha)) \mid r(e) \notin H\} \quad \text{and} \quad F'_\alpha := \{e \in s^{-1}(r(\alpha)) \mid r(e) \in H\}.$$

Thus  $F_\alpha$  is a finite set,  $F'_\alpha$  is an infinite set and  $s^{-1}(r(\alpha)) = F_\alpha \cup F'_\alpha$ .

Define, as in [2]

$$\mathcal{T}_E := \{(H, S) \mid H \text{ is a hereditary and saturated set of vertices and } S \subseteq B_H\}.$$

Following [21, Definitions 3.3] for  $(H, S) \in \mathcal{T}_E$  we define

$$U_H := \{x \in G_E^{(0)} \mid r(x_n) \in H \text{ for some } n \geq 0\}, \quad (3.6)$$

$$U_S := \{\alpha \in \text{Path}(E) \mid r(\alpha) \in S\} \text{ and} \quad (3.7)$$

$$U_{H,S} := U_H \sqcup U_S. \quad (3.8)$$

(Note for a path  $x$ , we adopt the convention that  $r(x_0) := s(x)$ .) Then the map  $(H, S) \mapsto U_{H,S}$  from  $\mathcal{T}_E$  to the collection  $\mathcal{O}_E$  of open invariant subset of  $G_E^{(0)}$  is a bijection by [21, Theorem 3.3]. In the next lemma, we show how each  $U_{H,S}$  can be expressed as a disjoint union of basis elements. First, for a hereditary and saturated set of vertices  $H$ , define

$$F'_E(H) := \{\alpha \in F_E(H) \text{ such that } s(\alpha_{|\alpha|}) \notin B_H\}.$$

**Lema 3.29.** *Let  $E$  be an arbitrary graph and  $(H, S) \in \mathcal{T}_E$ . Then:*

$$U_{H,S} = \left( \bigsqcup_{v \in H} Z(v) \right) \sqcup \left( \bigsqcup_{\alpha \in F'_E(H)} Z(\alpha) \right) \sqcup \left( \bigsqcup_{\substack{\alpha \in \text{Path}(E) \\ r(\alpha) \in S}} Z(\alpha \setminus F_\alpha) \right).$$

*Proof.* Denote by  $A_1$ ,  $A_2$  and  $A_3$ , respectively, the three pairwise disjoint sets appearing in the last line of the statement of the lemma. First we show that  $U_{H,S} \subseteq A_1 \cup A_2 \cup A_3$ . Notice that  $U_S \subseteq A_3$ . Pick  $x \in U_H$ . Then there exists  $n$  such that  $r(x_n) \in H$ . If  $x$  is a sink or an infinite emitter, then it is

in  $H$  and we are done. If  $x$  is not a vertex, fix the smallest  $n \geq 0$  such that  $r(x_n) \in H$  and let

$$\alpha = \begin{cases} x_1 \dots x_n & \text{if } n > 0 \text{ and} \\ s(x) & \text{otherwise.} \end{cases}$$

Then  $\alpha \in F_E(H) \cup H$  and  $x = \alpha x_n x_{n+1} \dots \in Z(\alpha)$  or  $x \in Z(\alpha \setminus F_\alpha)$  and we are done. Thus we have  $U_{H,S} \subseteq A_1 \cup A_2 \cup A_3$ .

The reverse containment and that the sets involved are disjoint are both straightforward.  $\square$

**Definition 3.30.** Let  $H$  be a hereditary and saturated set of vertices in an arbitrary graph  $E$ . We say that  $H$  satisfies the *Finiteness Condition* (*Condition (F)* for short) if

$$|H \cup F'_E(H) \cup \{\alpha \in \text{Path}(E) \mid r(\alpha) \in B_H\}| < \infty.$$

**Proposition 3.31.** Let  $E$  be an arbitrary graph. Fix  $(H, S) \in \mathcal{T}_E$ . Then  $U_{H,S}$  is compact if and only if  $H$  satisfies Condition (F).

*Proof.* Suppose  $U_{H,S}$  is compact. By way of contradiction, suppose that

$$|H \cup F'_E(H) \cup \{\alpha \in \text{Path}(E) \mid r(\alpha) \in B_H\}| = \infty.$$

Since the sets involved in Lemma 3.29 provide a disjoint open cover of  $U_{H,S}$  with no finite subcover we get a contradiction to the fact that  $U_{H,S}$  is compact and therefore Condition (F) is satisfied.

For the reverse implication, suppose that  $H$  satisfies Condition (F). By [22, Lemma 2.1] we have that  $U_{H,S}$  is a union of a finite number of compact sets, thus  $U_{H,S}$  is compact.  $\square$

Where

Let  $\mathcal{T}_E^c$  denote the set of elements  $(H, S) \in \mathcal{T}_E$  such that  $S = B_H$  and  $U_{H,B_H}$  is compact (that is,  $H$  satisfies Condition (F) by Proposition 3.31). Further, let  $\mathcal{T}_E^{cm}$  denote the subset of  $\mathcal{T}_E^c$  consisting of those pairs  $(H, B_H)$  such that  $U_{H,B_H}$  is minimal as a compact set.

# Chapter 4

## Simplicity of some Lie algebras related to Leavitt path algebras

This last chapter is devoted to explain the advances in the subject concerning the Lie structure of some Lie algebras arising from Leavitt path algebras.

For any associative algebra  $A$  its *antisymmetrization* is the Lie algebra  $A^-$  defined as the same vector space and product given by

$$[a, b] = ab - ba,$$

for any  $a, b \in A$ .

It is well-known that  $A$  simple does not imply, necessarily,  $A^-$  simple. But, under some mild restrictions, the Lie algebra  $[A, A]/Z([A, A])$  turns out to be simple.

Simple Leavitt path algebras are characterized as those Leavitt path algebras whose associated graph is cofinal (there are no more hereditary and saturated sets than the total and the empty set) and satisfies Condition (L), i.e., every cycle has an exit. A natural question is to ask about the simplicity of the Lie algebra  $[L, L]/Z([L, L])$ , whenever  $L$  stands for the Leavitt path algebra  $L_K(E)$  of a graph  $E$  or, more generally, about the simplicity of the Lie algebra  $[M_d(L), M_d(L)]/Z([M_d(L), M_d(L)])$ , where  $M_d(L)$  is the algebra of matrices over  $L$ , for  $d \in \mathbb{N}$ .

In [6] the authors study the simplicity of such a Lie algebra whenever  $L$  is the algebra  $M_d(L_K(n))$ , for  $n, d \in \mathbb{N}$  and  $L_K(n)$  the Leavitt algebra of type  $(1, n)$ , which is known to be isomorphic to the Leavitt path algebra of the  $n$ -petal rose.

As a remark we recall the reader that it was proved in [1] the following result:

**Theorem 4.1.** ([1, Theorem 4.14]). (Abrams; Áhn; Pardo)

$$M_d(L_K(n)) \cong L_K(n) \text{ if and only if } \gcd(d, n-1) = 1.$$

On the other hand, it was also stated ([1, Proposition 2.1]) that if  $\gcd(d, n-1) > 1$  then the type of  $M_d(L_K(n))$  is

$$\left(1, \frac{n-1}{\gcd(d, n-1)} + 1\right),$$

hence for  $d, d' \in \mathbb{N}$  such that  $\gcd(d, n-1) \neq \gcd(d', n-1)$  the algebras  $M_d(L_K(n))$  and  $M_{d'}(L_K(n))$  are not isomorphic.

For a field  $K$  and a row-finite graph  $E$ , the authors of [7] study the elements of  $L_K(E)$  which are in the commutator  $[L_K(E), L_K(E)]$ , hence obtaining conditions under which this Lie algebra is simple, whenever the algebra  $L_K(E)$  is a simple associative algebra. Concretely,

**Lema 4.2.**

Let  $\alpha, \beta$  be paths.

- (i) If  $\alpha$  is not a closed path, then  $\alpha, \alpha^* \in [L_K(E), L_K(E)]$ .
- (ii) If  $\alpha \neq \beta\gamma$  and  $\beta \neq \alpha\gamma$  for any closed path  $\gamma$ , then  $\alpha\beta^*$  is in  $[L_K(E), L_K(E)]$ .

Why do they study elements in  $[L_K(E), L_K(E)]$ ? The reason is that the center of this Lie algebra is a Lie ideal and that for unital associative algebras (with some mild conditions), this center is nonzero if and only if 1 is an element of it, that is (see [7]):

**Theorem 4.3.** (Abrams; Mesyah)

Let  $E$  be a graph and  $K$  a field. Assume that  $L_K(E)$  is a simple algebra.

- (i) If  $E^0$  is infinite then the Lie algebra  $[L_K(E), L_K(E)]$  is simple;
- (ii) If  $E^0$  is finite, then  $[L_K(E), L_K(E)]$  is simple if and only if  $1 = \sum_{v \in E^0} v \notin [L_K(E), L_K(E)]$ .

There exist however no non-simple Leavitt path algebras  $L_K(E)$  such that the Lie algebra  $[L_K(E), L_K(E)]$  simple.

For the concrete results characterizing when  $[L_K(E), L_K(E)]$  is simple, for  $L_K(E)$  a unital purely infinite simple algebra, see [7, Theorem 36]. It will depend on the characteristic of the field and on the order of the unit inside  $K_0(L_K(E))$ . Concretely, the result says:

**Theorem 4.4.** (Abrams; Mesyah)

Let  $K$  be a field, let  $E$  be a finite graph for which  $L_K(E)$  is purely infinite simple, and let  $M = M_E$  denote the matrix  $I_m - A_E^t$  (for  $A_E$  the adjacency matrix of the graph  $E$ ).

- (i) Suppose that  $\text{char}(K) = 0$ . Then the Lie  $K$ -algebra  $[L_K(E), L_K(E)]$  is simple if and only if  $\bar{1}^m + \text{Im}(M_E^{\mathbb{Z}^m})$  has infinite order in  $\text{Coker}(M_E^{\mathbb{Z}^m})$ ; that is, if and only if  $[1_{L_K(E)}]$  has infinite order in  $K_0(L_K(E))$ .
- (ii) Suppose that  $\text{char}(K) = p \neq 0$ . Then the Lie  $K$ -algebra  $[L_K(E), L_K(E)]$  is simple if and only if  $\bar{1}^m + \text{Im}(M_E^{\mathbb{Z}^m})$  is not  $p$ -divisible in  $\text{Coker}(M_E^{\mathbb{Z}^m})$ ; that is, if and only if  $[1_{L_K(E)}]$  is not  $p$ -divisible in  $K_0(L_K(E))$ .

Subsequents results studying the simplicity of the Lie algebra  $[L_K(E), L_K(E)]$ , for any field  $K$  and any row-finite graph  $E$  are those contained in [9]. Concretely,

**Theorem 4.5.** (Alahmedi; Alsulami)

Let  $E$  be a row-finite graph. The Lie algebra  $[L_K(E), L_K(E)]$  is simple if and only if either  $L_K(E)$  is simple (case covered by Theorem 4.3) or  $E$  contains a simple subgraph  $F$  such that every vertex  $v \in E^0 \setminus F^0$  is a balloon over  $F$  and  $\sum_{w \in X} w \in [L_K(E), L_K(E)]$ , for  $X$  the set of all ranges of edges in  $E^1$  connecting  $v$  to a vertex in  $F$ .



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